# Week5 monday



$$(\{S, T, V, W\}, \{a, b\}, \{S \to aT, T \to bV, V \to bW, W \to a\}, S)$$

$$(\{Q\},\{a,b\},\{Q\rightarrow abba\},Q)$$

$$(\{X,Y\},\{a,b\},\{X\to aYa,Y\to bb\},X)$$

Design a CFG to generate the language  $\{a^nb^n \mid n \geq 0\}$ 

Sample derivation:

Design a CFG to generate the language  $\{a^ib^j\mid j\geq i\geq 0\}$ 

Sample derivation:

**Theorem 2.20**: A language is generated by some context-free grammar if and only if it is recognized by some push-down automaton.

Definition: a language is called **context-free** if it is the language generated by a context-free grammar. The class of all context-free language over a given alphabet  $\Sigma$  is called **CFL**.

#### Consequences:

- Quick proof that every regular language is context free
- To prove closure of the class of context-free languages under a given operation, we can choose either of two modes of proof (via CFGs or PDAs) depending on which is easier
- To fully specify a PDA we could give its 6-tuple formal definition or we could give its input alphabet, stack alphabet, and state diagram. An informal description of a PDA is a step-by-step description of how its computations would process input strings; the reader should be able to reconstruct the state diagram or formal definition precisely from such a descripton. The informal description of a PDA can refer to some common modules or subroutines that are computable by PDAs:
  - PDAs can "test for emptiness of stack" without providing details. *How?* We can always push a special end-of-stack symbol, \$, at the start, before processing any input, and then use this symbol as a flag.
  - PDAs can "test for end of input" without providing details. *How?* We can transform a PDA to one where accepting states are only those reachable when there are no more input symbols.

Over  $\Sigma = \{a, b\}$ , let  $L = \{a^n b^m \mid n \neq m\}$ . Goal: Prove L is context-free.

Suppose  $L_1$  and  $L_2$  are context-free languages over  $\Sigma$ . Goal:  $L_1 \cup L_2$  is also context-free.

Approach 1: with PDAs

Let  $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, F_1)$  and  $M_2 = (Q_2, \Sigma, \Gamma_2, \delta_2, q_2, F_2)$  be PDAs with  $L(M_1) = L_1$  and  $L(M_2) = L_2$ .

Define M =

 $Approach\ 2:\ with\ CFGs$ 

Let  $G_1 = (V_1, \Sigma, R_1, S_1)$  and  $G_2 = (V_2, \Sigma, R_2, S_2)$  be CFGs with  $L(G_1) = L_1$  and  $L(G_2) = L_2$ .

Define G =

Suppose  $L_1$  and  $L_2$  are context-free languages over  $\Sigma$ . Goal:  $L_1 \circ L_2$  is also context-free.

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 $Approach\ 2:\ with\ CFGs$ 

Let  $G_1 = (V_1, \Sigma, R_1, S_1)$  and  $G_2 = (V_2, \Sigma, R_2, S_2)$  be CFGs with  $L(G_1) = L_1$  and  $L(G_2) = L_2$ .

Define G =

### Summary

Over a fixed alphabet  $\Sigma$ , a language L is **regular** 

iff it is described by some regular expression iff it is recognized by some DFA iff it is recognized by some NFA

Over a fixed alphabet  $\Sigma$ , a language L is **context-free** 

iff it is generated by some CFG iff it is recognized by some PDA

Fact: Every regular language is a context-free language.

Fact: There are context-free languages that are not nonregular.

**Fact**: There are countably many regular languages.

Fact: There are countably inifnitely many context-free languages.

Consequence: Most languages are **not** context-free!

### Examples of non-context-free languages

$$\begin{aligned} &\{a^nb^nc^n\mid 0\leq n, n\in\mathbb{Z}\}\\ &\{a^ib^jc^k\mid 0\leq i\leq j\leq k, i\in\mathbb{Z}, j\in\mathbb{Z}, k\in\mathbb{Z}\}\\ &\{ww\mid w\in\{0,1\}^*\} \end{aligned}$$

(Sipser Ex 2.36, Ex 2.37, 2.38)

There is a Pumping Lemma for CFL that can be used to prove a specific language is non-context-free: If A is a context-free language, there there is a number p where, if s is any string in A of length at least p, then s may be divided into five pieces s = uvxyz where (1) for each  $i \geq 0$ ,  $uv^ixy^iz \in A$ , (2) |uv| > 0, (3)  $|vxy| \leq p$ . We will not go into the details of the proof or application of Pumping Lemma for CFLs this quarter.

## Week5 wednesday

A set X is said to be **closed** under an operation OP if, for any elements in X, applying OP to them gives an element in X.

True/False	Closure claim						
True	The set of integers is closed under multiplication.						
	$\forall x \forall y  ( (x \in \mathbb{Z} \land y \in \mathbb{Z}) \to xy \in \mathbb{Z} )$						
True	For each set $A$ , the power set of $A$ is closed under intersection.						
	$\forall A_1 \forall A_2 ( (A_1 \in \mathcal{P}(A) \land A_2 \in \mathcal{P}(A) \in \mathbb{Z}) \to A_1 \cap A_2 \in \mathcal{P}(A) )$						
	The class of regular languages over $\Sigma$ is closed under complementation.						
	The class of regular languages over $\Sigma$ is closed under union.						
	The class of regular languages over $\Sigma$ is closed under intersection.						
	The class of regular languages over $\Sigma$ is closed under concatenation.						
	The class of regular languages over $\Sigma$ is closed under Kleene star.						
	The class of context-free languages over $\Sigma$ is closed under complementation.						
	The class of context-free languages over $\Sigma$ is closed under union.						
	The class of context-free languages over $\Sigma$ is closed under intersection.						
	The class of context-free languages over $\Sigma$ is closed under concatenation.						
	The class of context-free languages over $\Sigma$ is closed under Kleene star.						

Assume  $\Sigma = \{0, 1, \#\}$ 

**Turing machines**: unlimited read + write memory, unlimited time (computation can proceed without "consuming" input and can re-read symbols of input)

- Division between program (CPU, state diagram) and data
- Unbounded memory gives theoretical limit to what modern computation (including PCs, supercomputers, quantum computers) can achieve
- State diagram formulation is simple enough to reason about (and diagonalize against) while expressive enough to capture modern computation

For Turing machine  $M=(Q,\Sigma,\Gamma,\delta,q_0,q_{accept},q_{reject})$  the **computation** of M on a string w over  $\Sigma$  is:

- Read/write head starts at leftmost position on tape.
- Input string is written on |w|-many leftmost cells of tape, rest of the tape cells have the blank symbol. **Tape alphabet** is  $\Gamma$  with  $\bot \in \Gamma$  and  $\Sigma \subseteq \Gamma$ . The blank symbol  $\bot \notin \Sigma$ .
- Given current state of machine and current symbol being read at the tape head, the machine transitions to next state, writes a symbol to the current position of the tape head (overwriting existing symbol), and moves the tape head L or R (if possible). Formally, **transition function** is

$$\delta: Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$$

• Computation ends if and when machine enters either the accept or the reject state. This is called halting. Note:  $q_{accept} \neq q_{reject}$ .

The language recognized by the Turing machine M, is

 $\{w \in \Sigma^* \mid \text{computation of } M \text{ on } w \text{ halts after entering the accept state}\} = \{w \in \Sigma^* \mid w \text{ is accepted by } M\}$ 

An example Turing machine:  $\Sigma =$ 

$$,\Gamma =$$

$$\delta((q0,0)) =$$



Formal definition:

Sample computation:

$q0\downarrow$								
0	0	0	J	J	J			
		•			•			

The language recognized by this machine is ...

Extra practice:





Formal definition:

Sample computation: