Week9 friday

In practice, computers (and Turing machines) don't have infinite tape, and we can't afford to wait unboundedly long for an answer. "Decidable" isn't good enough - we want "Efficiently decidable".

For a given algorithm working on a given input, how long do we need to wait for an answer? How does the running time depend on the input in the worst-case? average-case? We expect to have to spend more time on computations with larger inputs.

A language is **recognizable** if _____

A language is **decidable** if _____

A language is **efficiently decidable** if _____

A function is **computable** if _____

A function is **efficiently computable** if _____

Definition (Sipser 7.1): For M a deterministic decider, its **running time** is the function $f : \mathbb{N} \to \mathbb{N}$ given by

$$f(n) = \text{max number of steps } M \text{ takes before halting, over all inputs of length } n$$

Definition (Sipser 7.7): For each function $t(n)$, the **time complexity class** $TIME(t(n))$, is defined by

$$TIME(t(n)) = \{L \mid L \text{ is decidable by a Turing machine with running time in } O(t(n))\}$$

An example of an element of $TIME(1)$ is

An example of an element of $TIME(n)$ is

Note: $TIME(1) \subseteq TIME(n) \subseteq TIME(n^2)$

Definition (Sipser 7.12) : P is the class of languages that are decidable in polynomial time on a deterministic 1-tape Turing machine

$$P = \bigcup_k TIME(n^k)$$

Compare to exponential time: brute-force search.

Theorem (Sipser 7.8): Let $t(n)$ be a function with $t(n) \geq n$. Then every $t(n)$ time deterministic multitape Turing machine has an equivalent $O(t^2(n))$ time deterministic 1-tape Turing machine.

Definition (Sipser 7.9): For N a nodeterministic decider. The **running time** of N is the function $f : \mathbb{N} \to \mathbb{N}$ given by

$$f(n) = \text{max number of steps } N \text{ takes on any branch before halting, over all inputs of length } n$$

Definition (Sipser 7.21): For each function $t(n)$, the **nondeterministic time complexity class** $NTIME(t(n))$, is defined by

$NTIME(t(n)) = \{L \mid L \text{ is decidable by a nondeterministic Turing machine with running time in } O(t(n))\}$

$$NP = \bigcup_k NTIME(n^k)$$

True or **False**: $TIME(n^2) \subseteq NTIME(n^2)$

True or **False**: $NTIME(n^2) \subseteq DTIME(n^2)$

Examples in P

Can't use nondeterminism; Can use multiple tapes; Often need to be "more clever" than naïve / brute force approach

$$PATH = \{\langle G, s, t\rangle \mid G \text{ is digraph with } n \text{ nodes there is path from s to t}\}$$

Use breadth first search to show in P

$$RELPRIME = \{\langle x, y\rangle \mid x \text{ and } y \text{ are relatively prime integers}\}$$

Use Euclidean Algorithm to show in P

$$L(G) = \{w \mid w \text{ is generated by } G\}$$

(where G is a context-free grammar). Use dynamic programming to show in P.

Examples in NP

"Verifiable" i.e. NP, Can be decided by a nondeterministic TM in polynomial time, best known deterministic solution may be brute-force, solution can be verified by a deterministic TM in polynomial time.

$HAMPATH = \{\langle G, s, t\rangle \mid G \text{ is digraph with } n \text{ nodes, there is path from } s \text{ to } t \text{ that goes through every node exa}$

$VERTEX - COVER = \{\langle G, k\rangle \mid G \text{ is an undirected graph with } n \text{ nodes that has a } k\text{-node vertex cover}\}$

$$CLIQUE = \{\langle G, k\rangle \mid G \text{ is an undirected graph with } n \text{ nodes that has a } k\text{-clique}\}$$

$$SAT = \{\langle X\rangle \mid X \text{ is a satisfiable Boolean formula with } n \text{ variables}\}$$

Week4 wednesday

Language	$s \in L$	$s \notin L$	Is the language regular or nonregular?
$\{a^n b^n \mid 0 \leq n \leq 5\}$			
$\{b^n a^n \mid n \geq 2\}$			
$\{a^m b^n \mid 0 \leq m \leq n\}$			
$\{a^m b^n \mid m \geq n+3, n \geq 0\}$			
$\{b^m a^n \mid m \geq 1, n \geq 3\}$			
$\{w \in \{a,b\}^* \mid w = w^{\mathcal{R}}\}$			
$\{ww^{\mathcal{R}} \mid w \in \{a,b\}^*\}$			

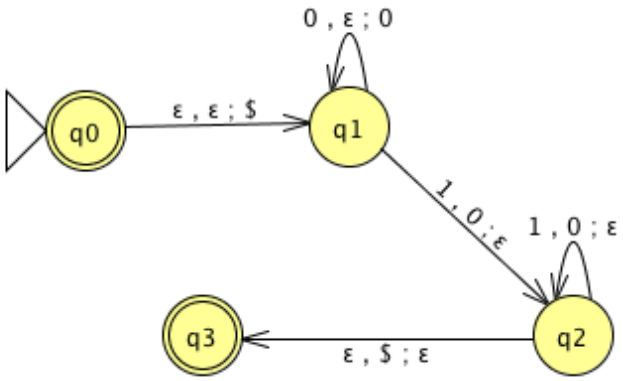
Regular sets are not the end of the story

- Many nice / simple / important sets are not regular
- Limitation of the finite-state automaton model: Can't "count", Can only remember finitely far into the past, Can't backtrack, Must make decisions in "real-time"
- We know actual computers are more powerful than this model...

The **next** model of computation. Idea: allow some memory of unbounded size. How?

- To generalize regular expressions: **context-free grammars**

- To generalize NFA: **Pushdown automata**, which is like an NFA with access to a stack: Number of states is fixed, number of entries in stack is unbounded. At each step (1) Transition to new state based on current state, letter read, and top letter of stack, then (2) (Possibly) push or pop a letter to (or from) top of stack. Accept a string iff there is some sequence of states and some sequence of stack contents which helps the PDA processes the entire input string and ends in an accepting state.



Trace the computation of this PDA on the input string 01.

Trace the computation of this PDA on the input string 011.

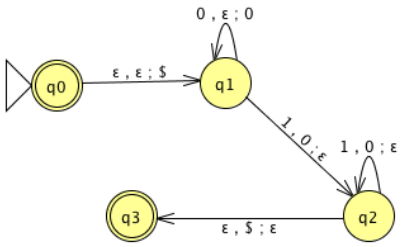
Week4 friday

Definition A **pushdown automaton** (PDA) is specified by a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where Q is the finite set of states, Σ is the input alphabet, Γ is the stack alphabet,

$$\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \to \mathcal{P}(Q \times \Gamma_\varepsilon)$$

is the transition function, $q_0 \in Q$ is the start state, $F \subseteq Q$ is the set of accept states.

Formal definition



Draw the state diagram of a PDA with $\Sigma = \Gamma$.

Draw the state diagram of a PDA with $\Sigma \cap \Gamma = \emptyset$.

A PDA recognizing the set { } can be informally described as:

> Read symbols from the input. As each 0 is read, push it onto the stack. As soon as 1s are seen, pop a 0 off the stack for each 1 read. If the stack becomes empty and there is exactly one 1 left to read, read that 1 and accept the input. If the stack becomes empty and there are either zero or more than one 1s left to read, or if the 1s are finished while the stack still contains 0s, or if any 0s appear in the input following 1s, reject the input.

