Odometry Derivation

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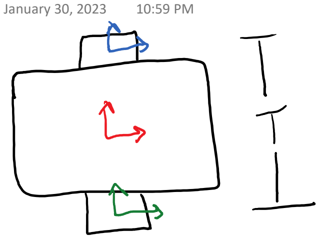


$$T_{BL}\left(0,\ 0,\ \frac{T}{2}\right) \qquad T_{BR}\left(0,\ 0,\ -\frac{T}{2}\right)$$

$$A_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ y & 1 & 0 \\ -x & 0 & 1 \end{bmatrix}$$

$$A_{BL} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{T}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A_{BR} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{T}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_{LB} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{T}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A_{RB} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{T}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$V_B = \begin{bmatrix} \omega \\ V_x \\ V_y \end{bmatrix} \qquad V_i = \begin{bmatrix} \omega_i \\ V_{xi} \\ V_{yi} \end{bmatrix}$$

Body twist   Twist in wheel i frame

$$V_L = A_{LB} V_B \qquad V_R = A_{RB} V_B$$

$$\begin{bmatrix} \omega_L \\ V_{xL} \\ V_{yL} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & 0 & 1 \end{bmatrix} \begin{bmatrix} \omega \\ V_x \\ V_y \end{bmatrix} = \begin{bmatrix} \omega \\ -\frac{T}{2}\omega + V_x \\ V_y \end{bmatrix}$$

$$\begin{bmatrix} \omega_R \\ V_{xR} \\ V_{yR} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{T}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \omega \\ V_x \\ V_y \end{bmatrix} = \begin{bmatrix} \omega \\ \frac{T}{2}\omega + V_x \\ V_y \end{bmatrix}$$

Conventional wheels:

$$V_{x_i} = r\dot{\phi}_i = r u_i$$

$$V_{y_i} = 0$$

$$\begin{bmatrix} \omega_L \\ r\dot{\phi}_L \\ 0 \end{bmatrix} = \begin{bmatrix} \omega \\ -\frac{T}{2}\omega + V_x \\ V_y \end{bmatrix} \qquad \begin{bmatrix} \omega_R \\ r\dot{\phi}_R \\ 0 \end{bmatrix} = \begin{bmatrix} \omega \\ \frac{T}{2}\omega + V_x \\ V_y \end{bmatrix}$$

$$\begin{bmatrix} r\phi_L \\ 0 \end{bmatrix} = \begin{bmatrix} \cdots \\ V_y \end{bmatrix} \qquad \begin{bmatrix} r\phi_R \\ 0 \end{bmatrix} \begin{bmatrix} \cdots \\ V_y \end{bmatrix}$$

$$\omega_L = \omega \qquad \omega_R = \omega$$

$$V_y = 0 \qquad V_y = 0$$

$$r\dot{\phi}_L = -\frac{T}{2}\omega + V_x \qquad r\dot{\phi}_R = \frac{T}{2}\omega + V_x$$

$$\dot{\phi}_L = \frac{1}{r}\left(-\frac{T}{2}\omega + V_x\right) \qquad \dot{\phi}_R = \frac{1}{r}\left(\frac{T}{2}\omega + V_x\right)$$

$$\begin{bmatrix} \dot{\phi}_L \\ \dot{\phi}_R \end{bmatrix} = \underbrace{\frac{1}{r}\begin{bmatrix} -\frac{T}{2} & 1 & 0 \\ \frac{T}{2} & 1 & 0 \end{bmatrix}}_{H} \begin{bmatrix} \omega \\ V_x \\ V_y \end{bmatrix}$$

$$\dot{\phi} = H \, V_B$$

H has a rank of 2 and is 2×3

$\therefore$ it has a nullspace with a dimension of 1

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ spans this nullspace, so

V_y does not affect wheel speed.

