CS1231S

AY24/25 Sem 1

by ngmh

1. Speaking Mathematically

Important Sets

- $\mathbb{N}$: the set of all natural numbers (include 0, i.e. $\mathbb{Z}_{\geq 0}$)
- $\mathbb{Z}$: the set of all integers
- $\mathbb{Q}$: the set of all rational numbers
- $\mathbb{R}$: the set of all real numbers
- $\mathbb{C}$: the set of all complex numbers

Statements

- Universal Statement $\forall$: A certain property is true for **all** elements in a set
- Conditional statement $\rightarrow$: If one thing is true then some other thing has to be true
- Existential Statement $\exists$: There is **at least one** thing for which a certain property is true
- Universal Conditional Statement
- Universal Existential Statement
- Existential Universal Statement (not the same!)

Terms used in Proofs

- Definition: Precise and unambiguous description of mathematical term.
- Axiom / Postulate: A statement that is assumed to be true without proof.
- Theorem: A mathematical statement that is proved using rigorous mathematical reasoning.
- Lemma: A small theorem; a minor result which helps to prove a theorem.
- Corollary: A result that is a simple deduction from a theorem.
- Conjecture: A statement believed to be true, but for which there is no proof (yet).

Properties of Integers on Addition and Multiplication

- Closure: $x + y \in \mathbb{Z}$
- Commutativity: $x + y = y + x$
- Associativity: $x + y + z = (x + y) + z = x + (y + z)$
- Distributivity: $x(y + z) = xy + xz$
- Trichotomy: $x = y$ or $x < y$ or $x > y$

Number Definitions

- Even and Odd Integers (Lecture 1 Slide 27): An integer is even iff $\exists k s.t. n = 2k$. An integer is odd iff $\exists k s.t. n = 2k + 1$. Every integer is even or odd, but not both (Assumption 1).
- Without Loss Of Generality (WLOG): Used before an assumption in a proof which narrows the premise to some special case, and implies that the proof for that case can be easily applied to all other cases.
- Counter-Example: Shows that a statement is not always true.
- Divisibility (Lecture 1 Slide 32): If n and d are integers with $d \neq 0$, $d \mid n$ iff $\exists k \in \mathbb{Z}$ s.t. $n = dk$
- Theorem 4.7.1: $\sqrt{2}$ is irrational
- Rational and Irrational Numbers: r is rational iff $\exists a, b \in \mathbb{Z}$ s.t. $r = \frac{a}{b}, b \neq 0$
- Fraction in lowest term (Lecture 1 Slide 37): A fraction $\frac{a}{b}$ is said to be in lowest terms if the largest integer that divides both a and b is 1. Every rational can be reduced to a fraction in its lowest term. (Assumption 2)
- Proposition 4.6.4: For all integers n, if n^2 is even then n is even. (Proof by contraposition)
- Colorful (CS1231S): An integer n is colorful if $\exists k$ s.t. $n = 3k$

2. The Logic of Compound Statements

Definitions

- Defn 2.1.1 (Statement): A statement (or proposition) is a sentence that is true or false, but not both.
- Defn 2.1.2 (Negation): The negation of p is "not p" and is denoted $\sim p$.
- Defn 2.1.3 (Conjunction): The conjunction of p and q is "p and q", denoted $p \wedge q$.
- Defn 2.1.4 (Disjunction): The disjunction of p and q is "p or q", denoted $p \vee q$.
- Defn 2.1.5 (Statement Form / Propositional Form): A statement form (or propositional form) is an expression made up of statement variables and logical connectives.
- Defn 2.1.6 (Logical Equivalence): Two statement forms are logically equivalent iff they have identical truth values for each possible substitution of statements for their statement variables. The logical equivalence of P and Q is denoted by $P \equiv Q$.
- Defn 2.1.7 (Tautology): A tautology is a statement form that is always true regardless of the truth values of its statement variables.
- Defn 2.1.8 (Contradiction): A contradiction is a statement form that is always false regardless of the truth values of its statement variables.
- Defn 2.2.1 (Conditional): The conditional of q by p is "if p then q", denoted $p \rightarrow q$. p is the hypothesis (antecedent), and q is the conclusion (consequent).
- Defn 2.2.2 (Contrapositive): The contrapositive of a conditional statement $p \rightarrow q$ is $\sim q \rightarrow \sim p$
- Defn 2.2.3 (Converse): The converse of a conditional statement $p \rightarrow q$ is $q \rightarrow p$
- Defn 2.2.4 (Inverse): The inverse of a conditional statement $p \rightarrow q$ is $\sim p \rightarrow \sim q$
- Defn 2.2.5 (Only if): "p only if q" means $\sim q \rightarrow \sim p$ or "if p then q" $p \rightarrow q$
- Defn 2.2.6 (Biconditional): The biconditional of p and q is "p if and only if q" and is denoted "$p \leftrightarrow q$"
- Defn 2.2.7 (Necessary and Sufficient Conditions): "r is a sufficient condition for s" means $r \rightarrow s$, "r is a necessary condition for s" means $s \rightarrow r$
- Defn 2.3.1 (Argument): An argument is a sequence of statements. All statements except for the final one are called premises, while the final statement is called the conclusion. The symbol $\therefore$ is normally placed before the conclusion. An argument form is valid if the conclusion is true when all the premises are true.
- Defn 2.3.2 (Sound and Unsound Arguments): an argument is sound iff it is valid and all its premises are true. An argument that is not sound is unsound.

Theorem 2.1.1 Logical Equivalences

Implication Law

$$p \rightarrow q \equiv \sim p \vee q$$

Commutative Laws

$$p \wedge q \equiv q \wedge p$$

$$p \vee q \equiv q \vee p$$

Associative Laws

$$p \wedge q \wedge r \equiv (p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$

$$p \vee q \vee r \equiv (p \vee q) \vee r \equiv p \vee (q \vee r)$$

Distributive Laws

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

Identity Laws

$$p \wedge \textbf{true} \equiv p$$

$$p \vee \textbf{false} \equiv p$$

Negation Laws

$$p \vee \sim p \equiv \textbf{true}$$

$$p \wedge \sim p \equiv \textbf{false}$$

Double Negative Law

$$\sim(\sim p) \equiv p$$

Idempotent Laws

$$p \wedge p \equiv p$$

$$p \vee p \equiv p$$

Universal Bound Laws

$$p \vee \textbf{true} \equiv \textbf{true}$$

$$p \wedge \textbf{false} \equiv \textbf{false}$$

De Morgan's Laws

$$\sim(p \wedge q) \equiv \sim p \vee \sim q$$

$$\sim(p \vee q) \equiv \sim p \wedge \sim q$$

Absorption Laws

$$p \vee (p \wedge q) \equiv p$$

$$p \wedge (p \vee q) \equiv p$$

Negations of **true** and **false**

$$\sim\textbf{true} \equiv \textbf{false}$$

$$\sim\textbf{false} \equiv \textbf{true}$$

Argument Forms and Fallacies

- Modus Ponens:
  - If p then q
  - p
  - $\therefore q$
- Modus Tollens:
  - If p then q
  - $\sim q$
  - $\therefore \sim p$
- Generalization:
  - p
  - $p \vee q$
- Specialization:
  - $p \wedge q$
  - $\therefore p$
- Conjunction:
  - p
  - q
  - $\therefore p \wedge q$
- Elimination:
  - $p \vee q$
  - $\sim q$
  - $\therefore p$
- Transitivity:
  - $p \rightarrow q$
  - $q \rightarrow r$
  - $\therefore p \rightarrow r$
- Division into Cases
  - $p \vee q$
  - $p \rightarrow r$
  - $q \rightarrow r$
  - $\therefore r$
- Contradiction Rule
  - $\sim p \rightarrow false$
  - $\therefore p$
- Converse Error
  - $p \rightarrow q$
  - q
  - p
- Inverse Error
  - $p \rightarrow q$
  - $\sim p$
  - $\sim q$

Compund Statements Notes

- Order of Operations: $\sim, \wedge/\vee, \rightarrow$ / $\leftrightarrow$ (use parentheses if ambiguous)
- Show logical unequivalence by (i) finding a different row in truth table, (ii) finding a counter example.
- A conditional statement is vacuously true when the hypothesis is false.
- A conditional is logically equivalent with its contrapositive, which is the negation of its converse / inverse.
- If r is a sufficient condition for s, then r is sufficient to guarantee the occurrence of r.
- If r is a necessary condition for s, s cannot occur without r.
- To test an argument form for validity, construct the truth table. A critical row is a row in the truth table in which all premises are true. If there is a critical row in which the conclusion is false, the argument form is invalid. If there are no critical rows, the argument is vacuously valid.

3. The Logic of Quantified Statements

Definitions

- Defn 3.1.1 (Predicate): A predicate is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables. The domain of a predicate variable is the set of all values that may be substituted in place of the variable.
- Defn 3.1.2 (Truth Set): If $P(x)$ is a predicate and x has domain D, the truth set has all elements of D that make $P(x)$ true when substituted for x, denoted by $\{x \in D \mid P(x)\}$
- Defn 3.1.3 (Universal Statement): Let $Q(x)$ be a predicate and D the domain of x. A universal statement has the form $\forall x \in D, Q(x)$. A value for x for which $Q(x)$ is false is a counterexample.

- Defn 3.1.4 (Existential Statement): Let $Q(x)$ be a predicate and D the domain of x. An existential statement has the form $\exists x \in D$ s.t. $Q(x)$. The symbol $\exists!$ is used to denote uniqueness.
- Theorem 3.2.1 Negation of a Universal Statment: The negation of the statement $\forall x \in D, P(x)$ is equivalent to $\exists x \in D$ s.t. $\sim P(x)$
- Theorem 3.2.2 Negation of an Existential Statement: The negation of the statment $\exists x \in D$ s.t. $P(x)$ is equivalent to $\forall x \in D, \sim P(x)$
- Defn 3.2.1 (Contrapositive, Converse, Inverse): These terms can also be applied on universal conditional statements.
- Defn 3.2.2 (Necessary and Sufficient conditions, Only if): These terms can also be applied on universal conditional statements.
- Defn 3.4.1 (Valid Argument Form): An argument form is valid if no matter what predicates are substituted in its premises, if the premise statements are all true, then the conclusion is also true. An argument is valid iff its form is valid.

Arguments with Quantified Statements

- Universal Modus Ponens:
  - $\forall x(P(x) \to Q(x))$
  - $P(a)$ for a particular a
  - $\therefore Q(a)$
- Universal Modus Tollens
  - $\forall x, (P(x) \to Q(x)$
  - $\sim Q(a)$ for a particular a
  - $\therefore \sim P(a)$
- Universal Transitivity
  - $\forall x(P(x) \to Q(x)$
  - $\forall x(Q(x) \to R(x)$
  - $\therefore \forall x(P(x) \to R(x)$
- Universal Instantiation
  - $\forall x \in DP(x)$
  - $\therefore P(a)$ if $a \in D$
- Universal Generalisation
  - $P(a)$ for every $a \in D$
  - $\therefore \forall x \in DP(x)$
- Existential Instantiation
  - $\exists x \in DP(x)$
  - $\therefore P(a)$ for some $a \in D$
- Existential Generalisation
  - $P(a)$ for some $a \in D$
  - $\exists x \in DP(x)$
- Converse Error
  - $\forall x(P(x) \to Q(x))$
  - $Q(a)$ for a particular a
  - $\therefore P(a)$
- Inverse Error
  - $\forall x(P(x) \to Q(x))$
  - $\sim P(a)$ for a particular a
  - $\therefore \sim Q(a)$

Quantified Statements Notes

- To show a universal statement is true, we use exhaustion. To show it is false, we use a counterexample.
- To show an existential statement is true, we use an example. To show it is false, we use exhaustion.
- $\forall x \in D, Q(x) \equiv Q(x_1) \land Q(x_2) \land ... \land Q(x_n)$, and $\exists x \in D, Q(x) \equiv Q(x_1) \lor Q(x_2) \lor ... \lor Q(x_n)$
- Universal statements can be vacuously true if the predicate of a conditional is false for every element in the domain, which also happens if the domain is empty.
- In a statement with multiple quantifiers, the order of quantifiers with the **SAME** type can be interchanged.

4. Methods of Proof

Definitions

- Prime: An integer n is prime iff $n > 1$ and $\forall r, s$, if $n = rs$ then $r = n$ or $s = n$.
- Composite: An integer n is composite iff $n > 1$ and $\exists r, s$ s.t. $n = rs, 1 < r < n$, $1 < s < n$.
- Theorem 4.2.1: Every integer is a rational number. (Prove with $\frac{a}{1}$)
- Theorem 4.2.2: The sum of any two rational numbers is rational. (Proof by algebra)
- Corollary 4.2.3: The double of a rational number is rational.
- Theorem 4.3.1: $\forall a, b$, if $a \mid b$ then $a \leq b$ (Proof by algebra)
- Theorem 4.3.2: The only divisors of 1 are 1 and -1 (Proof by cases)
- Theorem 4.3.3: $\forall a, b, c$ if $a \mid b$ and $b \mid c$ then $a \mid c$
- Theorem 4.6.1: There is no greatest integer (Proof by contradiction)

Proofs

Methods of Proof

- Direct Proof: Show a series of steps leading from start to end. Deduction is a type of direct proof.
- Divison into Cases: Split the statement into cases and show it is true in all of them.
- Constructive Proof of Existential statements: Provide an example where the statement is true.
- Disproving Universal Statements by Counterexample: Give a counterexample where the negation is true, or where the antecedent is true and the consequent is false.
- Proving Universal Statements by Exhaustion: Show the statement is true for every element in the domain.
- Proving Universal Statements by Generalizing from the Generic Particular: Suppose x is a particular but arbitrarily chosen element of the set, show x satisfies the property.
- Proof by Contradiction:
  - Suppose the statement to be proved, S, is false. That is, the negation of the statement, $\sim S$, is true.
  - Show that this supposition leads logically to a contradiction.
  - Conclude that the statement S is true.
- Proof by Contraposition: Prove the contraposition of the statement instead.

5. Set Theory

Definitions

- Set: An **unordered** collection of objects (members / elements)
- Set-Roster Notation: Write all elements of the set between braces.
- Membership of a set: $x \in S$ means x is an element of S. Similarly, $x \notin S$ means x is not an element of S.
- Cardinality of a set: $\mid S \mid$ is the size of the set S, or the number of elements in S.
- Set-Builder Notation: Let U be a set and $P(x)$ be a predicate over U. Then the set of $x \in U$ s.t. $P(x)$ is true is $\{x \in U : P(x)\}$.
- Replacement Notation: Let A be a set and $t(x)$ be a term in a variable x. Then the set of all objects of the form $t(x)$ where $x \in A$ is $\{t(x) : x \in A\}$.
- Subset and Superset: Let A and B be sets. A is a subset of B, $A \subseteq B$, iff every element in A is also an element of B. We can also write $B \supseteq A$ or B is a superset of A.
- Proper Subset: A is a proper subset of B, $A \subsetneq B$ iff $A \subseteq B$ and $A \neq B$.
- Empty Set: $\emptyset$
- Theorem 6.2.4: An empty set is a subset of every set, $\forall A, \emptyset \subseteq A$
- Singleton: Set with exactly one element.
- Ordered Pair: An ordered pair has the form (x, y). Two Ordered pairs (a, b) and (c, d) are equal iff $a = c$ and $b = d$. This can also be extended to ordered n-tuples.
- Cartesian Product: The Cartesian product of sets A and B, $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$. This can also be extended to more than 2 sets.
- Set Equality: $A = B$ iff every element of A is B and every element of B is in A, or $A \subseteq B$ and $B \subseteq A$.
- Union: The union of A and B, $A \cup B$ is the set of all elements that are in at least one of A or B.
- Intersection: The intersection of A and B, $A \cap B$ is the set of all elements that are in both A and B.
- Difference / Relative Complement: The difference of $B - A$, $B \backslash A$ is the set of elements that are in B and not A.
- Complement: The complement of A, $\bar{A}$ is the set of elemnts in U that are not in A.
- Interval Notation: $(a, b) = \{x \in \mathbb{R} : a < x < b\}$, $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$. [] means closed intervals, while () means open intervals.
- Disjoint: Two sets are disjoint iff they have no elements in common, $A \cap B = \emptyset$.
- Mutually / Pairwise Disjoint / Nonoverlapping: Multiple sets are mutually disjoint iff $\forall A_i, A_j$, A_i and A_j are disjoint.
- Partition: Collection of mutually disjoint sets.
- Theorem 4.4.1 Quotient-Remainder Theorem: Given n, d, $\exists! q, r$ s.t. $n = dq + r$, $0 \leq r < d$
- Power Set: $\mathcal{P}(A)$ is the set of all subsets of A. By power set axiom, any element of $\mathcal{P}(A)$ is a set.
- Theorem 6.3.1 Cardinality of Power Set of a Finite Set: If $\mid A \mid = n$, $\mid \mathcal{P}(A) \mid = 2^n$.

Set Properties

- Inclusion of Intersection: $A \cap B \subseteq A$
- Inclusion in Union: $A \subseteq A \cup B$
- Transitive Property of Subsets: $A \subseteq B \land B \subseteq C \to A \subseteq C$.
- Procedural Definitions: Define using $\land, \lor$ instead
  - $a \in X \cup Y \leftrightarrow a \in X \lor a \in Y$
  - $a \in X \cap Y \leftrightarrow a \in X \land a \in Y$
  - $a \in X - Y \leftrightarrow a \in X \land a \notin Y$
  - $a \in \bar{X} \leftrightarrow a \notin X$
  - $(a, b) \in X \times Y \leftrightarrow a \in X \land b \in Y$

Theorem 6.2.2 Set Identities

Commutative Laws

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

Associative Laws

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

Distributive Laws:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Identity Laws

$$A \cup \emptyset = A$$

$$A \cap U = A$$

Complement Laws

$$A \cup \bar{A} = U$$

$$A \cap \bar{A} = \emptyset$$

Double Complement Law

$$\bar{\bar{A}} = A$$

Idempotent Laws

$$A \cup A = A$$

$$A \cap A = A$$

Universal Bound Laws

$$A \cup U = U$$

$$A \cap \emptyset = \emptyset$$

De Morgan's Laws

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

Absorption Laws

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

Complements of U and $\emptyset$

$$\bar{U} = \emptyset$$

$$\bar{\emptyset} = U$$

Set Difference Law

$$A \backslash B = A \cap \bar{B}$$

6. Relations

Definitions

- Relation: A binary relation from A to B is a subset of $A \times B$. Given an ordered pair (x, y) in $A \times B$, x is related to y by R, xRy, iff $(x, y) \in R$
- Domain, Co-Domain, Range: Let R be a relation from A to B. Domain is the set $\{a \in A : aRb$ for some $b \in B\}$. Co-Domain is B. Range is the set $\{b \in B : aRb$ for some $a \in A\}$

- Inverse Relation: The inverse relation R^{-1} is $\{(y,x) \in B \times A : (x,y) \in R\}$
- Relation on a Set: A relation on a set A is a subset of $A \times A$.
- Composite Relations: Let $R \subseteq A \times B$ and $S \subseteq B \times C$ be relations. The composition of R with S, denoted $S \circ R$ is the relation from A to C s.t. $\forall x \in A, \forall Z \in C, xS \circ Rz \leftrightarrow (\exists y \in B, xRy \wedge ySz)$.
- Proposition: Composition is Associative
- Proposition: Inverse of Composition: $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.
- n-ary Relation: Subset of A^n
- Reflexivity: R is reflexive iff $\forall x \in A, xRx$.
- Symmetry: R is symmetric iff $\forall x, y \in A, xRy \rightarrow yRx$
- Antisymmetry: R is antisymmetric iff $\forall x, y \in A, xRy \wedge yRx \rightarrow x = y$.
- Assymetry: R is asymmetric iff $\forall x, y \in A, xRy \rightarrow yRx$
- Transitivity: R is transitive iff $\forall x, y, z \in A, xRy \wedge yRz \rightarrow xRz$.
- Transitive Closure: The transitive closure of R on A is R^t on A s.t. (i) R^t is transitive, (ii) $R \subseteq R^t$, (iii) If S is another transitive relation containing R, $R^t \supseteq S$. Reflexive and Symmetric closure are similarly defined.
- Partition: $\mathscr{C}$ is a partition of A if (i) $\mathscr{C}$ is a set where all elements are non-empty subsets of A, (ii) Every element of A is in exactly one element of $\mathscr{C}$. Elements of a partition are components. Alternatively, $\forall x \in A, \exists! S \in \mathscr{C}(x \in S)$.
- Relation Induced by Partition: The relation R induced by a partition on A is $\forall x, y, \in A, xRy \leftrightarrow \exists$ a component S of $\mathscr{C}$ s.t. $x, y, \in S$.
- Theorem 8.3.1 Relation Induced by a Partition: R is reflexive, symmetric, and transitive.
- Equivalence Relation: R is an equivalence relation iff R is reflexive, symmetric and transitive. The symbol $\sim$ is commonly used.
- Equivalence Class: Suppose A is a set and $\sim$ is an equivalence relation on A. For each $a \in A$, the equivalence class of a, $[a]$, is the set of elements $x \in A$ s.t. $a \sim x$. $[a]_\sim = \{x \in A : a \sim x\}$
- Theorem 8.3.4: The partition Induced by an Equivalence Relation: If R is an equivalence relation on A, then the distinct equivalence classes of R form a partition of A.
- Congruence: a is congruent to b modulo n iff $a - b = nk$ for some $k \in \mathbb{Z}$.
- Congruence mod n is an equivalence relation on $\mathbb{Z}$ for every $n \in \mathbb{Z}^+$
- Set of equivalence classes: $A/\sim$ is the set of all equivalence classes with respect to $\sim$, $A/\sim = \{[x]_\sim : x \in A\}$.
- Partial Order Relation: R is a partial order iff R is reflexive, antisymmetric and transitive.
- Partially Ordered Set: A is a poset with respect to a partial order relation R on A, (A, R).
- Curly Less Than Equals: $\preccurlyeq$ is used for partial orders to prevent confusion with $\leq$.
- Hasse Diagrams: Simplification of the directed graph of a partial order relation. Place vertices s.t. all arrows point upwards, then remove all self-loops, remove arrows implied by transitivity, and remove direction indicators.
- Comparable: a and b are comparable iff $a \preccurlyeq b$ or $b \preccurlyeq a$. If not, they are noncomparable.
- Compatible: a and b are compatible iff $\exists c \in A, a \preccurlyeq c, b \preccurlyeq c$. (There are other possible definitions, be careful.)
- Maximal Element: c is maximal iff $\forall x \in A, c \preccurlyeq x \rightarrow c = x$.
- Minimal Element: c is minimal iff $\forall x \in A, x \preccurlyeq c \rightarrow c = x$.
- Largest Element: c is the largest iff $\forall x \in A, x \preccurlyeq c$.
- Smallest Element: c is the smallest iff $\forall x \in A, c \preccurlyeq x$.
- Total Order Relations: R is a total order relation if R is a partial order and $\forall x, y \in A, xRy \vee yRx$.
- Linearization: A total order s.t. $x \preccurlyeq y \rightarrow x \preccurlyeq^* y$.
- Well-Ordered Set: A is well-ordered iff every non-empty subset of A contains a smallest (not minimal) element.

Lemma Rel.1 Equivalence Classes

The following are equivalent:

- (i) $x \sim y$
- (ii) $[x] = [y]$
- (iii) $[x] \cap [y] \neq \emptyset$

Proof

- (i) $\rightarrow$ (ii)
  - Suppose $x \sim y$
  - $y \sim x$ by symmetry
  - For every $z \in [x]$
  - $x \sim z$ by definition of $[x]$
  - $y \sim z$ by transitivity
  - $z \in [y]$ by definition of $[y]$
  - $\therefore [x] \subseteq [y]$
  - Similarly for $[y] \subseteq [x]$
  - $\therefore [x] = [y]$
- (ii) $\rightarrow$ (iii)
  - Suppose $[x] = [y]$
  - $[x] \cap [y] = [x]$ by Idempotent Law
  - However, $x \sim x$ by reflexivity
  - This shows $x \in [x] = [x] \cap [y]$
  - $\therefore [x] \cap [y] \neq \emptyset$
- (iii) $\rightarrow$ (i)
  - Suppose $[x] \cap [y] \neq \emptyset$
  - Take $z \in [x] \cap [y]$
  - Then $z \in [x] and z \in [y]$ by definition of $\cap$
  - Then $x \sim z$ and $y \sim z$ by definition of $[x]$ and $[y]$
  - $y \sim z$ implies $z \sim y$ by symmetry
  - $\therefore x \sim y$ by transitivity

Theorem Rel.2 Equivalence classes form a partition

$A/\sim$ is a partition of A Proof Steps

- $A/\sim$ is a set by definition
- Show every element of $A/\sim$ is a nonempty subset of A.
  - Let $S \in A/\sim$
  - Find $x \in A$ s.t. $S = [x]$ by definition of $A/\sim$
  - Then $S = [x] \subseteq A$ by definition of equivalence classes
  - $x \sim x$ by reflexivity of $\sim$
  - Hence $x \in [x] = S$ by definition of $[x]$
  - $\therefore S$ is non-empty
- Show every element of A is in at least one element of $A/\sim$.
  - Let $x \in A$
  - $x \sim x$ by reflexivity of $\sim$
  - $\therefore x \in [x] \in A/\sim$
- Show every element of A is in at most one element of $A/\sim$.
  - Let $x \in A$ s.t. $x \in S_1, x \in S_2$
  - Find $y_1, y_2 \in A$ s.t. $S_1 = [y_1]$ and $S_2 = [y_2]$
  - $x \in [y_1] \cap [y_2]$
  - $[y_1] \cap [y_2] \neq \emptyset$
  - $\therefore S_1 = [y_1] = [y_2] = S_2$ by equivalence classes lemma

Relations Notes

- Relations can be represented with arrow diagrams or directed graphs.
- Composite relations can be found by walking on the directed graph.
- We can view partitions as a "in the same component" relation.
- Properties such as reflexivity and symmetry can be vacuously true. A set can be both symmetric and antisymmetric at the same time.
- Any smallest element is minimal, and any largest element is maximal.
- Kahn's Algorithm
  - Find a minimal element c of A
  - Remove c from A and add it to the linearization
  - Repeat until A is empty

7. Functions

Definitions

- Function: $\forall x \in X \; \exists! y \in Y \; (x,y) \in f$.
- Function Property 1 (Every input in Domain has an output): $\forall x \in X \; \exists y \in Y \; (x,y) \in f$.
- Function Property 2 (The output is unique): $\forall x \in X \; \forall y_1, y_2 \in Y \; (x,y_1) \in f \wedge (x,y_2) \in f \rightarrow y_1 = y_2$.
- Argument: Input x to the function in $f : x \mapsto y$.
- Image: Output y of the function, $f(x)$.
- Preimage: If $f(x) = y$, x is a preimage of y.
- Setwise Image: If $A \subseteq X$, then $f(A) = \{f(x) : x \in A\}$.
- Setwise Preimage: If $B \subseteq Y$, then $f^{-1}(B) = \{x \in X : f(x) \in B\}$.
- Domain, Co-Domain, Range: $f : A \rightarrow B$, A is domain, B, is co-domain, range is a subset of B such that $\{b \in B : b = f(a)$ for some $a \in A\}$.
- Sequence: $a_0, a_1, a_2, ...$ which can be represented by a function a with domain $\mathbb{Z}_{\geq 0}$ such that $a(n) = a_n$ for $n \in \mathbb{Z}_{\geq 0}$. A^∞ denotes the set of all finite sequences over A.
- Fibonacci Sequence: $F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n$.
- String or Word over A: $a_0, a_1, a_2, ...a_{l-1}$ where $l \in \mathbb{Z}_{\geq 0}$ and $a_0, a_1, a_2, ...a_{l-1} \in A$. l is the length of the string, and A is the alphabet. ϵ represents an empty string with $l = 0$. A^* denotes the set of all finite strings over A.
- Equality of Sequences: Two sequences are equal if $a(n) = b(n)$ for every $n \in \mathbb{Z}_{\geq 0}$.
- Equality of Strings: $s1 = s_2$ iff $a_i = b_i$ for all $i \in \{0, 1, 2, ...l-1\}$.
- Theorem 7.1.1 Function Equality: $f : A \rightarrow B$ and $g : C \rightarrow D$ are equal iff $A = C$, $B = D$, and $f(x) = g(x) \forall x \in A$.
- Injective (one-to-one): $\forall x_1, x_2 \in X, f(x_1) = f(x_2) \rightarrow x_1 = x_2$.
- Surjective (onto): $\forall y \in Y \; \exists x \in X \; y = f(x)$. $Range = Codomain$ for a surjective function.
- Bijective (one-to-one correspondence): f is bijective iff f is injective and surjective. $\forall y \in Y \; \exists! x \in X \; y = f(x)$.
- Inverse Function: g is an inverse of f iff $\forall x \in X \; \forall y \in Y \; y = f(x) \leftrightarrow x = g(y)$. The inverse of f is denoted as f^{-1}.
- Proposition: Uniqueness of inverses.
- Theorem 7.2.3: If f is a bijection, f^{-1} is also a bijection. f is a bijection iff f has an inverse.
- Composition of Functions: $(g \circ f)(x) = g(f(x)) \; \forall x \in X$. $g \circ f$ is the composition of f and g.
- Identity Function: $id_X(x) = x \; \forall x \in X$.
- Theorem 7.3.1 Composition with an Identity Function: $f \circ id_X = f, id_Y \circ f = f$.
- Theorem 7.3.2 Composition of a Function with Its Inverse: $f^{-1} \circ f = id_X$ and $f \circ f^{-1} = id_Y$.
- Associativity of Function Composition Theorem: $(h \circ g) \circ f = h \circ (g \circ f)$
- Theorem 7.3.3 Composition of Injections: If f and g are both injective, $g \circ f$ is also injective.
- Theorem 7.3.4 Composition of Surjections: If f and g are both surjective, $g \circ f$ is also surjective.
- $\mathbb{Z}_n$: The quotient $\mathbb{Z}/\sim_n$ is the congruence mod n relation on $\mathbb{Z}$.
- Addition and Multiplication on $\mathbb{Z}_n$: For $[x], [y] \in \mathbb{Z}_n$, $[x] + [y] = [x + y]$ and $[x] \cdot [y] = [x \cdot y]$.

- General Well-Defined Function Property: $\forall x_1, x_2 \in X$ $x_1 = x_2 \rightarrow f(x_1) = f(x_2)$.
- Well-Defined Property for Equivalence Relations: $\forall x_1, x_2 \in X$ $x_1 \sim x_2 \rightarrow f(x_1) \sim f(x_2)$.
- Well-Defined Property for Equivalence Classes: $\forall x_1, x_2 \in X$ $[x_1] = [x_2] \rightarrow [f(x_1)] = [f(x_2)]$.
- Order of Bijection: The smallest n such that $f \circ f \circ ... \circ f = id_A$, where there are n-many f's.
- Type Signature: Function : (Parameter 1 Domain, Parameter 2 Domain) $\rightarrow$ Output Co-Domain e.g. $+ : (\mathbb{Q}, \mathbb{Q}) \rightarrow \mathbb{Q}$

Addition on $\mathbb{Z}_n$ is well-defined

- Show that $[x_1] = [x_2]$ and $[y_1] = [y_2] \rightarrow [x_1] + [y_1] = [x_2] + [y_2]$.
- Let $[x_1], [y_1], [x_2], [y_2] \in \mathbb{Z}_n$ such that $[x_1] = [x_2]$ and $[y_1] = [y_2]$.
- Then $x_1 \equiv x_2 (mod n)$ and $y_1 \equiv y_2 (mod n)$ by congruence.
- Find $k, l \in \mathbb{Z}$ such that $x_1 - x_2 = nk$ and $y_1 - y_2 = nl$.
- Then $(x_1 + y_1) - (x_2 + y_2) = (x_1 - x_2) + (y_1 - y_2) = nk + nl = n(k + l)$.
- $[x_1 + y_1] = [x_2 + y_2]$ by congruence.
- $[x_1] + [y_1] = [x_1 + y_1] = [x_2 + y_2] = [x_2] + [y_2]$.

Multiplication on $\mathbb{Z}_n$ is well-defined

- Show that $[x_1] = [x_2]$ and $[y_1] = [y_2] \rightarrow [x_1] \cdot [y_1] = [x_2] \cdot [y_2]$.
- Let $[x_1], [y_1], [x_2], [y_2] \in \mathbb{Z}_n$ such that $[x_1] = [x_2]$ and $[y_1] = [y_2]$.
- Then $x_1 \equiv x_2 (mod n)$ and $y_1 \equiv y_2 (mod n)$ by congruence.
- Find $k, l \in \mathbb{Z}$ such that $x_1 - x_2 = nk$ and $y_1 - y_2 = nl$.
- Then $(x_1 \cdot y_1) - (x_2 \cdot y_2) = (nk + x_2) \cdot (nl + y_2) - (x_2 \cdot y_2) = n(nkl + ky_2 + lx_2)$.
- $[x_1 \cdot y_1] = [x_2 \cdot y_2]$ by congruence.
- $[x_1] \cdot [y_1] = [x_1 \cdot y_1] = [x_2 \cdot y_2] = [x_2] \cdot [y_2]$.

Functions Notes

- DO NOT CONFUSE FUNCTION AND FUNCTION INVERSE WITH SETWISE IMAGE AND PREIMAGE!!!
- Input to setwise image and preimage is a **set**, not an **element**

8. Mathematical Induction

Definitions

- Sequence: Ordered set with members called terms. Can have infinite terms.
- Summation: $\sum_{k=m}^{n} a_k = a_m + a_{m+1} + ... + a_n$. k is the index, m is the lower limit, and n is the upper limit.
- Product: $\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot ... \cdot a_n$.
- Theorem 5.1.1: Adding 2 summations is the same as the summation of adding both their terms. Multiplying a constant to all terms of a summation is equal to multiplying a constant to the result of the summation. Multiplying 2 products is the same as the product of multiplying both their terms. (Rewrite this if I'm not lazy)
- Arithmetic Sequence: A sequence where there is a constant d such that $a_k = a_{k-1} + d$. Then $a_n = a_0 + dn$.
- Geometric Sequence: A sequence where there is a constant r such that $a_k = ra_{k-1}$. Then $a_n = a_0 r^n$.
- Principle of Mathematical Induction: Let $P(n)$ be a property defined for integers and a be a fixed integer. Then if $P(a)$ is true and $\forall k \geq a, P(k) \rightarrow P(k+1)$ then $\forall n \geq a, P(n)$.
- Closed Form: If a sum with variable number of terms is equal to a formula that does not have $(...)$ or $\sum$, the formula without those is the closed form.
- Theorem 5.2.3 Sum of a Geometric Sequence: For $r \neq 1$, $\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}$.
- Proposition 5.3.1: $\forall n \geq 0, 2^{2n} - 1 \equiv 0$ (mod 3).
- Proposition 5.3.2: $\forall n \geq 3, 2n + 1 < 2^n$.
- Weak Mathematical Induction (1PI): Similar to PMI.
- Strong Mathematical Induction (2PI): Replace with $P(a) \wedge P(a+1) \wedge ... \wedge P(k) \rightarrow P(k+1)$ instead.
- Lecture 8 Slide 47: Any integer greater than 1 is divisible by a prime number.
- Lecture 8 Slide 48: Any amount $\geq$ \$12 can be formed by \$4 and \$5 coins.
- Well-Ordering Principle for Integers: Every nonempty subset of $\mathbb{Z}_{\geq 0}$ has a smallest element.
- Recurrence Relation: Formula that relates each term a_k to certain predecessors. Initial conditions specify the values of those elements without predecessors.
- Recursive Definition of a Set:
  - Base Clause: Specify certain elements in S (founders).
  - Recursion Clause: Specify certain functions under which S is closed (constructors).
  - Minimality Clause: Membership for S can be demonstrated by finitely many successsive applications of the above clauses.
- Structural Induction over Recursive Set:
  - Basis Step: Show $P(c)$ is true for every founder c.
  - Induction Step: Show that $\forall x \in S\ P(x) \rightarrow P(f(x))$ is true for every constructor f.

Theorem 5.2.2 Sum of the First n Integers

- Proof by Mathematical Induction
- Let $P(n) \equiv (1 + 2 + ... + n = \frac{n(n+1)}{2}), \forall n \in \mathbb{Z}^+$.
- Basis Step: $1 = \frac{1(1+1)}{2}$, therefore $P(1)$ is true.
- Assume $P(k)$ is true for some $k \geq 1$. That is, $1 + 2 + ... + k = \frac{k(k+1)}{2}$.
- Inductive Step:
  - $1 + 2 + ... + k + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)((k+1)+1)}{2}$
  - $P(k+1)$ is true.
- Therefore, $P(n)$ is true for $n \in \mathbb{Z}^+$.

Proof of Well-Ordering Principle for Non-Negative Integers

- Proof by Contradiction
- Suppose not, let $S \subseteq \mathbb{Z}_{\geq 0}$ be non-empty with no smallest element.
- For each $n \in \mathbb{Z}_{\geq 0}$ let $P(n)$ be $n \notin S$.
- Inductive Step:
  - Let $k \in \mathbb{Z}_{\geq 0}$ such that $P(0), P(1), ...P(k-1)$ are true.
  - If $k \in S$, then k is the smallest element of S, which is a contradiction.
  - So $k \notin S$, and $P(k)$ is true.
- $P(n)$ is true $\forall n \in \mathbb{Z}_{\geq 0}$.
- $S = \emptyset$ which is a contradiction.

9. Cardinality

Definitions

- Pigeonhole Principle: If there is an injection between 2 finite sets A and B, then $|A| \leq |B|$. Contrapositive: If m pigeons are put into n pigeonholes and $m > n$, then there must be at least one pigeonhole with at least 2 pigeons.
- Dual Pigeonhole Principle: If there is a surjection between 2 finite sets A and B, then $|A| \geq |B|$. Contrapositive: If m pigeons are put into n pigeonholes and $m < n$, then there must be at least one pigeonhole with no pigeons.
- Finite and Infinite Sets: S is finite iff S is empty, or there is a bijection from S to some $\mathbb{Z}_n$ where $\mathbb{Z}_n$ is the set of positive integers from 1 to n. S is infinite if it is not finite.
- Cardinality: The cardinality of a finite set S, $|S|$ is 0 if S is empty, or n if there is a bijection from S to $\mathbb{Z}_n$.
- Equality of Cardinality of Finite Sets Theorem: Finite A and B have the same cardinality iff there is a bijection from A to B.
- Theorem Cardinality.1: Subset of a Finite Set: If B is finite, $A \subseteq B$ is also finite.
- Same Cardinality (Cantor): Any A and B have the same cardinality iff there is a bijection from A to B.
- Theorem 7.4.1 Properties of Cardinality: Same-cardinality is an equivalence relation.
- Cardinal Numbers: Define $\aleph_0 = |\mathbb{Z}^+|$. This is the first transfinite cardinal number.
- Countably Infinite: S is countably infinite iff $|S| = \aleph_0$.
- Countable and Uncountable Sets: A set is countable iff it is finite or countably infinite. A set is uncountable if it is not countable.
- Theorem 9.2.5 Cartesian Product: The cartesian product of 2 countably infinite sets is also countably infinite.
- Corollary 9.2.5: The cartesian product of a finite number of countably infinite sets is also countably infinite.
- Theorem 9.2.5 Unions: The union of countably many countable sets is countable.
- Proposition 9.1: An infinite set B is countable iff there is a sequence $b_0, b_1, ... \in B$ where every element of B appears exactly once.
- Lemma 9.2 Countability via Sequence: An infinite set B is countable iff there is a sequence $b_0, b_1, ...$ where every element of B appears.
- Theorem 7.4.2 (Cantor): The set of real numbers between 0 and 1, $(0, 1)$ is uncountable.
- Theorem 7.4.3: Any subset of any countable set is countable.
- Corollary 7.4.4: Any set with an uncountable subset is uncountable.
- Proposition 9.3: Every infinite set has a countably infinite subset.
- Lemma 9.4 Union of Countably Infinite Sets: For countably infinite A and B, $A \cup B$ is countable.

Cantor's Diagonalisation Argument

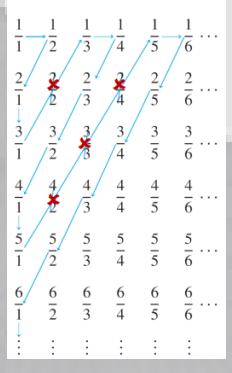
- Suppose $(0, 1)$ is countable.
- Since it is not finite, it is countably infinite.
- List the elments x_i of $(0, 1)$ in a sequence as follows, where a_{ij} is a digit:
  - $x_1 = 0.a_{11}a_{12}a_{13}...a_{1n}...$
  - $x_2 = 0.a_{21}a_{22}a_{23}...a_{2n}...$
  - $x_n = 0.a_{n1}a_{n2}a_{n3}...a_{nn}...$
- Now construct $d = 0.d_1d_2d_3...d_n...$ where $d_n \neq a_{nn}$.
- $d \in (0, 1)$, but $d \neq x_n$.
- We can always construct a new d no matter the value of n, so $(0, 1)$ is uncountable.

Cardinality Notes

- $\mathbb{Z}$ is countable: Start from middle then work outwards, alternating between negative and positive integers. This is the function

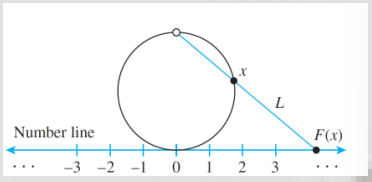
$$f(x) = \begin{cases} n/2 & even \\ -(n-1)/2 & odd \end{cases}$$

- $\mathbb{Q}^+$ is countable: Define a function to count all fractions in the following manner.



This function will be a bijection from $|\mathbb{Z}^+|$ to $|\mathbb{Q}^+|$.

- $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable: Similar to the previous proof, but it is (x, y) instead of $\frac{x}{y}$. The function $f(x, y) = \frac{(x+y-2)(x+y-1)}{2} + x$ can be used.
- Lemma 9.4 Proof: Use sequence argument, alternatively take an element from each set.
- Cardinality of $|\mathbb{R}| = |(0, 1)|$: Let $F(x)$ be the projection of x from the topmost point of the circle onto the number line.



If points on the circle represent $(0, 1)$, then $F(x)$ is a bijection from $(0, 1)$ to $\mathbb{R}$.

10./11. Counting and Probability I and II

Definitions

- Random Process: One outcome from a set of outcomes is guaranteed to occurr, but it is impossible to predict with certainty which outcome that will be.
- Sample Space: Set of all possible outcomes of a random process.
- Event: Subset of sample space.
- Equally Likely Probability: $P(E) = \frac{|E|}{|S|}$ where E is an event in a finite sample space S.
- Theorem 9.1.1 Number of Elements in a List: There are $n - m + 1$ integers from m to n inclusive, where $m \leq n$.
- Theorem 9.2.1 Multiplication / Product Rule: If an operation has k steps and each step i can be completed in n_i ways, the entire operation can be completed in $n_1 \cdot n_2 \cdot ... \cdot n_k$ ways.
- Theorem 9.2.2 Permutations: A set of n elements has $n!$ permutations.
- r-permutation: An r-permutation of a set with n elements is an ordered selection of r elements taken from the set, a.k.a nPr.
- Theorem 9.2.3 r-permutations from a set of n elements: $nPr = n(n-1)(n-2)...(n-r+1) = \frac{n!}{(n-r)!}$.
- Theorem 9.3.1 Addition Rule: A set A which is the union of k mutually disjoint subsets fulfills $|A| = |A_1| + |A_2| + ... + |A_k|$.
- Theorem 9.3.2 Difference Rule: If $B \subseteq A$, $|A \setminus B| = |A| - |B|$.
- Probability of Complement: $P(\bar{A}) = 1 - P(A)$.
- Theorem 9.3.3 Principle of Inclusion and Exclusion: $|A \cup B| = |A| + |B| - |A \cap B|$, $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$. This can be extended to any finite number of sets.
- Pigeonhole Principle: A function from one finite set to another smaller finite set cannot be one-to-one. There must be at least 2 elements in the domain that share the same image in the co-domain.
- Generalised Pigeonhole Principle: For a function f from a set X with n elements to a set Y with m elements, for $k < n/m$, there is some $y \in Y$ such that y is the image of at least $k + 1$ distinct $x \in X$.
- r-combination: An r-combination of a set with n elements is a subset of r of the n elements, a.k.a $\binom{n}{r}$ or nCr.
- Formula for $\binom{n}{r}$: $\binom{n}{r} = \frac{nPr}{r!} = \frac{n!}{r!(n-r)!}$. In other words, $nPr = nCr \cdot r!$.
- Theorem 9.5.2 Permutations with Sets of Indistinguishable Objects: Suppose there are n objects, with n_i of k different types i, and $n_1 + n_2 ... + n_k = n$. The number of distinguishable permutations is $\frac{n!}{n_1!n_2!...n_k!}$.
- Multiset: An r-combination with repetition allowed. We can use [] instead of $\{\}$ to represent a multiset.
- Theorem 9.6.1 Number of r-combinations with Repetition: The number of r-combinations with repetition from a set of n elements is $\binom{r+n-1}{r}$.
- Theorem 9.7.1 Pascal's Formula: $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$.
- Theorem 9.7.2 Binomial Theorem: $(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$.
- Probability Axioms:
  - $0 \leq P(A) \leq 1$
  - $P(\emptyset) = 0$ and $P(S) = 1$ where S is the sample space.
  - If A and B are disjoint events, $P(A \cup B) = P(A) + P(B)$.
- Probability of General Union of Two Events: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- Expected Value: The expected value of a process is $\sum_{k=1}^{n} a_k p_k$ where a_k is a real number and p_k is the probability of a_k happening.
- Linearity of Expectation: $E[X + Y] = E[X] + E[Y]$, regardless of whether X and Y are independent.
- Conditional Probability: The probability of B given A is $P(B|A) = \frac{P(A \cap B)}{P(A)}$.
- Theorem 9.9.1 Baye's Theorem: Suppose S is the union of mutually disjoint events $B_1, B_2, ..., B_n$. If A is an event in S, $P(B_k|A) = \frac{P(A|B_k) \cdot P(B_k)}{P(A|B_1) \cdot P(B_1) + P(A|B_2) \cdot P(B_2) + ... + P(A|B_n) \cdot P(B_n)}$
- Independent Events: A and B are independent iff $P(A \cap B) = P(A) \cdot P(B)$. Then $P(A|B) = P(A)$ and $P(B|A) = P(B)$.
- Pairwise Independent: Any 2 chosen events are independent.
- Mutually Independent: ALL events together are independent. $P(A_1 \cap A_2 \cap ... \cap A_n) = P(A_1) \cdot P(A_2) \cdot ... \cdot P(A_n)$.

Medical Testing

- False Positive: Patient does not have disease but tests positive.
- False Negative: Patient has disease but tests negative.
- True Positive Rate / Sensitivity: $P(TestPostive|Infected)$.
- True Negative Rate / Specificity: $P(TestNegative|NotInfected)$.
- $P(Disease|+) = \frac{P(+|Disease) \cdot P(Disease)}{P(+)}$.
- $P(+|\overline{Disease}) = \frac{P(\overline{Disease}|+) \cdot P(+)}{P(\overline{Disease})}$.

Stars and Bars

- The number of solutions to $x_1 + x_2 + .. + x_k = n$ where all x_i are positive is $\binom{n-1}{k-1}$. There are $n - 1$ gaps between stars, $k - 1$ of which need to contain bars.
- The number of solutions to $x_1 + x_2 + .. + x_k = n$ where all x_i are non-negative is $\binom{n+k-1}{k-1}$. There are $n + k - 1$ total stars and bars, out of which $k - 1$ have to be bars.

12. Graphs

Definitions

- Undirected Graph: An undirected graph G consists of 2 finite sets: A nonempty set V of vertices and a set E of edges where each edge is a set consisting of either 1 or 2 vertices as endpoints. An edge connects 2 endpoints. 2 vertices connected by an edge are adjacent, and a vertex with a loop is adjacent to itself. An edge can be represented as $e = \{v, w\}$.
- Directed Graph: Similar to undirected graph, but each edge has an ordered pair of endpoints. A directed edge can be represented as $e = (v, w)$.
- Four-Colour Theorem: Four colours are sufficient to colour any map in a plane such that regions with a common boundary do not share the same colour.
- Simple Graph: An undirected graph with no loops are parallel edges.
- Complete Graph: A complete graph on n vertices, K_n, is a simple graph with with exactly one edge connecting each pair of distinct vertices.
- Bipartite Graph: A simple graph whose vertices can be divided into two disjoin sets such that every edge connects 2 vertices from different sets.
- Complete Bipartite Graph: A bipartite graph between U and V such that every vertex in U connects to every vertex in V. If $|U| = m$ and $|V| = n$, it is known as $K_{m,n}$.
- Subgraph: H is a subgraph of G iff every vertex and edge in H is also in G.
- Degree: The degree of a vertex is the number of edges incident on it. An edge which is a loop is counted twice.
- Theorem 10.1.1 Handshake Theorem: The total degree of $G = 2 \cdot |E|$.
- Corollary 10.1.1: The total degree of a graph is even.
- Proposition 10.1.3: In any graph there are an even number of vertices of odd degree.
- Indegree and Outdegree of Directed Graph: The indegree of v, $deg^-(v)$ is the number of directed edges ending at v. The outdegree of v, $deg^+(v)$ is the number of directed edges starting at v. The total indegree must be the same as the total outdegree and the number of edges.
- Walk: An alternating sequence of adjacent vertices and edges of a graph. The number of edges is the length of the walk.
- Trivial Walk: A walk from a vertex to itself.
- Trail: A walk without a repeated edge.
- Path: A trail without repeated vertices.
- Closed Walk: Starts and ends at the same vertex.
- Circuit / Cycle: A closed walk of at least length 3 with no repeated edges.
- Simple Cycle: Cycle without any repeated vertex except first and last.
- Cyclic: Consists a loop or cycle, if not it is acyclic.
- Connectedness: 2 vertices are connected iff there is a walk between them. A graph is connected iff any 2 vertices are connected.
- Lemma 10.2.1: Let G be a graph.
  - If G is connected, any two distinct vertices of G can be connected by a path.
  - If two vertices are part of a cycle in G and one edge is removed from the cycle, there still exists a trail between them.
  - If G is connected and G contains a cycle, an edge of the cycle can be removed without disconnecting G.
- Connected Component: H is a connected component of G iff H is a subgraph of G, H is connected, and there is no connected subgraph of G with H as a subgraph that $\neq H$.
- Euler Circuit: An Euler circuit is a circuit that contains every vertex and traverses every edge exactly once.
- Eulerian Graph: A graph with an Euler circuit.
- Theorem 10.2.2: If an graph has an Euler circuit, then every vertex of the graph has a positive even degree.
- Contrapositive of Theorem 10.2.2: If a vertex in a graph has odd degree, the graph does not have an Euler circuit.
- Theorem 10.2.3: If a graph is connected and every vertex has a positive even degree, the graph has an Euler circuit.
- Theorem 10.2.4: A graph has an Euler circuit iff the graph is connected and every vertex has a positive even degree.
- Euler Trail: An Euler trail/path is a sequence of edges and vertices between 2 nodes which passes through every vertex at least once, and every edge exactly once.
- Corollary 10.2.5: There is an Euler trail between 2 vertices iff the graph is connected, the start and endpoints have odd degreee while all other vertices have a positive even degree.
- Hamiltonian Circuit: A simple circuit that includes every vertex of a graph. Every vertex appears exactly once except for the start and endpoint which are the same.
- Hamiltonian Graph: A graph with a Hamiltonian circuit.
- Proposition 10.2.6: If a graph G has a Hamiltonian circuit, then G has a subgraph H where
  - H contains every vertex of G

  - H is connected
  - H has the same number of edges as vertices
  - Every vertex of H has degree 2.
- Matrix: An m by n matrix has m rows and n columns. a_{ij} is the element in the ith row and jth column.
- Adjacency Matrix: The n by n matrix where a_{ij} is the number of edges from v_i to v_j.
- Symmetric Matrix: $a_{ij} = a_{ji}$.
- Scalar Product: $\sum_{k=1}^{n} a_{ik}b_{kj}$.
- Matrix Multiplication: If A is an m by k matrix and B is an k by n matrix, AB is the matrix where $c_{ij} = \sum_{r=1}^{k} a_{ir}b_{rj}\ \forall i \leq m$ and $\forall j \leq n$.
- Identity Matrix: A square matrix where all entries are 0 except those on the main diagonal which are 1.
- nth Power of a Matrix: $A^0 = I$ where I is the identity matrix, $A^n = AA^{n-1}$.
- Theorem 10.3.2: The ijth entry of $A^n =$ number of walks of length n from v_i to v_j.
- Isomorphic Graph: 2 graphs are isomorphic iff there exists bijections between their vertices and edges which preserve the edge-endpoint functions. For simple graphs, 2 graphs are isomorphic iff there exists a permutation of vertices which preserves edges.
- Theorem 10.4.1 Graph Isomorphism is an Equivalence Relation.
- Planar Graph: A graph that can be drawn on a two dimensional plane without edges crossing.
- Kuratowski's Theorem: A finite graph is planar iff it does not contain a subgraph that is a subdivision of the complete graph K_5 or the complete bipartite graph $K_{3,3}$.
- Euler's Formula: A connected planar graph has $f = e - v + 2$ faces.

