MA3205

AY24/25 Sem 2

by ngmh

Chapter 2 - 8

EP2.36 There is a bijection $F : \mathcal{P}(X) \to \{0,1\}^X$.

$F(a)(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \notin A \end{cases}$ which is $1-1$ and onto.

D2.37 Cartesian Product Let F be a function with $dom(F)$ as a set. $\prod F = \{f : f \text{ is a function} \wedge dom(f) = dom(F) \wedge \forall x \in dom(F)\ [f(x) \in F(x)]\}$. If $F = \langle A_i : i \in I \rangle$, then $\prod F = \prod_{i \in I} A_i = \{f : f \text{ is a function} \wedge dom(f) = I \wedge \forall i \in I\ [f(i) \in A_i]\}$

A2.38 Axiom of Choice If $\langle A_i : i \in I \rangle$ is a sequence of sets such that $\forall i \in I\ [A_i \neq \emptyset]$, then $\prod_{i \in I} A_i \neq \emptyset$

T5.11 Schröder-Bernstein If $A \precsim B$ and $B \precsim A$, then $A \approx B$.

D5.19 A set is finite if there exits $n \in \mathbb{N}$ such that $n \approx A$. A is infinite if it is not finite. A is countable if $A \precsim \mathbb{N}$. A is uncountable if it is not countable.

D6.45/13 Partial/Linear/Well Order

1. $\forall x \in X\ [x \not< x]$
2. $\forall x, y, z \in X\ [(x < y \wedge y < z) \Rightarrow x < z]$
3. $\forall x, y \in X\ [x = y \vee x \lhd y \vee y \lhd x]$ (linear order)
4. $\forall A \subseteq X\ [A \neq \emptyset \Rightarrow \exists a \in A\ \forall a' \in A\ [a \leq a']]$ (well order)

D6.9 Maximal / Minimal Element $x \in X$ is maximal if $\forall y \in X\ [x \not< y]$.

D6.11 $C \subseteq X$ is a chain if $\forall x, y \in C\ [x \text{ and } y \text{ are comparable}]$. $A \subseteq X$ is an antichain if $\forall x, y \in A\ [x \neq y \Rightarrow x \text{ and } y \text{ are incomparable}]$. A chain is maximal if there is no chain $C' \subseteq X$ where $C \subsetneq C'$. $\emptyset$ and singletons are chains and antichains.

L6.12 For a finite partial order, every chain or antichain is contained in a maximal chain or antichain.

D6.16 For a linear order $\langle X, < \rangle$ $pred_{\langle X,<\rangle}(x) = \{x' \in X : x' < x\}$, or the set of predecessors of x in X for the ordering $<$. A subset $A \subseteq X$ is downwards closed if $\forall a \in A\ \forall x \in X\ [x < a \Rightarrow x \in A]$. The predecessor subset is downwards closed along with the entire set.

D6.33/34 If $\langle X, \lhd \rangle$ and $\langle Y, \prec \rangle$ are linear orders, $f : X \to Y$ is an isomorphism between them if f is $(1-1$ and) onto and $\forall x, y \in X\ [x \lhd y \Leftrightarrow f(x) \prec f(y)]$. Two linear orders are isomorphic if f exists which is an isomorphism.

D6.35 $\langle X, < \rangle$ and $\langle Y, \prec \rangle$ are linear orders. $f : X \to Y$ is an embedding it is $1-1$ and order preserving. $\langle X, < \rangle \hookrightarrow \langle Y, \prec \rangle$.

D6.42 A linear order $\langle X, \lhd \rangle$ has type omega ω if X is infinite and $\forall x \in X$, $pred_{\langle X,\lhd\rangle}(x)$ is finite.

L7.6 (AC) A countable union of countable sets is countable.

L7.8 $\mathbb{Q} \approx \mathbb{N}$.

F7.12 If $x, y \in \mathbb{R}$ and $x < y$, there is a $q \in \mathbb{Q}$ with $x < q < y$.

T7.14/15 $\approx$: $2^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}, \mathcal{P}(\mathbb{N} \times \mathbb{N}), \mathcal{P}(\mathbb{N}), \mathcal{P}(\mathbb{Q}), \mathbb{R}, \mathfrak{c}, \mathbb{R}^2\ \mathbb{R}^{\mathbb{N}}$.

C7.24 If $B \approx C$, then $A^B \approx A^C$.

C7.25 If $A \precsim D$, $B \precsim C$, and $B \neq \emptyset$, then $A^B \precsim D^C$.

L7.27 Suppose A, B, and C are sets with $B \cap C = \emptyset$. $A^B \times A^C \approx A^{B \cup C}$.

C7.28 $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \approx \mathbb{N}^{\mathbb{N}}$ using $A \cup B = \mathbb{N}, A \cap B = \emptyset$.

L7.31/32 Let A, B, C be sets. $A^{(B \times C)} \approx (A^B)^C$. $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \approx \mathbb{N}^{\mathbb{N}}$.

E7.37 $\mathbb{R}^{\mathbb{R}} \approx 2^{\mathbb{R}}$.

E7.39 There are only $\mathfrak{c}$ many increasing functions.

D8.8 Let $\langle X, < \rangle$ be a partial order and $A \subseteq X$. $x \in X$ is an upper bound of A if $\forall a \in A\ [a \leq x]$. x is a lower bound if $\forall a \in A\ [x \leq a]$. Let U be the set of upper bounds of A and L be the set of lower bounds of A. If there exists $u \in U$ such that $\forall x \in U\ [u \leq x]$, then u is the supremum of A in X or $sup_X(A)$ or minimal upper bound. For L and $[x \leq l]$, it is called the infimum or $inf_X(A)$ or greatest lower bound. There can only be at most one supremum or infimum.

D8.13 A linear order $\langle X, < \rangle$ is dense if $\forall x, y \in X\ \exists z \in X\ [x < y \Rightarrow x < z < y]$.

D8.14 A linear order is without endpoints if it has neither a maximal nor minimal element. A countable dense linear order with no endpoints is universal for all countable linear orders; every countable linear order embeds into such an order.

T8.15 (Cantor, AC) Suppose $\langle X, < \rangle$ is a non-empty dense linear order without endpoints. Let $\langle Y, \prec \rangle$ be any countable linear order. Then $\langle Y, \prec \rangle \hookrightarrow \langle X, < \rangle$.

T8.16 (Cantor) Let $\langle X, < \rangle$ and $\langle Y, \prec \rangle$ be non-empty countable dense linear orders without endpoints. Then they are isomorphic.

9. Well-Ordered Sets

F9.1 If $\langle X, < \rangle$ is a linear order of type ω, then it is a well-order.

L9.2 Suppose A and B are downwards closed subsert of X where $\langle X, < \rangle$ is a well-order. If $\langle A, < \rangle \cong \langle B, < \rangle$, then $A = B$.

C9.3 Suppose $\langle X, < \rangle$ is a well-order, and $x < x' \in X$. Then $\langle pred_{\langle X,<\rangle}(x'), < \rangle \ncong \langle pred_{\langle X,<\rangle}(x), < \rangle$.

C9.4 Suppose $\langle X, < \rangle$ is a well-order. Then for any $x \in X$, $\langle pred_{\langle X,<\rangle}(x), < \rangle \ncong \langle X, < \rangle$.

L9.5 If $\langle X, < \rangle$ and $\langle Y, \prec \rangle$ are isomorphic well-orders, then the isomorphism between them is unique.

T9.6 Suppose $\langle X, < \rangle$ and $\langle Y, \prec \rangle$ are well-orders. Then exactly one of the following holds:

1. $\langle X, < \rangle \cong \langle Y, \prec \rangle$
2. $\exists x \in X\ [\langle pred_{\langle X,<\rangle}(x), < \rangle \cong \langle Y, \prec \rangle]$
3. $\exists y \in Y\ [\langle X, < \rangle \cong \langle pred_{\langle Y,\prec\rangle}(y), \prec \rangle]$

E9.11 Define the product of X and Y to be $Z = Y \times X$. The dictionary order $\lhd$ on Z is a well-order.

E9.13 Given well-orders $\langle X, <_X \rangle \cong \langle A, <_A \rangle, \langle Y, <_Y \rangle \cong \langle B, <_B \rangle$, then the product and sum of $\langle X, <_X \rangle, \langle Y, <_Y \rangle \cong \langle A, <_A \rangle, \langle B, <_B \rangle$.

10. Ordinals

$\mathbf{WO} = \{\langle X, < \rangle : X \text{ is a set} \wedge < \text{is a well-ordering of } X\}$

10.1 Basic Properties of Ordinals

D10.1 A set x is transitive if every element of x is a subset of x, or $\forall y\ [y \in x \Rightarrow y \subseteq x]$.

D10.2 Ordinals A set α is an ordinal if it is transitive and well-ordered by $\in$. Let $\in_\alpha = \{\langle \beta, \gamma \rangle \in \alpha \times \alpha : \beta \in \gamma\}$. α is an ordinal if α is transitive and $\langle \alpha, \in_\alpha \rangle$ is a well-order. The subscript of $\in_\alpha$ is often omitted.

F10.3 $\mathbb{N}$ is an ordinal. Every $n \in \mathbb{N}$ is also an ordinal.

T10.4 Let x be an ordinal. The following hold:

1. $\forall y \in x\ [y \text{ is an ordinal} \wedge y = pred_{\langle x, \in \rangle}(y)]$
2. if y is any ordinal and $\langle x, \in \rangle \cong \langle y, \in \rangle$ then $x = y$
3. if y is any ordinal, then exactly one of the following things hold: $x \in y, x = y, y \in x$
4. if y, z are ordinals and $x \in y$ and $y \in z$, then $x \in z$
5. if $\mathbf{C}$ is a non-empty class of ordinals, then $\exists y \in \mathbf{C}\ \forall z \in \mathbf{C}\ [y \in z \vee y = z]$

D10.5 $\mathbf{ORD} = \{\alpha : \alpha \text{ is an ordinal}\}$ is the class of all ordinals.

T10.6 Burali-Forti $\mathbf{ORD}$ is not a set.

L10.7 Every transitive set of ordinals is an ordinal.

T10.8 Let $\langle X, < \rangle$ be a well-ordered set. Then there exists a unique α such that $\langle X, < \rangle \cong \langle \alpha, \in_\alpha \rangle$.

D10.11 If $\langle X, < \rangle$ is any well-ordered set, then $otp(\langle X, < \rangle)$, or the order type of $\langle X, < \rangle$ is the unique ordinal α such that $\langle X, < \rangle$ is isomorphic to $\langle \alpha, \in_\alpha \rangle$.

L10.13 For ordinals α, β, $\alpha \leq \beta$ iff $\alpha \subseteq \beta$.

L10.14 If A is a non-empty set of ordinals, then $min(A) = \bigcap A$. If A is any set of ordinals, then $sup_{\mathbf{ORD}}(A) = \bigcup A$.

L10.15 For any α, $S(\alpha)$ is an ordinal, $\alpha < S(\alpha)$, and $\forall \beta\ [\beta < S(\alpha) \iff \beta \leq \alpha]$.

D10.16 α is a successor ordinal if $\exists \beta\ [\alpha = S(\beta)]$. α is a limit ordinal if $\alpha \neq 0$ and α is not a successor ordinal.

L10.17 An ordinal α is a natural number iff $\forall \beta \leq \alpha\ [\beta = 0 \vee \beta \text{ is a successor ordinal}]$.

CV10.18 $\omega = \mathbb{N}$.

Induction and Recursion on the Ordinals

T10.19 Let $P(\alpha)$ be some property. If $\forall \alpha \in \mathbf{ORD}\ [\forall \beta < \alpha\ [P(\beta)] \implies P(\alpha)]$ then $\forall \alpha \in \mathbf{ORD}\ [P(\alpha)]$.

D10.20 Let $\mathbf{FOD} = \{\sigma : \sigma \text{ is a function } \wedge \exists \alpha \in \mathbf{ORD}\ [dom(\sigma) = a]\}$ denote the class of all functions whose domain is some ordinal. An ordinal extender is a function $\mathbf{E} : \mathbf{FOD} \to \mathbf{V}$. When you plug in a function with domain α into an ordinal extender, the output tells you what the value of the function at α ought to be.

T10.21 Suppose $\mathbf{E} : \mathbf{FOD} \to \mathbf{V}$ is any extender. Then there exists a unique function $\mathbf{F} : \mathbf{ORD} \to \mathbf{V}$ satisfying the condition that $\forall \alpha \in \mathbf{ORD}\ [\mathbf{F}(\alpha) = \mathbf{E}(\mathbf{F} \restriction \alpha)]$. The function generated is a proper class and not a set. $\mathbf{F} \restriction \alpha$ is a function with $dom(\mathbf{F} \restriction \alpha) = \alpha$ because $\alpha \subseteq \mathbf{ORD}$.

