Monday

For Turing machine $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$ where δ is the **transition function**
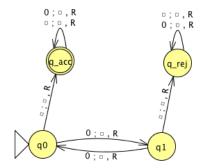
$$\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$$

the **computation** of M on a string w over Σ is:

- Read/write head starts at leftmost position on tape.
- Input string is written on $|w|$-many leftmost cells of tape, rest of the tape cells have the blank symbol. **Tape alphabet** is Γ with $\sqcup \in \Gamma$ and $\Sigma \subseteq \Gamma$. The blank symbol $\sqcup \notin \Sigma$.
- Given current state of machine and current symbol being read at the tape head, the machine transitions to next state, writes a symbol to the current position of the tape head (overwriting existing symbol), and moves the tape head L or R (if possible).
- Computation ends **if and when** machine enters either the accept or the reject state. This is called **halting**. Note: $q_{accept} \neq q_{reject}$.

The **language recognized by the Turing machine** M, is $L(M) = \{w \in \Sigma^* \mid w \text{ is accepted by } M\}$, which is defined as

$$\{w \in \Sigma^* \mid \text{computation of } M \text{ on } w \text{ halts after entering the accept state}\}$$

0 ; □ , R
□ ; □ , R

□ ; □ , R
0 ; □ , R

q_acc

q_rej

□ , □ , R

□ , □ , R

q0

q1

0 ; □ , R

0 ; □ , R

Formal definition:

Sample computation:

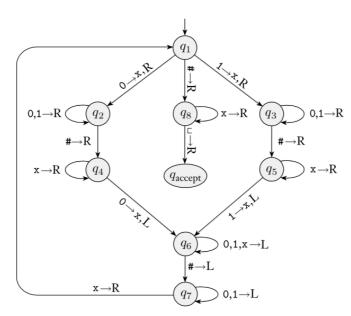| $q0 \downarrow$						
0	0	0	␣	␣	␣	␣

The language recognized by this machine is . . .

To define a Turing machine, we could give a

- **Formal definition**, namely the 7-tuple of parameters including set of states, input alphabet, tape alphabet, transition function, start state, accept state, and reject state; or,

- **Implementation-level definition**: English prose that describes the Turing machine head movements relative to contents of tape, and conditions for accepting / rejecting based on those contents.

Conventions for drawing state diagrams of Turing machines: (1) omit the reject state from the diagram (unless it's the start state), (2) any missing transitions in the state diagram have value $(q_{reject}, ␣, R)$.

Sipser Figure 3.10

Computation on input string 01#01

The state diagram of the Turing machine with states q_1 through q_8, q_{accept}, with transitions:
- $q_1 \xrightarrow{0 \to x, R} q_2$
- $q_1 \xrightarrow{\# \to R} q_8$
- $q_1 \xrightarrow{1 \to x, R} q_3$
- q_2: $0,1 \to R$ (self loop)
- $q_8 \xrightarrow{x \to R} q_3$ ... $q_8 \xrightarrow{\sqcup \to R} q_{accept}$
- q_3: $0,1 \to R$ (self loop)
- $q_2 \xrightarrow{\# \to R} q_4$
- $q_3 \xrightarrow{\# \to R} q_5$
- $q_4 \xrightarrow{x \to R} q_4$; $q_5 \xrightarrow{x \to R} q_5$
- $q_4 \xrightarrow{0 \to x, L} q_6$
- $q_5 \xrightarrow{1 \to x, L} q_6$
- q_6: $0,1,x \to L$ (self loop)
- $q_6 \xrightarrow{\# \to L} q_7$
- q_7: $0,1 \to L$ (self loop); $q_7 \xrightarrow{x \to R} q_1$

$q_1 \downarrow$						
0	1	#	0	1	␣	␣

Implementation level description of this machine:

Zig-zag across tape to corresponding positions on either side of # to check whether the characters in these positions agree. If they do not, or if there is no #, reject. If they do, cross them off.

Once all symbols to the left of the # are crossed off, check for any un-crossed-off symbols to the right of #; if there are any, reject; if there aren't, accept.

The language recognized by this machine is

$$\{w\#w \mid w \in \{0,1\}^*\}$$

Extra practice

Computation on input string 01#1

$q_1 \downarrow$						
0	1	#	1	␣	␣	␣

Review: Week 6 Monday

Recall: Review quizzes based on class material are assigned each day. These quizzes will help you track and confirm your understanding of the concepts and examples we work in class. Quizzes can be submitted on Gradescope as many times (with no penalty) as you like until the quiz deadline: the three quizzes each week are all due on Friday (with no penalty late submission open until Sunday).

Please complete the review quiz questions on Gradescope about formal and implementation-level descriptions of Turing machines.