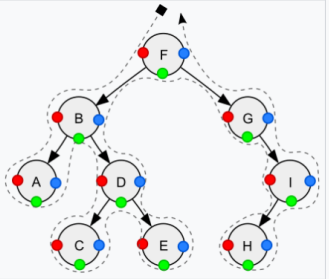
Graph Notes

- An Euler circuit may visit some vertices more than once so it might not be a Hamiltonian circuit.
- A Hamiltonian circuit might not include all edges and hence may not be an Euler circuit.
- For an undirected graph, the adjacency matrix is symmetric.
- For matrix multiplication, c_{ij} is the scalar product of the ith row of A and the jth column of B.
- Matrix multiplication is NOT commutative!

13. Trees

Definitions

- Tree: A tree is a simple graph iff it is simple, acyclic, and connected.
- Trivial Tree: Tree with one vertex.
- Forest: A graph is a forest iff it is simple, acyclic, and not connected.
- Lemma 10.5.1: Any non-trivial tree has at least one vertex f degree 1.
- Terminal Vertex (Leaf) and Internal Vertex: A vertex in a tree with degree 1 is a leaf, a vertex with degree more than 1 is an internal vertex.
- Theorem 10.5.2: Any tree with n vertices has $n - 1$ edges.
- Lemma 10.5.3: If a graph is connected, removing an edge from a cycle in the graph keeps the graph still connnected.
- Theorem 10.5.4: If a connected graph has n vertices and $n - 1$ edges, the graph is a tree.
- Rooted Tree: A rooted tree has a specific vertex designated as the root.
- Level: The level of a vertex is the number of edges along the unique path between it and the root.
- Height: The height of a rooted tree is the maximum level of any vertex of the tree.
- Child: The children of a vertex are adjacent to it and are one level farther from the root.
- Parent: If w is a child of v, v is the parent of w.
- Siblings: Two vertices with the same parent are known as siblings.
- Ancestor and Descendant: If v lies on the unique path between w and the root, v is an ancestor of w, and w is a descendant of v.
- Binary Tree: A binary tree is a rooted tree where every parent has at most two children. Each child is either a left or right child.
- Full Binary Tree: A binary tree where each parent has exactly two children.
- Left / Right Subtree: The left subtree of a vertex is the binary tree whose root is the left child of the vertex, with vertices which are descendants of the left child and edges connecting vertices in the left subtree. The right subtree is defined analogously.
- Theorem 10.6.1 Full Binary Tree: A full binary tree with k internal vertices has $2k + 1$ total vertices and $k + 1$ terminal vertices.
- Theorem 10.6.2 Height of Binary Tree: For a binary tree with height h and t leaves, $t \leq 2^h$ and $log_2 t \leq h$.
- Breadth-First-Search: Start at the root, visit adjacent vertices, then move to the next level.
- Depth-First Search:
  - Pre-Order: Print Self, Traverse Left, Traverse Right
  - In-Order: Traverse Left, Print Self, Traverse Right
  - Post-Order: Traverse Left, Traverse Right, Print Self
  - Follow the path around the nodes, and write down a node when the circle on it is met.
  - Given only pre-order and post-order, there might be more than one tree that satisifes.
- Spanning Tree: A subgraph that contains every vertex, and is a tree.
- Proposition 10.7.1: Every connected graph has a spanning tree, and any two spanning trees for a graph have the same number of edges.
- Weighted Graph: A graph where every edge has a positive real number weight.
- Minimum Spanning Tree: A spanning tree with the least possible total weight.
- Kruskal's Algorithm:
  - Process edges in order of increasing weight.
  - If an edge does not produce a cycle, add it to the minimum spanning tree.
- Prim's Algorithm:
  - Create a graph with a starting vertex, and consider the set of vertices of the graph excluding the start point.
  - Find an edge connecting the current graph to one of the remaining vertexes with least weight.
  - Add this edge and its endpoints to the current graph.



Depth-first traversal (dotted path) of a binary tree:
Pre-order (node visited at position red ●):
F, B, A, D, C, E, G, I, H;
In-order (node visited at position green ●):
A, B, C, D, E, F, G, H, I;
Post-order (node visited at position blue ●):
A, C, E, D, B, H, I, G, F.

Useful Tutorial Questions

- Tutorial 1 Question 10: The product of any two odd integers is an odd integer.
- Tutorial 1 Question 11: n^2 is odd iff n is odd. n^2 is even iff n is even.
- Tutorial 2 Question 4: Rational numbers are closed under addition. Integers and rational numbers are not closed under division.
- Tutorial 2 Question 11: If $n = ab$ where a and b are positive, then $a \leq n^{\frac{1}{2}}$ or $b \leq n^{\frac{1}{2}}$.
- Tutorial 3 Question 5: $A \cap (B \backslash C) = (A \cap B) \backslash C$.
- Tutorial 3 Question 6: $A \backslash (B \backslash C) = (A \backslash B) \cup (A \cap C)$.
- Tutorial 3 Question 8: $A \subseteq B$ iff $A \cup B = B$.
- Tutorial 3 Question 9: $(A \backslash B) \cup (B \backslash A) = (A \cup B) \backslash (A \cap B)$.
- Tutorial 4 Question 2: Show the following are logically equivalent: (i) R is symmetric, (ii) $\forall x, y \in A, xRy \leftrightarrow yRx$, (iii) $R = R^{-1}$.
  - (i) $\rightarrow$ (ii)
    - Suppose R is symmetric
    - Let $x, y \in A$
    - If xRy then yRx by symmetry of R
    - If yRx then xRy by symmetry of R
    - $\therefore xRy \leftrightarrow yRx$
  - (ii) $\rightarrow$ (iii)
    - Suppose $\forall x, y \in A, xRy \leftrightarrow yRx$
    - $\forall x, y \in A$
    - $(x, y) \in R \leftrightarrow xRy$ by definition of xRy
    - $\leftrightarrow yRx$
    - $\leftrightarrow xR^{-1}y$ by definition of R^{-1}
    - $\leftrightarrow (x, y) \in R^{-1}$ by definition of $xR^{-1}y$
    - $\therefore R = R^{-1}$
  - (iii) $\rightarrow$ (i)
    - Suppose $R = R^{-1}$
    - Let $x, y \in A, xRy$
    - Then $xR^{-1}y$ as $R = R^{-1}$
    - yRx by definition of R^{-1}
    - $\therefore$ R is symmetric
- Tutorial 4 Question 5: For an equivalence relation R,
  - (i) $R^{-1} \circ R = R \circ R^{-1}$
  - (ii) $R \subseteq R \circ R$
  - (iii) $R \circ R \subseteq R$
  - (iv) $R \circ R^{-1} = R$
  - (v) $R = R \circ R$ from (ii) and (iii)
- Tutorial 4 Question 7: Composition of relations is associative, $T \circ (S \circ R) = (T \circ S) \circ R$.
- Tutorial 5 Question 5: $\subseteq$ on $\mathcal{P}(A)$ is a partial order.
- Tutorial 5 Question 8: Every asymmetric relation is antisymmetric.
- Tutorial 5 Question 11: Any two comparable elements are compatible. Any two compatible elements are not necessarily compatible.
- Tutorial 6 Question 4: $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
- Tutorial 6 Question 6:
  - Given $f : A \rightarrow B$
  - $g : B \rightarrow A$ is a left inverse of f iff $g(f(a)) = a\ \forall a \in A$
  - $h : B \rightarrow A$ is a left inverse of f iff $f(h(b)) = b\ \forall b \in B$
  - The statement if a function has a left inverse, then it has a right inverse, is false. The converse is also false.
- Tutorial 6 Question 7: Let $f : B \rightarrow C$. Given a function g with domain C such that $g \circ f$ is injective, f is injective.
- Tutorial 6 Question 9: Considering preimage and image:

  - $X \subseteq f^{-1}(f(X))$, but not necessarily $f^{-1}(f(X)) \subseteq X$.
  - $Y \subseteq f(f^{-1}(Y))$, but not necessarily $f(f^{-1}(Y)) \subseteq Y$.
- Tutorial 6 Question 10: For the relation $x \sim y \leftrightarrow x - y \in \mathbb{Z}$ on $\mathbb{Q}$, red addition is well-defined while red-multiplication is not well-defined.
- Tutorial 7 Question 2: $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$.
- Tutorial 7 Question 3: $1 + nx \leq (1+x)^n$ for $n \in \mathbb{Z}^+$ and $x \in \mathbb{R}_{\geq -1}$.
- Tutorial 7 Question 4: Given odd a, $2^{n+2} | (a^{2^n} - 1)$.
- Tutorial 7 Question 5: Any number ≥ 8 can be made up of multiple 3's and 5's.
- Tutorial 7 Question 6: Every positive integer can be written as a sum of distinct non-negative integer powers of 2.
- Tutorial 7 Question 7: Given $a_0 = 0, a_1 = 2, a_2 = 7, a_{n+3} = a_{n+2} + a_{n+1} + a_n, a_n < 3^n$.
- Tutorial 7 Question 8: For the Fibonacci function, $F(a+b) = F(a+1) \cdot F(b) + F(a) \cdot F(b-1)$. To prove this, use the base cases $P(0,b)$ and $P(1,b)$.
- Tutorial 7 Question 9: Hamming numbers defined by $1 \in H$ and $n \in H \rightarrow 2n \in H, 3n \in H, 5n \in H$, have a canonical representation; a unique way of representing each member of the set as $2^i 3^j 5^k$.
- Tutorial 8 Question 1: $\mathbb{Z}$ is countable using the formula $g(n) = (-1)^n \lfloor x \rfloor$.
- Tutorial 8 Question 2: The union of a countably infinite set and a finite set is countable. Create a new sequence with all elements of the finite set in front.
- Tutorial 8 Question 3: The union of infinitely many finite sets is infinite.
- Tutorial 8 Question 4: The union of a finite number of countable sets is countable.
- Tutorial 8 Question 5: Let S_i be a countably infinite set for each $i \in \mathbb{Z}^+$. The union of all S is countable. This can be proven by using the fact that $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.
- Tutorial 8 Question 6: A bijection exists between $X \cup Y$ and X where X is an infinite set and Y is a finite set.
- Tutorial 8 Question 7: The power set of a countably infinite set is uncountable. This can be proven by a diagonalisation argument.
  - Suppose $\mathcal{P}(A)$ is countable.
  - $\mathcal{P}(A)$ is infinite as A is infinite.
  - There is a sequence a where every element of A appears exactly once.
  - There is a sequence B where every element of $\mathcal{P}(A)$ appears exactly once.
  - Now define $B = \{a_i : a_i \notin b_i\}$.
  - Note that $B \in \mathcal{P}(A)$.
  - Then $B \neq B_i$ for all i.
  - This is because if $a_i \notin B_i$, then $a_i \in B$, and if $a_i \in B_i$, then $a_i \notin B$. So in all cases, $B \neq B_i$.
  - Since B is not in the original sequence, this contradicts the claim that $\mathcal{P}(A)$ is countable.
  - Therefore $\mathcal{P}(A)$ is uncountable.
- Tutorial 8 Question 8: If R is reflexive, then $|A| \leq |R|$. The same i snot necessarily true for symmetry and transitivity.
- Tutorial 8 Question 9: $Even(F_n) \leftrightarrow Even(F_{n+3})$.
- Tutorial 8 Question 10: Given the equivalence relation $g(a) = g(b) \leftrightarrow a \sim b$, the function $f : X/\sim \rightarrow Y$ given by $f([x]) = g(x)$ is well-defined and injective.
- Tutorial 9 Question 4: Given n boxes with white or blue balls, such that at least one box contains a white ball, and boxes with white balls are consecutive, this can be done in $\frac{n(n+1)}{2}$ ways. For $1 \leq k \leq n$, there are $n - k + 1$ ways.
- Tutorial 9 Question 5: The number of circular permutations of n objects is $(n-1)!$.
- Tutorial 9 Question 9:
  - If you randomly put 51 points in a unit square, there are always 3 points that can be covered by a circle with radius 1/7.
  - Divide the unit square into 25 equal smaller squares. One small square must have at least three points.
  - The diagonal of the square is less than the diameter of the circle we want to place, so it can be fully contained.
- Tutorial 9 Question 10: Given any 5 distinct non-negative integers, two of them have a difference divisible by 4.
- Tutorial 9 Question 11:
  - A chessmaster has 11 weeks to prepare for a tournament. They play at least 1 game every day, but no more than 12 games in any week. There is a succession of consecutive days where the chess master will have played exactly 21 games.
  - Consider 77 days, and P_i as the cumulative number of games played including day i. All $1 \leq P_i \leq 132$, and are distinct.
  - Consider $Q_i = P_i + 21$. The largest Q_i possible is 153, and they are also distinct.
  - There are 154 possible numbers, but the range of P_i and Q_i is 153. There must be some $P_j = Q_i = P_i + 21$.
  - The chess master plays exactly 21 games from day $i+1$ to j.
- Tutorial 10 Question 2: From Tutorial 9 Question 4: Put crosses on the side of all boxes. Choose 2 out of $n+1$ crosses to be the start and end of boxes with white balls. The number of ways is $\binom{n+1}{2} = \frac{n(n+1)}{2}$.
- Tutorial 10 Question 6: For a random relation on a power set, what is the probability that the relation is:
  - Reflexive: Consider the n by n matrix. For a reflexive relation, all entries in the diagonal must be 1. For the remaining $n^2 - n$ entries, they can be anything we wish. So $P(Reflexive) = \frac{2^{n^2-n}}{2^{n^2}} = \frac{1}{2^n}$.
  - Symmetric: Consider the n by n matrix. For a symmetric relation, $a_{ij} = a_{ji}$. We can consider the lower triangular half of the matrix and the main diagonal, and let these be anything we want. There are $\frac{n^2+n}{2}$ entries, so $P(Symmetric) = \frac{2^{\frac{n^2+n}{2}}}{2^{n^2}} = \frac{1}{2^{\frac{n^2-n}{2}}}$.
  - Antisymmetric: Entries on the main diagonal can be either 0 or 1. For a a_{ij} in the lower triangular half of the matrix, there can be 3 possibilities: i and j are unrelated, iRj, or jRi. This gives $2^n \cdot 3^{\frac{n(n-1)}{2}}$ relations, and a probability of $2^{n-n^2} \cdot 3^{\frac{n(n-1)}{2}}$.
- Tutorial 10 Question 8: There are 4 non-isomorphic undirected graphs with two vertices and two edges.
- Tutorial 10 Question 10: Given group of people who shake hands including a host who shook hand with everyone else, there are at least two people who shook hands the same number of times.
- Tutorial 10 Question 11: A dominating set is a set of vertices in an undirected simple graph where every vertex outside of this set is adjacent to at least one vertex in this set. A minimal dominating set does not have a dominating proper subset.
- Tutorial 10 Question 11: For all simple graphs with 6 vertices, either the graph or its complement contains a triangle.
- Tutorial 11 Question 4: If a simple undirected graph is connected, then $|E| \geq |V| - 1$. The converse is NOT true.
- Tutorial 11 Question 5: If a simple undirected graph is acyclic, then $|E| \leq |V| - 1$. The converse is NOT true.
- Tutorial 11 Question 6: For a simple undirected graph, it is a tree iff there is exactly one path betewen every pair of vertices.
- Tutorial 11 Question 7: If separating stones into 2 piles gives $k_1 \times k_2$ money, the maximum (only) amount of money you can earn is $\frac{n(n-1)}{2}$. Imagine splitting as cutting all edges in the bipartite graph between the 2 resulting sets. Since every edge has to be cut, the total is the number of edges in the complete graph.
- Tutorial 11 Question 9: Number of binary trees with n nodes is the nth Catalan Number. $C_n = \frac{1}{n+1}\binom{2n}{n}$. $C_i = 1, 2, 5, 14, 42, 132$.

Useful Assignment Questions

- Assignment 1 Question 1: Variant Absorption Laws
  - $p \wedge (\sim p \vee q) \equiv p \wedge q$
  - $p \vee (\sim p \wedge q) \equiv p \vee q$
- Assignment 1 Question 6(b): $A \subseteq \emptyset \rightarrow A = \emptyset$.
- Assignment 1 Question 6(c): $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.
- Assignment 2 Question 2(c): On average how many times must a 6-sided fair dice be rolled until a 3 turns up? $E(X) = \frac{1}{6} \cdot 1 + \frac{5}{6}(E(X) + 1)$, giving $E(X) = 6$.
- Assignment 2 Question 5: $14 | 2^{4n} - 2^n$.
- Assignment 2 Question 6: Let $f : \mathcal{P}(\mathbb{N}) \backslash \{\emptyset\} \rightarrow \mathbb{N}$ map S to its smallest element. Then, f does not have an inverse, and $f^{-1}(\{n\})$ is uncountable.

pls give me A+ prof thanks