MA3205

AY24/25 Sem 2

by ngmh

1. Sets and Operations

A1.1 Axiom of Extensionality $\forall x\,[x \in A \iff x \in B]$

D1.6 Subcollection $A \subseteq B$ if $\forall x\,[x \in A \Rightarrow x \in B]$

D1.7 Empty Set x is empty if $\forall y\,[y \notin x]$

F1.8 If $x = \emptyset$ and A is a collection then $x \subseteq A$

F1.9 If $x = \emptyset$ and $y = \emptyset$, $x = y$

D1.11 Basic Operations

1. $x \cup y = \{z : z \in x \vee z \in y\}$
2. $x \cap y = \{z : z \in x \wedge z \in y\}$
3. $x \setminus y = \{z : z \in x \wedge z \notin y\}$
4. $x \triangle y = (x \setminus y) \cup (y \setminus x)$
5. $\mathcal{P}(x) = \{z : z \subseteq x\}$

L1.12 Properties

1. $x \cup y = y \cup x$
2. $x \cap y = y \cap x$
3. $x \cup (y \cup z) = (x \cup y) \cup z$
4. $x \cap (y \cap z) = (x \cap y) \cap z$
5. $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$
6. $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$
7. $x \setminus (y \cup z) = (x \setminus y) \cap (x \setminus z)$
8. $x \setminus (y \cap z) = (x \setminus y) \cup (x \setminus z)$

D1.13 Union and Intersection

$\bigcup A = \{x : \exists y\,[y \in A \wedge x \in y]\}$

$$\bigcap A = \begin{cases} \emptyset & \text{if } A = \emptyset \\ \{x : \forall y\,[y \in A \Rightarrow x \in y]\} & \text{otherwise} \end{cases}$$

E1.16 Symmetric Difference

1. $(X \triangle Y) \triangle Z = X \triangle (Y \triangle Z)$
2. $X \triangle X = \emptyset$
3. $X \triangle Y = Y \triangle X$
4. $X \triangle \emptyset = X$

E1.18 $X \cap (Y \triangle Z) = (X \cap Y) \triangle (X \cap Z)$

2. Pairing, Products, and Relations

D2.1 Ordered Pair $\langle a, b\rangle = \{\{a\}, \{a, b\}\}$

L2.2 $\langle x, y\rangle = \langle a, b\rangle$ iff $x = a \wedge y = b$

D2.3 Cartesian Product $A \times B = \{z : \exists a \in A\, \exists b \in B\, [z = \langle a, b\rangle]\}$, $A^2 = A \times A$

E2.5 Define $pair(a, b) = \{a, \{a, b\}\}$. Assuming we cannot have $A \in B \in A$, $pair(a, b) = pair(x, y)$ iff $a = x \wedge b = y$

D2.6 Relation A relation is a collection of ordered pairs.

1. R is a relation if $\forall x \in R\, \exists a\, \exists b\, [x = \langle a, b\rangle]$
2. R is a relation on A if $R \subseteq A \times A$
3. $dom(R) = \{a : \exists b\, [\langle a, b\rangle \in R]\}$
4. $ran(R) = \{b : \exists a\, [\langle a, b\rangle \in R]\}$
5. $R^{-1} = \{x : \exists a\, \exists b\, [\langle a, b\rangle \in R \wedge x = \langle b, a\rangle]\}$

D2.8 Function A function is a relation where no two elements have the same first coordinate.

1. $\forall a, b, c\, [(\langle a, b\rangle \in f \wedge \langle a, c\rangle \in f) \Rightarrow b = c]$
2. $f : A \to B$ if f is a function, $dom(f) = A$ and $ran(f) \subseteq B$

F2.9 If R is a relation and $S \subseteq R$, then S is a relation. If f is a function and $g \subseteq f$, then g is a function.

