# Week5 monday

To fully specify a PDA we could give its 6-tuple formal definition or we could give its input alphabet, stack alphabet, and state diagram. An informal description of a PDA is a step-by-step description of how its computations would process input strings; the reader should be able to reconstruct the state diagram or formal definition precisely from such a descripton. The informal description of a PDA can refer to some common modules or subroutines that are computable by PDAs:

- PDAs can "test for emptyness of stack" without providing details. How? We can always push a special end-of-stack symbol, \$, at the start, before processing any input, and then use this symbol as a flag.
- PDAs can "test for end of input" without providing details. *How?* We can transform a PDA to one where accepting states are only those reachable when there are no more input symbols.

Big picture: PDAs were motivated by wanting to add some memory of unbounded size to NFA. How do we accomplish a similar enhancement of regular expressions to get a syntactic model that is more expressive?

DFA, NFA, PDA: Machines process one input string at a time; the computation of a machine on its input string reads the input from left to right.

Regular expressions: Syntactic descriptions of all strings that match a particular pattern; the language described by a regular expression is built up recursively according to the expression's syntax

Context-free grammars: Rules to produce one string at a time, adding characters from the middle, beginning, or end of the final string as the derivation proceeds.

| Term                      | Typical symbol                 | Definition  |
|---------------------------|--------------------------------|---|
| Context-free grammar      | G                              | $G = (V, \Sigma, R, S)$   |
| (CFG)                     |                                |   |
| Variables                 | V                              | Finite set of symbols that represent phases in production                             |
|                           |                                | pattern   |
| Terminals                 | $\Sigma$                       | Alphabet of symbols of strings generated by CFG                                       |
|                           |                                | $V \cap \Sigma = \emptyset$   |
| Rules                     | R                              | Each rule is $A \to u$ with $A \in V$ and $u \in (V \cup \Sigma)^*$                   |
| Start variable            | S                              | Usually on LHS of first / topmost rule  |
| Derivation                |                                | Sequence of substitutions in a CFG  |
|                           | $S \implies \cdots \implies w$ | Start with start variable, apply one rule to one occurrence                           |
|                           |                                | of a variable at a time   |
| Language generated by the | L(G)                           | $\{w \in \Sigma^* \mid \text{ there is derivation in } G \text{ that ends in } w\} =$ |
| CFG G                     |                                | $\{w \in \Sigma^* \mid S \implies {}^*w\}$  |
| Context-free language     |                                | A language that is the language generated by some CFG                                 |
| Sipser pages 102-103      |                                |   |

| Examples of context-free grammars, | derivations i | n those | grammars, | and | $\mathbf{the}$ | languages | gen- |
|------------------------------------|---------------|---------|-----------|-----|----------------|-----------|------|
| erated by those grammars           |               |         |           |     |                |           |      |

$$G_1 = (\{S\}, \{0\}, R, S)$$
 with rules

$$S \to 0S$$

$$S \to 0$$

In  $L(G_1)$  ...

Not in  $L(G_1)$  ...

$$G_2 = (\{S\}, \{0, 1\}, R, S)$$

$$S \to 0S \mid 1S \mid \varepsilon$$

In  $L(G_2)$  ...

Not in  $L(G_2)$  ...

 $(\{S,T\},\{0,1\},R,S)$  with rules

$$\begin{split} S &\to T1T1T1T \\ T &\to 0T \mid 1T \mid \varepsilon \end{split}$$

In  $L(G_3)$  ...

Not in  $L(G_3)$  ...

 $G_4 = (\{A, B\}, \{0, 1\}, R, A)$  with rules

$$A \rightarrow 0A0 \mid 0A1 \mid 1A0 \mid 1A1 \mid 1$$

In  $L(G_4)$  ...

Not in  $L(G_4)$  ...

