Week5 monday

Three different CFGs that each generate the language $\{abba\}$

$$(\{S, T, V, W\}, \{a, b\}, \{S \to aT, T \to bV, V \to bW, W \to a\}, S)$$

$$(\{Q\},\{a,b\},\{Q\rightarrow abba\},Q)$$

$$(\{X,Y\},\{a,b\},\{X\to aYa,Y\to bb\},X)$$

Design a CFG to generate the language $\{a^nb^n \mid n \geq 0\}$

Sample derivation:

Design a CFG to generate the language $\{a^ib^j\mid j\geq i\geq 0\}$

Sample derivation:

Theorem 2.20: A language is generated by some context-free grammar if and only if it is recognized by some push-down automaton.

Definition: a language is called **context-free** if it is the language generated by a context-free grammar. The class of all context-free language over a given alphabet Σ is called **CFL**.

Consequences:

- Quick proof that every regular language is context free
- To prove closure of the class of context-free languages under a given operation, we can choose either of two modes of proof (via CFGs or PDAs) depending on which is easier
- To fully specify a PDA we could give its 6-tuple formal definition or we could give its input alphabet, stack alphabet, and state diagram. An informal description of a PDA is a step-by-step description of how its computations would process input strings; the reader should be able to reconstruct the state diagram or formal definition precisely from such a descripton. The informal description of a PDA can refer to some common modules or subroutines that are computable by PDAs:
 - PDAs can "test for emptiness of stack" without providing details. *How?* We can always push a special end-of-stack symbol, \$, at the start, before processing any input, and then use this symbol as a flag.
 - PDAs can "test for end of input" without providing details. *How?* We can transform a PDA to one where accepting states are only those reachable when there are no more input symbols.

Over $\Sigma = \{a, b\}$, let $L = \{a^n b^m \mid n \neq m\}$. Goal: Prove L is context-free.

Suppose L_1 and L_2 are context-free languages over Σ . Goal: $L_1 \cup L_2$ is also context-free.

Approach 1: with PDAs

Let $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \Gamma_2, \delta_2, q_2, F_2)$ be PDAs with $L(M_1) = L_1$ and $L(M_2) = L_2$.

Define M =

 $Approach\ 2:\ with\ CFGs$

Let $G_1 = (V_1, \Sigma, R_1, S_1)$ and $G_2 = (V_2, \Sigma, R_2, S_2)$ be CFGs with $L(G_1) = L_1$ and $L(G_2) = L_2$.

Define G =

Suppose L_1 and L_2 are context-free languages over Σ . Goal: $L_1 \circ L_2$ is also context-free.

Approach 1: with PDAs

Let $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \Gamma_2, \delta_2, q_2, F_2)$ be PDAs with $L(M_1) = L_1$ and $L(M_2) = L_2$.

Define M =

Approach 2: with CFGs

Let $G_1 = (V_1, \Sigma, R_1, S_1)$ and $G_2 = (V_2, \Sigma, R_2, S_2)$ be CFGs with $L(G_1) = L_1$ and $L(G_2) = L_2$.

Define G =

Summary

Over a fixed alphabet Σ , a language L is **regular**

iff it is described by some regular expression iff it is recognized by some DFA iff it is recognized by some NFA

Over a fixed alphabet Σ , a language L is **context-free**

iff it is generated by some CFG iff it is recognized by some PDA

Fact: Every regular language is a context-free language.

Fact: There are context-free languages that are not nonregular.

Fact: There are countably many regular languages.

Fact: There are countably inifnitely many context-free languages.

Consequence: Most languages are **not** context-free!

Examples of non-context-free languages

$$\begin{aligned} &\{a^nb^nc^n\mid 0\leq n, n\in\mathbb{Z}\}\\ &\{a^ib^jc^k\mid 0\leq i\leq j\leq k, i\in\mathbb{Z}, j\in\mathbb{Z}, k\in\mathbb{Z}\}\\ &\{ww\mid w\in\{0,1\}^*\} \end{aligned}$$

(Sipser Ex 2.36, Ex 2.37, 2.38)

There is a Pumping Lemma for CFL that can be used to prove a specific language is non-context-free: If A is a context-free language, there there is a number p where, if s is any string in A of length at least p, then s may be divided into five pieces s = uvxyz where (1) for each $i \geq 0$, $uv^ixy^iz \in A$, (2) |uv| > 0, (3) $|vxy| \leq p$. We will not go into the details of the proof or application of Pumping Lemma for CFLs this quarter.

Week5 wednesday

A set X is said to be **closed** under an operation OP if, for any elements in X, applying OP to them gives an element in X.

True/False	Closure claim						
True	The set of integers is closed under multiplication.						
	$\forall x \forall y ((x \in \mathbb{Z} \land y \in \mathbb{Z}) \to xy \in \mathbb{Z})$						
True	For each set A , the power set of A is closed under intersection.						
	$\forall A_1 \forall A_2 ((A_1 \in \mathcal{P}(A) \land A_2 \in \mathcal{P}(A) \in \mathbb{Z}) \to A_1 \cap A_2 \in \mathcal{P}(A))$						
	The class of regular languages over Σ is closed under complementation.						
	The class of regular languages over Σ is closed under union.						
	The class of regular languages over Σ is closed under intersection.						
	The class of regular languages over Σ is closed under concatenation.						
	The class of regular languages over Σ is closed under Kleene star.						
	The class of context-free languages over Σ is closed under complementation.						
	The class of context-free languages over Σ is closed under union.						
	The class of context-free languages over Σ is closed under intersection.						
	The class of context-free languages over Σ is closed under concatenation.						
	The class of context-free languages over Σ is closed under Kleene star.						

Assume $\Sigma = \{0, 1, \#\}$

Turing machines: unlimited read + write memory, unlimited time (computation can proceed without "consuming" input and can re-read symbols of input)

- Division between program (CPU, state diagram) and data
- Unbounded memory gives theoretical limit to what modern computation (including PCs, supercomputers, quantum computers) can achieve
- State diagram formulation is simple enough to reason about (and diagonalize against) while expressive enough to capture modern computation

For Turing machine $M=(Q,\Sigma,\Gamma,\delta,q_0,q_{accept},q_{reject})$ the **computation** of M on a string w over Σ is:

- Read/write head starts at leftmost position on tape.
- Input string is written on |w|-many leftmost cells of tape, rest of the tape cells have the blank symbol. **Tape alphabet** is Γ with $\bot \in \Gamma$ and $\Sigma \subseteq \Gamma$. The blank symbol $\bot \notin \Sigma$.
- Given current state of machine and current symbol being read at the tape head, the machine transitions to next state, writes a symbol to the current position of the tape head (overwriting existing symbol), and moves the tape head L or R (if possible). Formally, **transition function** is

$$\delta: Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$$

• Computation ends if and when machine enters either the accept or the reject state. This is called **halting**. Note: $q_{accept} \neq q_{reject}$.

The language recognized by the Turing machine M, is

 $\{w \in \Sigma^* \mid \text{computation of } M \text{ on } w \text{ halts after entering the accept state}\} = \{w \in \Sigma^* \mid w \text{ is accepted by } M\}$

An example Turing machine: $\Sigma =$

$$,\Gamma =$$

$$\delta((q0,0)) =$$



Formal definition:

Sample computation:

$q0\downarrow$								
0	0	0	J	J	J			
		•			•			

The language recognized by this machine is ...

Extra practice:





Formal definition:

Sample computation: