Karlsruher Institut für Technologie

Game theoretical approach of pricing a perpetual option

Annabel Malle, Minh Hieu Nguyen | 9. Februar 2022

Table of contents

Introduction

Motivation

- Financial markets are affected by every agent's decision
- Agent's goal: maximize their economical (monetary) benefit
- Valuation of specific market instruments (e.g. perpetual options) is challenging

Which is the fair price P of the option?

- Standard method is the maximizing expected utility method
- Our approach: Combination of game theory and option pricing

- Wide spectrum for applications of this method

Basics

Put option

Put option: Investor may sell asset S to the intermediary at $t > 0$ (without obligation) [1]

Value

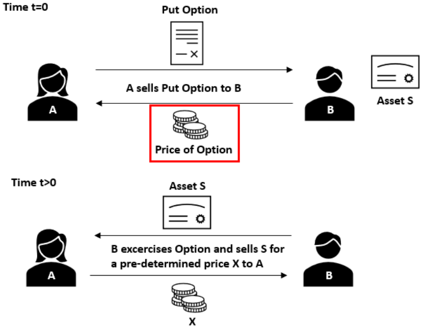
The value of a put option at time $t > 0$ of its exercise is

$$P(S, t) = max(X - S(t), 0). \quad (1)$$

Perpetual put option: Option without expiry date and no restrictions on when it can be exercised

$S(t)$: current value of underlying asset
X: pre-defined Strikeprice

$P(S, t)$: value of put option
t: time



[1] The following chapter is based on the lecture notes of [3].

Arbitrage-free market

Our option valuation model is based on the principle of *no-arbitrage*.

- **No** strategy of an agent can yield to risk-free profit
- No free lottery
- No free lunch
- Law of the one price (identical future cash flow $\rightarrow$ identical price)

Options - Value

Karlsruher Institut für Technologie

Options

The value of an option with underlying asset S follows a geometric Brownian motion

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t,\, t \in [0, T],\, S(0) = S_0 \tag{2}$$

where μ denotes the drift, σ the volatility and $\mathrm{d}W_t$ the Wiener process.

By Merton [1] it is shown that the value of any option $P(S, t)$ must satisfy the linear partial differential equation:

$$\frac{1}{2}\sigma^2 S^2 P_{SS} + (rS - a)P_S + P_t - rP + b = 0 \tag{3}$$

a: dividend payout to holder of asset
b: payout to holder of contigent claim

r: interest rate
P_S, P_t, P_{SS}: partial derivatives of P

Options - Boundary condition

- Boundary conditions specify each option
- Excercise of option if S reaches a specific level $\bar{S}$:

$$P(S) = \bar{P}(\bar{S}), \text{where } \bar{P}(\bar{S}) \text{ is payoff received by owner of option upon excercise} \tag{4}$$

- Agents may influence this boundary condition (parameter $\bar{S}$) to find optimal strategy and maximize their payoff
- Then payoff isn't depending on time t anymore, thus the partial differential equation (3) is shortend:

$$\frac{1}{2}\sigma^2 S^2 P_{SS} + (rS - a)P_S - rP + b = 0 \tag{5}$$

Game theory - Basics

- Modelling strategic interactions among agents
- Agents can influence the parameters and boundary conditions of the options by their strategies
- Assumptions:
  - The agents act rational
  - Each agent wants to maximize their profit respectivly their payoff

Game theory - Normal form

- Two agents A (intermediary) and B (investor)
- Strategies of agents are collected in sets $S_A = \{a_1, ..., a_{n_A}\}$ and $S_B = \{b_1, ..., b_{n_B}\}$ where n_A, n_B are the numbers of strategies
- Game is decribed by the set of all strategies $S_{all} := S_A \times S_B$
- The option will be excercised as soon as S reaches $\bar{S}$ for the first time (BD (4))
- The uncertain payoff $u_x : S_{all} \to \mathbb{R}$ at $t > 0$ of an agent will be valued by option pricing theory
- First agent A chooses a strategy, then chooses B a strategy and each agents knows which decision was made and what the payoff will be (*game of perfect information*)

[1] Those basics are retrieved from the lecture notes of [2] and adjusted to our specific model.

Game theoretical approach to option pricing

Determining the price of a perpetual put option

- **Which price P_∞ should the intermediary A ask for?**

Three-step procedure[2]

1. Definition of game by setting agent's action set, sequence of choices and resulting payoffs
2. Valuation of the option considering all possible actions of the agents as parameters
3. Finding the agent's optimal strategies and solving the game

- Estimated value of option $\approx$ agent's expected utility (increasing monotonic relationship)
- **Maximizing the value of the option will maximize the agents utility** (and vice-versa)

[2] Method of [4].

Step 1: Structure of the game

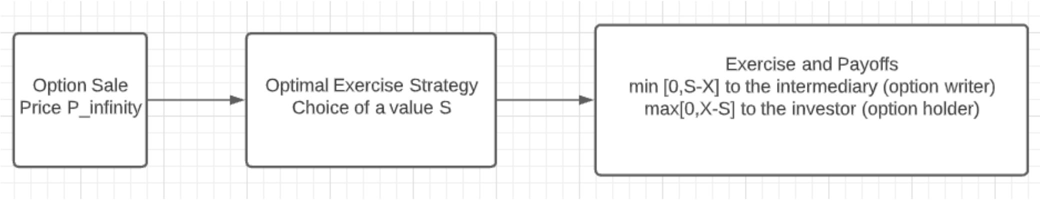


Abbildung: Structure of the option pricing game.

- First intermediary sells a perpetual put option to the investor at a price P_∞
- Then investor then chooses his optimal exercise strategy $\bar{S}$
- Finally if the investor chooses to exercise the option, he receives $X - \bar{S}$ from the intermediary

Step 2: Valuing the Option for a Given Exercise Strategy

The second step in the method is to determine the arbitrage-free value of the perpetual put option, $P_\infty(S)$, given the investor's exercise strategy $\bar{S}$.

- Since the option is perpetual, its value does not depend explicitly on t and must satisfy the ordinary differential equation (5)

$$\frac{1}{2}\sigma^2 S^2 P''_\infty + rSP'_\infty - rP_\infty = 0 \tag{6}$$

subject to the boundary conditions:

$$P_\infty(\infty) = 0 \tag{7}$$

means the option is worthless if the value of the underlying asset is very large.

$$P_\infty(\bar{S}) = X - \bar{S} \tag{8}$$

means the payoff from the option upon exercise.

Step 2: Valuing the Option for a Given Exercise Strategy

The general solution to (6) is

$$P_\infty(S) = \alpha_1 S + \alpha_2 S^{-\gamma}, \tag{9}$$

where

$$\gamma \equiv \frac{2r}{\sigma^2}. \tag{10}$$

From boundary condition (7), we have $\alpha_1 = 0$. On the other hand, boundary condition. (8) requires that

$$P_\infty(\bar{S}) = X - \bar{S} = \alpha_2 \bar{S}^{-\gamma}. \tag{11}$$

Solving for α_2 yields

$$\alpha_2 = (X - \bar{S})\bar{S}^{\alpha}. \tag{12}$$

Step 2: Valuing the Option for given Exercise Strategy

The value of the perpetual put option, given the investor's exercise strategy $\bar{S}$

$$P_\infty(S) = (X - \bar{S})\bar{S}^\gamma S^{-\gamma} = (X - \bar{S})(\frac{S}{\bar{S}})^{-\gamma}. \tag{13}$$

Step 3: Solving the Game

- At this point, the investor's exercise strategy $\bar{S}$ is still unknown (*free boundary*)
- Finding free boundary $\bar{S}$: Investor exercises, if value of the option is maximal, thus setting

$$\frac{\partial P_\infty}{\partial \bar{S}} = -(\frac{S}{\bar{S}})^{-\gamma} + \frac{\gamma}{\bar{S}}(X - \bar{S})(\frac{S}{\bar{S}})^{-\gamma} = 0. \tag{14}$$

Simplifying, this expression becomes

$$\bar{S} = \gamma(X - \bar{S}) \tag{15}$$

The underlying asset value $\bar{S}$ at which exercising the perpetual put option

$$\bar{S} = \frac{\gamma}{1+\gamma}X. \tag{16}$$

The Solution

- Intermediary anticipates that investor will exercise the option when it is optimal to do so
- She asks for a price equal to the value of the option assuming that the investor exercises optimally, which is given by (13)
- The optimal excercise strategy is given by (16)

The market price of the perpetual put option

$$P_\infty(S) = (X - \bar{S})(\frac{S}{\bar{S}})^{-\gamma} = \frac{X}{1+\gamma}(\frac{(1+\gamma)S}{\gamma S})^{-\gamma}. \tag{17}$$

Simulation

Simulation

The price of the underlying asset is modeled by the geometric Brownian motion (GBM)

$$S(t) = S_0 \exp(at + \sigma W(t)), \text{ where } a = \mu - \sigma^2/2,\ t \in [0, T] \tag{18}$$

with $\mu, \sigma > 0$, initial condition $S_0 \geq 0$ and $W(t)$ denoting the Wiener process. $S(t)$ is the solution of the stochastic differential equation (2).