Pre class reading for next time: pages 176-177 on variants of Turing machines

Wednesday

Fix $\Sigma = \{0, 1\}$, $\Gamma = \{0, 1, \sqcup\}$ for the Turing machines with the following state diagrams:

<table>
<tr>
<td>

$\sqcup ; \sqcup , R$ (self-loop on q0)

q0, q_acc

Implementation level description:

Example of string accepted:
Example of string rejected:

</td>
<td>

q_rej, q_acc

Implementation level description:

Example of string accepted:
Example of string rejected:

</td>
</tr>
<tr>
<td>

q0 → q_acc with $\sqcup ; \sqcup , R$

Implementation level description:

Example of string accepted:
Example of string rejected:

</td>
<td>

$1 ; \sqcup , R$
$0 ; \sqcup , R$
$\sqcup ; \sqcup , R$ (self-loop on q0)

q0, q_acc

Implementation level description:

Example of string accepted:
Example of string rejected:

</td>
</tr>
</table>

Two models of computation are called **equally expressive** when every language recognizable with the first model is recognizable with the second, and vice versa.

True / False: NFAs and PDAs are equally expressive.

True / False: Regular expressions and CFGs are equally expressive.

To say a language is **Turing-recognizable** means that there is some Turing machine that recognizes it.

*Some examples of models that are **equally expressive** with deterministic Turing machines:*

$\boxed{\textbf{May-stay} \text{ machines}}$ The May-stay machine model is the same as the usual Turing machine model, except that on each transition, the tape head may move L, move R, or Stay.

Formally: $(Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$ where

$$\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R, S\}$$

Claim: Turing machines and May-stay machines are equally expressive. *To prove ...*

To translate a standard TM to a may-stay machine:

To translate one of the may-stay machines to standard TM: any time TM would Stay, move right then left.

Formally: suppose $M_S = (Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej})$ has $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R, S\}$. Define the Turing-machine

$$M_{new} = (\qquad\qquad\qquad\qquad\qquad\qquad\qquad)$$

$\boxed{\textbf{Multitape Turing machine}}$ A multitape Turing macihne with k tapes can be formally represented as $(Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej})$ where Q is the finite set of states, Σ is the input alphabet with $\textvisiblespace \notin \Sigma$, Γ is the tape alphabet with $\Sigma \subsetneq \Gamma$, $\delta : Q \times \Gamma^k \to Q \times \Gamma^k \times \{L, R\}^k$ (where k is the number of states)

If M is a standard TM, it is a 1-tape machine.

To translate a k-tape machine to a standard TM: Use a new symbol to separate the contents of each tape and keep track of location of head with special version of each tape symbol. <small>Sipser Theorem 3.13</small>

<small>FIGURE</small> **3.14**
Representing three tapes with one

Extra practice: $\boxed{\textbf{Wikipedia Turing machine}}$ Define a machine $(Q, \Gamma, b, \Sigma, q_0, F, \delta)$ where Q is the finite set of states Γ is the tape alphabet, $b \in \Gamma$ is the blank symbol, $\Sigma \subsetneq \Gamma$ is the input alphabet, $q_0 \in Q$ is the start state, $F \subseteq Q$ is the set of accept states, $\delta : (Q \setminus F) \times \Gamma \nrightarrow Q \times \Gamma \times \{L, R\}$ is a partial transition function If computation enters a state in F, it accepts If computation enters a configuration where δ is not defined, it rejects . <small>Hopcroft and Ullman, cited by Wikipedia</small>

Each enumerator machine has finite state control, unlimited work tape, and a printer. The computation proceeds according to transition function; at any point machine may "send" a string to the printer.
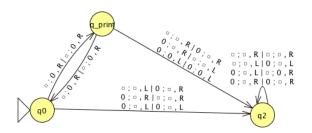
$$E = (Q, \Sigma, \Gamma, \delta, q_0, q_{print})$$

Q is the finite set of states, Σ is the output alphabet, Γ is the tape alphabet ($\Sigma \subsetneq \Gamma, \_ \in \Gamma \setminus \Sigma$),

$$\delta : Q \times \Gamma \times \Gamma \to Q \times \Gamma \times \Gamma \times \{L, R\} \times \{L, R\}$$

where in state q, when the working tape is scanning character x and the printer tape is scanning character y, $\delta((q, x, y)) = (q', x', y', d_w, d_p)$ means transition to control state q', write x' on the working tape, write y' on the printer tape, move in direction d_w on the working tape, and move in direction d_p on the printer tape. The computation starts in q_0 and each time the computation enters q_{print} the string from the leftmost edge of the printer tape to the first blank cell is considered to be printed.

The language **enumerated** by E, $L(E)$, is $\{w \in \Sigma^* \mid E \text{ eventually, at finite time, prints } w\}$.

Theorem 3.21 A language is Turing-recognizable iff some enumerator enumerates it.

Proof:

Assume L is enumerated by some enumerator, E, so $L = L(E)$. We'll use E in a subroutine within a high-level description of a new Turing machine that we will build to recognize L.

Goal: build Turing machine M_E with $L(M_E) = L(E)$.

Define M_E as follows: $M_E = $ "On input w,

1. Run E. For each string x printed by E.

2.   Check if $x = w$. If so, accept (and halt); otherwise, continue."

Assume L is Turing-recognizable and there is a Turing machine M with $L = L(M)$. We'll use M in a subroutine within a high-level description of an enumerator that we will build to enumerate L.

Goal: build enumerator E_M with $L(E_M) = L(M)$.

Idea: check each string in turn to see if it is in L.

How? Run computation of M on each string. *But*: need to be careful about computations that don't halt.

Recall String order for $\Sigma = \{0, 1\}$: $s_1 = \varepsilon$, $s_2 = 0$, $s_3 = 1$, $s_4 = 00$, $s_5 = 01$, $s_6 = 10$, $s_7 = 11$, $s_8 = 000$, ...

Define E_M as follows: $E_M = $ " *ignore any input.* Repeat the following for $i = 1, 2, 3, \ldots$

1. Run the computations of M on s_1, s_2, ..., s_i for (at most) i steps each

2. For each of these i computations that accept during the (at most) i steps, print out the accepted string."

Review: Week 6 Wednesday

Please complete the review quiz questions on Gradescope about variants of Turing machines.

Pre class reading for next time: Theorem 3.16 on page 178 (nondeterministism)

Friday

Nondeterministic Turing machine

At any point in the computation, the nondeterministic machine may proceed according to several possibilities: $(Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej})$ where

$$\delta : Q \times \Gamma \to \mathcal{P}(Q \times \Gamma \times \{L, R\})$$

The computation of a nondeterministic Turing machine is a tree with branching when the next step of the computation has multiple possibilities. A nondeterministic Turing machine accepts a string exactly when some branch of the computation tree enters the accept state.

Given a nondeterministic machine, we can use a 3-tape Turing machine to simulate it by doing a breadth-first search of computation tree: one tape is "read-only" input tape, one tape simulates the tape of the nondeterministic computation, and one tape tracks nondeterministic branching. <small>Sipser page 178</small>

Two models of computation are called **equally expressive** when every language recognizable with the first model is recognizable with the second, and vice versa.

Church-Turing Thesis (Sipser p. 183): The informal notion of algorithm is formalized completely and correctly by the formal definition of a Turing machine. In other words: all reasonably expressive models of computation are equally expressive with the standard Turing machine.

A language L is **recognized by** a Turing machine M means

A Turing machine M **recognizes** a language L if means

A Turing machine M is a **decider** means

A language L is **decided by** a Turing machine M means

A Turing machine M **decides** a language L means

Fix $\Sigma = \{0, 1\}$, $\Gamma = \{0, 1, \square\}$ for the Turing machines with the following state diagrams:

$\square ; \square , R$

q0   q_acc

Decider? Yes   /   No

q_rej   q_acc

Decider? Yes   /   No

q0 $\square ; \square , R$ q_acc

Decider? Yes   /   No

$1 ; \square , R$
$0 ; \square , R$
$\square ; \square , R$

q0   q_acc

Decider? Yes   /   No

Claim: If two languages (over a fixed alphabet Σ) are Turing-recognizable, then their union is as well.

Proof using Turing machines:

Proof using nondeterministic Turing machines:

Proof using enumerators:

Describing Turing machines (Sipser p. 185)

To define a Turing machine, we could give a

- **Formal definition**: the 7-tuple of parameters including set of states, input alphabet, tape alphabet, transition function, start state, accept state, and reject state; or,

- **Implementation-level definition**: English prose that describes the Turing machine head movements relative to contents of tape, and conditions for accepting / rejecting based on those contents.

- **High-level description**: description of algorithm (precise sequence of instructions), without implementation details of machine. As part of this description, can "call" and run another TM as a subroutine.

The Church-Turing thesis posits that each algorithm can be implemented by some Turing machine

High-level descriptions of Turing machine algorithms are written as indented text within quotation marks.

Stages of the algorithm are typically numbered consecutively.

The first line specifies the input to the machine, which must be a string. This string may be the encoding of some object or list of objects.

Notation: $\langle O \rangle$ is the string that encodes the object O. $\langle O_1, \ldots, O_n \rangle$ is the string that encodes the list of objects $O_1, \ldots, O_n$.

Assumption: There are Turing machines that can be called as subroutines to decode the string representations of common objects and interact with these objects as intended (data structures).

For example, since there are algorithms to answer each of the following questions, by Church-Turing thesis, there is a Turing machine that accepts exactly those strings for which the answer to the question is "yes"

- Does a string over $\{0, 1\}$ have even length?

- Does a string over $\{0, 1\}$ encode a string of ASCII characters?[1]

- Does a DFA have a specific number of states?

- Do two NFAs have any state names in common?

- Do two CFGs have the same start variable?

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[1] An introduction to ASCII is available on the w3 tutorial here.

Review: Week 6 Friday

Please complete the review quiz questions on Gradescope about descriptions of Turing machines.

Pre class reading for next time: Page 184-185 Terminology for describing Turing machines