Extra practice: Is there a CFG G with  $L(G) = \emptyset$ ?



$$(\{S,T,V,W\},\{a,b\},\{S\rightarrow aT,T\rightarrow bV,V\rightarrow bW,W\rightarrow a\},S)$$

$$(\{Q\}, \{a, b\}, \{Q \rightarrow abba\}, Q)$$

$$(\{X,Y\},\{a,b\},\{X\to aYa,Y\to bb\},X)$$

Design a CFG to generate the language  $\{a^nb^n\mid n\geq 0\}$ 

Sample derivation:

Design a CFG to generate the language  $\{a^ib^j\mid j\geq i\geq 0\}$ 

Sample derivation:

# Week5 wednesday

**Theorem 2.20**: A language is generated by some context-free grammar if and only if it is recognized by some push-down automaton.

Definition: a language is called **context-free** if it is the language generated by a context-free grammar. The class of all context-free language over a given alphabet  $\Sigma$  is called **CFL**.

### Consequences:

- Quick proof that every regular language is context free
- To prove closure of the class of context-free languages under a given operation, we can choose either of two modes of proof (via CFGs or PDAs) depending on which is easier

Over  $\Sigma = \{a, b\}$ , let  $L = \{a^n b^m \mid n \neq m\}$ . Goal: Prove L is context-free.

Suppose  $L_1$  and  $L_2$  are context-free languages over  $\Sigma$ . Goal:  $L_1 \cup L_2$  is also context-free.

Approach 1: with PDAs

Let  $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, F_1)$  and  $M_2 = (Q_2, \Sigma, \Gamma_2, \delta_2, q_2, F_2)$  be PDAs with  $L(M_1) = L_1$  and  $L(M_2) = L_2$ .

Define M =

 $Approach\ 2:\ with\ CFGs$ 

Let  $G_1 = (V_1, \Sigma, R_1, S_1)$  and  $G_2 = (V_2, \Sigma, R_2, S_2)$  be CFGs with  $L(G_1) = L_1$  and  $L(G_2) = L_2$ .

Define G =

Suppose  $L_1$  and  $L_2$  are context-free languages over  $\Sigma$ . Goal:  $L_1 \circ L_2$  is also context-free.

Approach 1: with PDAs

Let  $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, F_1)$  and  $M_2 = (Q_2, \Sigma, \Gamma_2, \delta_2, q_2, F_2)$  be PDAs with  $L(M_1) = L_1$  and  $L(M_2) = L_2$ .

Define M =

 $Approach\ 2:\ with\ CFGs$ 

Let  $G_1 = (V_1, \Sigma, R_1, S_1)$  and  $G_2 = (V_2, \Sigma, R_2, S_2)$  be CFGs with  $L(G_1) = L_1$  and  $L(G_2) = L_2$ .

Define G =

#### Summary

Over a fixed alphabet  $\Sigma$ , a language L is **regular** 

iff it is described by some regular expression iff it is recognized by some DFA iff it is recognized by some NFA

Over a fixed alphabet  $\Sigma$ , a language L is **context-free** 

iff it is generated by some CFG iff it is recognized by some PDA

Fact: Every regular language is a context-free language.

Fact: There are context-free languages that are not nonregular.

Fact: There are countably many regular languages.

Fact: There are countably inifnitely many context-free languages.

Consequence: Most languages are **not** context-free!

### Examples of non-context-free languages

$$\begin{aligned} &\{a^nb^nc^n\mid 0\leq n, n\in\mathbb{Z}\}\\ &\{a^ib^jc^k\mid 0\leq i\leq j\leq k, i\in\mathbb{Z}, j\in\mathbb{Z}, k\in\mathbb{Z}\}\\ &\{ww\mid w\in\{0,1\}^*\} \end{aligned}$$

(Sipser Ex 2.36, Ex 2.37, 2.38)

There is a Pumping Lemma for CFL that can be used to prove a specific language is non-context-free: If A is a context-free language, there there is a number p where, if s is any string in A of length at least p, then s may be divided into five pieces s = uvxyz where (1) for each  $i \geq 0$ ,  $uv^ixy^iz \in A$ , (2) |uv| > 0, (3)  $|vxy| \leq p$ . We will not go into the details of the proof or application of Pumping Lemma for CFLs this quarter.