EP10.24 Define $V_0 = \emptyset$. Fix $\alpha \in \mathbf{ORD}$ and suppose V_β is given for all $\beta < \alpha$. If $\alpha = S(\beta)$ for some β let $V_\alpha = \mathcal{P}(V_\beta)$. If α is a limit ordinal, then $V_\alpha = \bigcup\{V_\beta : \beta < \alpha\}$. Define $\mathbf{E} : \mathbf{FOD} \to \mathbf{V}$ as follows. Fix $\sigma \in \mathbf{FOD}$. Let $\alpha = dom(\sigma) \in \mathbf{ORD}$. If $\alpha = 0, \mathbf{E}(\sigma) = \emptyset$. If α is a successor ordinal, $\exists! \beta, S(\beta) = \alpha$. Then $\beta \in \alpha$, so $\sigma(\beta)$ is defined and in $\mathbf{V}$. Let $\mathbf{E}(\sigma) = \mathcal{P}(\sigma(\beta))$. If α is a limit ordinal, then let $\mathbf{E}(\sigma) = \bigcup ran(\sigma)$.

E10.26 Call $\mathbf{C}$ trans-finitely inductive if:

1. $0 \in \mathbf{C}$
2. $\forall x \in \mathbf{C}\ [S(x) \in \mathbf{C}]$
3. for any set $X \subseteq \mathbf{C}, \bigcup X \in \mathbf{C}$

$\mathbf{ORD}$ is the smallest trans-finitely inductive class.

E10.27 Let α be any ordinal. If $X \subseteq \alpha$, then $otp(\langle X, \in \rangle) \leq \alpha$.

E10.28 Let α be any ordinal. α is a limit ordinal iff $\bigcup \alpha = \alpha$.

E10.29 For EP10.24, for each $\alpha \in \mathbf{ORD}$, V_α is transitive and $\bigcup_{\alpha \in \mathbf{ORD}} V_\alpha$ is transitive, $\alpha \subseteq V_\alpha$ and $\alpha \notin V_\alpha$.

11. Ordinal Arithmetic

11.1 Addition and Multiplication

D11.1 Let $\langle X, <_X \rangle$ and $\langle Y, <_Y \rangle$ be well-orders. Define $X \oplus Y = (\{0\} \times X) \cup (\{1\} \times Y)$. Define $<_{X \oplus Y}$ to be:

1. $\forall x, x' \in X\ [\langle 0, x \rangle <_{X \oplus Y} \langle 0, x' \rangle \iff x <_X x']$
2. $\forall y, y' \in Y\ [\langle 1, y \rangle <_{X \oplus Y} \langle 1, y' \rangle \iff y <_Y y']$
3. $\forall x \in X \forall y \in Y\ [\langle 0, x \rangle <_{X \oplus Y} \langle 1, y \rangle]$

Then it is a well-order.

D11.2 Suppose α and β are ordinals. Define $\alpha + \beta$ to be the order-type of the well-order $\langle \alpha \oplus \beta, <_{\alpha \oplus \beta} \rangle$, where $<_\alpha = \in_\alpha$ and $<_\beta = \in_\beta$.

L11.4 Let $\langle X, <_X \rangle, \langle Y, <_Y \rangle, \langle Z, <_Z \rangle$ be well-orders. Suppose $A, B \subseteq Z$. Assume $A \cup B = Z$ and $\forall a \in A\ \forall b \in B\ [a <_Z b]$. Then if $\langle A, <_Z \rangle \cong \langle X, <_X \rangle$ and $\langle B, <_Z \rangle \cong \langle Y, <_Y \rangle$, then $\langle Z, <_Z \rangle \cong \langle X \oplus Y, <_{X \oplus Y} \rangle$.

L11.5 For any α, β, γ:

1. $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$
2. $\alpha + 0 = \alpha$
3. $\alpha + 1 = S(\alpha)$
4. $\alpha + S(\beta) = S(\alpha + \beta)$
5. if β is a limit ordinal, then $\alpha + \beta = sup\{\alpha + \xi : \xi < \beta\}$

R11.6 $(2), (3), (5)$ can be used to give an inductive definition of $+$. For a fixed α, we can define $\dot{+}$ which is equivalent to $+$ on $\mathbf{ORD}$ by:

1. $\alpha \dot{+} 0 = \alpha$
2. $\alpha \dot{+} S(\beta) = S(\alpha \dot{+} \beta)$
3. if β is a limit ordinal, then $\alpha \dot{+} \beta = sup\{\alpha \dot{+} \xi : \xi < \beta\}$

D11.7 Let α and β be ordinals. Let $<_{\alpha\cdot\beta}$ be the dictionary order on $\beta\times\alpha$. That is, for $\langle\zeta,\xi\rangle,\langle\zeta',\xi'\rangle\in\beta\times\alpha$, $\langle\zeta,\xi\rangle<_{\alpha\cdot\beta}\langle\zeta',\xi'\rangle$ iff either $\zeta<\zeta'$ or $\zeta=\zeta'$ and $\xi<\xi'$. Then it is a well-order and $\alpha\cdot\beta=otp(\langle\beta\times\alpha,<_{\alpha\cdot\beta}\rangle)$ which is β copies of α.

L11.8 Suppose α,β,γ are ordinals. Suppose $A\subseteq\gamma$ and $\langle A,\in\rangle\cong\langle\beta,\in\rangle$. Then $\langle A\times\alpha,<_{\alpha\cdot\gamma}\rangle\cong\langle\beta\times\alpha,<_{\alpha\cdot\beta}\rangle$.

L11.9 For any α,β,γ:

1. $\alpha\cdot(\beta\cdot\gamma)=(\alpha\cdot\beta)\cdot\gamma$
2. $\alpha\cdot 0=0$
3. $\alpha\cdot 1=\alpha$
4. $\alpha\cdot S(\beta)=a\cdot\beta+\alpha$
5. if β is a limit ordinal, $\alpha\cdot\beta=sup\{a\cdot\xi:\xi<\beta\}$
6. $\alpha\cdot(\beta+\gamma)=\alpha\cdot\beta+\alpha\cdot\gamma$

Also, $\cdot$ is not commutative on $\mathbf{ORD}$ since $2\cdot\omega\neq\omega\cdot 2$. (6) fails for multiplication on the right since $(1+1)\cdot\omega=\omega\neq 1\cdot\omega+1\cdot\omega$.

Exponentiation

D11.10 For a fixed α, define α^β by recursion on β using the following clauses:

1. if $\alpha=0$,then $\alpha^0=0$; if $\alpha>0$, then $\alpha^0=1$
2. $\alpha^{\beta+1}=\alpha^\beta\cdot\alpha$
3. if β is a limit ordinal, then $\alpha^\beta=sup\{a^\xi:\xi<\beta\}$

E11.11 Define the extender $\mathbf{E}^+_\alpha:\mathbf{FOD}\to\mathbf{V}$ as follows. For any $\sigma\in\mathbf{FOD}$,

$$\mathbf{E}^+_\alpha(\sigma)=\begin{cases}\alpha & \text{if } dom(\sigma)=0\\ S(\sigma(\beta)) & \text{if } dom(\sigma)=S(\beta)\\ \bigcup ran(\sigma) & \text{if } dom(\sigma) \text{ is a limit ordinal}\end{cases}$$

For other operations need to check the case where $\sigma(\beta)\notin\mathbf{ORD}$.

E11.12 For any ordinal $\alpha>0,\alpha\cdot\omega>\alpha$.

E11.13 $\alpha<\beta\Rightarrow\gamma+\alpha<\gamma+\beta\wedge\alpha+\gamma\le\beta+\gamma$ but not $<$ on the second clause.

E11.14 If $\alpha\ge\omega$ is an ordinal, then $1+\alpha=\alpha$.

E11.15 If $\gamma>0$, then $\alpha<\beta\Rightarrow\gamma\cdot\alpha<\gamma\cdot\beta\wedge\alpha\cdot\gamma\le\beta\cdot\gamma$ but not $<$ on the second clause.

E11.16 Let $0<\alpha\le\beta$ be ordinals. There exist unique δ,ξ such that $\xi<\alpha$ and $\alpha\cdot\delta+\xi=\beta$.

E11.17 $\alpha^{(\beta+\gamma)}=\alpha^\beta\cdot\alpha^\gamma$ for ordinals $\alpha>0$.

E11.18 Define $\alpha_0=\omega$ and $\forall n\in\omega, a_{n+1}=\omega^{\alpha_n}$. Let $\epsilon_0=sup\{\alpha_n:n\in\omega\}$. Then $\omega^{\epsilon_0}=\epsilon_0$.

Cardinals and Cardinal Arithmetic

D12.1 A set X is said to be well-orderable if there exists a relation $<\subseteq X\times X$ such that $\langle X,<\rangle$ is a well-order.

D12.2 Let X be a well-orderable set. Define the cardinality of X, $|X|$, to be the minimal element of $\{\alpha\in\mathbf{ORD}:\alpha\approx X\}$. $|\alpha|$ is defined for every $\alpha\in\mathbf{ORD}$, and $|\alpha|\le\alpha$.

D12.3 α is a cardinal if $|\alpha|=\alpha$.

F12.4 If $n\in\omega$, then n is a cardinal. ω is a cardinal.

L12.5 If $|\alpha|\le\beta\le\alpha$, then $|\beta|=|\alpha|$.

L12.6 A se is finite iff $|X|<\omega$. A set is countable iff $|X|\le\omega$.

D12.7 Let κ and λ be cardinals. These are well-orderable:

1. $\kappa\boxplus\lambda=|(\{0\}\times\kappa)\cup(\{1\}\times\lambda)|$
2. $\kappa\boxtimes\lambda=|\kappa\times\lambda|$

L12.8 Every infinite cardinal is a limit ordinal.

T12.9 If κ is an infinite cardinal, then $\kappa\boxtimes\kappa=\kappa$.

C12.10 Let κ and λ be infinite cardinals. Then $\kappa\boxplus\lambda=\kappa\boxtimes\lambda=max\{\kappa,\lambda\}$.

T12.11 For every set X there is a cardinal α such that there is no 1-1 function $f:\alpha\to X$.

D12.16 For each $\alpha\in\mathbf{ORD}$, α^+ is the least cardinal strictly greater than α.

L12.17 Suppose $\mathbf{F}:\mathbf{ORD}\to\mathbf{ORD}$ is a function such that $\forall\alpha,\beta\in\mathbf{ORD}\,[\alpha<\beta\Rightarrow\mathbf{F}(\alpha)<\mathbf{F}(\beta)]$. Then $\forall\beta\in\mathbf{ORD}\,[\beta\le\mathbf{F}(\beta)]$.

D12.18 Define a sequence $\langle\omega_\alpha:\alpha\in\mathbf{ORD}\rangle$ by induction using the following clauses:

1. $\omega_0=\omega$
2. $\omega_{S(\alpha)}=\omega_\alpha^+$
3. if α is a limit ordinal, then $\omega_\alpha=sup\{\omega_\xi:\xi<\alpha\}$

R12.19 ω_α is sometimes deonted as $\aleph_\alpha$.

L12.20 $\alpha<\beta\implies\aleph_\alpha<\aleph_\beta$ and every infinite cardinal is equal to $\aleph_\alpha$ for some $\alpha\in\mathbf{ORD}$.

12.1 Choice and Cardinality

D12.21 Let X be any set. F is a choice function on X if F is a function, $dom(F)=X\backslash\{0\}$, and $\forall a\in X\backslash\{0\}[F(a)\in a]$.

T12.22 Zermelo TFAE for a set X:

1. X is well-orderable
2. there exists a choice function on $\mathcal{P}(X)$

T12.26 AC TFAE:

1. the Cartesian product of non-empty sets is non-empty
2. for every set X there exists a choice function on X
3. every set is well-orderable
4. for any two sets X and Y, either $X\lessapprox Y$ or $Y\lessapprox X$
5. for every set X there is an ordinal α and a 1-1 function $f:X\to\alpha$
6. for every set X there is a cardinal κ such that $X\approx\kappa$

Cardinal Exponentiation and König's Theorem

D12.28 (AC) Let κ and λ be cardinals. Define $\kappa^\lambda=|\{f:f \text{ is a function}\wedge dom(f)=\lambda\wedge ran(f)\subseteq\kappa\}|$.

L12.30 Let κ,λ,θ be cardinals. The following hold:

1. $(\kappa^\lambda)^\theta=\kappa^{(\lambda\boxtimes\theta)}$
2. $(\kappa^\lambda)\boxtimes(\kappa^\theta)=\kappa^{(\lambda\boxplus\theta)}$

D12.31 Define a sequence of cardinals $\langle\beth_\alpha:\alpha\in\mathbf{ORD}\rangle$ by induction using the following clauses:

1. $\beth_0=\omega$
2. $\beth_{S(\alpha)}=2^{\beth_\alpha}$
3. if α is a limit ordinal, then $\beth_\alpha=sup\{\beth_\xi:\xi<\alpha\}$

D12.32 The Generalised Continuum Hypothesis is the statement that $\forall\alpha\in\mathbf{ORD}\,[\beth_\alpha=\aleph_\alpha]$. The Continuum Hypothesis is the statement that $\beth_1=\aleph_1$. Note $\beth_1=2^{\beth_0}=2^{\aleph_0}$, so CH says $2^{\aleph_0}=\aleph_1$.

T12.34 König $(\aleph_\omega)^{\aleph_0}>\aleph_\omega$.

C12.35 $2^{\aleph_0}\neq\aleph_\omega$.

E12.36 Let κ,λ be infinite cardinals where $\lambda\le\kappa$. Then $\kappa^\lambda=|\{X\subseteq\kappa:|X|=\lambda\}|$.

E12.37 Let $\kappa,\lambda,\theta,\chi$ be cardinals. If $\kappa\le\lambda$, then $\kappa^\theta\le\lambda^\theta$. If $\kappa\le\chi,\lambda\le\theta$ and $\lambda\neq 0$, then $\kappa^\lambda\le\chi^\theta$.

E12.38 Let α be an ordinal. Let $W=\{\langle Y,\lhd\rangle:Y\subseteq\alpha\wedge\langle Y,\lhd\rangle \text{ is a well-order}\}$. $\alpha^+=\{otp(\langle Y,\lhd\rangle):\langle Y,\lhd\rangle\in W\}$.

E12.39 There is a cardinal $\kappa=\aleph_\kappa$ and $\kappa=\beth_\kappa$.

E12.40 Suppose $\mathbf{F}:\mathbf{ORD}\to\mathbf{ORD}$ and $\forall\alpha,\beta\in\mathbf{ORD}\,[\alpha<\beta\implies\mathbf{F}(\alpha)<\mathbf{F}(\beta)]$ and for any limit ordinal β, $\mathbf{F}(\beta)=sup\{\mathbf{F}(\alpha):\alpha<\beta\}$. Then $\forall\alpha\in\mathbf{ORD}\,\exists\beta>\alpha\,[\mathbf{F}(\beta)=\beta]$.

E12.41 $(\aleph_{\omega_1})^{\aleph_1}>\aleph_{\omega_1}$ and $2^{\aleph_1}\neq\aleph_\omega,\aleph_{\omega_1}$.

13. Some applications of AC

D13.1 Let A be any set. $\mathcal{F}\subseteq\mathcal{P}(A)$ is of finite character iff $\forall X\subseteq A, X\in\mathcal{F}\iff\forall Y\subseteq X\,[|Y|<\omega\implies Y\in\mathcal{F}]$. All of X's finite subsets are in $\mathcal{F}$.

L13.2 Suppose $\mathcal{F}\subseteq\mathcal{P}(A)$ is of finite character. Then for any $X\in\mathcal{F}$ and any $Y\subseteq X, Y\in\mathcal{F}$.

T13.3 TFAE:

1. AC
2. for any set A and any $\mathcal{F}\subseteq\mathcal{P}(A)$, if F has finite character, then for every $X\in\mathcal{F}$, there exists $Y\in\mathcal{F}$ such that $X\subseteq Y$ and Y is maximal in $\langle\mathcal{F},\subsetneq\rangle$ (Teichmüller-Tukey Lemma)
3. every chain in every partial order is contained in a maximal chain (Hausdorff's maximal chain theorem)
4. if $\langle X,<\rangle$ is any partial order where every chain in $\langle X,<\rangle$ has an upper bound in $\langle X,<\rangle$, then $\langle X,<\rangle$ has a maximal element (Zorn's lemma)

$(4)\implies(1)$: Prove the standard version of AC. Let I be any set and suppose $\langle X_i:i\in I\rangle$ is any sequence of non-empty sets. Consider $A=\{\sigma:\sigma \text{ is a function}\wedge dom(\sigma)\subseteq I\wedge\forall i\in dom(\sigma)\,[\sigma(i)\in X_i]\}$. Partially order A by $\subsetneq$. Let $C\subseteq A$ be any chain. C is a directed collection of functions. So $\bigcup C=\tau$ is a function and $dom(\tau)=\bigcup\{dom(\sigma):\sigma\in C\}\subseteq I$. $\tau(i)=\sigma(i)\in X_i$. Therefore $\tau\in A$ and $\forall\sigma\in C\,[\sigma\subseteq\tau]$. So τ is an upper bound for C. So every chain has an upper bound and there is a maximal $\sigma\in A$ by Zorn's lemma. We claim that $dom(\sigma)=I$. If not, there exists $i\in I\backslash dom(\sigma)$. Since $X_i\neq 0$, choose $x_i\in X_i$. Put $\tau=\sigma\cup\{\langle i,x_i\rangle\}$. Then $\tau\in A$ and $\sigma\subsetneq\tau$, contradicting maximality of σ.

Using Zorn's Lemma $\langle X,<\rangle$ is a partial order. If every chain in $\langle X,<\rangle$ has an upper bound in $\langle X,<\rangle$, then $\langle X,<\rangle$ has a maximal element.

1. Find a relevant $\langle X,<\rangle$, e.g. $\langle\mathcal{P}(X),\subsetneq\rangle$ which might be given.
2. Take a chain $\zeta\subseteq X$. Show ζ has an upper bound in $\langle X,<\rangle$. Usually this involves taking unions of things in ζ. But you have to check that these unions belong to X. Also, $\zeta=\emptyset$ is always a chain.
3. By Zorn's lemma $\exists x\in X$ maximal in $\langle X,<\rangle$. Now maximality will imply x has some special property. Most of the time you check x has the relevant property, because if it did not it would contradict maximality in $\langle X,<\rangle$.

E13.16 Let $\langle X,<\rangle$ be a partial order. Every antichain in $\langle X,<\rangle$ is contained in a maximal antichain. Let $A\subseteq X$ be an antichain.

1. $P=\{B\subseteq X:A\subseteq B\wedge B \text{ is an antichain}\}$. $\langle P,\subsetneq\rangle$ is a partial order.
2. Let $\zeta\subseteq P$ be a chain in $\langle P,\subsetneq\rangle$. Case I: $\zeta=\emptyset$. Then $A\in P$ and $\forall B\in\zeta\,[B\subseteq A]$. So A is an upper bound for $\zeta\in P$. Case II: $\zeta\neq\emptyset$. Let $D=\bigcup\zeta$. For any $B\in\zeta, B\subseteq D$. If $D\in P$, then D would be an upper bound of ζ as $\forall B\in\zeta\,[B\subseteq D]$. So We want to show $D\in P$. $D\subseteq X$ as $\forall B\in\zeta\,[B\subseteq X]$. As $\zeta\neq\emptyset,\exists B\in\zeta$ such that $A\subseteq B\subseteq D$. So $A\subseteq D$. Show D is an antichain. Suppose $x,y\in D,x\neq y$. $\exists B,B'\in\zeta$ with $x\in B,y\in B'$. As ζ is a chain, WLOG $B\subseteq B'$. So $x,y\in B'$. Since B' is an antichain, $x\not<y,y\not<x$. So D is an antichain and $D\in P$.
3. By Zorn's, $\exists B\in P$ which is maximal in $\langle P,\subsetneq\rangle$. Now $A\subseteq B$, B is an antichain. If $\exists D,B\subsetneq D$, then $D\in P$ as $A\subseteq B\subseteq D$, contradicting maximality of B in $\langle P,\subsetneq\rangle$.

23/24 Q4 $\mathbf{F}(0)=1,\mathbf{F}(\beta)=\mathbf{F}(\alpha)\cdot\beta$ if $\beta=\alpha+1$, $\mathbf{F}(\beta)=sup\{\mathbf{F}(\alpha):\alpha<\beta\}$ if β is a limit ordinal. The extender is

$$\mathbf{E}(\sigma)=\begin{cases}1 & \text{if } dom(\sigma)=0\\ \sigma(\bigcup dom(\sigma))\cdot dom(\sigma) & \text{if } dom(\sigma)=S(\beta)\\ \bigcup ran(\sigma) & \text{if } dom(\sigma) \text{ is a limit ordinal}\\ \emptyset & \text{otherwise}\end{cases}$$

Swap out $\beta=dom(\sigma)$, $\mathbf{F}(\alpha)=\sigma(\bigcup dom(\sigma))$ where $\beta=\alpha+1$, $sup...=\bigcup ran(\sigma)$