The process $S(t)$ is the solution of the SDE (2)

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t),\ t \in [0, T],\ S(0) = S_0 \tag{19}$$

Wiener process - Standard Brownian Motion

Brownian motion can be described by a continuous-time stochastic process called the Wiener process.
Let $W(t)$ be a random variable that depends continuously on $t \in [0, T]$. The random variable is characterized by:

- $W(0) = 0$ with probability 1.
- For $0 \leq s < t \leq T$, the increment:

  $$W(t) - W(s) \sim \sqrt{(t-s)}\mathcal{N}(\mu, \sigma^2)$$

  is normally distributed, where $\mathcal{N}(\mu, \sigma^2)$ denotes the normal distribution with mean μ and variance σ.
- $W(t)$ has independent increments, which means that if $0 \leq s < t < u < v \leq T$, then $W(t) - W(s)$ and $W(v) - W(u)$ are also independent.

Wiener process - Standard Brownian Motion

To implement the scalar standard Brownian motion in Python, suppose we discrete the time interval $[0, T]$ into $N - 1$ equidistant sub-intervals or N points:

$$0 < t_1 < t_2 < ... < t_{j-1} < t_j < t_{j+1} < ... < t_{N-1} < t_N = T$$

The time step, $\triangle t = t_{j+1} - t_j$, is obtained from $\triangle t = \frac{T}{N-1}$. The increment can be given by

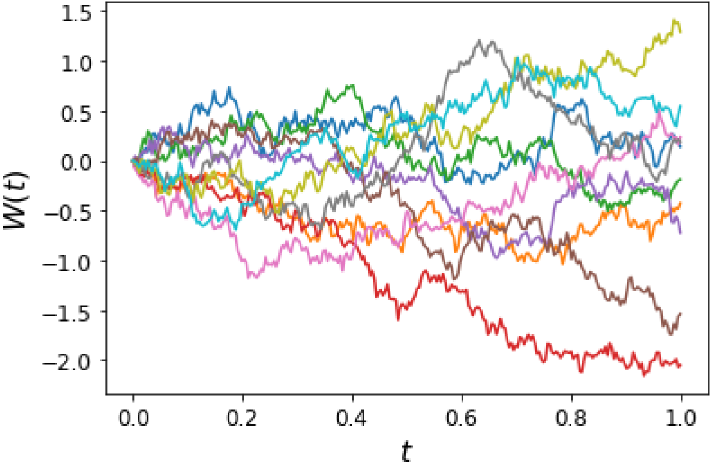
$$dW(t_j) = W(t_j) - W(t_{j-1}) = \sqrt{(\triangle t)\mathcal{N}(0,1)} \tag{20}$$

for standard normal distribution with $\mu = 0$ and $\sigma^2 = 1$.

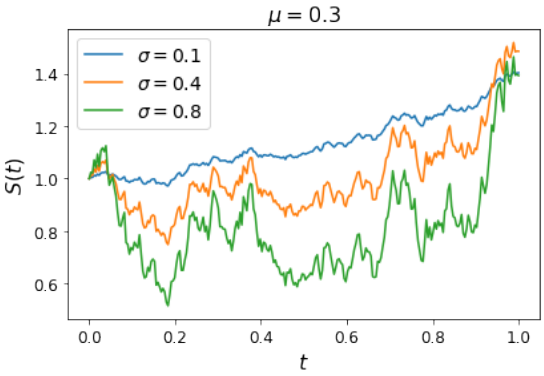
Wiener process $W(t)$ can be obtained by rearranging equation (20) to:

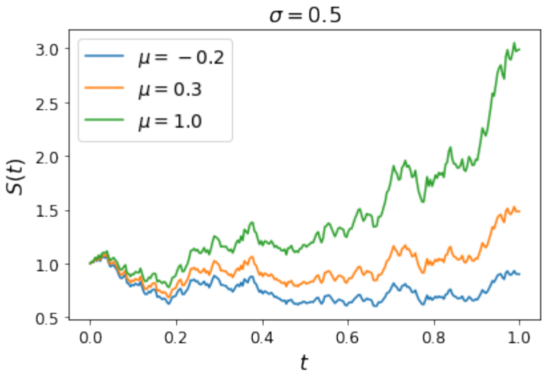
$$W(t_j) = W(t_{j-1}) + dW(t_j). \tag{21}$$

10 Wiener example paths



Geometric Brownian motion





Euler-Maruyama method

A numerical simulation of this SDE can be obtained by employing the Euler-Maruyama method:

- Define step-size $\tau = \frac{T}{N}$, $N \in N$ and for $n = 1, \ldots, N$ let $t_n = \tau n$

