MA3205
AY24/25 Sem 2
by ngmh

1. Sets and Operations

A1.1 Axiom of Extensionality $\forall x\, [x \in A \iff x \in B]$
D1.6 Subcollection $A \subseteq B$ if $\forall x\, [x \in A \Rightarrow x \in B]$
D1.7 Empty Set x is empty if $\forall y\, [y \notin x]$
F1.8 If $x = \emptyset$ and A is a collection then $x \subseteq A$
F1.9 If $x = \emptyset$ and $y = \emptyset$, $x = y$
D1.11 Basic Operations
1. $x \cup y = \{z : z \in x \vee z \in y\}$
2. $x \cap y = \{z : z \in x \wedge z \in y\}$
3. $x \setminus y = \{z : z \in x \wedge z \notin y\}$
4. $x \triangle y = (x \setminus y) \cup (y \setminus x)$
5. $\mathcal{P}(x) = \{z : z \subseteq x\}$

L1.12 Properties
1. $x \cup y = y \cup x$
2. $x \cap y = y \cap x$
3. $x \cup (y \cup z) = (x \cup y) \cup z$
4. $x \cap (y \cap z) = (x \cap y) \cap z$
5. $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$
6. $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$
7. $x \setminus (y \cup z) = (x \setminus y) \cap (x \setminus z)$
8. $x \setminus (y \cap z) = (x \setminus y) \cup (x \setminus z)$

D1.13 Union and Intersection
$\bigcup A = \{x : \exists y\, [y \in A \wedge x \in y]\}$
$$\bigcap A = \begin{cases} \emptyset & \text{if } A = \emptyset \\ \{x : \forall y\, [y \in A \Rightarrow x \in y]\} & \text{otherwise} \end{cases}$$
E1.16 Symmetric Difference
1. $(X \triangle Y) \triangle Z = X \triangle (Y \triangle Z)$
2. $X \triangle X = \emptyset$
3. $X \triangle Y = Y \triangle X$
4. $X \triangle \emptyset = X$

E1.18 $X \cap (Y \triangle Z) = (X \cap Y) \triangle (X \cap Z)$

2. Pairing, Products, and Relations

D2.1 Ordered Pair $\langle a, b\rangle = \{\{a\}, \{a, b\}\}$
L2.2 $\langle x, y\rangle = \langle a, b\rangle$ iff $x = a \wedge y = b$
D2.3 Cartesian Product $A \times B = \{z : \exists a \in A\, \exists b \in B\, [z = \langle a, b\rangle]\}$, $A^2 = A \times A$
E2.5 Define $pair(a, b) = \{a, \{a, b\}\}$. Assuming we cannot have $A \in B \in A$, $pair(a, b) = pair(x, y)$ iff $a = x \wedge b = y$
D2.6 Relation A relation is a collection of ordered pairs.
1. R is a relation if $\forall x \in R\, \exists a\, \exists b\, [x = \langle a, b\rangle]$
2. R is a relation on A if $R \subseteq A \times A$
3. $dom(R) = \{a : \exists b\, [\langle a, b\rangle \in R]\}$
4. $ran(R) = \{b : \exists a\, [\langle a, b\rangle \in R]\}$
5. $R^{-1} = \{x : \exists a\, \exists b\, [\langle a, b\rangle \in R \wedge x = \langle b, a\rangle]\}$

D2.8 Function A function is a relation where no two elements have the same first coordinate.
1. $\forall a, b, c\, [(\langle a, b\rangle \in f \wedge \langle a, c\rangle \in f) \Rightarrow b = c]$
2. $f : A \to B$ if f is a function, $dom(f) = A$ and $ran(f) \subseteq B$

F2.9 If R is a relation and $S \subseteq R$, then S is a relation. If f is a function and $g \subseteq f$, then g is a function.
D2.10 R restricted to A: $R \restriction A = R \cap (A \times ran(R))$
F2.11 $f \restriction A$ is a function. If $A \subseteq dom(f)$, then $dom(f \restriction A) = A$
D2.12 Image of A under R: $Im_R(A) = \{b : \exists a \in A\, [\langle a, b\rangle \in R]\}$. If f is a function, for any $a \in dom(f)$ $f(a)$ denotes the unique b such that $\langle a, b\rangle \in f$
D2.14 $Im_{f^{-1}}(B) = \{a : \exists b \in B\, [\langle b, a\rangle \in f^{-1}]\} = \{a : \exists b \in B\, [\langle a, b\rangle \in f]\} = \{a : a \in dom(f) \wedge f(a) \in B\}$
L2.15 $Im_R(\bigcup A) = \bigcup\{I : \exists a \in A\, [I = Im_R(a)]\}$
L2.16 If for any x and z, if $x \neq z$ then $Im_R(\{x\}) \cap Im_R(\{z\}) = \emptyset$, then
1. $Im_R(\bigcap A) = \bigcap\{I : \exists a \in A\, [I = Im_R(a)]\}$
2. $Im_R(B \setminus A) = Im_R(B) \setminus Im_R(A)$

C2.17 For any function and sets,
1. $Im_{f^{-1}}(\bigcup A) = \bigcup\{I : \exists a \in A\, [I = Im_{f^{-1}}(a)]\}$
2. $Im_{f^{-1}}(\bigcap A) = \bigcap\{I : \exists a \in A\, [I = Im_{f^{-1}}(a)]\}$
3. $Im_{f^{-1}}(B \setminus A) = Im_{f^{-1}}(B) \setminus Im_{f^{-1}}(A)$

D2.18 f as composed with g:
$g \circ f = \{x : \exists a\, \exists b\, \exists c\, [\langle a, b\rangle \in f \wedge \langle b, c\rangle \in g \wedge x = \langle a, c\rangle]\}$
L2.19 If f, g, h are functions then
1. $g \circ f$ is a function
2. If $f : A \to B$ and $g : B \to C$, then $g \circ f : A \to C$
3. $h \circ (g \circ f) = (h \circ g) \circ f$

D2.20 Injection / Surjection / Bijection Consider $f : A \to B$
1. $1-1$ / Injection: $\forall a, a' \in A\, [f(a) = f(a') \Rightarrow a = a']$
2. Onto / surjection: $ran(f) = B$
3. Bijection: $1-1$ and Onto

D2.21 $X^Y = \{f : f \text{ is a function} \wedge f : Y \to X\}$
L2.22 If $f : A \to B$ is $1-1$ and onto, then $f^{-1} : B \to A$ is $1-1$ and onto.
E2.23 It is possible that $Im_f(a \cap b) \neq Im_f(a) \cap Im_f(b)$
E2.24 If f is $1-1$, $Im_f(\bigcap A) = \bigcap\{Im_f(a) : a \in A\}$ and $Im_f(B \setminus A) = Im_f(B) \setminus Im_f(A)$
2.27 The following are equivalent
1. $\forall x, z\, [x \neq z \Rightarrow Im_R(\{x\}) \cap Im_R(\{z\}) = \emptyset]$
2. R^{-1} is a function

CV2.28 Functions as sequences Suppose $dom(f) = I$. $f = \langle A_i : i \in I\rangle = \{x : \exists i \in I\, [x = \langle i, A_i\rangle]\}$. $\forall i \in I, f(i) = A_i$.
CV2.29
1. $Im_f(A) = \{f(a) : a \in A \cap dom(f)\}$. If $A \subseteq dom(f)$, then $Im_f(A) = \{f(a) : a \in A\}$
2. If $dom(f) = A \times B$, $f(\langle a, b\rangle) = f(a, b)$

CV2.30 Suppose F is a function, $x \in dom(F)$, and $F(x)$ is also a function. Then if $y \in dom(F(x))$, $F(x)(y) = (F(x))(y)$.
CV2.32 If $F = \langle A_i : i \in I\rangle$, then $\bigcup ran(F) = \bigcup_{i \in I} A_i$, similarly for $\bigcap ran(F)$
CV2.33 To specify a function f with domain I, it is enough to specify $f(i)$ for each $i \in I$. $f = \{z : \exists i \in I\, \exists x\, [z = \langle i, x\rangle \wedge x \text{ satisfies property } P \text{ w.r.t } i]\}$. If there is a unique object satisfying P for each i, then f is a function and $dom(f) = I$.
EP2.34 Define $F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$. For $f \in \mathbb{N}^{\mathbb{N}}$, we must specify $F(f) \in \mathbb{N}^{\mathbb{N}}$. We must specify $F(f)(n) \in \mathbb{N}$ for each $n \in \mathbb{N}$. For example, $F(f)(n) = f(n) + 1$. Then $F(f) = \{\langle n, f(n) + 1\rangle : n \in \mathbb{N}\}$ and $F = \{\langle f, \{\langle n, f(n) + 1\rangle : n \in \mathbb{N}\}\rangle : f \in \mathbb{N}^{\mathbb{N}}\}$. Similarly, define $\mathcal{F} : (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$. For $F \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$, we must specify $\mathcal{F}(F) \in \mathbb{N}^{\mathbb{N}}$ by specifying $\mathcal{F}(F)(n)$ for each n. Since F is a function with domain $\mathbb{N}$, $F(i)$ is defined for all $i \leq n$ and $F(i) \in \mathbb{N}^{\mathbb{N}}$. So $F(i)(n) \in \mathbb{N}$. Set $\mathcal{F}(F)(n) = max\{F(i)(n) : i \leq n\}$. $\mathcal{F}(F)$ eventually dominates $\{F(n) : n \in \mathbb{N}\}$.
EP2.35 Let I be a set, $\langle J_i : i \in I\rangle$ be a sequence of sets, and $\langle A_{i,j} : j \in J_i$ be a sequence of sets. Define $X = \{\bigcup_{j \in J_i} A_{i,j} : i \in I\}$. First define F with $dom(F) = I$. For each $i \in I$, $F(i) = \bigcup_{j \in J_i} A_{i,j}$. $X = Im_F(I) = ran(F)$. $\bigcap X = \bigcap_{i \in I} \bigcup_{j \in J_i} A_{i,j}$.
EP2.36 There is a biection $F : \mathcal{P}(X) \to \{0, 1\}^X$. We must specify $F(a)$ for each $a \in \mathcal{P}(X)$; a function with $dom(F(a)) = X$ and $ran(F(a)) \subseteq \{0, 1\}$. It is enough to specify $F(a)(x)$ for each $x \in X$.
$$F(a)(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \notin A \end{cases} \quad \text{which is } 1-1 \text{ and onto.}$$
D2.37 Cartesian Product Let F be a function with $dom(F)$ as a set. $\prod F = \{f : f \text{ is a function} \wedge dom(f) = dom(F) \wedge \forall x \in dom(F)\, [f(x) \in F(x)]\}$. If $F = \langle A_i : i \in I\rangle$, then
$\prod F = \prod_{i \in I} A_i = \{f : f \text{ is a function} \wedge dom(f) = I \wedge \forall i \in I\, [f(i) \in A_i]\}$
A2.38 Axiom of Choice If $\langle A_i : i \in I\rangle$ is a sequence of sets such that $\forall i \in I\, [A_i \neq \emptyset]$, then $\prod_{i \in I} A_i \neq \emptyset$
D2.40 Directed Collection G is directed if $\forall a, b \in G\, \exists c \in G\, [a \subseteq c \wedge b \subseteq c]$
L2.41 If G is a directed collection of functions, $f = \bigcup G$ is a function. $dom(f) = \bigcup\{dom(\sigma) : \sigma \in G\}$ and $ran(f) = \bigcup\{ran(\sigma) : \sigma \in G\}$
T2.47 Generalised De Morgan's (Requires **Axiom of Choice**) Let I be a set, and $\langle J_i : i \in I\rangle$ be a sequence of sets. Suppose $I \neq \emptyset$ and $\forall i \in I\, [J_i \neq \emptyset]$. For each $i \in I$, let $\langle A_{i,j} : j \in J_i\rangle$ be a sequence of sets.
1. $\bigcup_{i \in I} \bigcap_{j \in J_i} A_{i,j} = \bigcap\{\bigcup_{i \in I} A_{i, f(i)} : f \in \prod_{i \in I} J_i\}$
2. $\bigcap_{i \in I} \bigcup_{j \in J_i} A_{i,j} = \bigcup\{\bigcap_{i \in I} A_{i, f(i)} : f \in \prod_{i \in I} J_i\}$
3. $\prod_{i \in I}(\bigcup_{j \in J_i} A_{i,j}) = \bigcup\{\prod_{i \in I} A_{i, f(i)} : f \in \prod_{i \in I} J_i\}$
4. $\prod_{i \in I}(\bigcap_{j \in J_i} A_{i,j}) = \bigcap\{\prod_{i \in I} A_{i, f(i)} : f \in \prod_{i \in I} J_i\}$

T2.48 Fix $n \geq 1$, Let X be a set and $A_1, \ldots, A_n$ be subsets of X. There are at most 2^{2^n} sets that can be formed using $X \setminus .$, $\cup$, and $\cap$.
1. Redefine $\bigcap \emptyset = X$
2. Let $S = \{0, 1\}^{\{1, \ldots, n\}}$, then $|S| = 2^n$
3. For each $\sigma \in S$ define $b_\sigma = (\bigcap\{A_i : \sigma(i) = 0\}) \cap (\bigcap\{X \setminus A_i : \sigma(i) = 1\})$
4. For each $a \in \mathcal{P}(S)$ let $c_a = \bigcup\{b_\sigma : \sigma \in a\}$
5. Let $\mathcal{B} = \{c_a : a \in \mathcal{P}(S)\}$. $|\mathcal{B}| \leq |\mathcal{P}(S)| = 2^{2^n}$
6. **CL2.49** For each $1 \leq i \leq n$, $A_i \in \mathcal{B}$
7. **CL2.50** For any $a, b \in \mathcal{P}(S), c_a \cup c_b = c_{(a,b)}$
8. **CL2.51** For any $a, b \in \mathcal{P}(S), X \setminus c_a = c_{(S \setminus a)}$
9. **Claim 2.52** For any $a, b \in \mathcal{P}(S), c_a \cap c_b = c_{(a \cap b)}$

E2.53 There exists $\langle A_n : n \in \mathbb{N}\rangle$ and $\langle B_n : n \in \mathbb{N}\rangle$ such that
1. $\forall n \in \mathbb{N}\, [B_n \subseteq A_n]$
2. $\forall n, m \in \mathbb{N}\, [n \neq m \Rightarrow B_n \cap B_m = \emptyset]$
3. $\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} B_n$

E2.55 If $I \neq \emptyset$ is a set and $\langle A_i : i \in I\rangle$ is a sequence of sets and X is a set then
1. $X \setminus (\bigcup_{i \in I} A_i) = \bigcap_{i \in I}(X \setminus A_i)$
2. $X \setminus (\bigcap_{i \in I} A_i) = \bigcup_{i \in I}(X \setminus A_i)$

3. Russell's Paradox and Proper Classes

T3.1 Russell $R = \{x : x \text{ is a set} \wedge x \notin x\}$ is not a set
Modified Morse-Kelley Rules
1. Everything is a class.
2. Every set is a class.
3. Every collection of sets is a class.
4. Axiom of Comprehension: If A is a class and x is a set, then $A \cap x$ is a set.
5. Axiom of Replacement: If F is a class which is a function and x a set, then $Im_F(x)$ is a set.
6. Axioms of Pairing / Union / Power-Set: If A and B are sets, then so are $\{A, B\}, \bigcup A, \mathcal{P}(A)$
7. Axiom of Choice
8. Axiom of infinity: $\mathbb{N}$ is a set
9. Axiom of Extensionality

T3.3 $\mathbf{V} = \{x : x \text{ is a set}\}$ is not a set, but a proper class
EP3.4 $A \times B$ is a set
E3.5 $dom(A), ran(A), \bigcap A, A^B$ are sets
E3.6 For I and $\langle A_i : i \in I\rangle$ which is a sequence of sets, $\prod_{i \in I} A_i$ is a set
E3.7 If R is a relation, $Im_R(A)$ is a set
E3.8 $\mathbf{U} = \{x : \exists a\, \exists b\, [x = \langle a, b\rangle]\}$ is not a set
E3.9 Let F be a class. If F is a function and $dom(F)$ is a set, F is a set

4. The Natural Numbers

F4.1 Peano Axioms

1. 0 is a natural number
2. If n is a natural number, there exists $S(n)$ which is also a natural number
3. If $n \neq m$, then $S(n) \neq S(m)$
4. $0 \neq S(n)$ for any natural number n
5. If X is a class of natural numbers where $0 \in X$ and $\forall n \in X\,[S(n) \in X]$, then $X = \mathbb{N}$

D4.2 $S(x) = x \cup \{x\}$

D4.3 0 is the empty set $\emptyset$

D4.4 A class is inductive if $0 \in A$ and $\forall x \in A\,[S(x) \in A]$. n is a natural number if it belongs to every inductive class.

F4.5 Axiom of infinity The class of all natural numbers $\mathbb{N} = \{n : n \text{ is a natural number}\}$ is a set.

L4.6 0 is a natural number, and if n is a natural number, then so is $S(n)$. $\mathbb{N}$ is an inductive class, and $\mathbb{N} \subseteq A$ for every inductive class A.

L4.7 If X is any set of natural number such that $0 \in X$ and $\forall n \in X\,[S(n) \in X]$, then $X = \mathbb{N}$

F4.8 Principle of Mathematical Induction Suppose P is a property, which 0 has, and $\forall n \in \mathbb{N}\,[n \text{ has property } P \Rightarrow S(n) \text{ has property } P]$. Then all natural numbers have property P.

L4.9 If n is a natural number then

1. $\forall x \in n\,[x \subseteq n]$
2. $n \subseteq \mathbb{N}$
3. $\forall x\,[(x \subseteq n \wedge x \neq \emptyset) \Rightarrow \exists m \in x\,[x \cap m = \emptyset]]$

L4.10 For natural numbers n, m, k

1. $n \notin n$
2. $m \subseteq n \Rightarrow (m \in n \vee m = n)$
3. $(m \subseteq n \wedge n \in k) \Rightarrow m \in k$
4. $m = n \vee m \in n \vee n \in m$

L4.11 For $X \subseteq \mathbb{N}$, if $X \neq \emptyset$, then $\exists n \in X\,[X \cap n = \emptyset]$

D4.12 We identify the relation $<$ on natural numbers with $\in$

F4.13 Principle of Strong Induction Suppose P is some property. Suppose that $\forall n \in \mathbb{N}\,[\text{if } P \text{ holds for all } m \in \mathbb{N} \text{ less than } n \text{ then } P \text{ holds for } n]$. Then P holds for all $n \in \mathbb{N}$.

L4.14 If $n, m \in \mathbb{N}$ and $n \neq m$, then $S(n) \neq S(m)$.

E4.15 For natural numbers n, m, k

1. $m \in n \in k$ implies $m \in k$
2. It is impossible to have $m \in n \in S(m)$
3. If $n \neq 0$ then $n = S(\bigcup n)$
4. $n \leq m$ iff $n \subseteq m$
5. $max\{n, m\} = n \cup m$
6. Either $n = 0$ or $\exists k \in n\,[S(k) = n]$

E4.16 If $X \subseteq \mathbb{N}$ and $\forall n \in X\,[n \subseteq X]$, then $X = \mathbb{N}$ or $\exists n \in \mathbb{N}\,[X = n]$

D4.17 Extenders Let $\mathbf{FN} = \{\sigma : \sigma \text{ is a function} \wedge \exists n \in \mathbb{N}\,[dom(\sigma) = n]\}$ be the proper class of all functions whose domain is some natural number. An extender is a function $\mathbf{E} : \mathbf{FN} \to \mathbf{V}$. When you input $\sigma = \{\langle 0, \sigma(0)\rangle, ..., \langle n, \sigma(n)\rangle\}$ into $\mathbf{E}$, $\mathbf{E}(\sigma)$ outputs the next value $\sigma(S(n))$.

T4.19 Suppose $\mathbf{E} : \mathbf{FN} \to \mathbf{V}$ is an extender. Then there exists a unique $f : \mathbb{N} \to \mathbf{V}$ satisfying $\forall n \in \mathbb{N}\,[f(n) = \mathbf{E}(f \restriction n)]$.

CL4.21 For each $n \in \mathbb{N}$ there is an approximation to f with domain equal to n.

CL4.22 Let $\sigma, \tau \in \mathbf{FN}$ be approximations to f. Either $\sigma \subseteq \tau$ or $\tau \subseteq \sigma$.

EP4.23 Consider $\mathbf{E}(\sigma) = \bigcup \sigma(m)$. $f(0) = \mathbf{E}(f \restriction 0) = \mathbf{E}(\emptyset) = X$ and $f(S(n)) = \mathbf{E}(f \restriction S(n)) = \bigcup f(n)$. $f(0) = X, f(1) = \bigcup X, f(2) \bigcup \bigcup X$. This set $\bigcup ran(f)$ is the transitive closure of X, $trcl(X)$.

EP4.24 Consider $f(0) = \emptyset, f(S(n)) = \mathcal{P}(f(n))$. Set $\mathbf{E}(\sigma) = \emptyset, \mathbf{E}(\sigma) = \mathcal{P}(\sigma(m))$. This gives $V_0 = f(0) = \emptyset, V_{S(n)} = f(S(n)) = \mathcal{P}(f(n)) = \mathcal{P}(V_n)$. $V_\omega = \bigcup ran(f) = \bigcup\{V_n : n \in \mathbb{N}\}$.

D2.45 Addition and Multiplication

- Define $\langle f_m : m \in \mathbb{N}\rangle$ such that $f_m : \mathbb{N} \to \mathbb{N}$ is the unique function such that $f_m(0) = m$ and $\forall n \in \mathbb{N}\,[f_m(S(n)) = S(f_m(n))]$
- In other words, define the extender $\mathbf{E} : \mathbf{FN} \to \mathbf{V}$ as follows. For any $\sigma \in \mathbf{FN}$, $$\mathbf{E}(\sigma) = \begin{cases} m & \text{if } dom(\sigma) = 0 \\ S(\sigma(\bigcup dom(\sigma))) & \text{if } dom(\sigma) \neq 0 \end{cases}$$
- $f_m : \mathbb{N} \to \mathbf{V}$ is the unique function satisfying $\forall n \in \mathbb{N}\,[f_m(n) = \mathbf{E}(f_m \restriction n)]$.
- Then $m + n = f_m(n)$, and $m + S(n) = (m + n) + 1$.
- Define $\langle g_m : m \in \mathbb{N}\rangle$ such that $g_m : \mathbb{N} \to \mathbb{N}$ is the unique function such that $g_m(0) = 0$ and $\forall n \in \mathbb{N}\,[g_m(S(n)) = f_{g_m(n)}(m)]$
- In other words, define the extender $\mathbf{E} : \mathbf{FN} \to \mathbf{V}$ as follows. For any $\sigma \in \mathbf{FN}$, $$\mathbf{E}(\sigma) = \begin{cases} 0 & \text{if } dom(\sigma) = 0 \\ f_{\sigma(\bigcup dom(\sigma))}(m) & \text{if } dom(\sigma) \neq 0 \text{ and } \sigma(\bigcup dom(\sigma)) \in \mathbb{N} \\ \emptyset & \text{if } dom(\sigma) \neq 0 \text{ and } \sigma(\bigcup dom(\sigma)) \notin \mathbb{N} \end{cases}$$
- $g_m : \mathbb{N} \to \mathbf{V}$ is the unique function satisfying $\forall n \in \mathbb{N}\,[g_m(n) = \mathbf{E}(g_m \restriction n)]$.
- Then $m \cdot n = g_m(n)$, and $m \cdot S(n) = (m \cdot n) + m$.

E4.26 For $n, m, k \in \mathbb{N}$

1. $n + 1 = S(n)$
2. $n + (m + k) = (n + m) + k$
3. $n + m = m + n$
4. $n + n = 2 \cdot n$
5. If $2 \cdot n = 2 \cdot m$ then $n = m$
6. $n \cdot (m + k) = n \cdot m + n \cdot k$
7. $n \cdot (m \cdot k) = (n \cdot m) \cdot k$
8. $n \cdot m = m \cdot n$

E4.27 For $m, n, k \in \mathbb{N}$

1. If $n < k$ then $m + n < m + k$
2. If $m \neq 0$ and $n < k$ then $m \cdot n < m \cdot k$

E4.28 A transitive set satisfies $\forall x \in X\,[x \subseteq X]$.

1. For each $n \in \mathbb{N}$, V_n is transitive and V_ω is transitive
2. For each $n \in \mathbb{N}$, $n \subseteq V_n$ and $n \notin V_n$
3. $\mathbb{N} \subseteq V_\omega$ and $\mathbb{N} \notin V_\omega$

E4.29 $f \in \mathbb{N}^{\mathbb{N}}$ is increasing if $\forall n \in \mathbb{N}\,[f(n) \leq f(n+1)]$. f is unbunded if $\forall k \in \mathbb{N}\,\exists n \in \mathbb{N}\,[f(n) > k]$. Let $H : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a function. For each $m \in \mathbb{N}$, let h_m be the function in $\mathbb{N}^{\mathbb{N}}$ such that $\forall n \in \mathbb{N}\,[h_m(n) = H(m, n)]$ which is increasing and unbounded. There exists an increasing and unbounded function f such that $\forall m \in \mathbb{N}\,\exists l \in \mathbb{N}\,\forall n \in \mathbb{N}\,[n \geq l \Rightarrow f(n) < h_m(n)]$.

E4.30 Let X be a set, $0_X \in X$ be some element, and $S_X : X \to X$ be some function. Suppose

1. $\forall x \in X\,[S_x(x) \neq 0_X]$
2. $\forall x, y \in X\,[x \neq y \Rightarrow S_X(x) \neq S_X(y)]$
3. For every $A \subseteq X$, if $0_X \in A$ and $\forall x \in A\,[S_X(x) \in A]$, then $X = A$

Then $\langle \mathbb{N}, S, 0\rangle$ is isomorphic to $\langle X, S_X, 0_X\rangle$. There is a $1-1$ and onto function $F : \mathbb{N} \to X$ such that $F(0) = 0_X$ and $\forall n \in \mathbb{N}\,[F(S(n)) = S_x(F(n))]$.

E4.31 Define $A_0 = \{\emptyset\}$, $A_1 = \mathbb{N}$, for $n \geq 1$, $A_{S(n)} = A_n \times \mathbb{N}$. There is an extender $\mathbf{E} : \mathbf{FN} \to \mathbf{V}$, $\mathbf{E}(\sigma) = \sigma(\bigcup dom(\sigma)) \times \mathbb{N}$, where $dom(\sigma) = 0 \Rightarrow \mathbf{E}(\sigma) = \{\emptyset\}$, $dom(\sigma) = 1 \Rightarrow \mathbf{E}(\sigma) = \{\mathbb{N}\}$ that generates $\langle A_n : n \in \mathbb{N}\rangle$.

E4.32

1. X is transitive iff $\bigcup X \subseteq X$
2. $trcl(X)$ is the smallest transitive set containing X as a subset

5. Comparing Sizes of Sets

D5.1 Equinumerosity $A \approx B$ if there exists $f : A \to B$ which is both $1-1$ and onto.

F5.2 $\mathcal{P}(A) \approx \{0, 1\}^A$

D5.4 $A \lesssim B$ means there exists $f : A \to B$ which is $1-1$ and B is at least as big as A. If $A \lesssim B$ but $A \not\approx B$, then $A \lnsim B$. It is not possible to find $g : A \to B$ that is both $1-1$ and onto. B is strictly bigger in size than A.

L5.5 If $f : A \to B$ and $g : B \to C$ are $1-1$ functions then $g \circ f : A \to C$ is $1-1$.

L5.6 For sets A, B, C

1. $A \lesssim A$
2. If $A \lesssim B$ and $B \lesssim C$ then $A \lesssim C$
3. If $A \approx B$ and $B \approx C$ then $A \approx C$

T5.7 Cantor For any set X, $X \lnsim \mathcal{P}(X)$.

5.2 The Schröder Bernstein Theorem

T5.8 If $f : A \to B$ and $g : B \to A$ are both $1-1$ functions, then there exists $I \subseteq A$ and $J \subseteq B$ such that $f \restriction I : I \to J$ is $1-1$ and onto, and $g \restriction (B \setminus J) : B \setminus J \to A \setminus I$ is $1-1$ and onto.

CL5.9 For each $b \in B \setminus J, g(b) \in A \setminus I$.

CL5.10 For each $a \in A \setminus I$, there exists $b \in B \setminus J$ with $g(b) = a$.

T5.11 Schröder-Bernstein For any sets A and B, if $A \lesssim B$ and $B \lesssim A$, then $A \approx B$.

E5.12 Suppose $f : X \to Y$ is a $1-1$ function. For any $Z \subseteq X$, $Z \approx Im_f(Z)$.

E5.13 Suppose $I \subseteq A$ and $J \subseteq B$. If $I \approx J$ and $(A \setminus I) \approx (B \setminus J)$, then $A \approx B$.

E5.14 If $n \in \mathbb{N}$ and $A \approx S(n)$, then $\forall a \in A, (A \setminus \{a\} \approx n)$.

E5.15 If $n \in \mathbb{N}$ and $A \approx n$, then if $a \notin A$, $(A \cup \{a\}) \approx S(n)$.

E5.16 Let $n, m \in \mathbb{N}$. Then

1. If $f : n \to n$ is $1-1$, then f is onto. There is no $1-1$ function from $S(n)$ to n.
2. If $m \in n$, then $m \lnsim n$.
3. If $x \subsetneq n$, then $x \lnsim n$.
4. $n \lnsim \mathbb{N}$
5. If $A \approx n$, $B \approx m$, and $A \cap B = \emptyset$, then $(A \cup B) \approx (n + m)$.

D5.19 A set is finite if there exits $n \in \mathbb{N}$ such that $n \approx A$. A is infinite if it is not finite. A is countable if $A \lesssim \mathbb{N}$. A is uncountable if it is not countable.

L5.20 If $f : A \to B$ is a $1-1$ function, then for any $X, Y \subseteq A$, if $Im_f(X) = Im_f(Y)$, then $X = Y$.

L5.21 For sets A, B, C, D

1. If $A \lesssim B$ then $\mathcal{P}(A) \lesssim \mathcal{P}(B)$
2. If $A \lesssim B$ then $A^C \lesssim B^C$
3. If $A \lesssim B, C \lesssim D$, and $B \cap D = \emptyset$, then $A \cup C \lesssim B \cup D$

L5.22 If $n \in \mathbb{N}$ and $A \lesssim n$, then A is finite.

L5.23 If $n \in \mathbb{N}$ and there exists an onto function $\sigma : n \to A$, then $A \lesssim n$

L5.24 If A and B are finite, then so is $A \cup B$.

T5.25 If A is a finite set and f is a function with $dom(f) = A$ then

1. If $X \subsetneq A$, then $X \lnsim A$
2. $ran(f)$ is finite and $ran(f) \lesssim A$
3. If $\forall a \in A\,[a \text{ is finite}]$ then $\cup A$ is finite
4. $\mathcal{P}(A)$ is finite

E5.26 If $A \subseteq \mathbb{N}$ is finite and nonempty, $max(A) = \bigcup A$

E5.27 If $A \lesssim C$ and $B \lesssim D$, then $A \times B \lesssim C \times D$. If A and B are finite, $A \times B$ and A^B are finite.

E5.28 If I is a finite set and $\langle A_i : i \in I\rangle$ is a sequence of sets such that $\forall i \in I\,[A_i \text{ is finite}]$, then $\prod_{i \in I} A_i$ is finite.

E5.29 Suppose $\langle A_n : n \in \mathbb{N}\rangle$ is a sequence of infinite subsets of $\mathbb{N}$. There exists an infinite set $A \subseteq \mathbb{N}$ such that $\forall n \in N\,[A \cap A_n \text{ is infinite and } (\mathbb{N} \setminus A) \cap A_n \text{ is infinite}]$.

E5.30 For any function, $dom(f) \approx f$.

6. Orders

Quasi, Partial, Linear, and Well-Orders

D6.2 Quasi Order Reflexive, Transitive

1. $\forall x \in X\ [x \leq x]$
2. $\forall x, y, z \in X\ [(x \leq y \wedge y \leq z) \Rightarrow x \leq z]$

D6.4 Partial Order Irreflexive, Transitive

1. $\forall x \in X\ [x \not< x]$
2. $\forall x, y, z \in X\ [(x < y \wedge y < z) \Rightarrow x < z]$

D6.5 Linear Order Irreflexive, Transitive, Comparable

1. $\forall x \in X\ [x \lhd x]$
2. $\forall x, y, z \in X\ [(x \lhd y \wedge y \lhd z) \Rightarrow x \lhd z]$
3. $\forall x, y \in X\ [x = y \vee x \lhd y \vee y \lhd x]$

F6.6 Suppose $\langle X, < \rangle$ is a partial order. Define a relation $\leq$ on X by $x \leq y$ iff $x < y$ or $x = y$. Then $\langle X, \leq \rangle$ is a quasi order where $\forall x, y \in X\ [(x \leq y \wedge y \leq x) \Rightarrow x = y]$.

C6.8 If $\langle X, < \rangle$ is a partial order and $Y \subseteq X$ then $(Y \times Y) \cap <$ is a partial order on Y, as a shorthand for $\langle Y, ((Y \times Y) \cap <) \rangle$. Restricted to Y then Z is the same as restricting directly to Z.

D6.9 Maximal / Minimal Element $x \in X$ is maximal if $\forall y \in X\ [x \not< y]$. $x \in X$ is minimal if $\forall y \in X\ [y \not< x]$. There could be multiple in a partial order.

L6.10 A finite non-empty partial order has both a maximal and minimal element.

D6.11 $C \subseteq X$ is a chain if $\forall x, y \in C$ [x and y are comparable]. $A \subseteq X$ is an antichain if $\forall x, y \in A$ [$x \neq y \Rightarrow x$ and y are incomparable]. A chain is maximal if there is no chain $C' \subseteq X$ where $C \subsetneq C'$. $\emptyset$ and singletons are chains and antichains.

L6.12 For a finite partial order, every chain or antichain is contained in a maximal chain or antichain.

D6.13 Well-Order Every non-empty subset has a minimal element. $\forall A \subseteq X\ [A \neq \emptyset \Rightarrow \exists a \in A\ \forall a' \in A\ [a \leq a']]$.

L6.15 (AC) A linear order $\langle X, < \rangle$ is a well-order iff there is no $f : \mathbb{N} \to X$ where $\forall n \in \mathbb{N}\ [f(n) > f(n+1)]$.

D6.16 For a linear order $\langle X, < \rangle$ $pred_{\langle X, < \rangle}(x) = \{x' \in X : x' < x\}$, or the set of predecessors of x in X for the ordering $<$. A subset $A \subseteq X$ is downwards closed if $\forall a \in A\ \forall x \in X\ [x < a \Rightarrow x \in A]$. The predecessor subset is downwards closed along with the entire set.

F6.17 For a linear order $\langle X, < \rangle$, if $A \subseteq X$ is downwards closed, $\forall a \in A\ pred_{\langle A, < \rangle}(a) = pred_{\langle X, < \rangle}(a)$.

F6.19 Let $\langle X, < \rangle$ be a well-order and A a downwards closed subset of X. Either $A = X$ or $\exists x \in X$ such that $A = pred_{\langle X, < \rangle}(x)$.

E6.20 If $\langle X, < \rangle$ is a well-order and $A \subseteq X$, $\langle A, < \rangle$ is a well-order.

E6.21 Let $\langle X, < \rangle$ be a linear order. $f : X \to X$ is expansive if $\forall x \in X\ [f(x) \geq x]$, and order-preserving if $\forall x, y \in X\ [x < y \Rightarrow f(x) < f(y)]$. If $\langle X, < \rangle$ is a well-order, every order-preserving f is expansive.

E6.22 Suppose $\langle X, < \rangle$ is a quasi-order. Define E on X by $\forall x, y \in X, x\ E\ y$ iff $x \leq y$ and $y \leq x$. E is an equivalence relation on X. Let $Z = \{[x] : x \in X\}$ where $[x]$ is the equivalence class of x under E. Define $\prec$ on Z by $[x] \prec [y]$ iff $x \leq y$ and $y \not\leq x$. This relation is well-defined and a partial order on Z.

New orders from old

L6.23 Suppose X is a set and $\langle Y, \prec \rangle$ and $\langle Z, \lhd \rangle$ are partial orders. Suppose $f : X \to Y$ and $g : X \to Z$ are functions. Define $<$ on X by stipulating that $\forall x, x' \in X, x < x' \leftrightarrow (f(x) \prec f(x')\ or\ (f(x) = f(x')\ and\ g(x) \lhd g(x')))$. Then,

1. $<$ is a partial order on X
2. if $\langle Y, \prec \rangle$ and $\langle Z, \lhd \rangle$ are both linear orders and $\forall x, x' \in X\ [(f(x) = f(x')\ and\ g(x) = g(x')) \Rightarrow x = x']$ then $<$ is also a linear order on X
3. similarly for well-orders

C6.24 Let X be a set and $\langle Y, \prec \rangle$ be a partial order. Suppose $f : X \to Y$ is a function. Define $<^*$ on X by $\forall x, x' \in X, x <^* x' \leftrightarrow f(x) \prec f(x')$. Then $<^*$ is a partial order on X. If f is $1-1$ and $\prec$ is a linear order on Y, then $<^*$ is a linear order on X. Similarly for well-orders. Use $Y = Z, f = g, \prec = \lhd$.

EP 6.25 Lexographic / Dictionary Order Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by setting $f(\langle m, n \rangle) = m$ and $g(\langle m, n \rangle) = n$ for all $\langle m, n \rangle \in \mathbb{N} \times \mathbb{N}$. By L6.23, the relation $<_{lex}$ defined on $\mathbb{N} \times \mathbb{N}$ by $\langle m, n \rangle <_{lex} \langle k, l \rangle \leftrightarrow (f(\langle m, n \rangle) \in f(\langle k, l \rangle)\ or\ (f(\langle m, n \rangle) = f(\langle k, l \rangle)\ and\ g(\langle m, n \rangle) \in g(\langle k, l \rangle))) \leftrightarrow (m \in k\ or\ (m = k\ and\ n \in l))$ is a well-order on $\mathbb{N} \times \mathbb{N}$.

D6.26 Suppose $\langle I, < \rangle$ is a well-order and X is a set. For $f, g \in X^I$, if $f \neq g$, define $\Delta(f, g) = min(\langle \{i \in I : f(i) \neq g(i)\}, < \rangle)$. It is well-defined for the conditions mentioned above.

L6.27 Suppose $\langle X, \lhd \rangle$ is a linear order and $\langle I, < \rangle$ is a well-order. Define $\prec$ on X^I $\forall f, g \in X^I$ by $f \prec g \leftrightarrow (f \neq g\ and\ f(\Delta(f, g)) \lhd g(\Delta(f, h)))$. Then $\prec$ is a linear order on X^I.

D6.28

- $[\mathbb{N}]^n = \{a \in \mathcal{P}(\mathbb{N}) : a \approx n\}$, or subsets of the naturals equinumerous to n.
- $\mathbb{N}^n = \{\sigma : \sigma$ is a function and $dom(\sigma) = n \wedge ran(\sigma) \subseteq \mathbb{N}\}$, or functions with domain n and range as a subset of the naturals.
- $[\mathbb{N}]^{<\omega} = \{a \in \mathcal{P}(\mathbb{N}) : a$ is finite$\}$, or subsets of the naturals equinumerous to finite n.
- $\mathbb{N}^{<\omega} = \{\sigma : \sigma$ is a function and $dom(\sigma) \in \mathbb{N} \wedge ran(\sigma) \subseteq \mathbb{N}\}$, or functions with a finite domain and range as a subset of naturals.
- $[\mathbb{N}]^{<\omega} = \bigcup_{n \in \mathbb{N}} [\mathbb{N}]^n$ and $\mathbb{N}^{<\omega} = \bigcup_{n \in \mathbb{N}} \mathbb{N}^n$.

EP6.29 $\langle \mathbb{N}, \in \rangle$ is a well-order. Define a linear order $<_{lex}$ on $\mathbb{N}^{\mathbb{N}}$ using L6.27. Since $2^{\mathbb{N}} \subseteq \mathbb{N}^{\mathbb{N}}$, $\langle 2^{\mathbb{N}}, <_{lex} \rangle$ is also a linear order. Define $F : \mathbb{N}^{<\omega} \to \mathbb{N}^{\mathbb{N}}$ as follows. $\sigma \in \mathbb{N}^{<\omega}$, $dom(\sigma) \in \mathbb{N}$ and $\sigma : dom(\sigma) \to \mathbb{N}$. Define $F(\sigma) : \mathbb{N} \to \mathbb{N}$ by $F(\sigma)(n) = \begin{cases} \sigma(n) & \text{if } n \in dom(\sigma) \\ dom(\sigma) & \text{if } n \notin dom(\sigma) \end{cases}$ for every $n \in \mathbb{N}$. F is $1-1$.

Since $<_{lex}$ is a linear order on $\mathbb{N}^{\mathbb{N}}$, by C6.24, $<_{lex}$ defined on $\mathbb{N}^{<\omega}$ by $\sigma <_{lex} \tau \leftrightarrow F(\sigma) <_{lex} F(\tau)$ is also a linear order. From EP2.36, we have a $1-1$ and onto $F : \mathcal{P}(\mathbb{N}) \to 2^{\mathbb{N}}$. We can induce $A <_{lex} B \leftrightarrow F(A) <_{lex} F(B)$ for $A, B \in \mathcal{P}(\mathbb{N})$. Since $[\mathbb{N}]^{<\omega} \subseteq \mathcal{P}(\mathbb{N})$, $\langle [\mathbb{N}]^{<\omega}, <_{lex} \rangle$ is also a linear order.

E6.30 Let $\langle X, < \rangle$ and $\langle Y, \prec \rangle$ be partial orders. Define $\langle X \times Y, \lhd \rangle$ by $\langle x, y \rangle \lhd \langle x', y' \rangle \leftrightarrow (x < x' \wedge y \prec y')$. This is a partial order.

E6.31 A linear order $\langle X, < \rangle$ is dense if $\forall x, z \in X$ with $x < z$, $\exists y \in X$ such that $x < y < z$. $\langle \mathbb{N}^{\mathbb{N}}, <_{lex} \rangle$ is dense while $\langle 2^{\mathbb{N}}, <_{lex} \rangle$ is not dense.

E6.32 $<_{lex}$ on $\mathcal{P}(\mathbb{N})$ is defined by the $1-1$ and onto F from $EP2.36$.

1. $\forall A, B \in \mathcal{P}(\mathbb{N})$, $A <_{lex} B$ iff $A \neq B \wedge min(A \triangle B) \in A$
2. $\langle [\mathbb{N}]^{<\omega} \backslash \{\emptyset\}, <_{lex} \rangle$ does not have any maximal or minimal elements
3. $\langle [\mathbb{N}]^{<\omega} \backslash \{\emptyset\}, <_{lex} \rangle$ is dense

6.3 Embeddings and Isomorphisms

D6.33 If $\langle X, \lhd \rangle$ and $\langle Y, \prec \rangle$ are linear orders, $f : X \to Y$ is an isomorphism between them if f is $1-1$ and onto and $\forall x, y \in X\ [x \lhd y \Leftrightarrow f(x) \prec f(y)]$. Two linear orders are isomorphic if f exists which is an isomorphism.

L6.34 $\langle X, \lhd \rangle$ and $\langle Y, \prec \rangle$ are linear orders. Suppose $f : X \to Y$ is an onto function such that $\forall x, y \in X\ [x \lhd y \Rightarrow f(x) \prec f(y)]$. f is an isomorphism.

D6.35 $\langle X, < \rangle$ and $\langle Y, \prec \rangle$ are linear orders. $f : X \to Y$ is an embedding if $\forall x, x' \in X\ [x < x' \Leftrightarrow f(x) \prec f(x')]$ and f is $1-1$. If there exists and embedding f, we say that $\langle X, < \rangle$ embeds into $\langle Y, \prec \rangle$ and $\langle X, < \rangle \hookrightarrow \langle Y, \prec \rangle$. $\langle X, < \rangle$ is isomorphic to $\langle Im_f(X), \prec \rangle$.

F6.36 $\langle X, < \rangle$ and $\langle Y, \prec \rangle$ are linear orders. If $f : X \to Y$ is a function such that $\forall x, x' \in X\ [x < x' \Rightarrow f(x) \prec f(x')]$, then f is an embedding.

F6.37 $\langle X, < \rangle$ and $\langle Y, \prec \rangle$ are linear orders. Suppose A and B are downwards closed subsets of X and Y. If $f : A \to B$ is an isomorphism from $\langle A, < \rangle$ to $\langle B, \prec \rangle$, then $\forall a \in A$, $f \restriction pred_{\langle X, < \rangle}(a)$ is an isomorphism from $\langle pred_{\langle X, < \rangle}(a), < \rangle$ to $\langle pred_{\langle Y, \prec \rangle}(f(a)), \prec \rangle$.

T6.38 Suppose $\langle X, \lhd \rangle$ is a finite linear order. $\exists! n \in \mathbb{N}$ such that $\langle X, \lhd \rangle$ is isomorphic to $\langle n, \in \rangle$. This isomorphism is unique.

T6.39 Suppose $\langle X, \lhd \rangle$ is an infinite linear order such that $\forall x \in X, pred_{\langle X, \lhd \rangle}$ is finite. $\langle X, \lhd \rangle$ is isomorphic to $\langle \mathbb{N}, \in \rangle$. The isomorphism from X to $\mathbb{N}$ is unique.

CL6.40 $\forall x, y \in X\ [x \lhd y \Rightarrow f(x) \in f(y)]$.

CL6.41 $ran(f)$ is a downwards closed subset of $\langle \mathbb{N}, \in \rangle$.

D6.42 A linear order $\langle X, \lhd \rangle$ has type omega ω if X is infinite and $\forall x \in X$, $pred_{\langle X, \lhd \rangle}(x)$ is finite.

7. Countable and Uncountable Sets

Countable Sets

C7.1 If $X \subseteq \mathbb{N}$ is infinite, $\langle X, \in \rangle$ is isomorphic to $\langle \mathbb{N}, \in \rangle$.

C7.2 If X is infinite and countable, then $X \approx \mathbb{N}$.

T7.3 There exist linear orders of type omega on $\mathbb{N} \times \mathbb{N}$, $[\mathbb{N}]^{<\omega}$, and $\mathbb{N}^{<\omega}$. Define $f : \mathbb{N} \to X$ to show infinite, then $g : X \to \mathbb{N}$ and $h : X \to X$. Define $\prec$ as normal, and use it to show $pred_{\langle X, \prec \rangle}(x)$ is finite.

C7.4

1. $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$, $[\mathbb{N}]^{<\omega} \approx \mathbb{N}$, and $\mathbb{N}^{<\omega} \approx \mathbb{N}$
2. $\forall n \in \mathbb{N}\ [n \geq 1 \Rightarrow \mathbb{N}^n \wedge [\mathbb{N}]^{<\omega} \approx \mathbb{N}]$

L7.5 Suppose that $\langle A_n : n \in \mathbb{N} \rangle$ and $\langle f_n : n \in \mathbb{N} \rangle$ are sequences such that $\forall n \in \mathbb{N}$, $f_n : A_n \to \mathbb{N}$ is $1-1$. Then $\bigcup_{n \in \mathbb{N}} A_n$ is countable.

L7.6 (AC) A countable union of countable sets is countable. If $\langle A_n : n \in \mathbb{N} \rangle$ is a sequence of countable sets, then $\bigcup_{n \in \mathbb{N}} A_n$ is countable.

L7.8 The set of rationals $\mathbb{Q}$ is countable, i.e. $\mathbb{Q} \to \mathbb{N}$.

E7.9 Let $\langle X, < \rangle$, $\langle Y, \lhd \rangle$ and $\langle Z, \prec \rangle$ be linear orders. Suppose $f : X \to Y$ and $g : Y \to Z$ are isomorphisms. $f^{-1} : Y \to X$ and $g \circ f : X \to Z$ also are.

E7.10 Define a sequence of relations $\langle \lhd_n : n \in \mathbb{N} \rangle$ by induction on $n \in \mathbb{N}$. Let $\lhd_0$ be $\emptyset$. Suppose $\lhd_n$ is a relation on $[\mathbb{N}]^n$. Define $\lhd_{n+1}$ as follows. Fix $a, b \in [\mathbb{N}]^{S(n)}$. If $max(a) < max(b)$, $a \lhd_{n+1} b$. If $max(a) = max(b)$, $a \cap max(b)$ and $b \cap max(b)$ are in $[\mathbb{N}]^n$. Then if $a \cap max(a) \lhd_n b \cap max(b)$, $a \lhd_{n+1} b$. Now define $\langle \prec_n : n \in \mathbb{N} \rangle$ by induction. $\prec_0 = \emptyset$. Suppose $\lhd_n$ is a relation on $\mathbb{N}^n$. Define $\prec_{n+1}$ as follows. Fix $\sigma, \tau \in \mathbb{N}^{n+1}$. $ran(\sigma)$ and $ran(\tau)$ are finite non-empty subsets of $\mathbb{N}$. If $max(ran(\sigma)) < max(ran(\tau))$, $\sigma \prec_{n+1} \tau$. $\sigma \restriction n$ and $\tau \restriction n$ are members of $\mathbb{N}^n$. If $max(ran(\sigma)) = max(ran(\tau))$ and if $\sigma \restriction n \prec_n \tau \restriction n$, then $\sigma \prec_{n+1} \tau$. If $max(ran(\sigma)) = max(ran(\tau))$, $\sigma \restriction n = \tau \restriction n$, and $\sigma(n) < \tau(n)$, then $\sigma \prec_{n+1} \tau$.

1. $\langle \lhd_n : n \in \mathbb{N} \rangle$ is well-defined, $\forall n \in \mathbb{N}$, $\langle [\mathbb{N}]^n, \lhd_n \rangle$ is a linear order of type ω.
2. $\langle \prec_n : n \in \mathbb{N} \rangle$ is well-defined, $\forall n \in \mathbb{N}$, $\langle \mathbb{N}^n, \prec_n \rangle$ is a linear order of type ω.
3. We can then prove (2) and (3) of T7.3 without AC.

E7.11 Extenders for the following:

1. $f(0) = f(1) = 1, f(n) = f(n-1) + f(n-2)\ \forall n > 1$

$$\mathbf{E}(\sigma) = \begin{cases} 1 & dom(\sigma) \in 2\} \\ \sigma(\bigcup dom(\sigma)) + \sigma(\bigcup\bigcup dom(\sigma)) & \text{for valid } dom(\sigma) \text{ etc.} \\ \emptyset & \text{otherwise} \end{cases}$$

2. $f(n) = \lhd_n$ on $[\mathbb{N}]^n$
3. $f(n) = \prec_n$ on $\mathbb{N}^n$

$$\begin{cases} \emptyset & dom(\sigma) = 0 \\ \{\langle s, t \rangle \in [\mathbb{N}]^{dom(\sigma)} \times [\mathbb{N}]^{dom(\sigma)} : [(\bigcup s) \in (\bigcup t)]\ or\ [(\bigcup s) = (\bigcup t)\ and\ \langle s \cap (\bigcup s), t \cap (\bigcup t) \rangle \in \sigma(\bigcup dom(\sigma))]\} & dom(\sigma) \neq 0. \end{cases}$$

$$E(\sigma) = \begin{cases} \emptyset & dom(\sigma) = 0 \\ \{\langle s, t \rangle \in \mathbb{N}^{dom(\sigma)} \times \mathbb{N}^{dom(\sigma)} : [\bigcup ran(s) \in \bigcup ran(t)]\ or\ [\bigcup ran(s) = \bigcup ran(t)\ and\ \langle s \restriction \bigcup dom(\sigma), t \restriction \bigcup dom(\sigma) \rangle \in \sigma(\bigcup dom(\sigma))]\ or\ [\bigcup ran(s) = \bigcup ran(t)\ and\ s \restriction \bigcup dom(\sigma) = t \restriction \bigcup dom(\sigma)\ and\ s(\bigcup dom(\sigma)) \in t(\bigcup dom(\sigma))]\} & \text{if } dom(\sigma) \neq 0 \end{cases}$$

7.2 Sets of Size Continuum

F7.12 If $x, y \in \mathbb{R}$ and $x < y$, there is a $q \in \mathbb{Q}$ with $x < q < y$.
L7.13 $2^{\mathbb{N}} \lesssim \mathbb{R} \lesssim \mathcal{P}(\mathbb{Q})$
T7.14 These sets are equinumerous: $2^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}, \mathcal{P}(\mathbb{N}\times\mathbb{N}), \mathcal{P}(\mathbb{N}), \mathcal{P}(\mathbb{Q}), \mathbb{R}$. $2^{\mathbb{N}} \subseteq \mathbb{N}^{\mathbb{N}} \subseteq \mathcal{P}(\mathbb{N}\times\mathbb{N})$. So $2^{\mathbb{N}} \lesssim \mathbb{N}^{\mathbb{N}} \lesssim \mathcal{P}(\mathbb{N}\times\mathbb{N})$. $\mathbb{N}\approx\mathbb{N}\times\mathbb{N}$. By L5.22, $\mathcal{P}(\mathbb{N}) \lesssim \mathcal{P}(\mathbb{N}\times\mathbb{N}) \lesssim \mathcal{P}(\mathbb{N})$ so $\mathcal{P}(\mathbb{N}) \approx \mathcal{P}(\mathbb{N}\times\mathbb{N})$. By L7.8, $\mathbb{Q}\lesssim\mathbb{N}$. By F5.2, $2^{\mathbb{N}} \lesssim \mathbb{R} \lesssim \mathcal{P}(\mathbb{Q}) \lesssim \mathcal{P}(\mathbb{N}) \lesssim 2^{\mathbb{N}}$. We also have $2^{\mathbb{N}} \lesssim \mathbb{N}^{\mathbb{N}} \lesssim \mathcal{P}(\mathbb{N}\times\mathbb{N}) \lesssim 2^{\mathbb{N}}$.
D7.15 A set X has size continuum or size $\mathfrak{c}$ if $X \approx \mathcal{P}(\mathbb{N})$.
L7.16 $(r,s) = \{x\in\mathbb{R} : r<x<s\}$ has size $\mathfrak{c}$. The function
$$f(x) = \begin{cases} \frac{x-t}{s-x} & \text{if } x \geq t \\ \frac{x-t}{x-r} & \text{if } x<t\end{cases}$$
is well-defined and $1-1$ and onto.
E7.17 Let $l \subseteq \mathbb{R}^2$ be a line. Define $\phi:\mathbb{R}\to l$ as $\phi(x) = \langle x, mx+c\rangle$ and $\phi(y) = \langle c, y\rangle$ for the different line cases. $l\approx\mathbb{R}$. For $a<b$ and $c<d$, let $m=\frac{d-c}{b-a}$ and $p = c-ma$. Then $f:(a,b)\to(c,d)$ is $1-1$ and onto.
E7.19 Prove $\mathbb{N}\times\mathbb{N}\approx\mathbb{N}$. Define $H:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ by $H(m,n) = \frac{(m+n)(m+n+1)}{2}+m$. H is $1-1$ and onto by the following steps.
1. If $i+j=n$, then $H(i,j) = H(0,n)+i < H(0,n+1)$
2. If $i+j=n, x+y=n, i<x$, then $H(i,j)<H(x,y)$
3. If $i+j=n, x+y=m, n<m$, then $H(i,j)<H(x,y)$
4. H is $1-1$
5. Let $i\in\mathbb{N}$, and $n = min\{k\in\mathbb{N} : 2i < k(k+1)\}$. If $x = i - \frac{n(n-1)}{2}$ and $y = n-1-x$, then $\langle x,y\rangle \in \mathbb{N}\times\mathbb{N}$ and $H(x,y)=i$, and H is onto

L7.21 If $A\lesssim B$ and $A\neq\emptyset$, then there exists an onto $g:B\to A$.
L7.22 (AC) Suppose A and B are sets and $f:B\to A$ is onto. Then $A\lesssim B$.
L7.23 Let A,B,C be sets and suppose $f:C\to B$ is onto. Then $A^B\lesssim A^C$.
C7.24 If $B\approx C$, then $A^B\approx A^C$.
C7.25 If $A\lesssim D$, $B\lesssim C$, and $B\neq\emptyset$, then $A^B\lesssim D^C$.
L7.26 There exists a sequence $\langle A_n : n\in\mathbb{N}\rangle$ of pairwise disjoint infinite subsets of $\mathbb{N}$ such that $\bigcup_{n\in\mathbb{N}} A_n = \mathbb{N}$. Define $B_n = \{\langle n,m\rangle : m\in\mathbb{N}\}$, and let $A_n = f^{-1}(B_n)$.
L7.27 Suppose A, B, and C are sets with $B\cap C=\emptyset$. Then $A^B\times A^C\approx A^{B\cup C}$. Define $F:A^B\times A^C\to A^{B\cup C}$, $F(f,g) = f\cup g$, and show bijectivity.
C7.28 $\mathbb{N}^{\mathbb{N}}\times\mathbb{N}^{\mathbb{N}}\approx\mathbb{N}^{\mathbb{N}}$. Define $A\cup B=\mathbb{N}, A\cap B=\emptyset$. Then $\mathbb{N}^A\approx\mathbb{N}^{\mathbb{N}}\approx\mathbb{N}^B$ and $\mathbb{N}^{\mathbb{N}}\times\mathbb{N}^{\mathbb{N}}\approx\mathbb{N}^A\times\mathbb{N}^B\approx\mathbb{N}^{A\cup B}\approx\mathbb{N}^{\mathbb{N}}$.
C7.29 $\mathbb{R}^2$ has size $\mathfrak{c}$. This can be used to count lines and planes. Use $\mathbb{R}^2\approx\mathbb{R}^{\{0\}}\times\mathbb{R}^{\{1\}}\approx\mathbb{R}\times\mathbb{R}\approx\mathbb{N}^{\mathbb{N}}\times\mathbb{N}^{\mathbb{N}}\approx\mathbb{N}^{\mathbb{N}}\approx\mathbb{R}$.
EP7.30 Let $\mathfrak{L}$ denote the set of lines in $\mathbb{R}^2$. $\mathfrak{L} = \mathfrak{L}_0\cup\mathfrak{L}_1$ where $\mathfrak{L}_0 = \{l : l \text{ satisfies } y = mx+c, m,c,\in\mathbb{R}\}$ and $\mathfrak{L}_1 = \{l : l\text{ satisfies } x = c, c,\in\mathbb{R}\}$. Then $\mathfrak{L}_0\approx\mathbb{R}^2\approx\mathbb{R}$ and $\mathfrak{L}_1\approx\mathbb{R}$, so $\mathfrak{L}$ has size $\mathfrak{c}$.
L7.31 Let A,B,C be sets. $A^{(B\times C)}\approx(A^B)^C$. Define $F:A^{(B\times C)}\to(A^B)^C$ using $F(f):C\to A^B$ as $F(f)(c)(b) = f(\langle b,c\rangle)$.
C7.32 $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}\approx\mathbb{N}^{\mathbb{N}}$. Use $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}\approx\mathbb{N}^{(\mathbb{N}\times\mathbb{N})}$, and $\mathbb{N}\times\mathbb{N}\approx\mathbb{N}$.
C7.33 $\mathbb{R}^{\mathbb{N}}$ has size $\mathfrak{c}$. Use $\mathbb{R}\approx\mathbb{N}^{\mathbb{N}}$.
D7.34 A function $f:\mathbb{R}\to\mathbb{R}$ is continuous if for each $x\in\mathbb{R}$ and each $\epsilon>0$, there exists $\delta>0$ such that $Im_f((x-\delta,x+\delta))\subseteq(f(x)-\epsilon,f(x)+\epsilon)$. A set $U\subseteq\mathbb{R}$ is an open interval if there exists $r,s\in\mathbb{R}$ such that $U=(r,s)=\{x\in\mathbb{R}:r<x<s\}$. $U\subseteq\mathbb{R}$ is open if it is the union of a collection of open intervals.
L7.35 There are only $\mathfrak{c}$ many continuous functions from $\mathbb{R}$ to $\mathbb{R}$.
L7.36 There are only $\mathfrak{c}$ many open subsets of $\mathbb{R}$.
E7.37 $\mathbb{R}^{\mathbb{R}}\approx2^{\mathbb{R}}$. First, $2^{\mathbb{R}}\subseteq\mathbb{R}^{\mathbb{R}}$, so $2^{\mathbb{R}}\lesssim\mathbb{R}^{\mathbb{R}}$. Next, define $f:\mathbb{R}\to\mathbb{R}$. $f\in\mathbb{R}^{\mathbb{R}}$, $f\subseteq\mathbb{R}\times\mathbb{R}$, $f\in\mathcal{P}(\mathbb{R}\times\mathbb{R})$. So $\mathbb{R}^{\mathbb{R}}\subseteq\mathcal{P}(\mathbb{R}\times\mathbb{R})$, and $\mathbb{R}^{\mathbb{R}}\lesssim\mathcal{P}(\mathbb{R}\times\mathbb{R})\approx2^{\mathbb{R}\times\mathbb{R}}\approx2^{\mathbb{R}}$ as $\mathbb{R}\approx\mathbb{R}\times\mathbb{R}$.
E7.38 There are only countably many algebraic real numbers. Almost all real numbers are transcendental. $a\in\mathbb{R}$ is algebraic if there exists a non-zero polynomial $p(X)\in\mathbb{Z}[X]$ such that $p(a)=0$. If a is not algebraic, it is transcendental.
E7.39 A function $f:\mathbb{R}\to\mathbb{R}$ is increasing if $\forall x,y,\in\mathbb{R}\,[x\leq y\Rightarrow f(x)\leq f(y)]$. There are only $\mathfrak{c}$ many increasing functions.
E7.40 Let $X\subseteq2^{\mathbb{N}}$ be countable. Then $(2^{\mathbb{N}}\backslash X)\approx2^{\mathbb{N}}$. If T is the set of transcendental real numbers, $T\approx R$. Since $(2^{\mathbb{N}}\backslash X)\subseteq2^{\mathbb{N}}$, $(2^{\mathbb{N}}\backslash X)\lesssim2^{\mathbb{N}}$. We want to show that $2^{\mathbb{N}}\lesssim(2^{\mathbb{N}}\backslash X)$ so that we can apply Schröder Bernstein. Let $A,B\subseteq\mathbb{N}$ be infinite sets such that $A\cap B=\emptyset$ and $A\cup B=\mathbb{N}$. By C5.33, fix bijections $\psi:A\to\mathbb{N}$ and $\varphi:B\to\mathbb{N}$. Define $G:2^{\mathbb{N}}\to(2^{\mathbb{N}}\backslash X)$ as follows by defining $G(f):\mathbb{N}\to2$. Let $n\in\mathbb{N}$. If $n\in A$, define $G(f)(n)=f(\psi(n))\in2$. If $n\in B$, $e(\varphi(n))\in X\subseteq2^{\mathbb{N}}$, and $e(\varphi(n))(n)\in2$. If $e(\varphi(n))(n)=0$, then $G(f)(n)=1$, if $e(\varphi(n))(n)=1$, then $G(f)(n)=0$. Since either $n\in A$ or $n\in B$, $G(f)(n)\in2$ and $G(f)\in2^{\mathbb{N}}$. If $G(f)\in X$, then there exists $k\in\mathbb{N}$ with $e(k)=G(f)$ and $n\in B$ with $\varphi(n)=k$. But since $n\in B$, $G(f)(n)\neq e(\varphi(n))(n)=e(k)(n)=G(f)(n)$, a contradiction. This shows $G(f)\notin X$. Now we show that G is $1-1$. Fix $f\neq f'\in2^{\mathbb{N}}$. There exists $k\in\mathbb{N}$ with $f(k)\neq f'(k)$. There exists $n\in A$ with $\psi(n)=k$. $G(f)(n)=f(\psi(n))=f(k)\neq f'(k)=f'(\psi(n))=G(f')(n)$. Then $G(f)\neq G(f')$. We have shown that $2^{\mathbb{N}}\lesssim(2^{\mathbb{N}}\backslash X)$ as needed.

8. More about Partial and Linear Orders

8.1 Dilworth's Decomposition for Finite Partial Orders

T8.1 (Dilworth) Suppose $\langle X,<\rangle$ is a finite partial order. Let $k(X)=max\{m\in\mathbb{N}:\exists A\subseteq X\,[A \text{ is an antichain in } X\wedge A\approx m]\}$. X is a union of $k(X)$ disjoint chains.
CL8.2 For all $j,j'<n$, if $j\neq j'$, then x_j and $x_{j'}$ are incomparable.
CL8.3 $\langle Z,<\rangle$ does not have any n-element antichains.
E8.4 Suppose $k\in\mathbb{N}$ and $\langle X,<\rangle$ is a finite partial order such that all chains have at most k elements. X is a union of k many antichains.
E8.5 Suppose $\langle X,<\rangle$ is a partial order. Suppose $k,l\in\mathbb{N}$. Suppose $\langle X,<\rangle$ has the property that all chains have at most l elements and all antichains have at most k elements. X is finite or it has at most $k\cdot l$ elements.
E8.6 Suppose X is a finite set of women and Y is a set of men with $Y\approx n$ for some $n\in\mathbb{N}$. Let the sequence $\langle a_i:i<n\rangle$ enumerate the men. For each $i<n$, a_i chooses a set $S_i\subseteq X$ of women he likes. It is possible to marry each a_i to someone in S_i iff for all $k\leq n$ and all k-element subsets $F\subseteq n$, $\bigcup_{i\in F}S_i$ has at least k elements.

8.2 More about Linear Orders

D8.8 Let $\langle X,<\rangle$ be a partial order and $A\subseteq X$. $x\in X$ is an upper bound of A if $\forall a\in A\,[a\leq x]$. x is a lower bound if $\forall a\in A\,[x\leq a]$. Let U be the set of upper bounds of A and L be the set of lower bounds of A. If there exists $u\in U$ such that $\forall x\in U\,[u\leq x]$, then u is the supremum of A in X or $sup_X(A)$ or minimal upper bound. For L and $[x\leq l]$, it is called the infimum or $inf_X(A)$ or greatest lower bound. There can only be at most one supremum or infimum.
EP8.9 Let $A=\{p\in\mathbb{Q}:p^2<2\}$. $sup_{\mathbb{Q}}(A)$ and $inf_{\mathbb{Q}}(A)$ do not exist, but $sup_{\mathbb{R}}(A)=\sqrt{2}$ and $inf_{\mathbb{R}}(A)=-\sqrt{2}$.
D8.10 Let $\langle X,<\rangle$ be a linear order. A pair $\langle A,B\rangle$ is a cut of $\langle X,<\rangle$ if A is downwards closed, B is upwards closed, and A and B partition X i.e. $A\cap B=\emptyset$ and $A\cup B=X$.
F8.11 Suppose $\langle X,<\rangle$ is a linear order and $Y\subseteq X$. If $z\in X\backslash Y$, $A=\{a\in Y:a<z\}$, $B=\{b\in Y:z<b\}$, then $\langle A,B\rangle$ is a cut of $\langle Y,<\rangle$.
D8.13 A linear order $\langle X,<\rangle$ is dense if $\forall x,y\in X\,\exists z\in X\,[x<y\Rightarrow x<z<y]$.
D8.14 A linear order is without endpoints if it has neither a maximal nor minimal element. A countable dense linear order with no endpoints is universal for all countable linear orders; every countable linear order embeds into such an order.
T8.15 (Cantor, AC) Suppose $\langle X,<\rangle$ is a non-empty dense linear order without endpoints. Let $\langle Y,\prec\rangle$ be any countable linear order. Then $\langle Y,\prec\rangle\hookrightarrow\langle X,<\rangle$.
T8.16 (Cantor) Let $\langle X,<\rangle$ and $\langle Y,\prec\rangle$ be non-empty countable dense linear orders without endpoints. Then they are isomorphic.
E8.17 Embedding is a quasi-order (reflexive, transitive) on linear orders.
E8.19 $\langle X,<\rangle\hookrightarrow\langle Y,\prec\rangle\wedge\langle Y,\prec\rangle\hookrightarrow\langle X,<\rangle$ does not imply isomorphism. Take $X=\mathbb{Q}\cap[0,1]$ and $Y=\mathbb{Q}\cap(0,1)$.
E8.20 We can have $\langle X,<\rangle\not\hookrightarrow\langle Y,\prec\rangle\wedge\langle Y,\prec\rangle\not\hookrightarrow\langle X,<\rangle$ (incomparability). Take $X=\langle\mathbb{N},\in\rangle$ and $Y=\langle\mathbb{N},\ni\rangle$.

9. Well-Ordered Sets

F9.1 If $\langle X,<\rangle$ is a linear order of type ω, then it is a well-order.
L9.2 Suppose A and B are downwards closed subset of X where $\langle X,<\rangle$ is a well-order. If $\langle A,<\rangle\cong\langle B,<\rangle$, then $A=B$.
C9.3 Suppose $\langle X,<\rangle$ is a well-order, and $x<x'\in X$. Then $\langle pred_{\langle X,<\rangle}(x'),<\rangle\not\cong\langle pred_{\langle X,<\rangle}(x),<\rangle$.
C9.4 Suppose $\langle X,<\rangle$ is a well-order. Then for any $x\in X$, $\langle pred_{\langle X,<\rangle}(x),<\rangle\not\cong\langle X,<\rangle$.
L9.5 If $\langle X,<\rangle$ and $\langle Y,\prec\rangle$ are isomorphic well-orders, then the isomorphism between them is unique.
T9.6 Suppose $\langle X,<\rangle$ and $\langle Y,\prec\rangle$ are well-orders. Then exactly one of the following holds:
1. $\langle X,<\rangle\cong\langle Y,\prec\rangle$
2. $\exists x\in X\,[\langle pred_{\langle X,<\rangle}(x),<\rangle\cong\langle Y,\prec\rangle]$
3. $\exists y\in Y\,[\langle X,<\rangle\cong\langle pred_{\langle Y,\prec\rangle}(y),\prec\rangle]$

E9.11 Define the product of X and Y to be $Z=Y\times X$. The dictionary order $\lhd$ on Z is a well-order.
E9.13 Given well-orders $\langle X,<_X\rangle\cong\langle A,<_A\rangle,\langle Y,<_Y\rangle\cong\langle B,<_B\rangle$, then the product and sum of $\langle X,<_X\rangle,\langle Y,<_Y\rangle\cong\langle A,<_A\rangle,\langle B,<_B\rangle$.

10. Ordinals

$\mathbf{WO}=\{\langle X,<\rangle : X \text{ is a set}\wedge<\text{ is a well-ordering of } X\}$

10.1 Basic Properties of Ordinals

D10.1 A set x is transitive if every element of x is a subset of x, or $\forall y\,[y\in x\Rightarrow y\subseteq x]$.
D10.2 Ordinals A set α is an ordinal if it is transitive and well-ordered by $\in$. Let $\in_\alpha=\{\langle\beta,\gamma\rangle\in\alpha\times\alpha:\beta\in\gamma\}$. α is an ordinal if α is transitive and $\langle\alpha,\in_\alpha\rangle$ is a well-order. The subscript of $\in_\alpha$ is often omitted.
F10.3 $\mathbb{N}$ is an ordinal. Every $n\in\mathbb{N}$ is also an ordinal.
T10.4 Let x be an ordinal. The following hold:
1. $\forall y\in x\,[y\text{ is an ordinal}\wedge y=pred_{\langle x,\in\rangle}(y)]$
2. if y is any ordinal and $\langle x,\in\rangle\cong\langle y,\in\rangle$ then $x=y$
3. if y is any ordinal, then exactly one of the following things hold: $x\in y, x=y, y\in x$
4. if y,z are ordinals and $x\in y$ and $y\in z$, then $x\in z$
5. if $\mathbf{C}$ is a non-empty class of ordinals, then $\exists y\in\mathbf{C}\,\forall z\in\mathbf{C}\,[y\in z\vee y=z]$

D10.5 $\mathbf{ORD}=\{\alpha:\alpha\text{ is an ordinal}\}$ is the class of all ordinals.
T10.6 Burali-Forti $\mathbf{ORD}$ is not a set.
L10.7 Every transitive set of ordinals is an ordinal.
T10.8 Let $\langle X,<\rangle$ be a well-ordered set. Then there exists a unique α such that $\langle X,<\rangle\cong\langle\alpha,\in_\alpha\rangle$.
D10.11 If $\langle X,<\rangle$ is any well-ordered set, then $otp(\langle X,<\rangle)$, or the order type of $\langle X,<\rangle$ is the unique ordinal α such that $\langle X,<\rangle$ is isomorphic to $\langle\alpha,\in_\alpha\rangle$.
L10.13 For ordinals α,β, $\alpha\leq\beta$ iff $\alpha\subseteq\beta$.
L10.14 If A is a non-empty set of ordinals, then $min(A)=\bigcap A$. If A is any set of ordinals, then $sup_{\mathbf{ORD}}(A)=\bigcup A$.
L10.15 For any α, $S(\alpha)$ is an ordinal, $\alpha<S(\alpha)$, and $\forall\beta\,[\beta<S(\alpha)\iff\beta\leq\alpha]$.
D10.16 α is a successor ordinal if $\exists\beta\,[\alpha=S(\beta)]$. α is a limit ordinal if $\alpha\neq0$ and α is not a successor ordinal.

L10.17 An ordinal α is a natural number iff $\forall \beta \leq \alpha\, [\beta = 0 \vee \beta \text{ is a successor ordinal}]$.
CV10.18 $\omega = \mathbb{N}$.

Induction and Recursion on the Ordinals

T10.19 Let $P(\alpha)$ be some property. If $\forall \alpha \in \mathbf{ORD}\, [\forall \beta < \alpha\, [P(\beta)] \implies P(\alpha)]$ then $\forall \alpha \in \mathbf{ORD}\, [P(\alpha)]$.
D10.20 Let $\mathbf{FOD} = \{\sigma : \sigma \text{ is a function } \wedge \exists \alpha \in \mathbf{ORD}\, [dom(\sigma) = a]\}$ denote the class of all functions whose domain is some ordinal. An ordinal extender is a function $\mathbf{E} : \mathbf{FOD} \to \mathbf{V}$. When you plug in a function with domain α into an ordinal extender, the output tells you what the value of the function at α ought to be.
T10.21 Suppose $\mathbf{E} : \mathbf{FOD} \to \mathbf{V}$ is any extender. Then there exists a unique function $\mathbf{F} : \mathbf{ORD} \to \mathbf{V}$ satisfying the condition that $\forall \alpha \in \mathbf{ORD}\, [\mathbf{F}(\alpha) = \mathbf{E}(\mathbf{F} \restriction \alpha)]$. The function generated is a proper class and not a set. $\mathbf{F} \restriction \alpha$ is a function with $dom(\mathbf{F} \restriction \alpha) = \alpha$ because $\alpha \subseteq \mathbf{ORD}$.
EP10.24 Define $V_0 = \emptyset$. Fix $\alpha \in \mathbf{ORD}$ and suppose V_β is given for all $\beta < \alpha$. If $\alpha = S(\beta)$ for some β let $V_\alpha = \mathcal{P}(V_\beta)$. If α is a limit ordinal, then $V_\alpha = \bigcup\{V_\beta : \beta < \alpha\}$. Define $\mathbf{E} : \mathbf{FOD} \to \mathbf{V}$ as follows. Fix $\sigma \in \mathbf{FOD}$. Let $\alpha = dom(\sigma) \in \mathbf{ORD}$. If $\alpha = 0, \mathbf{E}(\sigma) = \emptyset$. If α is a successor ordinal, $\exists! \beta, S(\beta) = \alpha$. Then $\beta \in \alpha$, so $\sigma(\beta)$ is defined and in $\mathbf{V}$. Let $\mathbf{E}(\sigma) = \mathcal{P}(\sigma(\beta))$. If α is a limit ordinal, then let $\mathbf{E}(\sigma) = \bigcup ran(\sigma)$.
E10.26 Call $\mathbf{C}$ trans-finitely inductive if:

1. $0 \in \mathbf{C}$
2. $\forall x \in \mathbf{C}\, [S(x) \in \mathbf{C}]$
3. for any set $X \subseteq \mathbf{C}, \bigcup X \in \mathbf{C}$

$\mathbf{ORD}$ is the smallest trans-finitely inductive class.
E10.27 Let α be any ordinal. If $X \subseteq \alpha$, then $otp(\langle X, \in \rangle) \leq \alpha$.
E10.28 Let α be any ordinal. α is a limit ordinal iff $\bigcup \alpha = \alpha$.
E10.29 For EP10.24, for each $\alpha \in \mathbf{ORD}$, V_α is transitive and $\bigcup_{\alpha \in \mathbf{ORD}} V_\alpha$ is transitive, $\alpha \subseteq V_\alpha$ and $\alpha \notin V_\alpha$.

11. Ordinal Arithmetic

11.1 Addition and Multiplication

D11.1 Let $\langle X, <_X \rangle$ and $\langle Y, <_Y \rangle$ be well-orders. Define $X \oplus Y = (\{0\} \times X) \cup (\{1\} \times Y)$. Define $<_{X \oplus Y}$ to be:

1. $\forall x, x' \in X\, [\langle 0, x \rangle <_{X \oplus Y} \langle 0, x' \rangle \iff x <_X x']$
2. $\forall y, y' \in Y\, [\langle 1, y \rangle <_{X \oplus Y} \langle 1, y' \rangle \iff y <_Y y']$
3. $\forall x \in X \forall\, y \in Y\, [\langle 0, x \rangle <_{X \oplus Y} \langle 1, y \rangle]$

Then it is a well-order.
D11.2 Suppose α and β are ordinals. Define $\alpha + \beta$ to be the order-type of the well-order $\langle \alpha \oplus \beta, <_{\alpha \oplus \beta} \rangle$, where $<_\alpha = \in_\alpha$ and $<_\beta = \in_\beta$.
L11.4 Let $\langle X, <_X \rangle, \langle Y, <_Y \rangle, \langle Z, <_Z \rangle$ be well-orders. Suppose $A, B \subseteq Z$. Assume $A \cup B = Z$ and $\forall a \in A\, \forall b \in B\, [a <_Z b]$. Then if $\langle A, <_Z \rangle \cong \langle X, <_X \rangle$ and $\langle B, <_Z \rangle \cong \langle Y, <_Y \rangle$, then $\langle Z, <_Z \rangle \cong \langle X \oplus Y, <_{X \oplus Y} \rangle$.
L11.5 For any α, β, γ:

1. $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$
2. $\alpha + 0 = \alpha$
3. $\alpha + 1 = S(\alpha)$
4. $\alpha + S(\beta) = S(\alpha + \beta)$
5. if β is a limit ordinal, then $\alpha + \beta = sup\{\alpha + \xi : \xi < \beta\}$

R11.6 $(2), (3), (5)$ can be used to give an inductive definition of $+$. For a fixed α, we can define $\dot{+}$ which is equivalent to $+$ on $\mathbf{ORD}$ by:

1. $\alpha \dot{+} 0 = \alpha$
2. $\alpha \dot{+} S(\beta) = S(\alpha \dot{+} \beta)$
3. if β is a limit ordinal, then $\alpha \dot{+} \beta = sup\{\alpha \dot{+} \xi : \xi < \beta\}$

D11.7 Let α and β be ordinals. Let $<_{\alpha \cdot \beta}$ be the dictionary order on $\beta \times \alpha$. That is, for $\langle \zeta, \xi \rangle, \langle \zeta', \xi' \rangle \in \beta \times \alpha$, $\langle \zeta, \xi \rangle <_{\alpha \cdot \beta} \langle \zeta', \xi' \rangle$ iff either $\zeta < \zeta'$ or $\zeta = \zeta'$ and $\xi < \xi'$. Then it is a well-order and $\alpha \cdot \beta = otp(\langle \beta \times \alpha, <_{\alpha \cdot \beta} \rangle)$ which is β copies of α.
L11.8 Suppose α, β, γ are ordinals. Suppose $A \subseteq \gamma$ and $\langle A, \in \rangle \cong \langle \beta, \in \rangle$. Then $\langle A \times \alpha, <_{\alpha \cdot \gamma} \rangle \cong \langle \beta \times \alpha, <_{\alpha \cdot \beta} \rangle$.
L11.9 For any α, β, γ:

1. $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$
2. $\alpha \cdot 0 = 0$
3. $\alpha \cdot 1 = \alpha$
4. $\alpha \cdot S(\beta) = a \cdot \beta + \alpha$
5. if β is a limit ordinal, $\alpha \cdot \beta = sup\{a \cdot \xi : \xi < \beta\}$
6. $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$

Also, $\cdot$ is not commutative on $\mathbf{ORD}$ since $2 \cdot \omega \neq \omega \cdot 2$. (6) fails for multiplication on the right since $(1 + 1) \cdot \omega = \omega \neq 1 \cdot \omega + 1 \cdot \omega$.

Exponentiation

D11.10 For a fixed α, define α^β by recursion on β using the following clauses:

1. if $\alpha = 0$,then $\alpha^0 = 0$; if $\alpha > 0$, then $\alpha^0 = 1$
2. $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$
3. if β is a limit ordinal, then $\alpha^\beta = sup\{a^\xi : \xi < \beta\}$

E11.11 Define the extender $\mathbf{E}^+_\alpha : \mathbf{FOD} \to \mathbf{V}$ as follows. For any $\sigma \in \mathbf{FOD}$,

$$\mathbf{E}^+_\alpha(\sigma) = \begin{cases} \alpha & \text{if } dom(\sigma) = 0 \\ S(\sigma(\beta)) & \text{if } dom(\sigma) = S(\beta) \\ \bigcup ran(\sigma) & \text{if } dom(\sigma) \text{ is a limit ordinal} \end{cases}$$

For other operations need to check the case where $\sigma(\beta) \notin \mathbf{ORD}$.
E11.12 For any ordinal $\alpha > 0$, $\alpha \cdot \omega > \alpha$.
E11.13 $\alpha < \beta \Rightarrow \gamma + \alpha < \gamma + \beta \wedge \alpha + \gamma \leq \beta + \gamma$ but not $<$ on the second clause.
E11.14 If $\alpha \geq \omega$ is an ordinal, then $1 + \alpha = \alpha$.
E11.15 If $\gamma > 0$, then $\alpha < \beta \Rightarrow \gamma \cdot \alpha < \gamma \cdot \beta \wedge \alpha \cdot \gamma \leq \beta \cdot \gamma$ but not $<$ on the second clause.
E11.16 Let $0 < \alpha \leq \beta$ be ordinals. There exist unique δ, ξ such that $\xi < \alpha$ and $\alpha \cdot \delta + \xi = \beta$.
E11.17 $\alpha^{(\beta+\gamma)} = \alpha^\beta \cdot \alpha^\gamma$ for ordinals $\alpha > 0$.
E11.18 Define $\alpha_0 = \omega$ and $\forall n \in \omega, a_{n+1} = \omega^{\alpha_n}$. Let $\epsilon_0 = sup\{\alpha_n : n \in \omega\}$. Then $\omega^{\epsilon_0} = \epsilon_0$.

Cardinals and Cardinal Arithmetic

D12.1 A set X is said to be well-orderable if there exists a relation $< \subseteq X \times X$ such that $\langle X, < \rangle$ is a well-order.
D12.2 Let X be a well-orderable set. Define the cardinality of X, $|X|$, to be the minimal element of $\{\alpha \in \mathbf{ORD} : \alpha \approx X\}$. $|\alpha|$ is defined for every $\alpha \in \mathbf{ORD}$, and $|\alpha| \leq \alpha$.
D12.3 α is a cardinal if $|\alpha| = \alpha$.
F12.4 If $n \in \omega$, then n is a cardinal. ω is a cardinal.
L12.5 If $|\alpha| \leq \beta \leq \alpha$, then $|\beta| = |\alpha|$.
L12.6 A se is finite iff $|X| < \omega$. A set is countable iff $|X| \leq \omega$.
D12.7 Let κ and λ be cardinals. These are well-orderable:

1. $\kappa \boxplus \lambda = |(\{0\} \times \kappa) \cup (\{1\} \times \lambda)|$
2. $\kappa \boxtimes \lambda = |\kappa \times \lambda|$

L12.8 Every infinite cardinal is a limit ordinal.
T12.9 If κ is an infinite cardinal, then $\kappa \boxtimes \kappa = \kappa$.
C12.10 Let κ and λ be infinite cardinals. Then $\kappa \boxplus \lambda = \kappa \boxtimes \lambda = max\{\kappa, \lambda\}$.
T12.11 For every set X there is a cardinal α such that there is no 1-1 function $f : \alpha \to X$.
D12.16 For each $\alpha \in \mathbf{ORD}$, α^+ is the least cardinal strictly greater than α.
L12.17 Suppose $\mathbf{F} : \mathbf{ORD} \to \mathbf{ORD}$ is a function such that $\forall \alpha, \beta \in \mathbf{ORD}\, [\alpha < \beta \Rightarrow \mathbf{F}(\alpha) < \mathbf{F}(\beta)]$. Then $\forall \beta \in \mathbf{ORD}\, [\beta \leq \mathbf{F}(\beta)]$.

D12.18 Define a sequence $\langle \omega_\alpha : \alpha \in \mathbf{ORD} \rangle$ by induction using the following clauses:

1. $\omega_0 = \omega$
2. $\omega_{S(\alpha)} = \omega^+_\alpha$
3. if α is a limit ordinal, then $\omega_\alpha = sup\{\omega_\xi : \xi < \alpha\}$

R12.19 ω_α is sometimes deonted as $\aleph_\alpha$.
L12.20 $\alpha < \beta \implies \aleph_\alpha < \aleph_\beta$ and every infinite cardinal is equal to $\aleph_\alpha$ for some $\alpha \in \mathbf{ORD}$.

12.1 Choice and Cardinality

D12.21 Let X be any set. F is a choice function on X if F is a function, $dom(F) = X \backslash \{0\}$, and $\forall a \in X \backslash \{0\} [F(a) \in a]$.
T12.22 Zermelo TFAE for a set X:

1. X is well-orderable
2. there exists a choice function on $\mathcal{P}(X)$

T12.26 AC TFAE:

1. the Cartesian product of non-empty sets is non-empty
2. for every set X there exists a choice function on X
3. every set is well-orderable
4. for any two sets X and Y, either $X \precsim Y$ or $Y \precsim X$
5. for every set X there is an ordinal α and a 1-1 function $f : X \to \alpha$
6. for every set X there is a cardinal κ such that $X \approx \kappa$

Cardinal Exponentiation and König's Theorem

D12.28 (AC) Let κ and λ be cardinals. Define $\kappa^\lambda = |\{f : f \text{ is a function} \wedge dom(f) = \lambda \wedge ran(f) \subseteq \kappa\}|$.
L12.30 Let κ, λ, θ be cardinals. The following hold:

1. $(\kappa^\lambda)^\theta = \kappa^{(\lambda \boxtimes \theta)}$
2. $(\kappa^\lambda) \boxtimes (\kappa^\theta) = \kappa^{(\lambda \boxplus \theta)}$

D12.31 Define a squence of cardinals $\langle \beth_\alpha : \alpha \in \mathbf{ORD} \rangle$ by induction using the following clauses:

1. $\beth_0 = \omega$
2. $\beth_{S(\alpha)} = 2^{\beth_\alpha}$
3. if α is a limit ordinal, then $\beth_\alpha = sup\{\beth_\xi : \xi < \alpha\}$

D12.32 The Generalised Continuum Hypothesis is the statement that $\forall \alpha \in \mathbf{ORD}\, [\beth_\alpha = \aleph_\alpha]$. The Continuum Hypothesis is the statement that $\beth_1 = \aleph_1$. Note $\beth_1 = 2^{\beth_0} = 2^{\aleph_0}$, so CH says $2^{\aleph_0} = \aleph_1$.
T12.34 König $(\aleph_\omega)^{\aleph_0} > \aleph_\omega$.
C12.35 $2^{\aleph_0} \neq \aleph_\omega$.
E12.36 Let κ, λ be infinite cardinals where $\lambda \leq \kappa$. Then $\kappa^\lambda = |\{X \subseteq \kappa : |X| = \lambda\}|$.
E12.37 Let $\kappa, \lambda, \theta, \chi$ be cardinals. If $\kappa \leq \lambda$, then $\kappa^\theta \leq \lambda^\theta$. If $\kappa \leq \chi, \lambda \leq \theta$ and $\lambda \neq 0$, then $\kappa^\lambda \leq \chi^\theta$.
E12.38 Let α be an ordinal. Let $W = \{\langle Y, \lhd \rangle : Y \subseteq \alpha \wedge \langle Y, \lhd \rangle \text{ is a well-order}\}$. $\alpha^+ = \{otp(\langle Y, \lhd \rangle) : \langle Y, \lhd \rangle \in W\}$.
E12.39 There is a cardinal $\kappa = \aleph_\kappa$ and $\kappa = \beth_\kappa$.
E12.40 Suppose $\mathbf{F} : \mathbf{ORD} \to \mathbf{ORD}$ and $\forall \alpha, \beta \in \mathbf{ORD}\, [\alpha < \beta \implies \mathbf{F}(\alpha) < \mathbf{F}(\beta)]$ and for any limit ordinal β, $\mathbf{F}(\beta) = sup\{\mathbf{F}(\alpha) : \alpha < \beta\}$. Then $\forall \alpha \in \mathbf{ORD}\, \exists \beta > \alpha\, [\mathbf{F}(\beta) = \beta]$.
E12.41 $(\aleph_{\omega_1})^{\aleph_1} > \aleph_{\omega_1}$ and $2^{\aleph_1} \neq \aleph_\omega, \aleph_{\omega_1}$.

13. Some applications of AC

D13.1 Let A be any set. $\mathcal{F} \subseteq \mathcal{P}(A)$ is of finite character iff $\forall X \subseteq A, X \in \mathcal{F} \iff \forall Y \subseteq X\ [|Y| < \omega \implies Y \in \mathcal{F}]$. All of X's finite subsets are in $\mathcal{F}$.

L13.2 Suppose $\mathcal{F} \subseteq \mathcal{P}(A)$ is of finite character. Then for any $X \in \mathcal{F}$ and any $Y \subseteq X, Y \in \mathcal{F}$.

T13.3 TFAE:

1. AC
2. for any set A and any $\mathcal{F} \subseteq \mathcal{P}(A)$, if F has finite character, then for every $X \in \mathcal{F}$, there exists $Y \in \mathcal{F}$ such that $X \subseteq Y$ and Y is maximal in $\langle \mathcal{F}, \subsetneq \rangle$ (Teichmüller-Tukey Lemma)
3. every chain in every partial order is contained in a maximal chain (Hausdorff's maximal chain theorem)
4. if $\langle X, < \rangle$ is any partial order where every chain in $\langle X, < \rangle$ has an upper bound in $\langle X, < \rangle$, then $\langle X, < \rangle$ has a maximal element (Zorn's lemma)

$(4) \implies (1)$: Prove the standard version of AC. Let I be any set and suppose $\langle X_i : i \in I \rangle$ is any sequence of non-empty sets. Consider $A = \{\sigma : \sigma \text{ is a function} \wedge dom(\sigma) \subseteq I \wedge \forall i \in dom(\sigma)\ [\sigma(i) \in X_i]\}$. Partially order A by $\subsetneq$. Let $C \subseteq A$ be any chain. C is a directed collection of functions. So $\bigcup C = \tau$ is a function and $dom(\tau) = \bigcup\{dom(\sigma) : \sigma \in C\} \subseteq I$. $\tau(i) = \sigma(i) \in X_i$. Therefore $\tau \in A$ and $\forall \sigma \in C\ [\sigma \subseteq \tau]$. So τ is an upper bound for C. So every chain has an upper bound and there is a maximal $\sigma \in A$ by Zorn's lemma. We claim that $dom(\sigma) = I$. If not, there exists $i \in I \backslash dom(\sigma)$. Since $X_i \neq 0$, choose $x_i \in X_i$. Put $\tau = \sigma \cup \{\langle i, x_i \rangle\}$. Then $\tau \in A$ and $\sigma \subsetneq \tau$, contradicting maximality of σ.

Using Zorn's Lemma $\langle X, < \rangle$ is a partial order. If every chain in $\langle X, < \rangle$ has an upper bound in $\langle X, < \rangle$, then $\langle X, < \rangle$ has a maximal element.

1. Find a relevant $\langle X, < \rangle$, e.g. $\langle \mathcal{P}(X), \subsetneq \rangle$ which might be given.
2. Take a chain $\zeta \subseteq X$. Show ζ has an upper bound in $\langle X, < \rangle$. Usually this involves taking unions of things in ζ. But you have to check that these unions belong to X. Also, $\zeta = \emptyset$ is always a chain.
3. By Zorn's lemma $\exists x \in X$ maximal in $\langle X, < \rangle$. Now maximality will imply x has some special property. Most of the time you check x has the relevant property, because if it did not it would contradict maximality in $\langle X, < \rangle$.

E13.16 Let $\langle X, < \rangle$ be a partial order. Every antichain in $\langle X, < \rangle$ is contained in a maximal antichain. Let $A \subseteq X$ be an antichain.

1. $P = \{B \subseteq X : A \subseteq B \wedge B \text{ is an antichain}\}$. $\langle P, \subsetneq \rangle$ is a partial order.
2. Let $\zeta \subseteq P$ be a chain in $\langle P, \subsetneq \rangle$. Case I: $\zeta = \emptyset$. Then $A \in P$ and $\forall B \in \zeta\ [B \subseteq A]$. So A is an upper bound for $\zeta \in P$. Case II: $\zeta \neq \emptyset$. Let $D = \bigcup \zeta$. For any $B \in \zeta, B \subseteq D$. If $D \in P$, then D would be an upper bound of ζ as $\forall B \in \zeta\ [B \subseteq D]$. So We want to show $D \in P$. $D \subseteq X$ as $\forall B \in \zeta\ [B \subseteq X]$. As $\zeta \neq \emptyset$, $\exists B \in \zeta$ such that $A \subseteq B \subseteq D$. So $A \subseteq D$. Show D is an antichain. Suppose $x, y \in D, x \neq y$. $\exists B, B' \in \zeta$ with $x \in B, y \in B'$. As ζ is a chain, WLOG $B \subseteq B'$. So $x, y \in B'$. Since B' is an antichain, $x \not< y, y \not< x$. So D is an antichain and $D \in P$.
3. By Zorn's, $\exists B \in P$ which is maximal in $\langle P, \subsetneq \rangle$. Now $A \subseteq B$, B is an antichain. If $\exists D, B \subsetneq D$, then $D \in P$ as $A \subseteq B \subseteq D$, contradicting maximality of B in $\langle P, \subsetneq \rangle$.

E13.19 Show every vector space V has a basis. Let $P = \{B \in \mathcal{P}(V) : B \text{ is linearly independent}\}$. Then $\langle P, \subsetneq \rangle$ is a partial order. Take $\zeta \subseteq P$. If $\zeta = \emptyset$, then it is an upper bound. If $\zeta \neq \emptyset$, then take $B = \bigcup \zeta$. Suppose B is not linearly independent. Then there is a non-trivial solution, some $X \in \zeta$ must be linearly dependent, violating $X \in \zeta \in P$. So by Zorn's, there exists a maximal element in $B \in P$. Now show $span(B) = V$. Suppose otherwise. Then $\exists v \in V \notin span(B)$. Then take $B \cup \{v\}$ which is linearly independent, but this contradicts maximality of B. So $span(B) = V$, and B is a basis of V.

23/24 Q5 Call $X \subseteq \mathbb{R}$ an S-set if $\forall w, x, y, z \in X\ [w + x = y + z \implies \{w, x\} = \{y, z\}]$. Let $\mathcal{P} = \{X \subseteq \mathbb{R} : X \text{ is an S-}set\}$.

1. $\langle \mathcal{P}, \subsetneq \rangle$ is a partial order.
2. Let $\zeta \subseteq \mathcal{P}$ be a chain in $\langle \mathcal{P}, \subsetneq \rangle$. Let $Y = \bigcup \zeta$. $Y \subseteq \mathbb{R}$ as $\forall x \in \zeta[x \subseteq \mathbb{R}]$. Show Y is an S-set. Suppose $w, x, y, z \in Y$. Find $w \in X_1, x \in X_2, y \in X_3, z \in X_4$. As ζ is a chain, WLOG $X_1 \subseteq X_2 \subseteq X_3 \subseteq X_4$. Then $w, x, y, z \in X_4$. As $X_4 \in \mathcal{P}$, it is an S-set. So Y is an S-set and $Y \in \mathcal{P}$. Since $\forall x \in \zeta\ [x \subseteq Y]$, Y is an upper bound for ζ in $\langle \mathcal{P}, \subsetneq \rangle$.
3. Show there is an uncountable S-set. By Zorn's, let $X \in \mathcal{P}$ be maximal in $\langle \mathcal{P}, \subsetneq \rangle$. Assume X is countable. Let $Y = \{\frac{w+x}{2} : w, x \in X\}, Z = \{w + x - y : w, x, y \in X\}$. $X \cup Y \cup Z$ is countable. As $\mathbb{R}$ is uncountable, let $v \in \mathbb{R} \backslash (X \cup Y \cup Z)$. Then $X \cup \{v\}$ is an S-set by case bashing, and $X \cup \{v\} \in \mathcal{P}$, but this contradicts maximality of X.

23/24 Q4 $\mathbf{F}(0) = 1, \mathbf{F}(\beta) = \mathbf{F}(\alpha) \cdot \beta$ if $\beta = \alpha + 1$, $\mathbf{F}(\beta) = sup\{\mathbf{F}(\alpha) : \alpha < \beta\}$ if β is a limit ordinal. The extender is

$$\mathbf{E}(\sigma) = \begin{cases} 1 & \text{if } dom(\sigma) = 0 \\ \sigma(\bigcup dom(\sigma)) \cdot dom(\sigma) & \text{if } dom(\sigma) = S(\beta) \\ \bigcup ran(\sigma) & \text{if } dom(\sigma) \text{ is a limit ordinal} \\ \emptyset & \text{otherwise} \end{cases}$$

Swap out $\beta = dom(\sigma), \mathbf{F}(\alpha) = \sigma(\bigcup dom(\sigma))$ where $\beta = \alpha + 1$, $sup... = \bigcup ran(\sigma)$