State diagram for this PDA:

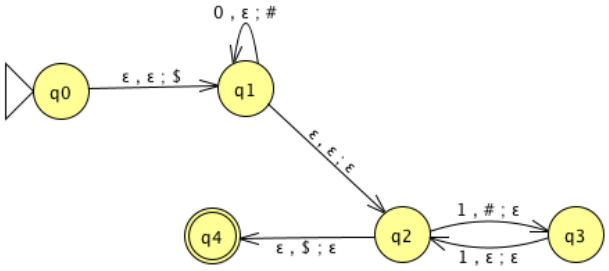
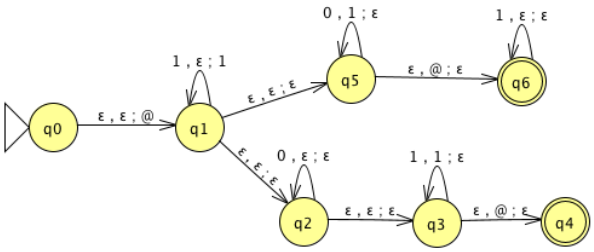
Consider the state diagram of a PDA with input alphabet Σ and stack alphabet Γ.

Label	means
$a, b; c$ when $a \in \Sigma$, $b \in \Gamma$, $c \in \Gamma$	
$a, \varepsilon; c$ when $a \in \Sigma$, $c \in \Gamma$	
$a, b; \varepsilon$ when $a \in \Sigma$, $b \in \Gamma$	
$a, \varepsilon; \varepsilon$ when $a \in \Sigma$	

How does the meaning change if a is replaced by ε?

Note: alternate notation is to replace ; with $\rightarrow$

For the PDA state diagrams below, $\Sigma = \{0, 1\}$.

Mathematical description of language	State diagram of PDA recognizing language
	
	
$\{0^i 1^j 0^k \mid i, j, k \geq 0\}$	

Week6 friday

To define a Turing machine, we could give a

- **Formal definition**: the 7-tuple of parameters including set of states, input alphabet, tape alphabet, transition function, start state, accept state, and reject state; or,
- **Implementation-level definition**: English prose that describes the Turing machine head movements relative to contents of tape, and conditions for accepting / rejecting based on those contents.
- **High-level description**: description of algorithm (precise sequence of instructions), without implementation details of machine. As part of this description, can "call" and run another TM as a subroutine.

Theorem 3.21 A language is Turing-recognizable iff some enumerator enumerates it.

Proof:

Assume L is enumerated by some enumerator, E, so $L = L(E)$. We'll use E in a subroutine within a high-level description of a new Turing machine that we will build to recognize L.

Goal: build Turing machine M_E with $L(M_E) = L(E)$.

Define M_E as follows: $M_E =$ "On input w,

1. Run E. For each string x printed by E.
2. Check if $x = w$. If so, accept (and halt); otherwise, continue."

Assume L is Turing-recognizable and there is a Turing machine M with $L = L(M)$. We'll use M in a subroutine within a high-level description of an enumerator that we will build to enumerate L.

Goal: build enumerator E_M with $L(E_M) = L(M)$.

Idea: check each string in turn to see if it is in L.

How? Run computation of M on each string. *But*: need to be careful about computations that don't halt.

Recall String order for $\Sigma = \{0, 1\}$: $s_1 = \varepsilon$, $s_2 = 0$, $s_3 = 1$, $s_4 = 00$, $s_5 = 01$, $s_6 = 10$, $s_7 = 11$, $s_8 = 000$, ...

Define E_M as follows: $E_M =$ " *ignore any input.* Repeat the following for $i = 1, 2, 3, \ldots$

1. Run the computations of M on s_1, s_2, ..., s_i for (at most) i steps each
2. For each of these i computations that accept during the (at most) i steps, print out the accepted string."

Nondeterministic Turing machine

At any point in the computation, the nondeterministic machine may proceed according to several possibilities: $(Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej})$ where

$$\delta : Q \times \Gamma \to \mathcal{P}(Q \times \Gamma \times \{L, R\})$$

The computation of a nondeterministic Turing machine is a tree with branching when the next step of the computation has multiple possibilities. A nondeterministic Turing machine accepts a string exactly when some branch of the computation tree enters the accept state.

Given a nondeterministic machine, we can use a 3-tape Turing machine to simulate it by doing a breadth-first search of computation tree: one tape is "read-only" input tape, one tape simulates the tape of the nondeterministic computation, and one tape tracks nondeterministic branching. Sipser page 178

Two models of computation are called **equally expressive** when every language recognizable with the first model is recognizable with the second, and vice versa.

Church-Turing Thesis (Sipser p. 183): The informal notion of algorithm is formalized completely and correctly by the formal definition of a Turing machine. In other words: all reasonably expressive models of computation are equally expressive with the standard Turing machine.

Claim: If two languages (over a fixed alphabet Σ) are Turing-recognizable, then their union is as well.

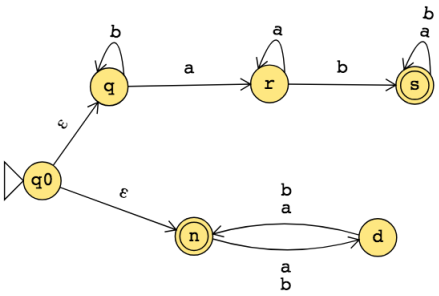
Proof using Turing machines:

Proof using nondeterministic Turing machines:

Proof using enumerators:

Week3 monday

The state diagram of an NFA over $\{a, b\}$ is below. The formal definition of this NFA is:



The language recognized by this NFA is:

Suppose A_1, A_2 are languages over an alphabet Σ. **Claim:** if there is a NFA N_1 such that $L(N_1) = A_1$ and NFA N_2 such that $L(N_2) = A_2$, then there is another NFA, let's call it N, such that $L(N) = A_1 \cup A_2$.

Proof idea: Use nondeterminism to choose which of N_1, N_2 to run.

Formal construction: Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ and assume $Q_1 \cap Q_2 = \emptyset$ and that $q_0 \notin Q_1 \cup Q_2$. Construct $N = (Q, \Sigma, \delta, q_0, F_1 \cup F_2)$ where

- $Q =$
- $\delta : Q \times \Sigma_\varepsilon \to \mathcal{P}(Q)$ is defined by, for $q \in Q$ and $a \in \Sigma_\varepsilon$:

Proof of correctness would prove that $L(N) = A_1 \cup A_2$ by considering an arbitrary string accepted by N, tracing an accepting computation of N on it, and using that trace to prove the string is in at least one of A_1, A_2; then, taking an arbitrary string in $A_1 \cup A_2$ and proving that it is accepted by N. Details left for extra practice.

Over the alphabet $\{a, b\}$, the language L described by the regular expression $\Sigma^* a \Sigma^* b$

includes the strings and excludes the strings

The state diagram of a NFA recognizing L is:

Suppose A_1, A_2 are languages over an alphabet Σ. **Claim:** if there is a NFA N_1 such that $L(N_1) = A_1$ and NFA N_2 such that $L(N_2) = A_2$, then there is another NFA, let's call it N, such that $L(N) = A_1 \circ A_2$.

Proof idea: Allow computation to move between N_1 and N_2 "spontaneously" when reach an accepting state of N_1, guessing that we've reached the point where the two parts of the string in the set-wise concatenation are glued together.

Formal construction: Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ and assume $Q_1 \cap Q_2 = \emptyset$. Construct $N = (Q, \Sigma, \delta, q_0, F)$ where

- $Q =$
- $q_0 =$
- $F =$
- $\delta : Q \times \Sigma_\varepsilon \to \mathcal{P}(Q)$ is defined by, for $q \in Q$ and $a \in \Sigma_\varepsilon$:

$$\delta((q,a)) = \begin{cases} \delta_1((q,a)) & \text{if } q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1((q,a)) & \text{if } q \in F_1 \text{ and } a \in \Sigma \\ \delta_1((q,a)) \cup \{q_2\} & \text{if } q \in F_1 \text{ and } a = \varepsilon \\ \delta_2((q,a)) & \text{if } q \in Q_2 \end{cases}$$

Proof of correctness would prove that $L(N) = A_1 \circ A_2$ by considering an arbitrary string accepted by N, tracing an accepting computation of N on it, and using that trace to prove the string can be written as the result of concatenating two strings, the first in A_1 and the second in A_2; then, taking an arbitrary string in $A_1 \circ A_2$ and proving that it is accepted by N. Details left for extra practice.

