MA3205
AY24/25 Sem 2
by ngmh

2. Pairing, Products, and Relations

EP2.36 There is a bijection $F : \mathcal{P}(X) \to \{0,1\}^X$.
$F(a)(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \notin A \end{cases}$ which is $1-1$ and onto.

D2.37 Cartesian Product Let F be a function with $dom(F)$ as a set. $\prod F = \{f : f \text{ is a function} \wedge dom(f) = dom(F) \wedge \forall x \in dom(F)\ [f(x) \in F(x)]\}$. If $F = \langle A_i : i \in I\rangle$, then $\prod F = \prod_{i\in I} A_i = \{f : f \text{ is a function} \wedge dom(f) = I \wedge \forall i \in I\ [f(i) \in A_i]\}$

A2.38 Axiom of Choice If $\langle A_i : i \in I\rangle$ is a sequence of sets such that $\forall i \in I\ [A_i \neq \emptyset]$, then $\prod_{i\in I} A_i \neq \emptyset$

4. The Natural Numbers

D4.17 Extenders Let $\mathbf{FN} = \{\sigma : \sigma \text{ is a function} \wedge \exists n \in \mathbb{N}\ [dom(\sigma) = n]\}$ be the proper class of all functions whose domain is some natural number. An extender is a function $\mathbf{E} : \mathbf{FN} \to \mathbf{V}$. When you input $\sigma = \{\langle 0, \sigma(0)\rangle, ..., \langle n, \sigma(n)\rangle\}$ into $\mathbf{E}$, $\mathbf{E}(\sigma)$ outputs the next value $\sigma(S(n))$.

D2.45 Addition

- Define $\langle f_m : m \in \mathbb{N}\rangle$ such that $f_m : \mathbb{N} \to \mathbb{N}$ is the unique function such that $f_m(0) = m$ and $\forall n \in \mathbb{N}\ [f_m(S(n)) = S(f_m(n))]$
- In other words, define the extender $\mathbf{E} : \mathbf{FN} \to \mathbf{V}$ as follows. For any $\sigma \in \mathbf{FN}$, $\mathbf{E}(\sigma) = \begin{cases} m & \text{if } dom(\sigma) = 0 \\ S(\sigma(\bigcup dom(\sigma))) & \text{if } dom(\sigma) \neq 0 \end{cases}$
- $f_m : \mathbb{N} \to \mathbf{V}$ is the unique function satisfying $\forall n \in \mathbb{N}\ [f_m(n) = \mathbf{E}(f_m \restriction n)]$.
- Then $m + n = f_m(n)$, and $m + S(n) = (m+n) + 1$.

5. Comparing Sizes of Sets

D5.1 Equinumerosity $A \approx B$ if there exists $f : A \to B$ which is both $1-1$ and onto.

F5.2 $\mathcal{P}(A) \approx \{0,1\}^A$

D5.4 $A \lesssim B$ means there exists $f : A \to B$ which is $1-1$ and B is at least as big as A. If $A \lesssim B$ but $A \not\approx B$, then $A \lnsim B$. It is not possible to find $g : A \to B$ that is both $1-1$ and onto. B is strictly bigger in size than A.

L5.5 If $f : A \to B$ and $g : B \to C$ are $1-1$ functions then $g \circ f : A \to C$ is $1-1$.

L5.6 For sets A, B, C

1. $A \lesssim A$
2. If $A \lesssim B$ and $B \lesssim C$ then $A \lesssim C$
3. If $A \approx B$ and $B \approx C$ then $A \approx C$

T5.7 Cantor For any set X, $X \lnsim \mathcal{P}(X)$.

5.2 The Schröder Bernstein Theorem

T5.11 Schröder-Bernstein For any sets A and B, if $A \lesssim B$ and $B \lesssim A$, then $A \approx B$.

E5.12 Suppose $f : X \to Y$ is a $1-1$ function. For any $Z \subseteq X$, $Z \approx Im_f(Z)$.

E5.13 Suppose $I \subseteq A$ and $J \subseteq B$. If $I \approx J$ and $(A \setminus I) \approx (B \setminus J)$, then $A \approx B$.

E5.14 If $n \in \mathbb{N}$ and $A \approx S(n)$, then $\forall a \in A, (A \setminus \{a\} \approx n)$.

E5.15 If $n \in \mathbb{N}$ and $A \approx n$, then if $a \notin A$, $(A \cup \{a\}) \approx S(n)$.

E5.16 Let $n, m \in \mathbb{N}$. Then

1. If $f : n \to n$ is $1-1$, then f is onto. There is no $1-1$ function from $S(n)$ to n.
2. If $m \in n$, then $m \lnsim n$.
3. If $x \subsetneq n$, then $x \lnsim n$.
4. $n \lnsim \mathbb{N}$
5. If $A \approx n$, $B \approx m$, and $A \cap B = \emptyset$, then $(A \cup B) \approx (n + m)$.

D5.19 A set is finite if there exits $n \in \mathbb{N}$ such that $n \approx A$. A is infinite if it is not finite. A is countable if $A \lesssim \mathbb{N}$. A is uncountable if it is not countable.

L5.20 If $f : A \to B$ is a $1-1$ function, then for any $X, Y \subseteq A$, if $Im_f(X) = Im_f(Y)$, then $X = Y$.