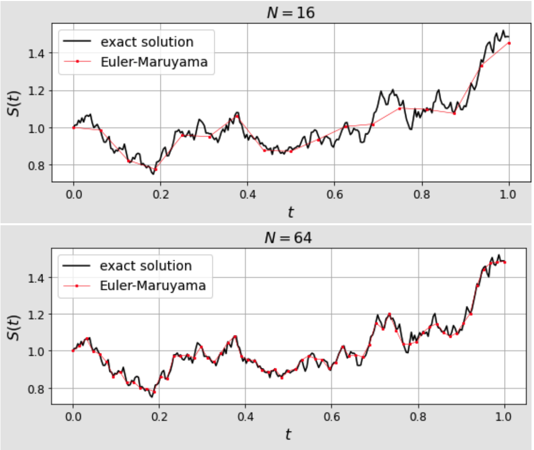
Compute the approximation $S_n \approx S(t_n)$ as

$$S_n = S_{n-1} + \tau\mu S_{n-1} + \sigma S_{n-1} \triangle W_{n-1} \tag{22}$$

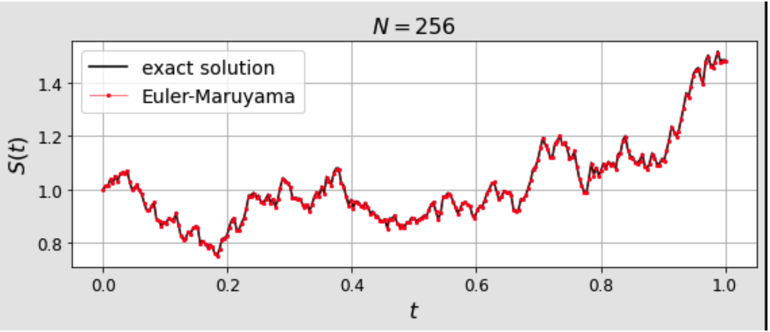
where $\triangle W_{n-1} = W(t_n) - W(t_{n-1})$

- Figure below shows approximation to the solution of above SDE with drift $\mu = 0.3$, volatility $\sigma = 0.4$ and initial condition $S_0 = 1$
- Using the Euler-Maruyama method with different time steps $\tau = T/N$, namely for $N \in \{16, 64, 256\}$

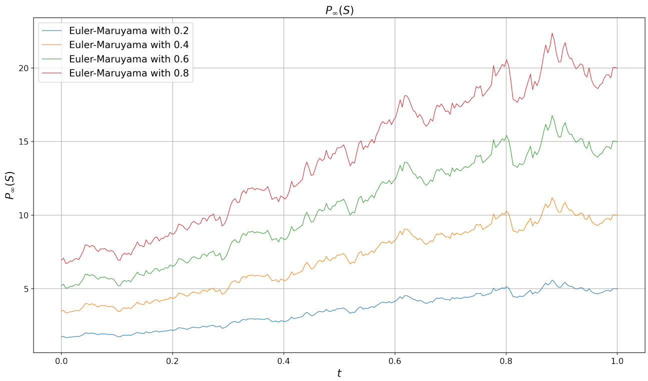
Euler-Maruyama



Euler-Maruyama



Price of perpetual put option



$$P_\infty(S) = (X - \bar{S})\left(\frac{S}{\bar{S}}\right)^{-\gamma} = \frac{X}{1+\gamma}\left(\frac{(1+\gamma)S}{\gamma S}\right)^{-\gamma}.$$

Conclusion

Conclusion

- Good alternative to the standard method for pricing such a specific option
- Main advantage: Separating valuation problem from analysis of strategic interactions of agents
- Best application on situations where
  - payoffs occur at random times (optimal stopping problem)
  - future decisions of the agent influences their payoffs
  - the risk-taking behaviour of agents are analyzed
- Limitations:
  - mathematical complexity $\rightarrow$ no general models
  - difficulty with stochastic optimal strategies
  - assumption that option values are a good proxy for agent's expected utility

Outlook

The game theoretical approach can be applied to other classical problems of corporate finance and financial intermediation: [3]

- Financial contracting
- The bankruptcy problem
- Analyzing the phenomenon ’Bank runs’
- Incentive effects of subordinated debt
- Deposit insurance

---

[3] Applications can be found in [4]

Literatur

[1] Robert C Merton. „On the pricing of contingent claims and the Modigliani-Miller theorem“. In: *Journal of Financial Economics* 5.2 (1977), S. 241–249.

[2] PD Dr. Gudrun Thaeter. *Lecture Notes: Mathematical Modelling and Simulation*. Winter 2020/21.

[3] Marliese Uhrig-Homburg. *Lecture Notes: Derivate*. Institut für Finanzwirtschaft, Banken und Versicherungen (FBV), Abteilung FInancial Engineering und Derivate, Sommer 2021.

[4] Alexandre C Ziegler. *A Game Theory Analysis of Options: Corporate Finance and Financial Intermediation in Continuous Time*. Springer Science & Business Media, 2012.