D2.10 R restricted to A: $R \upharpoonright A = R \cap (A \times ran(R))$

F2.11 $f \upharpoonright A$ is a function. If $A \subseteq dom(f)$, then $dom(f \upharpoonright A) = A$

D2.12 Image of A under R: $Im_R(A) = \{b : \exists a \in A\, [\langle a, b\rangle \in R]\}$. If f is a function, for any $a \in dom(f)$ $f(a)$ denotes the unique b such that $\langle a, b\rangle \in f$

D2.14 $Im_{f^{-1}}(B) = \{a : \exists b \in B\, [\langle b, a\rangle \in f^{-1}]\} = \{a : \exists b \in B\, [\langle a, b\rangle \in f]\} = \{a : a \in dom(f) \wedge f(a) \in B\}$

L2.15 $Im_R(\bigcup A) = \bigcup\{I : \exists a \in A\, [I = Im_R(a)]\}$

L2.16 If for any x and z, if $x \neq z$ then $Im_R(\{x\}) \cap Im_R(\{z\}) = \emptyset$, then

1. $Im_R(\bigcap A) = \bigcap\{I : \exists a \in A\, [I = Im_R(a)]\}$
2. $Im_R(B \setminus A) = Im_R(B) \setminus Im_R(A)$

C2.17 For any function and sets,

1. $Im_{f^{-1}}(\bigcup A) = \bigcup\{I : \exists a \in A\, [I = Im_{f^{-1}}(a)]\}$
2. $Im_{f^{-1}}(\bigcap A) = \bigcap\{I : \exists a \in A\, [I = Im_{f^{-1}}(a)]\}$
3. $Im_{f^{-1}}(B \setminus A) = Im_{f^{-1}}(B) \setminus Im_{f^{-1}}(A)$

D2.18 f as composed with g:
$g \circ f = \{x : \exists a\, \exists b\, \exists c\, [\langle a, b\rangle \in f \wedge \langle b, c\rangle \in g \wedge x = \langle a, c\rangle]\}$

L2.19 If f, g, h are functions then

1. $g \circ f$ is a function
2. If $f : A \to B$ and $g : B \to C$, then $g \circ f : A \to C$
3. $h \circ (g \circ f) = (h \circ g) \circ f$

D2.20 Injection / Surjection / Bijection Consider $f : A \to B$

1. $1-1$ / Injection: $\forall a, a' \in A\, [f(a) = f(a') \Rightarrow a = a']$
2. Onto / surjection: $ran(f) = B$
3. Bijection: $1-1$ and Onto

D2.21 $X^Y = \{f : f \text{ is a function} \wedge f : Y \to X\}$

L2.22 If $f : A \to B$ is $1-1$ and onto, then $f^{-1} : B \to A$ is $1-1$ and onto.

E2.23 It is possible that $Im_f(a \cap b) \neq Im_f(a) \cap Im_f(b)$

E2.24 If f is $1-1$, $Im_f(\bigcap A) = \bigcap\{Im_f(a) : a \in A\}$ and $Im_f(B \setminus A) = Im_f(B) \setminus Im_f(A)$

2.27 The following are equivalent

1. $\forall x, z\, [x \neq z \Rightarrow Im_R(\{x\}) \cap Im_R(\{z\}) = \emptyset]$
2. R^{-1} is a function

CV2.28 Functions as sequences Suppose $dom(f) = I$. $f = \langle A_i : i \in I\rangle = \{x : \exists i \in I\, [x = \langle i, A_i\rangle]\}$. $\forall i \in I, f(i) = A_i$.

CV2.29

1. $Im_f(A) = \{f(a) : a \in A \cap dom(f)\}$. If $A \subseteq dom(f)$, then $Im_f(A) = \{f(a) : a \in A\}$
2. If $dom(f) = A \times B$, $f(\langle a, b\rangle) = f(a, b)$

CV2.30 Suppose F is a function, $x \in dom(F)$, and $F(x)$ is also a function. Then if $y \in dom(F(x))$, $F(x)(y) = (F(x))(y)$.

CV2.32 If $F = \langle A_i : i \in I\rangle$, then $\bigcup ran(F) = \bigcup_{i \in I} A_i$, similarly for $\bigcap ran(F)$

CV2.33 To specify a function f with domain I, it is enough to specify $f(i)$ for each $i \in I$. $f = \{z : \exists i \in I\, \exists x\, [z = \langle i, x\rangle \wedge x \text{ satisfies property } P \text{ w.r.t } i]\}$. If there is a unique object satisfying P for each i, then f is a function and $dom(f) = I$.

