

## Week3 monday

The state diagram of an NFA over  $\{a, b\}$  is below. The formal definition of this NFA is:



The language recognized by this NFA is:

Suppose  $A_1, A_2$  are languages over an alphabet  $\Sigma$ . **Claim:** if there is a NFA  $N_1$  such that  $L(N_1) = A_1$  and NFA  $N_2$  such that  $L(N_2) = A_2$ , then there is another NFA, let's call it  $N$ , such that  $L(N) = A_1 \cup A_2$ .

**Proof idea:** Use nondeterminism to choose which of  $N_1, N_2$  to run.

**Formal construction:** Let  $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$  and  $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$  and assume  $Q_1 \cap Q_2 = \emptyset$  and that  $q_0 \notin Q_1 \cup Q_2$ . Construct  $N = (Q, \Sigma, \delta, q_0, F_1 \cup F_2)$  where

- $Q =$
- $\delta : Q \times \Sigma_\epsilon \rightarrow \mathcal{P}(Q)$  is defined by, for  $q \in Q$  and  $a \in \Sigma_\epsilon$ :

*Proof of correctness would prove that  $L(N) = A_1 \cup A_2$  by considering an arbitrary string accepted by  $N$ , tracing an accepting computation of  $N$  on it, and using that trace to prove the string is in at least one of  $A_1, A_2$ ; then, taking an arbitrary string in  $A_1 \cup A_2$  and proving that it is accepted by  $N$ . Details left for extra practice.*

Over the alphabet  $\{a, b\}$ , the language  $L$  described by the regular expression  $\Sigma^* a \Sigma^* b$

includes the strings \_\_\_\_\_ and excludes the strings \_\_\_\_\_

The state diagram of a NFA recognizing  $L$  is:

Suppose  $A_1, A_2$  are languages over an alphabet  $\Sigma$ . **Claim:** if there is a NFA  $N_1$  such that  $L(N_1) = A_1$  and NFA  $N_2$  such that  $L(N_2) = A_2$ , then there is another NFA, let's call it  $N$ , such that  $L(N) = A_1 \circ A_2$ .

**Proof idea:** Allow computation to move between  $N_1$  and  $N_2$  “spontaneously” when reach an accepting state of  $N_1$ , guessing that we’ve reached the point where the two parts of the string in the set-wise concatenation are glued together.

**Formal construction:** Let  $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$  and  $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$  and assume  $Q_1 \cap Q_2 = \emptyset$ . Construct  $N = (Q, \Sigma, \delta, q_0, F)$  where

- $Q =$
- $q_0 =$
- $F =$
- $\delta : Q \times \Sigma_\varepsilon \rightarrow \mathcal{P}(Q)$  is defined by, for  $q \in Q$  and  $a \in \Sigma_\varepsilon$ :

$$\delta((q, a)) = \begin{cases} \delta_1((q, a)) & \text{if } q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1((q, a)) & \text{if } q \in F_1 \text{ and } a \in \Sigma \\ \delta_1((q, a)) \cup \{q_2\} & \text{if } q \in F_1 \text{ and } a = \varepsilon \\ \delta_2((q, a)) & \text{if } q \in Q_2 \end{cases}$$

*Proof of correctness would prove that  $L(N) = A_1 \circ A_2$  by considering an arbitrary string accepted by  $N$ , tracing an accepting computation of  $N$  on it, and using that trace to prove the string can be written as the result of concatenating two strings, the first in  $A_1$  and the second in  $A_2$ ; then, taking an arbitrary string in  $A_1 \circ A_2$  and proving that it is accepted by  $N$ . Details left for extra practice.*

Suppose  $A$  is a language over an alphabet  $\Sigma$ . **Claim:** if there is a NFA  $N$  such that  $L(N) = A$ , then there is another NFA, let's call it  $N'$ , such that  $L(N') = A^*$ .

**Proof idea:** Add a fresh start state, which is an accept state. Add spontaneous moves from each (old) accept state to the old start state.

**Formal construction:** Let  $N = (Q, \Sigma, \delta, q_1, F)$  and assume  $q_0 \notin Q$ . Construct  $N' = (Q', \Sigma, \delta', q_0, F')$  where

- $Q' = Q \cup \{q_0\}$
- $F' = F \cup \{q_0\}$
- $\delta' : Q' \times \Sigma_\varepsilon \rightarrow \mathcal{P}(Q')$  is defined by, for  $q \in Q'$  and  $a \in \Sigma_\varepsilon$ :

$$\delta'((q, a)) = \begin{cases} \delta((q, a)) & \text{if } q \in Q \text{ and } q \notin F \\ \delta((q, a)) & \text{if } q \in F \text{ and } a \in \Sigma \\ \delta((q, a)) \cup \{q_1\} & \text{if } q \in F \text{ and } a = \varepsilon \\ \{q_1\} & \text{if } q = q_0 \text{ and } a = \varepsilon \\ \emptyset & \text{if } q = q_0 \text{ and } a \in \Sigma \end{cases}$$

*Proof of correctness would prove that  $L(N') = A^*$  by considering an arbitrary string accepted by  $N'$ , tracing an accepting computation of  $N'$  on it, and using that trace to prove the string can be written as the result of concatenating some number of strings, each of which is in  $A$ ; then, taking an arbitrary string in  $A^*$  and proving that it is accepted by  $N'$ . Details left for extra practice.*

**Application:** A state diagram for a NFA over  $\Sigma = \{a, b\}$  that recognizes  $L((\Sigma^*b)^*)$ :

**True or False:** The state diagram of any DFA is also the state diagram of a NFA.

**True or False:** The state diagram of any NFA is also the state diagram of a DFA.

**True or False:** The formal definition  $(Q, \Sigma, \delta, q_0, F)$  of any DFA is also the formal definition of a NFA.