L5.21 For sets A, B, C, D

1. If $A \lnsim B$ then $\mathcal{P}(A) \lnsim \mathcal{P}(B)$
2. If $A \lnsim B$ then $A^C \lnsim B^C$
3. If $A \lnsim B, C \lnsim D$, and $B \cap D = \emptyset$, then $A \cup C \lnsim B \cup D$

L5.22 If $n \in \mathbb{N}$ and $A \lesssim n$, then A is finite.

L5.23 If $n \in \mathbb{N}$ and there exists an onto function $\sigma : n \to A$, then $A \lesssim n$

L5.24 If A and B are finite, then so is $A \cup B$.

T5.25 If A is a finite set and f is a function with $dom(f) = A$ then

1. If $X \subsetneq A$, then $X \lnsim A$
2. $ran(f)$ is finite and $ran(f) \lesssim A$
3. If $\forall a \in A\ [a \text{ is finite}]$ then $\cup A$ is finite
4. $\mathcal{P}(A)$ is finite

E5.26 If $A \subseteq \mathbb{N}$ is finite and nonempty, $max(A) = \bigcup A$

E5.27 If $A \lesssim C$ and $B \lesssim D$, then $A \times B \lesssim C \times D$. If A and B are finite, $A \times B$ and A^B are finite.

E5.28 If I is a finite set and $\langle A_i : i \in I\rangle$ is a sequence of sets such that $\forall i \in I\ [A_i \text{ is finite}]$, then $\prod_{i\in I} A_i$ is finite.

E5.30 For any function, $dom(f) \approx f$.

6. Orders

Quasi, Partial, Linear, and Well-Orders

D6.2 Quasi Order Reflexive, Transitive

1. $\forall x \in X\ [x \leq x]$
2. $\forall x, y, z \in X\ [(x \leq y \wedge y \leq z) \Rightarrow x \leq z]$

D6.4 Partial Order Irreflexive, Transitive

1. $\forall x \in X\ [x \not< x]$
2. $\forall x, y, z \in X\ [(x < y \wedge y < z) \Rightarrow x < z]$

D6.5 Linear Order Irreflexive, Transitive, Comparable

1. $\forall x \in X\ [x \lhd x]$
2. $\forall x, y, z \in X\ [(x \lhd y \wedge y \lhd z) \Rightarrow x \lhd z]$
3. $\forall x, y \in X\ [x = y \vee x \lhd y \vee y \lhd x]$

F6.6 Suppose $\langle X, <\rangle$ is a partial order. Define a relation $\leq$ on X by $x \leq y$ iff $x < y$ or $x = y$. Then $\langle X, \leq\rangle$ is a quasi order where $\forall x, y \in X\ [(x \leq y \wedge y \leq x) \Rightarrow x = y]$.

C6.8 If $\langle X, <\rangle$ is a partial order and $Y \subseteq X$ then $(Y \times Y) \cap <$ is a partial order on Y, as a shorthand for $\langle Y, ((Y \times Y) \cap <)\rangle$. Restricted to Y then Z is the same as restricting directly to Z.

D6.9 Maximal / Minimal Element $x \in X$ is maximal if $\forall y \in X\ [x \not< y]$. $x \in X$ is minimal if $\forall y \in X\ [y \not< x]$. There could be multiple in a partial order.

L6.10 A finite non-empty partial order has both a maximal and minimal element.

D6.11 $C \subseteq X$ is a chain if $\forall x, y \in C\ [x \text{ and } y \text{ are comparable}]$. $A \subseteq X$ is an antichain if $\forall x, y \in A\ [x \neq y \Rightarrow x \text{ and } y \text{ are incomparable}]$. A chain is maximal if there is no chain $C' \subseteq X$ where $C \subsetneq C'$. $\emptyset$ and singletons are chains and antichains.

L6.12 For a finite partial order, every chain or antichain is contained in a maximal chain or antichain.

D6.13 Well-Order Every non-empty subset has a minimal element. $\forall A \subseteq X\ [A \neq \emptyset \Rightarrow \exists a \in A\ \forall a' \in A\ [a \leq a']]$.

L6.15 (AC) A linear order $\langle X, <\rangle$ is a well-order iff there is no $f : \mathbb{N} \to X$ where $\forall n \in \mathbb{N}\ [f(n) > f(n+1)]$.

D6.16 For a linear order $\langle X, <\rangle$ $pred_{\langle X,<\rangle}(x) = \{x' \in X : x' < x\}$, or the set of predecessors of x in X for the ordering $<$. A subset $A \subseteq X$ is downwards closed if $\forall a \in A\ \forall x \in X\ [x < a \Rightarrow x \in A]$. The predecessor subset is downwards closed along with the entire set.

F6.17 For a linear order $\langle X, <\rangle$, if $A \subseteq X$ is downwards closed, $\forall a \in A\ pred_{\langle A,<\rangle}(a) = pred_{\langle X,<\rangle}(a)$.

F6.19 Let $\langle X, <\rangle$ be a well-order and A a downwards closed subset of X. Etiher $A = X$ or $\exists x \in X$ such that $A = pred_{\langle X,<\rangle}(x)$.

E6.20 If $\langle X, <\rangle$ is a well-order and $A \subseteq X$, $\langle A, <\rangle$ is a well-order.

E6.21 Let $\langle X, <\rangle$ be a linear order. $f : X \to X$ is expansive if $\forall x \in X\ [f(x) \geq x]$, and order-preserving if $\forall x, y \in X\ [x < y \Rightarrow f(x) < f(y)]$. If $\langle X, <\rangle$ is a well-order, every order-preserving f is expansive.

New orders from old

L6.23 Suppose X is a set and $\langle Y, \prec\rangle$ and $\langle Z, \lhd\rangle$ are partial orders. Suppose $f : X \to Y$ and $g : X \to Z$ are functions. Define $<$ on X by stipulating that $\forall x, x' \in X, x < x' \leftrightarrow (f(x) \prec f(x')\ or\ (f(x) = f(x')\ and\ g(x) \lhd g(x')))$.
Then,

1. $<$ is a partial order on X
2. if $\langle Y, \prec\rangle$ and $\langle Z, \lhd\rangle$ are both linear orders and $\forall x, x' \in X\ [(f(x) = f(x')\ and\ g(x) = g(x')) \Rightarrow x = x']$ then $<$ is also a linear order on X
3. similarly for well-orders

C6.24 Let X be a set and $\langle Y, \prec\rangle$ be a partial order. Suppose $f : X \to Y$ is a function. Define $<^*$ on X by $\forall x, x' \in X, x <^* x' \leftrightarrow f(x) \prec f(x')$. Then $<^*$ is a partial order on X. If f is $1-1$ and $\prec$ is a linear order on Y, then $<^*$ is a linear order on X. Similarly for well-orders. Use $Y = Z, f = g, \prec = \lhd$.

EP 6.25 Lexographic / Dictionary Order Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by setting $f(\langle m, n\rangle) = m$ and $g(\langle m, n\rangle) = n$ for all $\langle m, n\rangle \in \mathbb{N} \times \mathbb{N}$. By L6.23, the relation $<_{lex}$ defined on $\mathbb{N} \times \mathbb{N}$ by $\langle m, n\rangle <_{lex} \langle k, l\rangle \leftrightarrow (f(\langle m, n\rangle) \in f(\langle k, l\rangle)\ or\ (f(\langle m, n\rangle) = f(\langle k, l\rangle)\ and\ g(\langle m, n\rangle) \in g(\langle k, l\rangle))) \leftrightarrow (m \in k\ or\ (m = k\ and\ n \in l))$ is a well-order on $\mathbb{N} \times \mathbb{N}$.

D6.26 Suppose $\langle I, <\rangle$ is a well-order and X is a set. For $f, g \in X^I$, if $f \neq g$, define $\Delta(f, g) = min(\langle\{i \in I : f(i) \neq g(i)\}, <\rangle)$. It is well-defined for the conditions mentioned above.

L6.27 Suppose $\langle X, \lhd\rangle$ is a linear order and $\langle I, <\rangle$ is a well-order. Define $\prec$ on X^I $\forall f, g \in X^I$ by $f \prec g \leftrightarrow (f \neq g\ and\ f(\Delta(f, g)) \lhd g(\Delta(f, h)))$. Then $\prec$ is a linear order on X^I.

D6.28

- $[\mathbb{N}]^n = \{a \in \mathcal{P}(\mathbb{N}) : a \approx n\}$, or subsets of the naturals equinumerous to n.
- $\mathbb{N}^n = \{\sigma : \sigma \text{ is a function and } dom(\sigma) = n \wedge ran(\sigma) \subseteq \mathbb{N}\}$, or functions with domain n and range as a subset of the naturals.
- $[\mathbb{N}]^{<\omega} = \{a \in \mathcal{P}(\mathbb{N}) : a \text{ is finite}\}$, or subsets of the naturals equinumerous to finite n.
- $\mathbb{N}^{<\omega} = \{\sigma : \sigma \text{ is a function and } dom(\sigma) \in \mathbb{N} \wedge ran(\sigma) \subseteq \mathbb{N}\}$, or functions with a finite domain and range as a subset of naturals.
- $[\mathbb{N}]^{<\omega} = \bigcup_{n\in\mathbb{N}} [\mathbb{N}]^n$ and $\mathbb{N}^{<\omega} = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$.

EP6.29 $\langle \mathbb{N}, \in\rangle$ is a well-order. Define a linear order $<_{lex}$ on $\mathbb{N}^{\mathbb{N}}$ using L6.27. Since $2^{\mathbb{N}} \subseteq \mathbb{N}^{\mathbb{N}}$, $\langle 2^{\mathbb{N}}, <_{lex}\rangle$ is also a linear order. Define $F : \mathbb{N}^{<\omega} \to \mathbb{N}^{\mathbb{N}}$ as follows. $\sigma \in \mathbb{N}^{<\omega}$, $dom(\sigma) \in \mathbb{N}$ and $\sigma : dom(\sigma) \to \mathbb{N}$. Define $F(\sigma) : \mathbb{N} \to \mathbb{N}$ by $F(\sigma)(n) = \begin{cases} \sigma(n) & \text{if } n \in dom(\sigma) \\ dom(\sigma) & \text{if } n \notin dom(\sigma) \end{cases}$ for every $n \in \mathbb{N}$. F is $1-1$.
Since $<_{lex}$ is a linear order on $\mathbb{N}^{\mathbb{N}}$, by C6.24, $<_{lex}$ defined on $\mathbb{N}^{<\omega}$ by $\sigma <_{lex} \tau \leftrightarrow F(\sigma) <_{lex} F(\tau)$ is also a linear order. From EP2.36, we have a $1-1$ and onto $F : \mathcal{P}(\mathbb{N}) \to 2^{\mathbb{N}}$. We can induce $A <_{lex} B \leftrightarrow F(A) <_{lex} F(B)$ for $A, B \in \mathcal{P}(\mathbb{N})$. Since $[\mathbb{N}]^{<\omega} \subseteq \mathcal{P}(\mathbb{N})$, $\langle [\mathbb{N}]^{<\omega}, <_{lex}\rangle$ is also a linear order.

E6.30 Let $\langle X, <\rangle$ and $\langle Y, \prec\rangle$ be partial orders. Define $\langle X \times Y, \lhd\rangle$ by $\langle x, y\rangle \lhd \langle x', y'\rangle \leftrightarrow (x < x' \wedge y \prec y')$. This is a partial order.

E6.31 A linear order $\langle X, <\rangle$ is dense if $\forall x, z \in X$ with $x < z$, $\exists y \in X$ such that $x < y < z$. $\langle \mathbb{N}^{\mathbb{N}}, <_{lex}\rangle$ is dense while $\langle 2^{\mathbb{N}}, <_{lex}\rangle$ is not dense.

E6.32 $<_{lex}$ on $\mathcal{P}(\mathbb{N})$ is defined by the $1-1$ and onto F from $EP2.36$.

1. $\forall A, B \in \mathcal{P}(\mathbb{N})$, $A <_{lex} B$ iff $A \neq B \wedge min(A \triangle B) \in A$
2. $\langle [\mathbb{N}]^{<\omega} \backslash \{\emptyset\}, <_{lex} \rangle$ does not have any maximal or minimal elements
3. $\langle [\mathbb{N}]^{<\omega} \backslash \{\emptyset\}, <_{lex} \rangle$ is dense

6.3 Embeddings and Isomorphisms

D6.33 If $\langle X, \lhd \rangle$ and $\langle Y, \prec \rangle$ are linear orders, $f : X \to Y$ is an isomorphism between them if f is $1-1$ and onto and $\forall x, y \in X\ [x \lhd y \Leftrightarrow f(x) \prec f(y)]$. Two linear orders are isomorphic if f exists which is an isomorphism.
L6.34 $\langle X, \lhd \rangle$ and $\langle Y, \prec \rangle$ are linear orders. Suppose $f : X \to Y$ is an onto function such that $\forall x, y \in X\ [x \lhd y \Rightarrow f(x) \prec f(y)]$. f is an isomorphism.
D6.35 $\langle X, < \rangle$ and $\langle Y, \prec \rangle$ are linear orders. $f : X \to Y$ is an embedding if $\forall x, x' \in X\ [x < x' \Leftrightarrow f(x) \prec f(x')]$ and f is $1-1$. If there exists and embedding f, we say that $\langle X, < \rangle$ embeds into $\langle Y, \prec \rangle$ and $\langle X, < \rangle \hookrightarrow \langle Y, \prec \rangle$. $\langle X, < \rangle$ is isomorphic to $\langle Im_f(X), \prec \rangle$.
F6.36 $\langle X, < \rangle$ and $\langle Y, \prec \rangle$ are linear orders. If $f : X \to Y$ is a function such that $\forall x, x' \in X\ [x < x' \Rightarrow f(x) \prec f(x')]$, then f is an embedding.
F6.37 $\langle X, < \rangle$ and $\langle Y, \prec \rangle$ are linear orders. Suppose A and B are downwards closed subsets of X and Y. If $f : A \to B$ is an isomorphism from $\langle A, < \rangle$ to $\langle B, \prec \rangle$, then $\forall a \in A, f \upharpoonright pred_{\langle X, < \rangle}(a)$ is an isomorphism from $\langle pred_{\langle X, < \rangle}(a), < \rangle$ to $\langle pred_{\langle Y, \prec \rangle}(f(a)), \prec \rangle$.
T6.38 Suppose $\langle X, \lhd \rangle$ is a finite linear order. $\exists! n \in \mathbb{N}$ such that $\langle X, \lhd \rangle$ is isomorphic to $\langle n, \in \rangle$. This isomorphism is unique.
T6.39 Suppose $\langle X, \lhd \rangle$ is an infinite linear order such that $\forall x \in X, pred_{\langle X, \lhd \rangle}$ is finite. $\langle X, \lhd \rangle$ is isomorphic to $\langle \mathbb{N}, \in \rangle$. The isomorphism from X to $\mathbb{N}$ is unique.
CL6.40 $\forall x, y \in X\ [x \lhd y \Rightarrow f(x) \in f(y)]$.
CL6.41 $ran(f)$ is a downwards closed subset of $\langle \mathbb{N}, \in \rangle$.
D6.42 A linear order $\langle X, \lhd \rangle$ has type omega ω if X is infinite and $\forall x \in X$, $pred_{\langle X, \lhd \rangle}(x)$ is finite.

