ELEC 573 HW 3

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nlcg2

2.

a) $\int_0^\infty \frac{\alpha x_0^\alpha}{x^{\alpha+1}} dx = \alpha x_0^\alpha \int_{x_0}^\infty x^{-\alpha-1} dx = \alpha x_0^\alpha \left(\frac{1}{-\alpha} x^{-\alpha}\right)\Big|_{x_0}^\infty$

$= \frac{\alpha x_0^\alpha}{-\alpha}\left(\frac{1}{\infty^\alpha} - \frac{1}{x_0^\alpha}\right)$

$= \frac{\alpha x_0^\alpha}{-\alpha}\left(-\frac{1}{x_0^\alpha}\right) = -x_0^\alpha\left(-\frac{1}{x_0^\alpha}\right)$

$= \boxed{1}$ => valid PDF

$F(X \geq x) = \int_x^\infty \frac{\alpha x_0^\alpha}{a^{\alpha+1}} da$

$= \alpha x_0^\alpha \left(\frac{1}{-\alpha} a^{-\alpha}\right)\Big|_x^\infty$

$= \frac{\alpha x_0^\alpha}{-\alpha}\left(\frac{1}{\infty} - \frac{1}{x^\alpha}\right) = -x_0^\alpha\left(-\frac{1}{x^\alpha}\right) = \left(\frac{x_0}{x}\right)^\alpha$

$\boxed{F(X \geq x) = \left(\frac{x_0}{x}\right)^\alpha}$

c) $l_n(\alpha) = \log\left(\prod_{i=1}^n \frac{\alpha x_0^\alpha}{x_i^{\alpha+1}}\right) = \sum_{i=1}^n \log\left(\frac{\alpha x_0^\alpha}{x_i^{\alpha+1}}\right) = \sum_{i=1}^n \left(\log(\alpha x_0^\alpha) - \log(x_i^{\alpha+1})\right)$

$\boxed{l_n(\alpha) = \sum_{i=1}^n \left(\log\alpha + \alpha\log x_0 - (\alpha+1)\log x_i\right)}$

$\frac{dl_n(\alpha)}{d\alpha} = \sum_{i=1}^n \left(\frac{1}{\alpha} + \log x_0 - \log x_i\right)$

$0 = \frac{n}{\alpha} + n\log x_0 - \sum_{i=1}^n \log x_i$

$\frac{n}{\alpha} = \sum_{i=1}^n \log x_i - n\log x_0 \Rightarrow \boxed{\hat{\alpha} = \frac{n}{\sum_{i=1}^n \log x_i - n\log x_0}}$

3.

a) $cl(G) = \frac{3 \cdot \#\text{ triangles}}{\#\text{ connected triples}}$, $d_i = 2k \ \forall i$

- # triangles per node, clockwise: choose 2 of the node's clockwise nbrs within k hops

avoid double-counting

$= \binom{k}{2} = \frac{k!}{2!(k-2)!} = \frac{k(k-1)}{2}$

- # connected triplets per node: $\binom{2k}{2} = \frac{2k(2k-1)}{2}$

$\Rightarrow cl(G) = \frac{3nk(k-1)}{2nk(2k-1)} = \frac{3(k-1)}{2(2k-1)} = \boxed{\frac{3k-3}{4k-2}}$

b) • # triangles;

- still $\frac{nk(k-1)}{2}$ original

- new:

• can be made by connecting nodes between $k+1$ and $2k$ hops away; prob that they're connected is $\underbrace{\frac{1}{2}n\,2k\cdot p}_{|E|} \Big/ \underbrace{\frac{n(n-1)}{2}}_{\binom{n}{2}} \approx \frac{2kp}{n}$

→ however, as $n\to\infty$ this term goes to 0, so it's negligible.

$\Rightarrow \#\Delta \approx \frac{nk(k-1)}{2}$

• # triples;

- still $\frac{2nk(2k-1)}{2}$ original

- each new edge creates triples w/ all of the original $2k$ nodes @ either of its ends $(=(2k)^2$ triples)

• n possible, w/ prob. p

$\Rightarrow (2k)^2np$

- new edges also create pairs w/ other new edges:

- ℓ new edges $\rightarrow \binom{\ell}{2} = \frac{\ell(\ell-1)}{2}$ new triples

- expected # of new edges to a node is $\frac{2|E|p}{n}$ (original E) $= \frac{2\left(\frac{2nk}{2}\right)p}{n} = 2kp$

$\Rightarrow$ exp. new triples is $n\frac{2kp(2kp-1)}{2} \approx (2k)^2p^2$

$$\Rightarrow cl(G) = \frac{\frac{1}{2}nk(k-1)\cdot 3}{\frac{1}{2}2nk(2k-1)+(2k)^2np+\underbrace{\frac{1}{2}n(2k^2)p^2}_{\text{ignore}}} = \frac{\frac{3}{2}nk(k-1)}{nk(2k-1)+(2k)^2np} = \frac{\frac{3}{2}nk(k-1)}{2nk^2-nk+4k^2np}$$

$$= \frac{3k(k-1)}{2(2k^2-k+4k^2p)}$$

$$= \frac{3k-3}{2(2k-1+4kp)}$$

$$\boxed{= \frac{3k-3}{4k-2+8kp}}$$

4. $i > m$:

a) $\underline{k_i(i) = m}$ (each node connects to m others @ birth; node i is born @ time i)

b) $\boxed{\frac{dk_i(t)}{dt} \sim \frac{m}{t}}$ (a ratio of $\frac{m}{t}$ nodes have new connections added from time $t \to t+1$)

$k_i(t) = \int \frac{m}{t} dt = \underline{m \log t} + c \to k_i(i) = m = m \log i + c$

$\Rightarrow \underline{c = m - m \log i}$

$\Rightarrow k_i(t) = m \log t + m - m \log i$

$= \underline{m + m \log\left(\frac{t}{i}\right)}$ $\square$

c) $k_i(t) = m + m \log\left(\frac{t}{i}\right) \geq d$

$m \log\left(\frac{t}{i}\right) \geq d - m$

$\frac{t}{i} \geq e^{\frac{d-m}{m}}$

$i \leq \frac{t}{e^{\frac{d-m}{m}}} \to$ t nodes by time t, so $\boxed{F(d) = e^{\frac{(m-d)}{d}}}$

$p(d) = -\frac{dF(d)}{dd} = -\frac{d}{dd} e^{\frac{m-d}{d}} = -\frac{d}{dd} e^{\frac{m}{d}-1} = -(e^{\frac{m}{d}-1}) \cdot -md^{-2}$

$p(d) = \boxed{\frac{1}{md^2} \cdot e^{\frac{m-d}{d}}}$