MA1521
AY24/25 Sem 1
by ngmh

0. Real Numbers and Functions

- Finding Maximal Domain: Exclude $Q(x) = 0$ if the function is $\frac{P(X)}{Q(X)}$.
- Finding Maximal Range: Express x in terms of y, and see range of y. e.g. If quadratic, y will be inside a square root.

1. Limits and Continuity

- Left Limit: $\lim_{x\to c^-} f(x)$ is the value $f(x)$ approaches when x approaches c from the left.
- Right Limit: $\lim_{x\to c^+} f(x)$ is the value $f(x)$ approaches when x approaches c from the right.
- Existence of Limit: If $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$, $\lim_{x\to c} f(x)$ exists with value L for c which is not an endpoint.
- Continuity at a Point: Limit at the point exists, only consider one side if endpoint.
- Continuity on an Interval: f is continuous on an interval if f is continuous at $x = c$ for all c in the interval.
- Evaluation of Limits: Addition, Scaling, Product of 2 functions, Fraction.
- Composition of Limit: If g is continuous at b and $\lim_{x\to c} f(x) = b$, then $lim_{x\to c} g(f(x)) = g(b) = g(lim_{x\to c} f(x))$.
- Limit at Infinity: $\lim_{x\to\infty} f(x)$ is the value $f(x)$ approaches as x tends to positive infinity. $\lim_{x\to-\infty} f(x)$ is the value $f(x)$ approaches as x tends to negative infinity. If the limit to infinity is c, $y = c$ is a horizontal asymptote of the graph $f(x)$.
- Indeterminate Form of Limit: A limit that evaluates to $\frac{0}{0}$ or $\frac{\infty}{\infty}$.
- Limit to infinity of Polynomial: $\lim_{x\to\pm\infty} \frac{P(x)}{Q(x)}$ depends on the leading term $\frac{Ax^c}{Bx^d}$. If $c < d$, the limit is 0. If $c > d$, the limit is $\pm\infty$. If $c = d$, the limit is $\frac{A}{B}$.
- Useful Trigo Limits: $\lim_{x\to c} \frac{sin(g(x))}{g(x)} = \lim_{x\to c} \frac{g(x)}{sin(g(x))} = 1$ if $\lim_{x\to c} g(x) = 0$ (also works for tan).
- Squeeze Theorem: If $g(x) \le f(x) \le h(x)$, then if $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$, $\lim_{x\to c} f(x) = L$ at $x = c$.
- Intermediate Value Theorem: If f is continuous on $[a, b]$, $f(c) = k$ for some $c \in [a, b]$ for all k between $f(a)$ and $f(b)$.

2. Derivatives

- Derivative: $f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$.
- Differentiability implies continuity at a point, but the converse is not necessarily true.
- Differentiability on Interval: A function f is differentiable on an interval if it is differentiable at every point in the interval.
- Standard Derivatives: For $f(x)$, multiply result by $f'(x)$.
- Product Rule: $\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$.
- Quotient Rule: $\frac{d}{dx}(\frac{u}{v}) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$.
- Derivative of Inverse Function: $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$. Let $b = f^{-1}(a)$ so that $a = f(b)$. $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))} = \frac{1}{f'(b)}$.
- Reciprocal of Derivative: If an inverse function exists, $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$.
- Derivative of Parametric Equation: $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$.
- Second Derivative of Parametric Equation: $\frac{d^2y}{dx^2} = \frac{d}{dt}(\frac{dy}{dx}) \div \frac{dx}{dt} = \frac{\frac{d}{dt}(g'(t))}{f'(t)} \div f'(t) = \frac{g''(t)f'(t)-g'(t)f''(t)}{f'(t)^3}$.
- Derivative of $y = f(x)^{g(x)}$: First find the derivative of lny then solve for $\frac{dy}{dx}$.
- Change of Base Formula: $log_a x = \frac{lnx}{lna}$.

3. Applications of Differentiation

- Tangent Equation: $y - f(x_0) = f'(x_0)(x - x_0)$.
- Normal Equation: $y - f(x_0) = -\frac{1}{f'(x_0)}(x - x_0)$.
- Increasing Function: $f'(x) > 0$ for all x in (a, b).
- Decreasing Function: $f'(x) < 0$ for all x in (a, b).
- Concave Upward: If $f''(c) > 0$ the graph is concave upward at $(c, f(c))$.
- Concave Downward: If $f''(c) < 0$ the graph is concave downward at $(c, f(c))$.
- Inflection Point: Graph has a tangent line and concavity changes. At this point $(c, f(c))$, $f''(c) = 0$ if it exists.
- Absolute Maxima: $x = c$ is an absolute maxima if $f(x) \le f(c)$ for all x in the domain of f. Minima is defined similarly.
- Local Maxima: $x = c$ is a local maxima at $x = c$ if $f(x) \le f(c)$ for x in some open interval containing it. Minima is defined similarly.
- Extreme Value Theorem: If f is continuous on a closed interval $[a, b]$, then f has an absolute maximum and minimum at some points in $[a, b]$.
- If f is differentiable on an open interval containing $x = c$ and f has a local extremum at $x = c$, then $f'(c) = 0$.
- Critical Point: Not an end-point and either $f'(c) = 0$ or $f'(c)$ does not exist.
- Local extrema and Critical Points: If f has a local extrema at $x = c$, then c is a critical point of f. The converse is not necessarily true.
- First Derivative Test for Absolute Extrema: For a critical point c, if $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then c is a maximum point. If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then c is a minimum point.
- First Derivative Test for Local Extrema: If f' does not change sign at $x = c$, there is no local extrema at c. If it changes from positive to negative, it is a maximum. If it changes from negative to positive, it is a minimum.
- Second Derivative Test: Suppose f is twice differentiable. Then if $f'(c) = 0$ and $f''(c) < 0$ then c is a local maximum. If $f''(c) > 0$ then c is a local minimum. If $f''(c) = 0$ then no conclusion can be drawn.
- L'Hôpital's Rule: If $lim_{x\to c} \frac{f(x)}{g(x)}$ is indeterminate, then it is equivalent to $lim_{x\to c} \frac{f'(x)}{g'(x)}$. This also holds for limits at infinity and one-sided limits.
- 0^0: Consider $\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{ln(x^x)} = 1$ by applying lopital's.
- ∞^0: Consider $\lim_{x\to 0^+} (\frac{2}{x})^x = 1$ using the previous result.
- 1^∞: Consider $\lim_{x\to 0^+} (1+x)^{\frac{1}{x}} = e$ using e^{ln} and lopital's.
- Rolle's Theorem: Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, there is at least one c in (a, b) with $f'(c) = 0$.
- Number of solutions using Rolle's Theorem: Consider $f(x)$ and $f'(x)$. If $f(x)$ has more than 1 solution, then there is $f'(c) = 0$ between the 2 solutions.
- Mean Value Theorem: Let f be continuous on $[a, b]$ and differentiable on (a, b). There is at least one c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

4. Integrals

- Integral: F is an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.
- Partial Fractions: Decompose $\frac{P(x)}{Q(x)}$ as a sub of simple fractions whose denominators are factors of $Q(x)$.
- Repeated Factor: Decompose into factor and factor squared.
- Quadratic Factor: Fraction will be $\frac{Ax+B}{ax^2+bx+c}$.
- Integration by Substitution: Let $u = g(x)$ be a differentiable function with range I and f is continuous on I. Then $\int f(g(x))g'(x)dx = \int f(u)du$.
- The Other Integration by Substitution: Let $x = g(t)$ be a differentiable function with range I and f is continuous on I. Then $\int f(x)dx = \int f(g(t))g'(t)dt$.
- Trigonometric Substitution:
  - $\sqrt{a^2 - (x+b)^2}$: $x + b = asin\theta$.
  - $\sqrt{a^2 + (x+b)^2}$: $x + b = atan\theta$.
  - $\sqrt{(x+b)^2 - a^2}$: $x + b = asec\theta$.
- Integration by Parts: $\int f'(x)g(x)dx = f(x) \cdot g(x) - \int f(x)g'(x)dx$.
- Order of Preference from Differentiate to Integration: Log, Inverse Trigo, Algebra, Trigo (Both), Exponential.
- Riemann Sum: $\int_a^b f(x)dx = \sum_{k=1}^n (\frac{b-a}{n}) f(a + k(\frac{b-a}{n}))$.
- Fundamental Theorem of Calculus 1: $\int_a^b f(x)dx = \int_a^b F'(x)dx = F(b) - F(a)$.
- Fundamental Theorem of Calculus 2: The function $g(x) = \int_a^x f(t)dt$ is continuous and differentiable, and $g'(x) = f(x)$. Then $\frac{d}{dx}\int_a^{g(x)} f(t)dt = f(g(x)) \cdot g'(x)$.
- Type I Improper Integral: Integral with an infinite boundary. Take the boundary of the integral replaced with a parameter. If both boundaries are infinite, split into 2 integrals about a point in between. The evaluate the limit of the integral as the parameter tends to infinity.
- Type II Improper Integral: Integral which is not continuous on a boundary. Similar to Type I, replace boundary with a limit.

5. Applications of Integration

- Area between Curves: If $f(x) \ge g(x)$, $A = \int_a^b f(x) - g(x)dx$. If not, $A = \int_a^b |f(x) - g(x)|dx$. The same can be said about curves defined on y instead.
- Volume of Solid of Revolution by Disk Method: $V = \pi\int_a^b f(x)^2dx$.
- Between 2 curves, need to distinguish between $V = \pi\int_a^b f(x)^2 - g(x)^2dx$ and $V = \pi\int_a^b (f(x) - g(x))^2dx$ (probably wrong).
- Cylindrical Shell Method: $V = 2\pi\int_a^b x|f(x)|dx$. Between 2 curves, $V = 2\pi\int_a^b x|f(x) - g(x)|dx$.
- Depending on how the function is defined and the axis to revolve around, use Disk or Cylindrical Shell Method.
- Arc Length of a Curve: $\int_a^b \sqrt{1 + f'(x)^2}dx$ for $y = f(x)$. (Or $g'(y)$ for $x = f(y)$).

6. Sequences and Series

- Limit of a Sequence: Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers. If $\lim_{n\to\infty} a_n = L$, the sequence converges to L. If it does not exist, the sequence diverges.
- Squeeze Theorem for Sequences: If $a_n \le b_n \le c_n$ for all n and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ then $\lim_{n\to\infty} b_n = L$.
- Convergent Series: If $\sum_{n=1}^\infty a_n$ is convergent. $\lim_{n\to\infty} a_n = 0$.
- n^{th} Term Test for Divergence: If $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n \ne 0$ then the series $\sum_{n=1}^\infty a_n$ is divergent. If it is 0, we cannot draw any conclusions.
- A series $\sum_{n=1}^\infty a_n$ of nonnegative terms converges iff its partial sums are bounded from above.
- Harmonic Series: $\sum_{n=1}^\infty \frac{1}{n}$ is divergent.
- Integral Test: The series $\sum_{n=1}^\infty a_n$ is convergent iff the improper integral $\int_1^\infty f(x)dx$ is convergent.
- P-Series: The p-series $\sum_{n=1}^\infty \frac{1}{n^p}$ is convergent iff $p > 1$.
- Comparison Test: Suppose $\sum_{n=1}^\infty a_n$ and $\sum_{n=1}^\infty b_n$ are series with nonnegative terms such that $0 \le a_n \le b_n$ for all n. If $\sum_{n=1}^\infty b_n$ is convergent, $\sum_{n=1}^\infty a_n$ is also convergent. If $\sum_{n=1}^\infty a_n$ is divergent, $\sum_{n=1}^\infty b_n$ is also divergent.
- Ratio Test: Suppose $\sum_{n=1}^\infty a_n$ is a series such that $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L$. If $0 \le L < 1$, the series is absolutely convergent. If $L > 1$, the series is divergent. If $L = 1$, the test is inconclusive.
- Root Test: Suppose $\sum_{n=1}^\infty a_n$ is a series such that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$. If $0 \le L < 1$, then the series is absolutely convergent. If $L > 1$, the series is divergent. If $L = 1$, the test is inconclusive.
- Alternating Series Test: If b_n is a sequence of positive numbers such that b_n is decreasing and $\lim_{n\to\infty} b_n = 0$, the alternating series $\sum_{n=1}^\infty (-1)^{n-1}b_n$ and $\sum_{n=1}^\infty (-1)^n b_n$ are convergent.

- Absolute Convergence: If $\sum_{n=1}^{\infty} |a_n|$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent. A conditionally convergent series is convergent but NOT absolutely convergent.
- Power Series: A power series has the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + ...$ A power series with $(x-a)$ instead is centered around a, and converges at $x = a$.
- Radius of Convergence of Power Series: For a power series, either the series converges at only $x = a$, all x, or there is a positive R such that the series converges absolutely if $|x-a| < R$ and diverges if $|x = a| > R$.
- Interval of Convergence: The interval $[a-R, a+R]$ where the power series converges. However, we need to manually check the endpoints of the interval.
- Finding Radius of Convergence: For the power series $\sum_{n=1}^{\infty} c_n (x-a)^n$, if $\lim_{n\to\infty} |\frac{c_{n+1}}{c_n}| = L$ or $\lim_{n\to\infty} \sqrt[n]{|c_n|} = L$. Then $L|x-a| < 1$, and $R = \frac{1}{L}$. For $(x-a)^{2n}$, $L|x-a|^2 < 1$. The radius is around the center of the power series.
- Order of Tests:
  - Don't look like they converge to zero: Divergence Test.
  - Check for p-series convergence.
  - Looks like p-series or geometric series: Comparison test.
  - Only has rationals, polynomials, radicals: Comparison or limit comparison test. However, all terms must be positive.
  - Factorials and powers of n: Ratio test, especially for factorials.
  - Can be written as $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$: Alternating series test.
  - Can be written as $a_n = (b_n)^n$: Root test.
  - If $a_n = f(n)$ for some positive, decreasing function and $\int_a^{\infty} f(x)dx$ is easy to calculate: Integral test.
- Standard Series:
  - $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + ...$
  - $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$
  - $ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - ...$
  - Others: Derive by subbing in for x (e.g. $-x^2$ for tan^{-1}) or integrating.
- Differentiation of Power Series: If a power series has radius of convergence $R > 0$, the function defined by the power series is differentiable on the interval $|x-a| < R$, $f'(x) = \sum_{n=1}^{\infty} nc_n (x-a)^{n-1}$ and $\int f(x)dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$.
- Taylor Series: If a power series has a representation at $x = a$, the coefficients are given by $c_n = \frac{f^{(n)}(a)}{n!}$ where $f^{(n)}$ is the n^{th} differentiation. It is unique and has the form $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$, (the Taylor Series of f at $x = a$).
- Maclaurin Series: Special case of Taylor Series when $a = 0$.

7. Vectors and Geometry of Space

- Distance Formula: $|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$.
- Equation of Sphere: A sphere with center $C(h, k, l)$ and radius r is $(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$.
- Triangle Law: The sum of 2 vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, $\boldsymbol{u} + \boldsymbol{v}$ is the vector from the start of $\boldsymbol{u}$ to the end of $\boldsymbol{v}$ if the start of $\boldsymbol{v}$ is put at $\boldsymbol{u}$.
- Scalar Multiplication: $c\boldsymbol{u}$ is the vector whose length is $|c|$ times of $\boldsymbol{u}$ and is in the same direction if $c > 0$ or opposite if $c < 0$.
- Vector Between 2 Points: The vector between $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, $\vec{AB}$ is $\boldsymbol{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$.
- Vector Properties: Addition is commutative and associative, 0 is the additive identity, addition with negation gives 0, scalar multiplication is commutative, distributive and associative, 1 is the multiplicative identity.
- Basis Vectors: Length 1 in the direction of each axis, $\boldsymbol{i} = \langle 1, 0, 0 \rangle$, $\boldsymbol{j} = \langle 0, 1, 0 \rangle$, $\boldsymbol{k} = \langle 0, 0, 1 \rangle$.
- Unit Vector: Vector of length 1, given by $\boldsymbol{u} = \frac{\boldsymbol{a}}{||\boldsymbol{a}||}$.
- Dot Product: For $\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$ and $\boldsymbol{b} = \langle b_1, b_2, b_3 \rangle$, $\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$.
- Dot Product Properties: Commutative, Distributive, Associative, 0 is identity, $\boldsymbol{a} \cdot \boldsymbol{a} = ||a||^2$.
- Angle Between Vectors (Dot Product): $\boldsymbol{a} \cdot \boldsymbol{b} = ||\boldsymbol{a}|| ||\boldsymbol{b}|| cos\theta$.
- Orthogonal Vectors: Two vectors are orthogonal iff their dot product is 0.
- Projection: Given $\boldsymbol{a} = \vec{PQ}$ and $\boldsymbol{b} = \vec{PR}$, let S be the foot of perpendicular from R to $\boldsymbol{a}$. $\vec{PS}$ is the vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$, $proj_{\boldsymbol{a}} \boldsymbol{b}$. The scalar projection is the signed magnitude of the vector projection (component of $\boldsymbol{b}$ along $\boldsymbol{a}$ is $comp_{\boldsymbol{a}} \boldsymbol{b} = ||\boldsymbol{b}|| cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{||\boldsymbol{a}||}$. $proj_{\boldsymbol{a}} \boldsymbol{b} = comp_{\boldsymbol{a}} \boldsymbol{b} \times \frac{\boldsymbol{a}}{||\boldsymbol{a}||} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{a} \cdot \boldsymbol{a}} \cdot \boldsymbol{a}$.
- Distance from Point to Plane: The shortest distance from $P(x_0, y_0, z_0)$ to the plane $ax + by + cz = d$ is $\frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2+b^2+c^2}}$.
- Cross Product: For $\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$ and $\boldsymbol{b} = \langle b_1, b_2, b_3 \rangle$, $\boldsymbol{a} \times \boldsymbol{b} = \langle a_2 b_3 - a_3 b_2, -(a_1 b_3 - a_3 b_1), a_1 b_2 - a_2 b_1 \rangle$.
- Orthogonal Vector to 2 other Vectors: $\boldsymbol{a} \times \boldsymbol{b}$ is orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{b}$.
- Angle between Vectors (Cross Product): $||\boldsymbol{a} \times \boldsymbol{b}|| = ||\boldsymbol{a}|| ||\boldsymbol{b}|| sin\theta$.
- Cross Product Properties: Multiplication with scalar is associative, it is distributive, and $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$.
- Parametric Equation of Line: $x = x_0 + at, y = y_0 + bt, z = z_0 + ct$.
- Skew Lines: Lines that are non-parallel and non-intersecting.
- Vector Equation of Plane: $\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r_0}) = 0$ where $\boldsymbol{n}$ is the normal vector.
- Linear Equation of Plane: $ax + by + cz = d$ where $\boldsymbol{n} = \langle a, b, c \rangle$.

8. Functions of Several Variables

- Derivative of Vector-Valued Function: If $\boldsymbol{r}(t) = \langle f(t), g(t), h(t) \rangle$ and f, g, h are differentiable at $t = a$, $\boldsymbol{r}'(a) = \langle f'(a), g'(a), h'(a) \rangle$. Normal derivative rules apply. This gives the tangent vector at $t = a$.
- Arc Length Formula: For the same $\boldsymbol{r}$, $s = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2} dt = \int_a^b ||\boldsymbol{r}'(t)|| dt$.
- Partial Derivative: f_x or $\frac{\partial f}{\partial x}$ is the partial derivative of f w.r.t x. That means all other variables are assumed to be constant. f_{xy} is a higher order derivative which means differentiating by x then y.
- Clairaut's Theorem: If f_{xy} and f_{yx} are both continuous, $f_{xy}(a, b) = f_{yx}(a, b)$.
- Equation of Tangent Plane: Suppose $z = f(x, y)$. $\boldsymbol{n} = \langle f_x(a, b), f_y(a, b), -1 \rangle$. The equation of the tangent plane is $f_x(a, b)(x-a) + f_y(a, b)(y-b) - (z - f(a, b)) = 0$.
- Chain Rule Case 1: $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$. Works with > 2 parameters as well.
- Chain Rule Case 2: If x and y have 2 parameters s and t, $\frac{dz}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}$ and $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$.
- Implicit Differentiation: Suppose $F(x, y, z) = 0$ where F is differentiable and $z = f(x, y)$ implicitly. Then $\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)}$ and $\frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)}$.
- Increment: If $z = f(x, y)$, $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$.
- Differentials: If $dx = \Delta x$ and $dy = \Delta y$, $dz = f_x(x, y)dx + f_y(x, y)dy$.
- Increment Approximation: For small increments, $\Delta z \approx dz = f_x(a, b)dx + f_y(a, b)dy = f_x(a, b)\Delta x + f_y(a, b)\Delta y$.
- 2D Directional Derivative: The Directional derivative of f in the direction of any UNIT vector $\boldsymbol{u} = \langle a, b \rangle$, $D_{\boldsymbol{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b = \langle f_x, f_y \rangle \cdot \boldsymbol{u}$.
- Gradient: $\nabla f(x, y) = \langle f_x, f_y \rangle$.
- 3D Directional Derivative: $D_{\boldsymbol{u}} f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) \cdot \boldsymbol{u}$ where $\nabla f = \langle f_x, f_y, f_z \rangle$.
- Level Curve v/s ∇f: Suppose $\nabla f(x_y, y_0) \neq 0$. Then $\nabla f(x_y, y_0)$ is perpendicular to the level curve $f(x, y) = k$ at the point (x_0, y_0) where $f(x_0, y_0) = k$.
- Level Surface v/s f: Suppose $\nabla F(x_0, y_0, z_0) \neq 0$. Then $\nabla F(x_0, y_0, z_0) \cdot \boldsymbol{r}'(t_0) = 0$, meaning they are orthogonal.
- Tangent Plane to Level Surface: $\nabla F(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$.
- Maximizing Rate of Increase: Assume $\nabla f(P) \neq 0$. Let $\boldsymbol{u}$ be a unit vector making an angle θ with ∇f. Then $D_{\boldsymbol{u}} f(P) = ||\nabla f(P)|| cos\theta$. $\nabla f(P)$ points in direction of maximum rate of increase, while $-\nabla f(P)$ points in direction of maximum rate of decrease.
- Local Extrema: If there is a local extrema at (a, b), then $f_x(a, b) = f_y(a, b) = 0$.
- Critical Point: $f_x(a, b) = 0$ and $f_y(a, b) = 0$ or a partial derivative does not exist.
- Saddle Point: Critical point where every disk centered at (a, b) contains points where $f(x, y) < f(a, b)$ and points where $f(x, y) > f(a, b)$.
- Second Derivative Test: $D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$.
  - $D > 0$, $f_{xx}(a, b) > 0$: Local Minimum.
  - $D > 0$, $f_{xx}(a, b) < 0$: Local Maximum.
  - $D < 0$: Saddle Point.
  - $D = 0$: No conclusion can be drawn.

9. Double Integrals

- Volume as Double Integral: The volume of the solid that lies above a rectangle and below the surface $z = f(x, y)$ is $V = \int\int_R f(x, y) dA$.
- Iterated Integral: $\int_a^b \int_c^d f(x, y) dy dx$ means integrating y from c to d first with x fixed then x from a to b.
- Fubini's Theorem: If f is continuous on R, then $\int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$.
- Special Case: $\int\int_R g(x)h(y) dA = (\int_a^b g(x))(\int_c^d h(y) dy)$.
- Type I Region: Region lying between graph of two continuous functions of x. Integrate by y first, then x.
- Type II Region: Region lying between graph of two continuous functions of y. Integrate by x first, then y.
- Polar Coordinates: $r^2 = x^2 + y^2$, $x = rcos\theta$, $y = rsin\theta$.
- Integration of Polar Coordinates: $\int\int_R f(x, y) dA = \int_c^d \int_a^b f(rcos\theta, rsin\theta) r dr d\theta$. Don't forget the extra r!
- Surface Area: $\int\int_D dS = \int\int_D \sqrt{f_x^2 + f_y^2 + 1} dA$ where f_x^2 and f_y^2 are partial derivatives.

10. Ordinary Differential Equations

- Separable ODE:
  - $y' = f(x)g(y)$
  - $\frac{1}{g(y)} y' = f(x)$
  - $\int \frac{1}{g(y)} dy = \int f(x) dx + C$.
- Reduction to Separable Form:
  - For $y' = g(\frac{y}{x})$, sub $v = \frac{y}{x}$.
  - Then $y = vx$ and $y' = v + xv'$.
  - The equation becomes $v + xv' = g(v)$ which is separable.
  - Another useful substitution is $u = ax + by + c$.
- Linear First Order ODE:
  - The standard form is $y' + P(x)y = Q(x)$.
  - Multiply by the integrating factor $I(x) = e^{\int P(x)dx}$.
  - The equation reduces to $(y \cdot I(x))' = Q(x) \cdot I(x)$, which we can solve by integrating both sides by x.
- The Bernoulli Equation:
  - $y' + p(x)y = q(x)y^n$, where $n \neq 0, 1$.
  - Sub $u = y^{1-n}$ and find y'.
  - Sub in for y' and divide by the coefficient of u' to get $u' + (1-n)p(x)u = (1-n)q(x)$ which is a first order linear ODE.
  - Solve the new ODE then finish the substitution.