From inspection (sliding direction aligned with y):

$$V_s = R \, V_B$$

$$\begin{bmatrix} V_{sL} \\ V_{sR} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \omega \\ V_x \\ V_y \end{bmatrix}$$

To satisfy $V_s = 0$ (i.e. robot does not slide), V_B must be in null space of R. R is rank 1 and 2×3, so null space is of dimension 2, and is spanned by

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

So arbitrary ω and v_x are permitted, but v_y must $= 0$.

Pseudo-inverse: https://en.wikipedia.org/w/index.php?title=Moore%E2%80%93Penrose_pseudoinverse&oldid=540619506#Special_cases

The rows of H are linearly independent, so we can find the right inverse:

$$H H^{+} = I_m$$

$$H^{+} = H^{*} (H H^{*})^{-1}$$

$$\dot{\phi}_L = \frac{1}{r}\left(-\frac{T}{2}\omega + v_x\right) \qquad \dot{\phi}_R = \frac{1}{r}\left(\frac{T}{2}\omega + v_x\right)$$

$$r\dot{\phi}_L = -\frac{T}{2}\omega + v_x \qquad r\dot{\phi}_R = \frac{T}{2}\omega + v_x$$

$$v_x = r\dot{\phi}_L + \frac{T}{2}\omega \qquad v_x = r\dot{\phi}_R - \frac{T}{2}\omega$$

$$r\dot{\phi}_L + \frac{T}{2}\omega = r\dot{\phi}_R - \frac{T}{2}\omega$$

$$T\omega = r\dot{\phi}_R - r\dot{\phi}_L$$

$$\omega = \frac{r\dot{\phi}_R - r\dot{\phi}_L}{T}$$

$$v_x = r\dot{\phi}_L + \frac{\cancel{T}}{2}\,\frac{r\dot{\phi}_R - r\dot{\phi}_L}{\cancel{T}}$$

$$v_x = \frac{r}{2}\left(\dot{\phi}_R + \dot{\phi}_L\right)$$

$$V_B = H^{+}\dot{\phi} = H^{+}u$$

$$\begin{bmatrix} \omega \\ V_x \\ V_y \end{bmatrix} = r \underbrace{\begin{bmatrix} -\frac{1}{T} & \frac{1}{T} \\ \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}} \begin{bmatrix} \dot{\phi}_L \\ \dot{\phi}_R \end{bmatrix}$$

Verified by calculating right inverse in sympy

Final Results for wheel speed to twist conversions:

(1.1) $$u = H V_B$$

(1.2) $$\begin{bmatrix} \dot{\phi}_L \\ \dot{\phi}_R \end{bmatrix} = \frac{1}{r} \begin{bmatrix} -\frac{T}{2} & 1 & 0 \\ \frac{T}{2} & 1 & 0 \end{bmatrix} \begin{bmatrix} \omega \\ V_x \\ V_y \end{bmatrix}$$

(1.3) $$\dot{\phi}_L = \frac{1}{r}\left(-\frac{T}{2}\omega + V_x\right)$$

(1.4) $$\dot{\phi}_R = \frac{1}{r}\left(\frac{T}{2}\omega + V_x\right)$$

Multiplying both sides by Δt:

(1.5) $$\begin{bmatrix} \Delta\phi_L \\ \Delta\phi_R \end{bmatrix} = \frac{1}{r} \begin{bmatrix} -\frac{T}{2} & 1 & 0 \\ \frac{T}{2} & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta\theta \\ \Delta x \\ \Delta y \end{bmatrix}$$

(1.6) $$\Delta\phi_L = \frac{1}{r}\left(-\frac{T}{2}\Delta\theta + \Delta x\right)$$

(1.7) $$\Delta\phi_L = \frac{1}{r}\left(\frac{T}{2}\Delta\theta + \Delta x\right)$$

(2.1) $V_B = H^\dagger u$

(2.2) $$\begin{bmatrix} \omega \\ v_x \\ v_y \end{bmatrix} = r \begin{bmatrix} -\frac{1}{T} & \frac{1}{T} \\ \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot\phi_L \\ \dot\phi_R \end{bmatrix}$$

(2.3) $$\omega = \frac{r}{T}\left(-\dot\phi_L + \dot\phi_R\right)$$

(2.4) $$v_x = \frac{r}{2}\left(\dot\phi_L + \dot\phi_R\right)$$

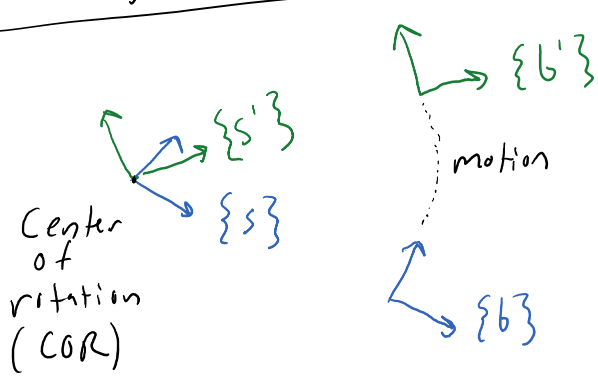
Multiplying both sides by Δt:

(2.5) $$\begin{bmatrix} \Delta\theta \\ \Delta x \\ \Delta y \end{bmatrix} = r \begin{bmatrix} -\frac{1}{T} & \frac{1}{T} \\ \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\phi_L \\ \Delta\phi_R \end{bmatrix}$$

(2.6) $$\Delta\theta = \frac{r}{T}\left(-\Delta\phi_L + \Delta\phi_R\right)$$

(2.7) $$\Delta x = \frac{r}{2}\left(\Delta\phi_L + \Delta\phi_R\right)$$

Integrating the twist



b : body frame before twist motion

b' : body frame after twist motion

s : COR frame aligned with b

s' : COR frame aligned with b'

$T_{sb}(0, x_s, y_s)$ pure translation since frames s and b are aligned

$T_{s'b'} = T_{sb}$ since s' and b' are aligned and (by definition of the COR) the b frame moves around a constant radius circle to b'

$$V_b = \begin{bmatrix} \omega \\ v_{xb} \\ v_{yb} \end{bmatrix} \qquad V_s = \begin{bmatrix} \omega \\ 0 \\ 0 \end{bmatrix}$$

since by definition the COR only experiences rotation

$$V_s = A_{sb} V_b$$

$$\begin{bmatrix} \omega \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ y_s & 1 & 0 \\ x_s & 0 & 1 \end{bmatrix} \begin{bmatrix} \omega \\ v_{xb} \\ v_{yb} \end{bmatrix}$$

Multiply both sides by Δt:

$$\begin{bmatrix} \Delta\theta \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ y_s & 1 & 0 \\ -x_s & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta\theta \\ \Delta x_b \\ \Delta y_b \end{bmatrix}$$

$$\Delta\theta = \Delta\theta$$

$$0 = \Delta\theta\, y_s + \Delta x_b \qquad 0 = -\Delta\theta\, x_s + \Delta y_b$$

$$y_s = \frac{-\Delta x_b}{\Delta\theta} \qquad x_s = \frac{\Delta y_b}{\Delta\theta}$$

$T_{ss'}(\Delta\theta, 0, 0)$ since by definition the COR only experiences rotation

$$T_{bb'} = T_{bs} T_{ss'} T_{s'b'}$$

All information to calculate $T_{bb'}$ is now known.

Finally, to get the new location in the world frame, simply calculate:

$$T_{wb'} = T_{wb} T_{bb'}$$

Where T_{wb} is the transformation to the previously known location in the world frame.

Final Results for integrating the twist:

(3.1) $T_{sb}\,(0, x_s, y_s)$

(3.2) $x_s = \dfrac{\Delta y_b}{\Delta \theta}$

(3.3) $y_s = -\dfrac{\Delta x_b}{\Delta \theta}$

(3.4) $T_{ss'}\,(\Delta\theta, 0, 0)$

(3.5) $T_{sb} = T_{s'b'}$

(3.6) $T_{bb'} = T_{bs} T_{ss'} T_{s'b'}$

(3.7) $T_{wb'} = T_{wb} T_{bb'}$