EP2.34 Define $F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$. For $f \in \mathbb{N}^{\mathbb{N}}$, we must specify $F(f) \in \mathbb{N}^{\mathbb{N}}$. We must specify $F(f)(n) \in \mathbb{N}$ for each $n \in \mathbb{N}$. For example, $F(f)(n) = f(n) + 1$. Then $F(f) = \{\langle n, f(n) + 1\rangle : n \in \mathbb{N}\}$ and $F = \{\langle f, \{\langle n, f(n) + 1\rangle : n \in \mathbb{N}\}\rangle : f \in \mathbb{N}^{\mathbb{N}}\}$. Similarly, define $\mathcal{F} : (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$. For $F \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$, we must specify $\mathcal{F}(F) \in \mathbb{N}^{\mathbb{N}}$ by specifying $\mathcal{F}(F)(n)$ for each n. Since F is a function with domain $\mathbb{N}$, $F(i)$ is defined for all $i \leq n$ and $F(i) \in \mathbb{N}^{\mathbb{N}}$. So $F(i)(n) \in \mathbb{N}$. Set $\mathcal{F}(F)(n) = max\{F(i)(n) : i \leq n\}$. $\mathcal{F}(F)$ eventually dominates $\{F(n) : n \in \mathbb{N}\}$.

EP2.35 Let I be a set, $\langle J_i : i \in I\rangle$ be a sequence of sets, and $\langle A_{i,j} : j \in J_i\rangle$ be a sequence of sets. Define $X = \{\bigcup_{j \in J_i} A_{i,j} : i \in I\}$. First define F with $dom(F) = I$. For each $i \in I$, $F(i) = \bigcup_{j \in J_i} A_{i,j}$. $X = Im_F(I) = ran(F)$. $\bigcap X = \bigcap_{i \in I} \bigcup_{j \in J_i} A_{i,j}$.

EP2.36 There is a biection $F : \mathcal{P}(X) \to \{0, 1\}^X$. We must specify $F(a)$ for each $a \in \mathcal{P}(X)$; a function with $dom(F(a)) = X$ and $ran(F(a)) \subseteq \{0, 1\}$. It is enough to specify $F(a)(x)$ for each $x \in X$.

$$F(a)(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \notin A \end{cases}$$ which is $1-1$ and onto.

D2.37 Cartesian Product Let F be a function with $dom(F)$ as a set. $\prod F = \{f : f \text{ is a function} \wedge dom(f) = dom(F) \wedge \forall x \in dom(F)\, [f(x) \in F(x)]\}$. If $F = \langle A_i : i \in I\rangle$, then
$\prod F = \prod_{i \in I} A_i = \{f : f \text{ is a function} \wedge dom(f) = I \wedge \forall i \in I\, [f(i) \in A_i]\}$

A2.38 Axiom of Choice If $\langle A_i : i \in I\rangle$ is a sequence of sets such that $\forall i \in I\, [A_i \neq \emptyset]$, then $\prod_{i \in I} A_i \neq \emptyset$

D2.40 Directed Collection G is directed if $\forall a, b \in G\, \exists c \in G\, [a \subseteq c \wedge b \subseteq c]$

L2.41 If G is a directed collection of functions, $f = \bigcup G$ is a function. $dom(f) = \bigcup\{dom(\sigma) : \sigma \in G\}$ and $ran(f) = \bigcup\{ran(\sigma) : \sigma \in G\}$

T2.47 Generalised De Morgan's (Requires **Axiom of Choice**) Let I be a set, and $\langle J_i : i \in I\rangle$ be a sequence of sets. Suppose $I \neq \emptyset$ and $\forall i \in I\, [J_i \neq \emptyset]$. For each $i \in I$, let $\langle A_{i,j} : j \in J_i\rangle$ be a sequence of sets.

1. $\bigcup_{i \in I} \bigcap_{j \in J_i} A_{i,j} = \bigcap\{\bigcup_{i \in I} A_{i,f(i)} : f \in \prod_{i \in I} J_i\}$
2. $\bigcap_{i \in I} \bigcup_{j \in J_i} A_{i,j} = \bigcup\{\bigcap_{i \in I} A_{i,f(i)} : f \in \prod_{i \in I} J_i\}$
3. $\prod_{i \in I} (\bigcup_{j \in J_i} A_{i,j}) = \bigcup\{\prod_{i \in I} A_{i,f(i)} : f \in \prod_{i \in I} J_i\}$
4. $\prod_{i \in I} (\bigcap_{j \in J_i} A_{i,j}) = \bigcap\{\prod_{i \in I} A_{i,f(i)} : f \in \prod_{i \in I} J_i\}$

T2.48 Fix $n \geq 1$, Let X be a set and $A_1, \ldots, A_n$ be subsets of X. There are at most 2^{2^n} sets that can be formed using $X \setminus .$, $\cup$, and $\cap$.

1. Redefine $\bigcap \emptyset = X$
2. Let $S = \{0, 1\}^{\{1, \ldots, n\}}$, then $|S| = 2^n$
3. For each $\sigma \in S$ define $b_\sigma = (\bigcap\{A_i : \sigma(i) = 0\}) \cap (\bigcap\{X \setminus A_i : \sigma(i) = 1\})$
4. For each $a \in \mathcal{P}(S)$ let $c_a = \bigcup\{b_\sigma : \sigma \in a\}$
5. Let $\mathcal{B} = \{c_a : a \in \mathcal{P}(S)\}$. $|\mathcal{B}| \leq |\mathcal{P}(S)| = 2^{2^n}$
6. **CL2.49** For each $1 \leq i \leq n$, $A_i \in \mathcal{B}$
7. **CL2.50** For any $a, b \in \mathcal{P}(S)$, $c_a \cup c_b = c_{(a,b)}$
8. **CL2.51** For any $a, b \in \mathcal{P}(S)$, $X \setminus c_a = c_{(S \setminus a)}$
9. **Claim 2.52** For any $a, b \in \mathcal{P}(S)$, $c_a \cap c_b = c_{(a \cap b)}$

E2.53 There exists $\langle A_n : n \in \mathbb{N}\rangle$ and $\langle B_n : n \in \mathbb{N}\rangle$ such that

1. $\forall n \in \mathbb{N}\, [B_n \subseteq A_n]$
2. $\forall n, m \in \mathbb{N}\, [n \neq m \Rightarrow B_n \cap B_m = \emptyset]$
3. $\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} B_n$

E2.55 If $I \neq \emptyset$ is a set and $\langle A_i : i \in I\rangle$ is a sequence of sets and X is a set then

1. $X \setminus (\bigcup_{i \in I} A_i) = \bigcap_{i \in I} (X \setminus A_i)$
2. $X \setminus (\bigcap_{i \in I} A_i) = \bigcup_{i \in I} (X \setminus A_i)$

3. Russell's Paradox and Proper Classes

T3.1 Russell $R = \{x : x \text{ is a set} \wedge x \notin x\}$ is not a set

Modified Morse-Kelley Rules

1. Everything is a class.
2. Every set is a class.
3. Every collection of sets is a class.
4. Axiom of Comprehension: If A is a class and x is a set, then $A \cap x$ is a set.
5. Axiom of Replacement: If F is a class which is a function and x a set, then $Im_F(x)$ is a set.
6. Axioms of Pairing / Union / Power-Set: If A and B are sets, then so are $\{A, B\}, \bigcup A, \mathcal{P}(A)$
7. Axiom of Choice
8. Axiom of infinity: $\mathbb{N}$ is a set
9. Axiom of Extensionality

T3.3 $\mathbf{V} = \{x : x \text{ is a set}\}$ is not a set, but a proper class

EP3.4 $A \times B$ is a set

E3.5 $dom(A), ran(A), \bigcap A, A^B$ are sets

E3.6 For I and $\langle A_i : i \in I\rangle$ which is a sequence of sets, $\prod_{i \in I} A_i$ is a set

E3.7 If R is a relation, $Im_R(A)$ is a set

E3.8 $\mathbf{U} = \{x : \exists a\, \exists b\, [x = \langle a, b\rangle]\}$ is not a set

E3.9 Let F be a class. If F is a function and $dom(F)$ is a set, F is a set

4. The Natural Numbers

F4.1 Peano Axioms

1. 0 is a natural number
2. If n is a natural number, there exists $S(n)$ which is also a natural number
3. If $n \neq m$, then $S(n) \neq S(m)$
4. $0 \neq S(n)$ for any natural number n
5. If X is a class of natural numbers where $0 \in X$ and $\forall n \in X\,[S(n) \in X]$, then $X = \mathbb{N}$

D4.2 $S(x) = x \cup \{x\}$

D4.3 0 is the empty set $\emptyset$

D4.4 A class is inductive if $0 \in A$ and $\forall x \in A\,[S(x) \in A]$. n is a natural number if it belongs to every inductive class.

F4.5 Axiom of infinity The class of all natural numbers $\mathbb{N} = \{n : n \text{ is a natural number}\}$ is a set.

L4.6 0 is a natural number, and if n is a natural number, then so is $S(n)$. $\mathbb{N}$ is an inductive class, and $\mathbb{N} \subseteq A$ for every inductive class A.

L4.7 If X is any set of natural number such that $0 \in X$ and $\forall n \in X\,[S(n) \in X]$, then $X = \mathbb{N}$

F4.8 Principle of Mathematical Induction Suppose P is a property, which 0 has, and $\forall n \in \mathbb{N}\,[n \text{ has property } P \Rightarrow S(n) \text{ has property } P]$. Then all natural numbers have property P.

L4.9 If n is a natural number then

1. $\forall x \in n\,[x \subseteq n]$
2. $n \subseteq \mathbb{N}$
3. $\forall x\,[(x \subseteq n \wedge x \neq \emptyset) \Rightarrow \exists m \in x\,[x \cap m = \emptyset]]$

L4.10 For natural numbers n, m, k

1. $n \notin n$
2. $m \subseteq n \Rightarrow (m \in n \vee m = n)$
3. $(m \subseteq n \wedge n \in k) \Rightarrow m \in k$
4. $m = n \vee m \in n \vee n \in m$

L4.11 For $X \subseteq \mathbb{N}$, if $X \neq \emptyset$, then $\exists n \in X\,[X \cap n = \emptyset]$

D4.12 We identify the relation $<$ on natural numbers with $\in$

F4.13 Principle of Strong Induction Suppose P is some property. Suppose that $\forall n \in \mathbb{N}\,[\text{if } P \text{ holds for all } m \in \mathbb{N} \text{ less than } n \text{ then } P \text{ holds for } n]$. Then P holds for all $n \in \mathbb{N}$.

L4.14 If $n, m \in \mathbb{N}$ and $n \neq m$, then $S(n) \neq S(m)$.

E4.15 For natural numbers n, m, k

1. $m \in n \in k$ implies $m \in k$
2. It is impossible to have $m \in n \in S(m)$
3. If $n \neq 0$ then $n = S(\bigcup n)$
4. $n \leq m$ iff $n \subseteq m$
5. $max\{n, m\} = n \cup m$
6. Either $n = 0$ or $\exists k \in n\,[S(k) = n]$

E4.16 If $X \subseteq \mathbb{N}$ and $\forall n \in X\,[n \subseteq X]$, then $X = \mathbb{N}$ or $\exists n \in \mathbb{N}\,[X = n]$

D4.17 Extenders Let $\mathbf{FN} = \{\sigma : \sigma \text{ is a function} \wedge \exists n \in \mathbb{N}\,[dom(\sigma) = n]\}$ be the proper class of all functions whose domain is some natural number. An extender is a function $\mathbf{E} : \mathbf{FN} \to \mathbf{V}$. When you input $\sigma = \{\langle 0, \sigma(0)\rangle, ..., \langle n, \sigma(n)\rangle\}$ into $\mathbf{E}$, $\mathbf{E}(\sigma)$ outputs the next value $\sigma(S(n))$.

T4.19 Suppose $\mathbf{E} : \mathbf{FN} \to \mathbf{V}$ is an extender. Then there exists a unique $f : \mathbb{N} \to \mathbf{V}$ satisfying $\forall n \in \mathbb{N}\,[f(n) = \mathbf{E}(f \upharpoonright n)]$.

CL4.21 For each $n \in \mathbb{N}$ there is an approximation to f with domain equal to n.

CL4.22 Let $\sigma, \tau \in \mathbf{FN}$ be approximations to f. Either $\sigma \subseteq \tau$ or $\tau \subseteq \sigma$.

EP4.23 Consider $\mathbf{E}(\sigma) = \bigcup \sigma(m)$. $f(0) = \mathbf{E}(f \upharpoonright 0) = \mathbf{E}(\emptyset) = X$ and $f(S(n)) = \mathbf{E}(f \upharpoonright S(n)) = \bigcup f(n)$. $f(0) = X, f(1) = \bigcup X, f(2) \bigcup\bigcup X$. This set $\bigcup ran(f)$ is the transitive closure of X, $trcl(X)$.

EP4.24 Consider $f(0) = \emptyset, f(S(n)) = \mathcal{P}(f(n))$. Set $\mathbf{E}(\sigma) = \emptyset, \mathbf{E}(\sigma) = \mathcal{P}(\sigma(m))$. This gives $V_0 = f(0) = \emptyset, V_{S(n)} = f(S(n)) = \mathcal{P}(f(n)) = \mathcal{P}(V_n)$. $V_\omega = \bigcup ran(f) = \bigcup\{V_n : n \in \mathbb{N}\}$.

D2.45 Addition and Multiplication

- Define $\langle f_m : m \in \mathbb{N}\rangle$ such that $f_m : \mathbb{N} \to \mathbb{N}$ is the unique function such that $f_m(0) = m$ and $\forall n \in \mathbb{N}\,[f_m(S(n)) = S(f_m(n))]$
- In other words, define the extender $\mathbf{E} : \mathbf{FN} \to \mathbf{V}$ as follows. For any $\sigma \in \mathbf{FN}$,

$$\mathbf{E}(\sigma) = \begin{cases} m & \text{if } dom(\sigma) = 0 \\ S(\sigma(\bigcup dom(\sigma))) & \text{if } dom(\sigma) \neq 0 \end{cases}$$

- $f_m : \mathbb{N} \to \mathbf{V}$ is the unique function satisfying $\forall n \in \mathbb{N}\,[f_m(n) = \mathbf{E}(f_m \upharpoonright n)]$.
- Then $m + n = f_m(n)$, and $m + S(n) = (m + n) + 1$.
- Define $\langle g_m : m \in \mathbb{N}\rangle$ such that $g_m : \mathbb{N} \to \mathbb{N}$ is the unique function such that $g_m(0) = 0$ and $\forall n \in \mathbb{N}\,[g_m(S(n)) = f_{g_m(n)}(m)]$
- In other words, define the extender $\mathbf{E} : \mathbf{FN} \to \mathbf{V}$ as follows. For any $\sigma \in \mathbf{FN}$,

$$\mathbf{E}(\sigma) = \begin{cases} 0 & \text{if } dom(\sigma) = 0 \\ f_{\sigma(\bigcup dom(\sigma))}(m) & \text{if } dom(\sigma) \neq 0 \text{ and } \sigma(\bigcup dom(\sigma)) \in \mathbb{N} \\ \emptyset & \text{if } dom(\sigma) \neq 0 \text{ and } \sigma(\bigcup dom(\sigma)) \notin \mathbb{N} \end{cases}$$

- $g_m : \mathbb{N} \to \mathbf{V}$ is the unique function satisfying $\forall n \in \mathbb{N}\,[g_m(n) = \mathbf{E}(g_m \upharpoonright n)]$.
- Then $m \cdot n = g_m(n)$, and $m \cdot S(n) = (m \cdot n) + m$.

E4.26 For $n, m, k \in \mathbb{N}$

1. $n + 1 = S(n)$
2. $n + (m + k) = (n + m) + k$
3. $n + m = m + n$
4. $n + n = 2 \cdot n$
5. If $2 \cdot n = 2 \cdot m$ then $n = m$
6. $n \cdot (m + k) = n \cdot m + n \cdot k$
7. $n \cdot (m \cdot k) = (n \cdot m) \cdot k$
8. $n \cdot m = m \cdot n$

E4.27 For $m, n, k \in \mathbb{N}$

1. If $n < k$ then $m + n < m + k$
2. If $m \neq 0$ and $n < k$ then $m \cdot n < m \cdot k$

E4.28 A transitive set satisfies $\forall x \in X\,[x \subseteq X]$.

1. For each $n \in \mathbb{N}$, V_n is transitive and V_ω is transitive
2. For each $n \in \mathbb{N}$, $n \subseteq V_n$ and $n \notin V_n$
3. $\mathbb{N} \subseteq V_\omega$ and $\mathbb{N} \notin V_\omega$

E4.29 $f \in \mathbb{N}^{\mathbb{N}}$ is increasing if $\forall n \in \mathbb{N}\,[f(n) \leq f(n+1)]$. f is unbunded if $\forall k \in \mathbb{N}\,\exists n \in \mathbb{N}\,[f(n) > k]$. Let $H : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a function. For each $m \in \mathbb{N}$, let h_m be the function in $\mathbb{N}^{\mathbb{N}}$ such that $\forall n \in \mathbb{N}\,[h_m(n) = H(m, n)]$ which is increasing and unbounded. There exists an increasing and unbounded function f such that $\forall m \in \mathbb{N}\,\exists l \in \mathbb{N}\,\forall n \in \mathbb{N}\,[n \geq l \Rightarrow f(n) < h_m(n)]$.

E4.30 Let X be a set, $0_X \in X$ be some element, and $S_X : X \to X$ be some function. Suppose

1. $\forall x \in X\,[S_x(x) \neq 0_X]$
2. $\forall x, y \in X\,[x \neq y \Rightarrow S_X(x) \neq S_X(y)]$
3. For every $A \subseteq X$, if $0_X \in A$ and $\forall x \in A\,[S_X(x) \in A]$, then $X = A$

Then $\langle \mathbb{N}, S, 0\rangle$ is isomorphic to $\langle X, S_X, 0_X\rangle$. There is a $1-1$ and onto function $F : \mathbb{N} \to X$ such that $F(0) = 0_X$ and $\forall n \in \mathbb{N}\,[F(S(n)) = S_x(F(n))]$.

E4.31 Define $A_0 = \{\emptyset\}$, $A_1 = \mathbb{N}$, for $n \geq 1$, $A_{S(n)} = A_n \times \mathbb{N}$. There is an extender $\mathbf{E} : \mathbf{FN} \to \mathbf{V}$, $\mathbf{E}(\sigma) = \sigma(\bigcup dom(\sigma)) \times \mathbb{N}$, where $dom(\sigma) = 0 \Rightarrow \mathbf{E}(\sigma) = \{\emptyset\}$, $dom(\sigma) = 1 \Rightarrow \mathbf{E}(\sigma) = \{\mathbb{N}\}$ that generates $\langle A_n : n \in \mathbb{N}\rangle$.

E4.32

1. X is transitive iff $\bigcup X \subseteq X$
2. $trcl(X)$ is the smallest transitive set containing X as a subset

5. Comparing Sizes of Sets

D5.1 Equinumerosity $A \approx B$ if there exists $f : A \to B$ which is both $1-1$ and onto.

F5.2 $\mathcal{P}(A) \approx \{0, 1\}^A$

D5.4 $A \lessapprox B$ means there exists $f : A \to B$ which is $1-1$ and B is at least as big as A. If $A \lessapprox B$ but $A \not\approx B$, then $A \lnapprox B$. It is not possible to find $g : A \to B$ that is both $1-1$ and onto. B is strictly bigger in size than A.

L5.5 If $f : A \to B$ and $g : B \to C$ are $1-1$ functions then $g \circ f : A \to C$ is $1-1$.

L5.6 For sets A, B, C

1. $A \lessapprox A$
2. If $A \lessapprox B$ and $B \lessapprox C$ then $A \lessapprox C$
3. If $A \approx B$ and $B \approx C$ then $A \approx C$

T5.7 Cantor For any set X, $X \lnapprox \mathcal{P}(X)$.

