

Week5 monday

To fully specify a PDA we could give its 6-tuple formal definition or we could give its input alphabet, stack alphabet, and state diagram. An informal description of a PDA is a step-by-step description of how its computations would process input strings; the reader should be able to reconstruct the state diagram or formal definition precisely from such a description. The informal description of a PDA can refer to some common modules or subroutines that are computable by PDAs:

- PDAs can “test for emptiness of stack” without providing details. *How?* We can always push a special end-of-stack symbol, \$, at the start, before processing any input, and then use this symbol as a flag.
- PDAs can “test for end of input” without providing details. *How?* We can transform a PDA to one where accepting states are only those reachable when there are no more input symbols.

Big picture: PDAs were motivated by wanting to add some memory of unbounded size to NFA. How do we accomplish a similar enhancement of regular expressions to get a syntactic model that is more expressive?

DFA, NFA, PDA: Machines process one input string at a time; the computation of a machine on its input string reads the input from left to right.

Regular expressions: Syntactic descriptions of all strings that match a particular pattern; the language described by a regular expression is built up recursively according to the expression’s syntax

Context-free grammars: Rules to produce one string at a time, adding characters from the middle, beginning, or end of the final string as the derivation proceeds.

| Term | Typical symbol | Definition |
|--|-------------------------------------|--|
| Context-free grammar (CFG) | G | $G = (V, \Sigma, R, S)$ |
| Variables | V | Finite set of symbols that represent phases in production pattern |
| Terminals | Σ | Alphabet of symbols of strings generated by CFG $V \cap \Sigma = \emptyset$ |
| Rules | R | Each rule is $A \rightarrow u$ with $A \in V$ and $u \in (V \cup \Sigma)^*$ |
| Start variable | S | Usually on LHS of first / topmost rule |
| Derivation | $S \Rightarrow \dots \Rightarrow w$ | Sequence of substitutions in a CFG Start with start variable, apply one rule to one occurrence of a variable at a time |
| Language generated by the CFG G | $L(G)$ | $\{w \in \Sigma^* \mid \text{there is derivation in } G \text{ that ends in } w\} = \{w \in \Sigma^* \mid S \Rightarrow^* w\}$ |
| Context-free language | | A language that is the language generated by some CFG |
| Sipser pages 102-103 | | |

Examples of context-free grammars, derivations in those grammars, and the languages generated by those grammars

$G_1 = (\{S\}, \{0\}, R, S)$ with rules

$$S \rightarrow 0S$$

$$S \rightarrow 0$$

In $L(G_1)$...

Not in $L(G_1)$...

$G_2 = (\{S\}, \{0, 1\}, R, S)$

$$S \rightarrow 0S \mid 1S \mid \varepsilon$$

In $L(G_2)$...

Not in $L(G_2)$...

$(\{S, T\}, \{0, 1\}, R, S)$ with rules

$$S \rightarrow T1T1T1T$$

$$T \rightarrow 0T \mid 1T \mid \varepsilon$$

In $L(G_3) \dots$

Not in $L(G_3) \dots$

$G_4 = (\{A, B\}, \{0, 1\}, R, A)$ with rules

$$A \rightarrow 0A0 \mid 0A1 \mid 1A0 \mid 1A1 \mid 1$$

In $L(G_4) \dots$

Not in $L(G_4) \dots$

Extra practice: Is there a CFG G with $L(G) = \emptyset$?

Design a CFG to generate the language $\{abba\}$

$$(\{S, T, V, W\}, \{a, b\}, \{S \rightarrow aT, T \rightarrow bV, V \rightarrow bW, W \rightarrow a\}, S)$$

$$(\{Q\}, \{a, b\}, \{Q \rightarrow abba\}, Q)$$

$$(\{X, Y\}, \{a, b\}, \{X \rightarrow aYa, Y \rightarrow bb\}, X)$$

Design a CFG to generate the language $\{a^n b^n \mid n \geq 0\}$

Sample derivation:

Design a CFG to generate the language $\{a^i b^j \mid j \geq i \geq 0\}$

Sample derivation:

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Theorem 2.20: A language is generated by some context-free grammar if and only if it is recognized by some push-down automaton.

Definition: a language is called **context-free** if it is the language generated by a context-free grammar. The class of all context-free languages over a given alphabet Σ is called **CFL**.

Consequences:

- Quick proof that every regular language is context free
- To prove closure of the class of context-free languages under a given operation, we can choose either of two modes of proof (via CFGs or PDAs) depending on which is easier

Over $\Sigma = \{a, b\}$, let $L = \{a^n b^m \mid n \neq m\}$. **Goal:** Prove L is context-free.

Suppose L_1 and L_2 are context-free languages over Σ . **Goal:** $L_1 \cup L_2$ is also context-free.

Approach 1: with PDAs

Let $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \Gamma_2, \delta_2, q_2, F_2)$ be PDAs with $L(M_1) = L_1$ and $L(M_2) = L_2$.

Define $M =$

Approach 2: with CFGs

Let $G_1 = (V_1, \Sigma, R_1, S_1)$ and $G_2 = (V_2, \Sigma, R_2, S_2)$ be CFGs with $L(G_1) = L_1$ and $L(G_2) = L_2$.

Define $G =$

Suppose L_1 and L_2 are context-free languages over Σ . **Goal:** $L_1 \circ L_2$ is also context-free.

Approach 1: with PDAs

Let $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \Gamma_2, \delta_2, q_2, F_2)$ be PDAs with $L(M_1) = L_1$ and $L(M_2) = L_2$.

Define $M =$

Approach 2: with CFGs

Let $G_1 = (V_1, \Sigma, R_1, S_1)$ and $G_2 = (V_2, \Sigma, R_2, S_2)$ be CFGs with $L(G_1) = L_1$ and $L(G_2) = L_2$.

Define $G =$

Summary

Over a fixed alphabet Σ , a language L is **regular**

- iff it is described by some regular expression
- iff it is recognized by some DFA
- iff it is recognized by some NFA

Over a fixed alphabet Σ , a language L is **context-free**

- iff it is generated by some CFG
- iff it is recognized by some PDA

Fact: Every regular language is a context-free language.

Fact: There are context-free languages that are not nonregular.

Fact: There are countably many regular languages.

Fact: There are countably infinitely many context-free languages.

Consequence: Most languages are **not** context-free!

Examples of non-context-free languages

$$\begin{aligned} &\{a^n b^n c^n \mid 0 \leq n, n \in \mathbb{Z}\} \\ &\{a^i b^j c^k \mid 0 \leq i \leq j \leq k, i \in \mathbb{Z}, j \in \mathbb{Z}, k \in \mathbb{Z}\} \\ &\{ww \mid w \in \{0, 1\}^*\} \end{aligned}$$

(Sipser Ex 2.36, Ex 2.37, 2.38)

There is a Pumping Lemma for CFL that can be used to prove a specific language is non-context-free: If A is a context-free language, there there is a number p where, if s is any string in A of length at least p , then s may be divided into five pieces $s = uvxyz$ where (1) for each $i \geq 0$, $uv^i xy^i z \in A$, (2) $|uv| > 0$, (3) $|vxy| \leq p$. *We will not go into the details of the proof or application of Pumping Lemma for CFLs this quarter.*