Suppose A is a language over an alphabet Σ. **Claim:** if there is a NFA N such that $L(N) = A$, then there is another NFA, let's call it N', such that $L(N') = A^*$.

Proof idea: Add a fresh start state, which is an accept state. Add spontaneous moves from each (old) accept state to the old start state.

Formal construction: Let $N = (Q, \Sigma, \delta, q_1, F)$ and assume $q_0 \notin Q$. Construct $N' = (Q', \Sigma, \delta', q_0, F')$ where

- $Q' = Q \cup \{q_0\}$
- $F' = F \cup \{q_0\}$
- $\delta' : Q' \times \Sigma_\varepsilon \to \mathcal{P}(Q')$ is defined by, for $q \in Q'$ and $a \in \Sigma_\varepsilon$:

$$\delta'((q,a)) = \begin{cases} \delta((q,a)) & \text{if } q \in Q \text{ and } q \notin F \\ \delta((q,a)) & \text{if } q \in F \text{ and } a \in \Sigma \\ \delta((q,a)) \cup \{q_1\} & \text{if } q \in F \text{ and } a = \varepsilon \\ \{q_1\} & \text{if } q = q_0 \text{ and } a = \varepsilon \\ \emptyset & \text{if } q = q_0 \text{ and } a \in \Sigma \end{cases}$$

Proof of correctness would prove that $L(N') = A^*$ by considering an arbitrary string accepted by N', tracing an accepting computation of N' on it, and using that trace to prove the string can be written as the result of concatenating some number of strings, each of which is in A; then, taking an arbitrary string in A^* and proving that it is accepted by N'. Details left for extra practice.

Application: A state diagram for a NFA over $\Sigma = \{a, b\}$ that recognizes $L((\Sigma^* b)^*)$:

True or **False**: The state diagram of any DFA is also the state diagram of a NFA.

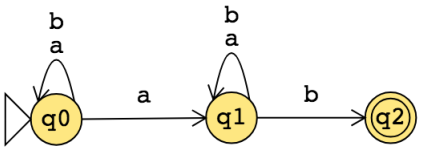
True or **False**: The state diagram of any NFA is also the state diagram of a DFA.

True or **False**: The formal definition $(Q, \Sigma, \delta, q_0, F)$ of any DFA is also the formal definition of a NFA.

True or **False**: The formal definition $(Q, \Sigma, \delta, q_0, F)$ of any NFA is also the formal definition of a DFA.

Week3 wednesday

Consider the state diagram of an NFA over $\{a, b\}$:



The language recognized by this NFA is

The state diagram of a DFA recognizing this same language is:

Suppose A is a language over an alphabet Σ. **Claim:** if there is a NFA N such that $L(N) = A$ then there is a DFA M such that $L(M) = A$.

Proof idea: States in M are "macro-states" – collections of states from N – that represent the set of possible states a computation of N might be in.

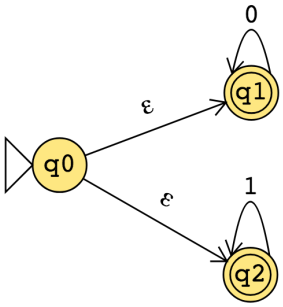
Formal construction: Let $N = (Q, \Sigma, \delta, q_0, F)$. Define

$$M = (\mathcal{P}(Q), \Sigma, \delta', q', \{X \subseteq Q \mid X \cap F \neq \emptyset\})$$

where $q' = \{q \in Q \mid q = q_0$ or is accessible from q_0 by spontaneous moves in $N\}$ and

$\delta'((X, x)) = \{q \in Q \mid q \in \delta((r, x))$ for some $r \in X$ or is accessible from such an r by spontaneous moves in $N\}$

Consider the state diagram of an NFA over $\{0, 1\}$. Use the "macro-state" construction to find an equivalent DFA.



Prune this diagram to get an equivalent DFA with only the "macro-states" reachable from the start state.

Suppose A is a language over an alphabet Σ. **Claim:** if there is a regular expression R such that $L(R) = A$, then there is a NFA, let's call it N, such that $L(N) = A$.

Structural induction: Regular expression is built from basis regular expressions using inductive steps (union, concatenation, Kleene star symbols). Use constructions to mirror these in NFAs.

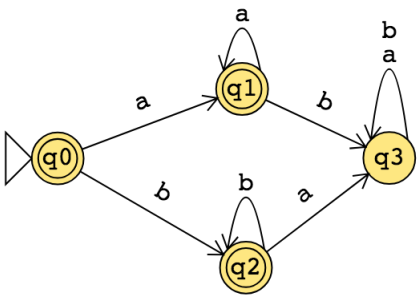
Application: A state diagram for a NFA over $\{a, b\}$ that recognizes $L(a^*(ab)^*)$:

Suppose A is a language over an alphabet Σ. **Claim:** if there is a DFA M such that $L(M) = A$, then there is a regular expression, let's call it R, such that $L(R) = A$.

Proof idea: Trace all possible paths from start state to accept state. Express labels of these paths as regular expressions, and union them all.

1. Add new start state with ε arrow to old start state.
2. Add new accept state with ε arrow from old accept states. Make old accept states non-accept.
3. Remove one (of the old) states at a time: modify regular expressions on arrows that went through removed state to restore language recognized by machine.

Application: Find a regular expression describing the language recognized by the DFA with state diagram



Conclusion: For each language L,

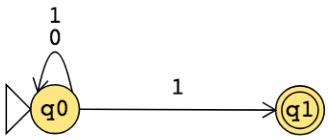
There is a DFA that recognizes L $\quad \exists M$ (M **is a DFA and** $L(M) = A$)
if and only if
There is a NFA that recognizes L $\quad \exists N$ (N **is a NFA and** $L(N) = A$)
if and only if
There is a regular expression that describes L $\exists R$ (R **is a regular expression and** $L(R) = A$)

A language is called **regular** when any (hence all) of the above three conditions are met.

Week2 friday

Nondeterministic finite automaton $M = (Q, \Sigma, \delta, q_0, F)$	
Finite set of states Q	Can be labelled by any collection of distinct names. Default: $q0, q1, \ldots$
Alphabet Σ	Each input to the automaton is a string over Σ.
Arrow labels Σ_ε	$\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$. Arrows in the state diagram are labelled either by symbols from Σ or by ε
Transition function δ	$\delta : Q \times \Sigma_\varepsilon \to \mathcal{P}(Q)$ gives the **set of possible next states** for a transition from the current state upon reading a symbol or spontaneously moving.
Start state q_0	Element of Q. Each computation of the machine starts at the start state.
Accept (final) states F	$F \subseteq Q$.
M accepts the input string	if and only if **there is** a computation of M on the input string that processes the whole string and ends in an accept state.
Page 53	

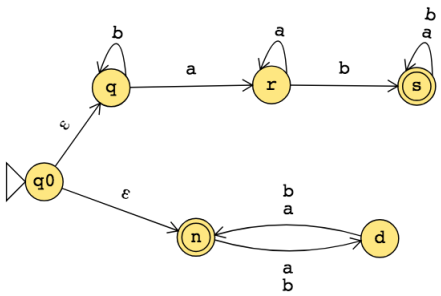
The formal definition of the NFA over $\{0, 1\}$ given by this state diagram is:



The language over $\{0, 1\}$ recognized by this NFA is:

Change the transition function to get a different NFA which accepts the empty string.

The state diagram of an NFA over $\{a, b\}$ is below. The formal definition of this NFA is:



The language recognized by this NFA is:

Week10 wednesday

Recall: For M a deterministic decider, its **running time** is the function $f : \mathbb{N} \to \mathbb{N}$ given by

$$f(n) = \text{max number of steps } M \text{ takes before halting, over all inputs of length } n$$

For each function $t(n)$, the **time complexity class** $TIME(t(n))$, is defined by

$$TIME(t(n)) = \{L \mid L \text{ is decidable by a Turing machine with running time in } O(t(n))\}$$

P is the class of languages that are decidable in polynomial time on a deterministic 1-tape Turing machine

$$P = \bigcup_k TIME(n^k)$$

Definition (Sipser 7.9): For N a nodeterministic decider. The **running time** of N is the function $f : \mathbb{N} \to \mathbb{N}$ given by

$$f(n) = \text{max number of steps } N \text{ takes on any branch before halting, over all inputs of length } n$$

Definition (Sipser 7.21): For each function $t(n)$, the **nondeterministic time complexity class** $NTIME(t(n))$, is defined by

$$NTIME(t(n)) = \{L \mid L \text{ is decidable by a nondeterministic Turing machine with running time in } O(t(n))\}$$

$$NP = \bigcup_k NTIME(n^k)$$

True or **False**: $TIME(n^2) \subseteq NTIME(n^2)$

True or **False**: $NTIME(n^2) \subseteq DTIME(n^2)$

Every problem in NP is decidable with an exponential-time algorithm

Nondeterministic approach: guess a possible solution, verify that it works.

Brute-force (worst-case exponential time) approach: iterate over all possible solutions, for each one, check if it works.

Examples in P

Can't use nondeterminism; Can use multiple tapes; Often need to be "more clever" than naïve / brute force approach

$$PATH = \{\langle G, s, t\rangle \mid G \text{ is digraph with } n \text{ nodes there is path from s to t}\}$$

Use breadth first search to show in P

$$RELPRIME = \{\langle x, y\rangle \mid x \text{ and } y \text{ are relatively prime integers}\}$$

Use Euclidean Algorithm to show in P

$$L(G) = \{w \mid w \text{ is generated by } G\}$$

(where G is a context-free grammar). Use dynamic programming to show in P.

Examples in NP

"Verifiable" i.e. NP, Can be decided by a nondeterministic TM in polynomial time, best known deterministic solution may be brute-force, solution can be verified by a deterministic TM in polynomial time.

$$HAMPATH = \{\langle G, s, t\rangle \mid G \text{ is digraph with } n \text{ nodes, there is path from } s \text{ to } t \text{ that goes through every node exactly once}\}$$

$$VERTEX - COVER = \{\langle G, k\rangle \mid G \text{ is an undirected graph with } n \text{ nodes that has a } k\text{-node vertex cover}\}$$

$$CLIQUE = \{\langle G, k\rangle \mid G \text{ is an undirected graph with } n \text{ nodes that has a } k\text{-clique}\}$$

$$SAT = \{\langle X\rangle \mid X \text{ is a satisfiable Boolean formula with } n \text{ variables}\}$$

Problems in P	**Problems in NP**
(Membership in any) regular language	Any problem in P
(Membership in any) context-free language	
A_{DFA}	SAT
E_{DFA}	$CLIQUE$
EQ_{DFA}	$VERTEX - COVER$
$PATH$	$HAMPATH$
$RELPRIME$	$\ldots$
$\ldots$	

Million-dollar question: Is $P = NP$?

One approach to trying to answer it is to look for *hardest* problems in NP and then (1) if we can show that there are efficient algorithms for them, then we can get efficient algorithms for all problems in NP so $P = NP$, or (2) these problems might be good candidates for showing that there are problems in NP for which there are no efficient algorithms.

Definition (Sipser 7.29) Language A is **polynomial-time mapping reducible** to language B, written $A \leq_P B$, means there is a polynomial-time computable function $f : \Sigma^* \to \Sigma^*$ such that for every $x \in \Sigma^*$

$$x \in A \qquad \text{iff} \qquad f(x) \in B.$$

The function f is called the polynomial time reduction of A to B.

Theorem (Sipser 7.31): If $A \leq_P B$ and $B \in P$ then $A \in P$.

Proof:

Definition (Sipser 7.34; based in Stephen Cook and Leonid Levin's work in the 1970s): A language B is **NP-complete** means (1) B is in NP **and** (2) every language A in NP is polynomial time reducible to B.

Theorem (Sipser 7.35): If B is NP-complete and $B \in P$ then $P = NP$.

Proof:

3SAT: A literal is a Boolean variable (e.g. x) or a negated Boolean variable (e.g. $\bar{x}$). A Boolean formula is a **3cnf-formula** if it is a Boolean formula in conjunctive normal form (a conjunction of disjunctive clauses of literals) and each clause has three literals.

$$3SAT = \{\langle \phi \rangle \mid \phi \text{ is a satisfiable 3cnf-formula}\}$$

Example strings in $3SAT$

Example strings not in $3SAT$

Cook-Levin Theorem: $3SAT$ is NP-complete.

Are there other NP-complete problems? To prove that X is NP-complete

- *From scratch*: prove X is in NP and that all NP problems are polynomial-time reducible to X.
- *Using reduction*: prove X is in NP and that a known-to-be NP-complete problem is polynomial-time reducible to X.

CLIQUE: A **k-clique** in an undirected graph is a maximally connected subgraph with k nodes.

$$CLIQUE = \{\langle G, k \rangle \mid G \text{ is an undirected graph with a } k\text{-clique}\}$$

Example strings in $CLIQUE$

Example strings not in $CLIQUE$

Theorem (Sipser 7.32):

$$3SAT \leq_P CLIQUE$$

Given a Boolean formula in conjunctive normal form with k clauses and three literals per clause, we will map it to a graph so that the graph has a clique if the original formula is satisfiable and the graph does not have a clique if the original formula is not satisfiable.

The graph has $3k$ vertices (one for each literal in each clause) and an edge between all vertices except

- vertices for two literals in the same clause
- vertices for literals that are negations of one another

Example: $(x \vee \bar{y} \vee \bar{z}) \wedge (\bar{x} \vee y \vee z) \wedge (x \vee y \vee z)$

Week10 friday

Model of Computation	**Class of Languages**
Deterministic finite automata: formal definition, how to design for a given language, how to describe language of a machine? **Nondeterministic finite automata**: formal definition, how to design for a given language, how to describe language of a machine? **Regular expressions**: formal definition, how to design for a given language, how to describe language of expression? *Also*: converting between different models.	**Class of regular languages**: what are the closure properties of this class? which languages are not in the class? using **pumping lemma** to prove nonregularity.
Push-down automata: formal definition, how to design for a given language, how to describe language of a machine? **Context-free grammars**: formal definition, how to design for a given language, how to describe language of a grammar?	**Class of context-free languages**: what are the closure properties of this class? which languages are not in the class?
Turing machines that always halt in polynomial time Nondeterministic Turing machines that always halt in polynomial time	P NP
Deciders (Turing machines that always halt): formal definition, how to design for a given language, how to describe language of a machine?	**Class of decidable languages**: what are the closure properties of this class? which languages are not in the class? using diagonalization and mapping reduction to show undecidability
Turing machines formal definition, how to design for a given language, how to describe language of a machine?	**Class of recognizable languages**: what are the closure properties of this class? which languages are not in the class? using closure and mapping reduction to show unrecognizability

Given a language, prove it is regular

Strategy 1: construct DFA recognizing the language and prove it works.

Strategy 2: construct NFA recognizing the language and prove it works.

Strategy 3: construct regular expression recognizing the language and prove it works.

"Prove it works" means ...

Example: $L = \{w \in \{0,1\}^* \mid w \text{ has odd number of 1s or starts with } 0\}$

Using NFA

Using regular expressions

Example: Select all and only the options that result in a true statement: "To show a language A is not regular, we can..."

a. Show A is finite

b. Show there is a CFG generating A

c. Show A has no pumping length

d. Show A is undecidable

Example: What is the language generated by the CFG with rules

$$S \to aSb \mid bY \mid Ya$$
$$Y \to bY \mid Ya \mid \varepsilon$$

Example: Prove that the language $T = \{\langle M \rangle \mid M \text{ is a Turing machine and } L(M) \text{ is infinite}\}$ is undecidable.

Example: Prove that the class of decidable languages is closed under concatenation.