5.2 The Schröder Bernstein Theorem

T5.8 If $f : A \to B$ and $g : B \to A$ are both $1-1$ functions, then there exists $I \subseteq A$ and $J \subseteq B$ such that $f \upharpoonright I : I \to J$ is $1-1$ and onto, and $g \upharpoonright (B \setminus J) : B \setminus J \to A \setminus I$ is $1-1$ and onto.

CL5.9 For each $b \in B \setminus J, g(b) \in A \setminus I$.

CL5.10 For each $a \in A \setminus I$, there exists $b \in B \setminus J$ with $g(b) = a$.

T5.11 Schröder-Bernstein For any sets A and B, if $A \lessapprox B$ and $B \lessapprox A$, then $A \approx B$.

E5.12 Suppose $f : X \to Y$ is a $1-1$ function. For any $Z \subseteq X, Z \approx Im_f(Z)$.

E5.13 Suppose $I \subseteq A$ and $J \subseteq B$. If $I \approx J$ and $(A \setminus I) \approx (B \setminus J)$, then $A \approx B$.

E5.14 If $n \in \mathbb{N}$ and $A \approx S(n)$, then $\forall a \in A, (A \setminus \{a\} \approx n)$.

E5.15 If $n \in \mathbb{N}$ and $A \approx n$, then if $a \notin A$, $(A \cup \{a\}) \approx S(n)$.

E5.16 Let $n, m \in \mathbb{N}$. Then

1. If $f : n \to n$ is $1-1$, then f is onto. There is no $1-1$ function from $S(n)$ to n.
2. If $m \in n$, then $m \lnapprox n$.
3. If $x \subsetneq n$, then $x \lnapprox n$.
4. $n \lnapprox \mathbb{N}$
5. If $A \approx n$, $B \approx m$, and $A \cap B = \emptyset$, then $(A \cup B) \approx (n + m)$.

D5.19 A set is finite if there exits $n \in \mathbb{N}$ such that $n \approx A$. A is infinite if it is not finite. A is countable if $A \lessapprox \mathbb{N}$. A is uncountable if it is not countable.

L5.20 If $f : A \to B$ is a $1-1$ function, then for any $X, Y \subseteq A$, if $Im_f(X) = Im_f(Y)$, then $X = Y$.

L5.21 For sets A, B, C, D

1. If $A \lessapprox B$ then $\mathcal{P}(A) \lessapprox \mathcal{P}(B)$
2. If $A \lessapprox B$ then $A^C \lessapprox B^C$
3. If $A \lessapprox B, C \lessapprox D$, and $B \cap D = \emptyset$, then $A \cup C \lessapprox B \cup D$

L5.22 If $n \in \mathbb{N}$ and $A \lessapprox n$, then A is finite.

L5.23 If $n \in \mathbb{N}$ and there exists an onto function $\sigma : n \to A$, then $A \lessapprox n$

L5.24 If A and B are finite, then so is $A \cup B$.

T5.25 If A is a finite set and f is a function with $dom(f) = A$ then

1. If $X \subsetneq A$, then $X \lnapprox A$
2. $ran(f)$ is finite and $ran(f) \lessapprox A$
3. If $\forall a \in A\,[a \text{ is finite}]$ then $\cup A$ is finite
4. $\mathcal{P}(A)$ is finite

E5.26 If $A \subseteq \mathbb{N}$ is finite and nonempty, $max(A) = \bigcup A$

E5.27 If $A \lessapprox C$ and $B \lessapprox D$, then $A \times B \lessapprox C \times D$. If A and B are finite, $A \times B$ and A^B are finite.

E5.28 If I is a finite set and $\langle A_i : i \in I\rangle$ is a sequence of sets such that $\forall i \in I\,[A_i \text{ is finite}]$, then $\prod_{i \in I} A_i$ is finite.

E5.29 Suppose $\langle A_n : n \in \mathbb{N}\rangle$ is a sequence of infinite subsets of $\mathbb{N}$. There exists an infinite set $A \subseteq \mathbb{N}$ such that $\forall n \in N\,[A \cap A_n \text{ is infinite and } (\mathbb{N} \setminus A) \cap A_n \text{ is infinite}]$.

E5.30 For any function, $dom(f) \approx f$.