## Week8 monday

Theorem: $A_T$	$_{M}$ is not	Turing-decida	able.
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**Proof**: Suppose towards a contradiction that there is a Turing machine that decides  $A_{TM}$ . We call this presumed machine  $M_{ATM}$ .

By assumption, for every Turing machine M and every string w

- If  $w \in L(M)$ , then the computation of  $M_{ATM}$  on  $\langle M, w \rangle$  \_\_\_\_\_
- If  $w \notin L(M)$ , then the computation of  $M_{ATM}$  on  $\langle M, w \rangle$  \_\_\_\_\_\_

Define a **new** Turing machine using the high-level description:

D = "On input  $\langle M \rangle$ , where M is a Turing machine:

- 1. Run  $M_{ATM}$  on  $\langle M, \langle M \rangle \rangle$ .
- 2. If  $M_{ATM}$  accepts, reject; if  $M_{ATM}$  rejects, accept."

Is D a Turing machine?

Is D a decider?

What is the result of the computation of D on  $\langle D \rangle$ ?

<b>Theorem</b> (Sipser Theorem 4.22): A language is Turing-decidable if and only if both it and its complement are Turing-recognizable.
<b>Proof, first direction:</b> Suppose language $L$ is Turing-decidable. WTS that both it and its complement are Turing-recognizable.
<b>Proof, second direction:</b> Suppose language $L$ is Turing-recognizable, and so is its complement. WTS that $L$ is Turing-decidable.
Give an example of a <b>decidable</b> set:
Give an example of a <b>recognizable undecidable</b> set:
Give an example of an <b>unrecognizable</b> set:

**True** or **False**: The class of Turing-decidable languages is closed under complementation?

Definition: A language L over an alphabet  $\Sigma$  is called **co-recognizable** if its complement, defined as  $\Sigma^* \setminus L = \{x \in \Sigma^* \mid x \notin L\}$ , is Turing-recognizable.

Notation: The complement of a set X is denoted with a superscript  $c, X^c$ , or an overline,  $\overline{X}$ .

## Week8 wednesday

### Mapping reduction

Motivation: Proving that  $A_{TM}$  is undecidable was hard. How can we leverage that work? Can we relate the decidability / undecidability of one problem to another?

If problem X is **no harder than** problem Y

- $\dots$  and if Y is easy,
- $\dots$  then X must be easy too.

If problem X is **no harder than** problem Y

- $\dots$  and if X is hard,
- $\dots$  then Y must be hard too.

"Problem X is no harder than problem Y" means "Can answer questions about membership in X by converting them to questions about membership in Y".

Definition: A is **mapping reducible to** B means there is a computable function  $f: \Sigma^* \to \Sigma^*$  such that for all strings x in  $\Sigma^*$ ,

 $x \in A$  if and only if  $f(x) \in B$ .

Notation: when A is mapping reducible to B, we write  $A \leq_m B$ .

Intuition:  $A \leq_m B$  means A is no harder than B, i.e. that the level of difficulty of A is less than or equal the level of difficulty of B.

#### Computable functions

Definition: A function  $f: \Sigma^* \to \Sigma^*$  is a **computable function** means there is some Turing machine such that, for each x, on input x the Turing machine halts with exactly f(x) followed by all blanks on the tape

Examples of computable functions:

The function that maps a string to a string which is one character longer and whose value, when interpreted as a fixed-width binary representation of a nonnegative integer is twice the value of the input string (when interpreted as a fixed-width binary representation of a non-negative integer)

$$f_1: \Sigma^* \to \Sigma^*$$
  $f_1(x) = x0$ 

To prove  $f_1$  is computable function, we define a Turing machine computing it.

High-level description

"On input w

- 1. Append 0 to w.
- 2. Halt."

 $Implementation-level\ description$ 

"On input w

- 1. Sweep read-write head to the right until find first blank cell.
- 2. Write 0.
- 3. Halt."

Formal definition ( $\{q0, qacc, qrej\}, \{0, 1\}, \{0, 1, \bot\}, \delta, q0, qacc, qrej$ ) where  $\delta$  is specified by the state diagram:

The function that maps a string to the result of repeating the string twice.

$$f_2: \Sigma^* \to \Sigma^* \qquad f_2(x) = xx$$

The function that maps strings that are not the codes of Turing machines to the empty string and that maps strings that code Turing machines to the code of the related Turing machine that acts like the Turing machine coded by the input, except that if this Turing machine coded by the input tries to reject, the new machine will go into a loop.

$$f_3: \Sigma^* \to \Sigma^* \qquad f_3(x) = \begin{cases} \varepsilon & \text{if } x \text{ is not the code of a TM} \\ \langle (Q \cup \{q_{trap}\}, \Sigma, \Gamma, \delta', q_0, q_{acc}, q_{rej}) \rangle & \text{if } x = \langle (Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej}) \rangle \end{cases}$$

where  $q_{trap} \notin Q$  and

$$\delta'((q,x)) = \begin{cases} (r,y,d) & \text{if } q \in Q, \ x \in \Gamma, \ \delta((q,x)) = (r,y,d), \ \text{and} \ r \neq q_{rej} \\ (q_{trap}, \neg, R) & \text{otherwise} \end{cases}$$

The	function	that r	naps stri	ngs that	are not	the	codes	of	CFGs	to the	e empty	string	and	that	maps	strings
that	code CF	Gs to	the code	of a PD	A that	recog	gnizes	$th\epsilon$	e langu	age g	enerated	l by th	e CF	G.		

Other examples?

# Week8 friday

Recall definition: A is **mapping reducible to** B means there is a computable function  $f: \Sigma^* \to \Sigma^*$  such that for all strings x in  $\Sigma^*$ ,

$$x \in A$$
 if and only if  $f(x) \in B$ .

Notation: when A is mapping reducible to B, we write  $A \leq_m B$ .

Intuition:  $A \leq_m B$  means A is no harder than B, i.e. that the level of difficulty of A is less than or equal the level of difficulty of B.

Example:  $A_{TM} \leq_m A_{TM}$ 

Example:  $A_{DFA} \leq_m \{ww \mid w \in \{0, 1\}^*\}$ 

Example:  $\{0^{i}1^{j} \mid i \geq 0, j \geq 0\} \leq_{m} A_{TM}$ 

**Theorem** (Sipser 5.22): If  $A \leq_m B$  and B is decidable, then A is decidable.

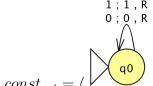
**Theorem** (Sipser 5.23): If  $A \leq_m B$  and A is undecidable, then B is undecidable.

#### Halting problem

 $HALT_{TM} = \{\langle M, w \rangle \mid M \text{ is a Turing machine, } w \text{ is a string, and } M \text{ halts on } w\}$ 

Define  $F: \Sigma^* \to \Sigma^*$  by

$$F(x) = \begin{cases} const_{out} & \text{if } x \neq \langle M, w \rangle \text{ for any Turing machine } M \text{ and string } w \text{ over the alphabet of } M \\ \langle M', w \rangle & \text{if } x = \langle M, w \rangle \text{ for some Turing machine } M \text{ and string } w \text{ over the alphabet of } M. \end{cases}$$



where  $const_{out} = \langle V, \varepsilon \rangle$  and M' is a Turing machine that computes like M except, if the computation ever were to go to a reject state, M' loops instead.





To use this function to prove that  $A_{TM} \leq_m HALT_{TM}$ , we need two claims: Claim (1): F is computable

Claim (2): for every  $x, x \in A_{TM}$  iff  $F(x) \in HALT_{TM}$ .