7. Countable and Uncountable Sets

Countable Sets

C7.1 If $X \subseteq \mathbb{N}$ is infinite, $\langle X, \in \rangle$ is isomorphic to $\langle \mathbb{N}, \in \rangle$.
C7.2 If X is infinite and countable, then $X \approx \mathbb{N}$.
T7.3 There exist linear orders of type omega on $\mathbb{N} \times \mathbb{N}$, $[\mathbb{N}]^{<\omega}$, and $\mathbb{N}^{<\omega}$. Define $f : \mathbb{N} \to X$ to show infinite, then $g : X \to \mathbb{N}$ and $h : X \to X$. Define $\prec$ as normal, and use it to show $pred_{\langle X, \prec \rangle}(x)$ is finite.
C7.4

1. $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$, $[\mathbb{N}]^{<\omega} \approx \mathbb{N}$, and $\mathbb{N}^{<\omega} \approx \mathbb{N}$
2. $\forall n \in \mathbb{N}\ [n \geq 1 \Rightarrow \mathbb{N}^n \ \wedge [\mathbb{N}]^{<\omega} \approx \mathbb{N}]$

L7.5 Suppose that $\langle A_n : n \in \mathbb{N} \rangle$ and $\langle f_n : n \in \mathbb{N} \rangle$ are sequences such that $\forall n \in \mathbb{N}$, $f_n : A_n \to \mathbb{N}$ is $1-1$. Then $\bigcup_{n \in \mathbb{N}} A_n$ is countable.
L7.6 (AC) A countable union of countable sets is countable. If $\langle A_n : n \in \mathbb{N} \rangle$ is a sequence of countable sets, then $\bigcup_{n \in \mathbb{N}} A_n$ is countable.
L7.8 The set of rationals $\mathbb{Q}$ is countable, i.e. $\mathbb{Q} \to \mathbb{N}$.
E7.9 Let $\langle X, < \rangle$, $\langle Y, \lhd \rangle$ and $\langle Z, \prec \rangle$ be linear orders. Suppose $f : X \to Y$ and $g : Y \to Z$ are isomorphisms. $f^{-1} : Y \to X$ and $g \circ f : X \to Z$ also are.
E7.10 Define a sequence of relations $\langle \lhd_n : n \in \mathbb{N} \rangle$ by induction on $n \in \mathbb{N}$. Let $\lhd_0$ be $\emptyset$. Suppose $\lhd_n$ is a relation on $[\mathbb{N}]^n$. Define $\lhd_{n+1}$ as follows. Fix $a, b \in [\mathbb{N}]^{S(n)}$. If $max(a) < max(b)$, $a \lhd_{n+1} b$. If $max(a) = max(b)$, $a \cap max(b)$ and $b \cap max(b)$ are in $[\mathbb{N}]^n$. Then if $a \cap max(a) \lhd_n b \cap max(b)$, $a \lhd_{n+1} b$. Now define $\langle \prec_n : n \in \mathbb{N} \rangle$ by induction. $\prec_0 = \emptyset$. Suppose $\lhd_n$ is a relation on $\mathbb{N}^n$. Define $\prec_{n+1}$ as follows. Fix $\sigma, \tau \in \mathbb{N}^{n+1}$. $ran(\sigma)$ and $ran(\tau)$ are finite non-empty subsets of $\mathbb{N}$. If $max(ran(\sigma)) < max(ran(\tau))$, $\sigma \prec_{n+1} \tau$. $\sigma \upharpoonright n$ and $\tau \upharpoonright n$ are members of $\mathbb{N}^n$. If $max(ran(\sigma)) = max(ran(\tau))$ and if $\sigma \upharpoonright n \prec_n \tau \upharpoonright n$, then $\sigma \prec_{n+1} \tau$. If $max(ran(\sigma)) = max(ran(\tau))$, $\sigma \upharpoonright n = \tau \upharpoonright n$, and $\sigma(n) < \tau(n)$, then $\sigma \prec_{n+1} \tau$.

1. $\langle \lhd_n : n \in \mathbb{N} \rangle$ is well-defined, $\forall n \in \mathbb{N}$, $\langle [\mathbb{N}]^n, \lhd_n \rangle$ is a linear order of type ω.
2. $\langle \prec_n : n \in \mathbb{N} \rangle$ is well-defined, $\forall n \in \mathbb{N}$, $\langle \mathbb{N}^n, \prec_n \rangle$ is a linear order of type ω.
3. We can then prove (2) and (3) of T7.3 without AC.

E7.11 Extenders for the following:

1. $f(0) = f(1) = 1$, $f(n) = f(n-1) + f(n-2)\ \forall n > 1$
$$\mathbf{E}(\sigma) = \begin{cases} 1 & dom(\sigma) \in 2\} \\ \sigma(\bigcup dom(\sigma)) + \sigma(\bigcup \bigcup dom(\sigma)) & \text{for valid } dom(\sigma) \text{ etc.} \\ \emptyset & \text{otherwise} \end{cases}$$
2. $f(n) = \lhd_n$ on $[\mathbb{N}]^n$
3. $f(n) = \prec_n$ on $\mathbb{N}^n$

$$\begin{cases} \emptyset & dom(\sigma)=0 \\ \{\langle s,t\rangle \in [\mathbb{N}]^{dom(\sigma)} \times [\mathbb{N}]^{dom(\sigma)} : [(\bigcup s) \in (\bigcup t)] \text{ or } [(\bigcup s)=(\bigcup t) \text{ and } \langle s\cap(\bigcup s), t\cap(\bigcup t)\rangle \in \sigma(\bigcup dom(\sigma))]\} & dom(\sigma) \neq 0. \end{cases}$$

$$E(\sigma) = \begin{cases} \emptyset & dom(\sigma)=0 \\ \{\langle s,t\rangle \in \mathbb{N}^{dom(\sigma)} \times \mathbb{N}^{dom(\sigma)} : [\bigcup ran(s) \in \bigcup ran(t)] \text{ or } [\bigcup ran(s) = \bigcup ran(t) \text{ and } \langle s\upharpoonright\bigcup dom(\sigma), t\upharpoonright\bigcup dom(\sigma)\rangle \in \sigma(\bigcup dom(\sigma))] \text{ or } [\bigcup ran(s) = \bigcup ran(t) \text{ and } s\upharpoonright\bigcup dom(\sigma) = t\upharpoonright\bigcup dom(\sigma) \text{ and } s(\bigcup dom(\sigma)) \in t(\bigcup dom(\sigma))]\} & \text{if } dom(\sigma) \neq 0 \end{cases}$$

7.2 Sets of Size Continuum

F7.12 If $x, y \in \mathbb{R}$ and $x < y$, there is a $q \in \mathbb{Q}$ with $x < q < y$.
L7.13 $2^{\mathbb{N}} \lesssim \mathbb{R} \lesssim \mathcal{P}(\mathbb{Q})$
T7.14 These sets are equinumerous: $2^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}}$, $\mathcal{P}(\mathbb{N} \times \mathbb{N})$, $\mathcal{P}(\mathbb{N})$, $\mathcal{P}(\mathbb{Q})$, $\mathbb{R}$. $2^{\mathbb{N}} \subseteq \mathbb{N}^{\mathbb{N}} \subseteq \mathcal{P}(\mathbb{N} \times \mathbb{N})$. So $2^{\mathbb{N}} \lesssim \mathbb{N}^{\mathbb{N}} \lesssim \mathcal{P}(\mathbb{N} \times \mathbb{N})$. $\mathbb{N} \approx \mathbb{N} \times \mathbb{N}$. By L5.22, $\mathcal{P}(\mathbb{N}) \lesssim \mathcal{P}(\mathbb{N} \times \mathbb{N}) \lesssim \mathcal{P}(\mathbb{N})$ so $\mathcal{P}(\mathbb{N}) \approx \mathcal{P}(\mathbb{N} \times \mathbb{N})$. By L7.8, $\mathbb{Q} \lesssim \mathbb{N}$. By F5.2, $2^{\mathbb{N}} \lesssim \mathbb{R} \lesssim \mathcal{P}(\mathbb{Q}) \lesssim \mathcal{P}(\mathbb{N}) \lesssim 2^{\mathbb{N}}$. We also have $2^{\mathbb{N}} \lesssim \mathbb{N}^{\mathbb{N}} \lesssim \mathcal{P}(\mathbb{N} \times \mathbb{N}) \lesssim 2^{\mathbb{N}}$.
D7.15 A set X has size continuum or size $\mathfrak{c}$ if $X \approx \mathcal{P}(\mathbb{N})$.
L7.16 $(r, s) = \{x \in \mathbb{R} : r < x < s\}$ has size $\mathfrak{c}$. The function
$$f(x) = \begin{cases} \frac{x-t}{s-x} & \text{if } x \geq t \\ \frac{x-t}{x-r} & \text{if } x < t \end{cases}$$
is well-defined and $1-1$ and onto.
E7.17 Let $l \subseteq \mathbb{R}^2$ be a line. Define $\phi : \mathbb{R} \to l$ as $\phi(x) = \langle x, mx + c \rangle$ and $\phi(y) = \langle c, y \rangle$ for the different line cases. $l \approx \mathbb{R}$. For $a < b$ and $c < d$, let $m = \frac{d-c}{b-a}$ and $p = c - ma$. Then $f : (a, b) \to (c, d)$ is $1-1$ and onto.
L7.21 If $A \lesssim B$ and $A \neq \emptyset$, then there exists an onto $g : B \to A$.
L7.22 (AC) Suppose A and B are sets and $f : B \to A$ is onto. Then $A \lesssim B$.
L7.23 Let A, B, C be sets and suppose $f : C \to B$ is onto. Then $A^B \lesssim A^C$.
C7.24 If $B \approx C$, then $A^B \approx A^C$.
C7.25 If $A \lesssim D$, $B \lesssim C$, and $B \neq \emptyset$, then $A^B \lesssim D^C$.
L7.26 There exists a sequence $\langle A_n : n \in \mathbb{N} \rangle$ of pairwise disjoint infinite subsets of $\mathbb{N}$ such that $\bigcup_{n \in \mathbb{N}} A_n = \mathbb{N}$. Define $B_n = \{\langle n, m \rangle : m \in \mathbb{N}\}$, and let $A_n = f^{-1}(B_n)$.
L7.27 Suppose A, B, and C are sets with $B \cap C = \emptyset$. Then $A^B \times A^C \approx A^{B \cup C}$. Define $F : A^B \times A^C \to A^{B \cup C}$, $F(f, g) = f \cup g$, and show bijectivity.
C7.28 $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \approx \mathbb{N}^{\mathbb{N}}$. Define $A \cup B = \mathbb{N}$, $A \cap B = \emptyset$. Then $\mathbb{N}^A \approx \mathbb{N}^{\mathbb{N}} \approx \mathbb{N}^B$ and $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \approx \mathbb{N}^A \times \mathbb{N}^B \approx \mathbb{N}^{A \cup B} \approx \mathbb{N}^{\mathbb{N}}$.
C7.29 $\mathbb{R}^2$ has size $\mathfrak{c}$. This can be used to count lines and planes. Use $\mathbb{R}^2 \approx \mathbb{R}^{\{0\}} \times \mathbb{R}^{\{1\}} \approx \mathbb{R} \times \mathbb{R} \approx \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \approx \mathbb{N}^{\mathbb{N}} \approx \mathbb{R}$.
EP7.30 Let $\mathfrak{L}$ denote the set of lines in $\mathbb{R}^2$. $\mathfrak{L} = \mathfrak{L}_0 \cup \mathfrak{L}_1$ where $\mathfrak{L}_0 = \{l : l \text{ satisfies } y = mx + c, c, m, c, \in \mathbb{R}\}$ and $\mathfrak{L}_1 = \{l : l \text{ satisfies } x = c, c, \in \mathbb{R}\}$. Then $\mathfrak{L}_0 \approx \mathbb{R}^2 \approx \mathbb{R}$ and $\mathfrak{L}_1 \approx \mathbb{R}$, so $\mathfrak{L}$ has size $\mathfrak{c}$.
L7.31 Let A, B, C be sets. $A^{(B \times C)} \approx (A^B)^C$. Define $F : A^{(B \times C)} \to (A^B)^C$ using $F(f) : C \to A^B$ as $F(f)(c)(b) = f(\langle b, c \rangle)$.
C7.32 $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \approx \mathbb{N}^{\mathbb{N}}$. Use $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \approx \mathbb{N}^{(\mathbb{N} \times \mathbb{N})}$, and $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$.
C7.33 $\mathbb{R}^{\mathbb{N}}$ has size $\mathfrak{c}$. Use $\mathbb{R} \approx \mathbb{N}^{\mathbb{N}}$.
D7.34 A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if for each $x \in \mathbb{R}$ and each $\epsilon > 0$, there exists $\delta > 0$ such that $Im_f((x - \delta, x + \delta)) \subseteq (f(x) - \epsilon, f(x) + \epsilon)$. A set $U \subseteq \mathbb{R}$ is an open interval if there exists $r, s \in \mathbb{R}$ such that $U = (r, s) = \{x \in \mathbb{R} : r < x < s\}$. $U \subseteq \mathbb{R}$ is open if it is the union of a collection of open intervals.
L7.35 There are only $\mathfrak{c}$ many continuous functions from $\mathbb{R}$ to $\mathbb{R}$.
L7.36 There are only $\mathfrak{c}$ many open subsets of $\mathbb{R}$.
E7.37 $\mathbb{R}^{\mathbb{R}} \approx 2^{\mathbb{R}}$. First, $2^{\mathbb{R}} \subseteq \mathbb{R}^{\mathbb{R}}$, so $2^{\mathbb{R}} \lesssim \mathbb{R}^{\mathbb{R}}$. Next, define $f : \mathbb{R} \to \mathbb{R}$. $f \in \mathbb{R}^{\mathbb{R}}$, $f \subseteq \mathbb{R} \times \mathbb{R}$, $f \in \mathcal{P}(\mathbb{R} \times \mathbb{R})$. So $\mathbb{R}^{\mathbb{R}} \subseteq \mathcal{P}(\mathbb{R} \times \mathbb{R})$, and $\mathbb{R}^{\mathbb{R}} \lesssim \mathcal{P}(\mathbb{R} \times \mathbb{R}) \approx 2^{\mathbb{R} \times \mathbb{R}} \approx 2^{\mathbb{R}}$ as $\mathbb{R} \approx \mathbb{R} \times \mathbb{R}$.
E7.38 There are only countably many algebraic real numbers. Almost all real numbers are transcendental. $a \in \mathbb{R}$ is algebraic if there exists a non-zero polynomial $p(X) \in \mathbb{Z}[X]$ such that $p(a) = 0$. If a is not algebraic, it is transcendental.
E7.39 A function $f : \mathbb{R} \to \mathbb{R}$ is increasing if $\forall x, y, \in \mathbb{R}\ [x \leq y \Rightarrow f(x) \leq f(y)]$. There are only $\mathfrak{c}$ many increasing functions.
E7.40 Let $X \subseteq 2^{\mathbb{N}}$ be countable. Then $(2^{\mathbb{N}} \backslash X) \approx 2^{\mathbb{N}}$. If T is the set of transcendental real numbers, $T \approx R$. Since $(2^{\mathbb{N}} \backslash X) \subseteq 2^{\mathbb{N}}$, $(2^{\mathbb{N}} \backslash X) \lesssim 2^{\mathbb{N}}$. We want to show that $2^{\mathbb{N}} \lesssim (2^{\mathbb{N}} \backslash X)$ so that we can apply Schröder Bernstein. Let $A, B \subseteq \mathbb{N}$ be infinite sets such that $A \cap B = \emptyset$ and $A \cup B = \mathbb{N}$. By C5.33, fix bijections $\psi : A \to \mathbb{N}$ and $\varphi : B \to \mathbb{N}$. Define $G : 2^{\mathbb{N}} \to (2^{\mathbb{N}} \backslash X)$ as follows by defining $G(f) : \mathbb{N} \to 2$. Let $n \in \mathbb{N}$. If $n \in A$, define $G(f)(n) = f(\psi(n)) \in 2$. If $n \in B$, $e(\varphi(n)) \in X \subseteq 2^{\mathbb{N}}$, and $e(\varphi(n))(n) \in 2$. If $e(\varphi(n))(n) = 0$, then $G(f)(n) = 1$, if $e(\varphi(n))(n) = 1$, then $G(f)(n) = 0$. Since either $n \in A$ or $n \in B$, $G(f)(n) \in 2$ and $G(f) \in 2^{\mathbb{N}}$. If $G(f) \in X$, then there exists $k \in \mathbb{N}$ with $e(k) = G(f)$ and $n \in B$ with $\varphi(n) = k$. But since $n \in B$, $G(f)(n) \neq e(\varphi(n))(n) = e(k)(n) = G(f)(n)$, a contradiction. This shows $G(f) \notin X$. Now we show that G is $1-1$. Fix $f \neq f' \in 2^{\mathbb{N}}$. There exists $k \in \mathbb{N}$ with $f(k) \neq f'(k)$. There exists $n \in A$ with $\psi(n) = k$. $G(f)(n) = f(\psi(n)) = f(k) \neq f'(k) = f'(\psi(n)) = G(f')(n)$. Then $G(f) \neq G(f')$. We have shown that $2^{\mathbb{N}} \lesssim (2^{\mathbb{N}} \backslash X)$ as needed.

8. More about Partial and Linear Orders

8.2 More about Linear Orders

D8.8 Let $\langle X, < \rangle$ be a partial order and $A \subseteq X$. $x \in X$ is an upper bound of A if $\forall a \in A\ [a \leq x]$. x is a lower bound if $\forall a \in A\ [x \leq a]$. Let U be the set of upper bounds of A and L be the set of lower bounds of A. If there exists $u \in U$ such that $\forall x \in U\ [u \leq x]$, then u is the supremum of A in X or $sup_X(A)$ or minimal upper bound. For L and $[x \leq l]$, it is called the infimum or $inf_X(A)$ or greatest lower bound. There can only be at most one supremum or infimum.
EP8.9 Let $A = \{p \in \mathbb{Q} : p^2 < 2\}$. $sup_{\mathbb{Q}}(A)$ and $inf_{\mathbb{Q}}(A)$ do not exist, but $sup_{\mathbb{R}}(A) = \sqrt{2}$ and $inf_{\mathbb{R}}(A) = -\sqrt{2}$.
D8.10 Let $\langle X, < \rangle$ be a linear order. A pair $\langle A, B \rangle$ is a cut of $\langle X, < \rangle$ if A is downwards closed, B is upwards closed, and A and B partition X i.e. $A \cap B = \emptyset$ and $A \cup B = X$.
F8.11 Suppose $\langle X, < \rangle$ is a linear order and $Y \subseteq X$. If $z \in X \backslash Y$, $A = \{a \in Y : a < z\}$, $B = \{b \in Y : z < b\}$, then $\langle A, B \rangle$ is a cut of $\langle Y, < \rangle$.
D8.13 A linear order $\langle X, < \rangle$ is dense if $\forall x, y \in X\ \exists z \in X\ [x < y \Rightarrow x < z < y]$.
D8.14 A linear order is without endpoints if it has neither a maximal nor minimal element. A countable dense linear order with no endpoints is universal for all countable linear orders; every countable linear order embeds into such an order.
T8.15 (Cantor, AC) Suppose $\langle X, < \rangle$ is a non-empty dense linear order without endpoints. Let $\langle Y, \prec \rangle$ be any countable linear order. Then $\langle Y, \prec \rangle \hookrightarrow \langle X, < \rangle$.
T8.16 (Cantor) Let $\langle X, < \rangle$ and $\langle Y, \prec \rangle$ be non-empty countable dense linear orders without endpoints. Then they are isomorphic.
E8.17 Embedding is a quasi-order (reflexive, transitive) on linear orders.
E8.19 $\langle X, < \rangle \hookrightarrow \langle Y, \prec \rangle \wedge \langle Y, \prec \rangle \hookrightarrow \langle X, < \rangle$ does not imply isomorphism. Take $X = \mathbb{Q} \cap [0, 1]$ and $Y = \mathbb{Q} \cap (0, 1)$.
E8.20 We can have $\langle X, < \rangle \not\hookrightarrow \langle Y, \prec \rangle \wedge \langle Y, \prec \rangle \not\hookrightarrow \langle X, < \rangle$ (incomparability). Take $X = \langle \mathbb{N}, \in \rangle$ and $Y = \langle \mathbb{N}, \ni \rangle$.