**True or False:** The formal definition  $(Q, \Sigma, \delta, q_0, F)$  of any NFA is also the formal definition of a DFA.

## Week3 wednesday

Consider the state diagram of an NFA over  $\{a, b\}$ :



The language recognized by this NFA is

The state diagram of a DFA recognizing this same language is:

Suppose  $A$  is a language over an alphabet  $\Sigma$ . **Claim:** if there is a NFA  $N$  such that  $L(N) = A$  then there is a DFA  $M$  such that  $L(M) = A$ .

**Proof idea:** States in  $M$  are “macro-states” – collections of states from  $N$  – that represent the set of possible states a computation of  $N$  might be in.

**Formal construction:** Let  $N = (Q, \Sigma, \delta, q_0, F)$ . Define

$$M = ( \mathcal{P}(Q), \Sigma, \delta', q', \{X \subseteq Q \mid X \cap F \neq \emptyset\} )$$

where  $q' = \{q \in Q \mid q = q_0 \text{ or is accessible from } q_0 \text{ by spontaneous moves in } N\}$  and

$\delta'((X, x)) = \{q \in Q \mid q \in \delta(r, x) \text{ for some } r \in X \text{ or is accessible from such an } r \text{ by spontaneous moves in } N\}$

Consider the state diagram of an NFA over  $\{0, 1\}$ . Use the “macro-state” construction to find an equivalent DFA.



Prune this diagram to get an equivalent DFA with only the “macro-states” reachable from the start state.

Suppose  $A$  is a language over an alphabet  $\Sigma$ . **Claim:** if there is a regular expression  $R$  such that  $L(R) = A$ , then there is a NFA, let's call it  $N$ , such that  $L(N) = A$ .

**Structural induction:** Regular expression is built from basis regular expressions using inductive steps (union, concatenation, Kleene star symbols). Use constructions to mirror these in NFAs.

**Application:** A state diagram for a NFA over  $\{a, b\}$  that recognizes  $L(a^*(ab)^*)$ :

Suppose  $A$  is a language over an alphabet  $\Sigma$ . **Claim:** if there is a DFA  $M$  such that  $L(M) = A$ , then there is a regular expression, let's call it  $R$ , such that  $L(R) = A$ .

**Proof idea:** Trace all possible paths from start state to accept state. Express labels of these paths as regular expressions, and union them all.

1. Add new start state with  $\varepsilon$  arrow to old start state.
2. Add new accept state with  $\varepsilon$  arrow from old accept states. Make old accept states non-accept.
3. Remove one (of the old) states at a time: modify regular expressions on arrows that went through removed state to restore language recognized by machine.

**Application:** Find a regular expression describing the language recognized by the DFA with state diagram



**Conclusion:** For each language  $L$ ,

There is a DFA that recognizes  $L$   $\iff \exists M$  ( $M$  is a DFA and  $L(M) = A$ )  
if and only if

There is a NFA that recognizes  $L$   $\iff \exists N$  ( $N$  is a NFA and  $L(N) = A$ )  
if and only if

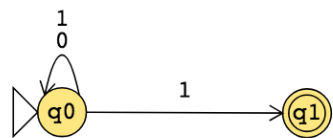
There is a regular expression that describes  $L$   $\iff \exists R$  ( $R$  is a regular expression and  $L(R) = A$ )

A language is called **regular** when any (hence all) of the above three conditions are met.

Week2 friday

<b>Nondeterministic finite automaton</b> $M = (Q, \Sigma, \delta, q_0, F)$	
Finite set of states $Q$	Can be labelled by any collection of distinct names. Default: $q_0, q_1, \dots$
Alphabet $\Sigma$	Each input to the automaton is a string over $\Sigma$ .
Arrow labels $\Sigma_\epsilon$	$\Sigma_\epsilon = \Sigma \cup \{\epsilon\}$ .
Transition function $\delta$	Arrows in the state diagram are labelled either by symbols from $\Sigma$ or by $\epsilon$ $\delta : Q \times \Sigma_\epsilon \rightarrow \mathcal{P}(Q)$ gives the <b>set of possible next states</b> for a transition from the current state upon reading a symbol or spontaneously moving.
Start state $q_0$	Element of $Q$ . Each computation of the machine starts at the start state.
Accept (final) states $F$	$F \subseteq Q$ .
$M$ accepts the input string	if and only if <b>there is</b> a computation of $M$ on the input string that processes the whole string and ends in an accept state.
Page 53	

The formal definition of the NFA over  $\{0, 1\}$  given by this state diagram is:



The language over  $\{0, 1\}$  recognized by this NFA is:

Change the transition function to get a different NFA which accepts the empty string.

The state diagram of an NFA over  $\{a, b\}$  is below. The formal definition of this NFA is:



The language recognized by this NFA is: