## Monday May 23

Recall definition: A is **mapping reducible to** B means there is a computable function  $f: \Sigma^* \to \Sigma^*$  such that for all strings x in  $\Sigma^*$ ,

$$x \in A$$
 if and only if  $f(x) \in B$ .

Notation: when A is mapping reducible to B, we write  $A \leq_m B$ .

**Theorem** (Sipser 5.23): If  $A \leq_m B$  and A is undecidable, then B is undecidable.

#### Halting problem

$$HALT_{TM} = \{ \langle M, w \rangle \mid M \text{ is a Turing machine, } w \text{ is a string, and } M \text{ halts on } w \}$$

We will define a computable function that witnesses the mapping reduction  $A_{TM} \leq_m HALT_{TM}$ .

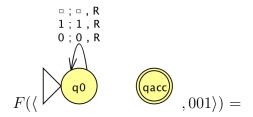
Using Theorem 5.23, we can then conclude that  $HALT_{TM}$  is undecidable.

Define  $F: \Sigma^* \to \Sigma^*$  by

$$F(x) = \begin{cases} const_{out} & \text{if } x \neq \langle M, w \rangle \text{ for any Turing machine } M \text{ and string } w \text{ over the alphabet of } M \\ \langle M', w \rangle & \text{if } x = \langle M, w \rangle \text{ for some Turing machine } M \text{ and string } w \text{ over the alphabet of } M. \end{cases}$$



where  $const_{out} = \langle V, \varepsilon \rangle$  and M' is a Turing machine that computes like M except, if the computation ever were to go to a reject state, M' loops instead.





To use this function to prove that  $A_{TM} \leq_m HALT_{TM}$ , we need two claims: Claim (1): F is computable Claim (2): for every  $x, x \in A_{TM}$  iff  $F(x) \in HALT_{TM}$ .

True or False:  $\overline{A_{TM}} \leq_m \overline{HALT_{TM}}$ 

True or False:  $HALT_{TM} \leq_m A_{TM}$ .

### Review: Week 9 Monday

Recall: Review quizzes based on class material are assigned each day. These quizzes will help you track and confirm your understanding of the concepts and examples we work in class. Quizzes can be submitted on Gradescope as many times (with no penalty) as you like until the quiz deadline: the three quizzes each week are all due on Friday (with no penalty late submission open until Sunday).

Please complete the review quiz questions on Gradescope about mapping reductions.

### Wednesday May 25

Recall: A is **mapping reducible to** B, written  $A \leq_m B$ , means there is a computable function  $f: \Sigma^* \to \Sigma^*$  such that for all strings x in  $\Sigma^*$ ,

 $x \in A$  if and only if  $f(x) \in B$ .

**Theorem** (Sipser 5.28): If  $A \leq_m B$  and B is recognizable, then A is recognizable.

**Proof**:

Corollary: If  $A \leq_m B$  and A is unrecognizable, then B is unrecognizable.

#### Strategy:

- (i) To prove that a recognizable language R is undecidable, prove that  $A_{TM} \leq_m R$ .
- (ii) To prove that a co-recognizable language U is undecidable, prove that  $\overline{A_{TM}} \leq_m U$ , i.e. that  $A_{TM} \leq_m \overline{U}$ .

| $E_{TM} = \{\langle$ | $\langle M \rangle$ | $\rangle \mid \Lambda$ | <i>l</i> is | a Tu | ring | machine | and | L(M) | $) = \emptyset$ | } |
|----------------------|---------------------|------------------------|-------------|------|------|---------|-----|------|-----------------|---|
|----------------------|---------------------|------------------------|-------------|------|------|---------|-----|------|-----------------|---|

| Example                | e string in $E_{TM}$ is |                            | Example string not in $E_{TM}$ is |
|------------------------|-------------------------|----------------------------|-----------------------------------|
| $E_{TM}$ is            | decidable / undecidable | and                        | recognizable / unrecognizable .   |
| $\overline{E_{TM}}$ is | decidable / undecidable | and                        | recognizable / unrecognizable .   |
| Claim:                 | <<                      | $_{n}$ $\overline{E_{TM}}$ | -<br>r.                           |

**Proof**: Need computable function  $F: \Sigma^* \to \Sigma^*$  such that  $x \in A_{TM}$  iff  $F(x) \notin E_{TM}$ . Define

F = "On input x,

- 1. Type-check whether  $x=\langle M,w\rangle$  for some TM M and string w. If so, move to step 2; if not, output
- 2. Construct the following machine  $M_x'$ :
- 3. Output  $\langle M_x' \rangle$ ."

Verifying correctness:

| Input string                                 | Output string |
|--|---------------|
| $\langle M, w \rangle$ where $w \in L(M)$    |               |
|  |               |
| $\langle M, w \rangle$ where $w \notin L(M)$ |               |
|  |               |
| x not encoding any pair of TM and string     |               |

|                         | $EQ_{TM} = \{ \langle M, M' \rangle \mid M$ | and and | M' are both Turing machines and $L(M) = L$ | $\mathcal{L}(M')$          |  |
|-------------------------|---|---------|--|----------------------------|--|
| Example s               | string in $EQ_{TM}$ is                      |         | Example string not in $EQ_{TM}$ is         |                            |  |
| $EQ_{TM}$ is            | decidable / undecidable                     | and     | recognizable / unrecognizable .            |                            |  |
| $\overline{EQ_{TM}}$ is | decidable / undecidable                     | and     | recognizable / unrecognizable .            |                            |  |
| To prove.               | show that                                   |         | $\leq_m EQ_{TM}$ and that                  | $<_m \overline{EQ_{TM}}$ . |  |

# Verifying correctness:

| Input string                              | Output string |
|---|---------------|
| $\langle M, w \rangle$ where M halts on w |               |
| $\langle M, w \rangle$ where M loops on w |               |
| x not encoding any pair of TM and string  |               |

# Review: Week 9 Wednesday

Please complete the review quiz questions on Gradescope about mapping reductions.

Pre class reading for next time: Introduction to Chapter 7.

### Friday May 27

In practice, computers (and Turing machines) don't have infinite tape, and we can't afford to wait unboundedly long for an answer. "Decidable" isn't good enough - we want "Efficiently decidable".

For a given algorithm working on a given input, how long do we need to wait for an answer? How does the running time depend on the input in the worst-case? average-case? We expect to have to spend more time on computations with larger inputs.

Definition (Sipser 7.1): For M a deterministic decider, its **running time** is the function  $f: \mathbb{N} \to \mathbb{N}$  given by

 $f(n) = \max$  number of steps M takes before halting, over all inputs of length n

Definition (Sipser 7.7): For each function t(n), the **time complexity class** TIME(t(n)), is defined by  $TIME(t(n)) = \{L \mid L \text{ is decidable by a Turing machine with running time in } O(t(n))\}$ 

An example of an element of TIME(1) is

An example of an element of TIME(n) is

Note:  $TIME(1) \subseteq TIME(n) \subseteq TIME(n^2)$ 

Definition (Sipser 7.12): P is the class of languages that are decidable in polynomial time on a deterministic 1-tape Turing machine

$$P = \bigcup_{k} TIME(n^k)$$

 $Compare\ to\ exponential\ time:\ brute-force\ search.$ 

Theorem (Sipser 7.8): Let t(n) be a function with  $t(n) \ge n$ . Then every t(n) time deterministic multitape Turing machine has an equivalent  $O(t^2(n))$  time deterministic 1-tape Turing machine.

Definition (Sipser 7.9): For N a nodeterministic decider. The **running time** of N is the function  $f : \mathbb{N} \to \mathbb{N}$  given by

 $f(n) = \max$  number of steps N takes on any branch before halting, over all inputs of length n

Definition (Sipser 7.21): For each function t(n), the **nondeterministic time complexity class** NTIME(t(n)), is defined by

 $NTIME(t(n)) = \{L \mid L \text{ is decidable by a nondeterministic Turing machine with running time in } O(t(n))\}$ 

$$NP = \bigcup_k NTIME(n^k)$$

True or False:  $TIME(n^2) \subseteq NTIME(n^2)$ 

True or False:  $NTIME(n^2) \subseteq DTIME(n^2)$ 

#### Examples in P

Can't use nondeterminism; Can use multiple tapes; Often need to be "more clever" than naïve / brute force approach

 $PATH = \{ \langle G, s, t \rangle \mid G \text{ is digraph with } n \text{ nodes there is path from s to t} \}$ 

Use breadth first search to show in P

 $RELPRIME = \{\langle x, y \rangle \mid x \text{ and } y \text{ are relatively prime integers} \}$ 

Use Euclidean Algorithm to show in P

$$L(G) = \{ w \mid w \text{ is generated by } G \}$$

(where G is a context-free grammar). Use dynamic programming to show in P.

#### Examples in NP

"Verifiable" i.e. NP, Can be decided by a nondeterministic TM in polynomial time, best known deterministic solution may be brute-force, solution can be verified by a deterministic TM in polynomial time.

 $HAMPATH = \{\langle G, s, t \rangle \mid G \text{ is digraph with } n \text{ nodes, there is path from } s \text{ to } t \text{ that goes through every node example of the example of the$ 

 $VERTEX-COVER=\{\langle G,k\rangle\mid G \text{ is an undirected graph with } n \text{ nodes that has a } k\text{-node vertex cover}\}$ 

 $CLIQUE = \{\langle G, k \rangle \mid G \text{ is an undirected graph with } n \text{ nodes that has a } k\text{-clique}\}$ 

 $SAT = \{\langle X \rangle \mid X \text{ is a satisfiable Boolean formula with } n \text{ variables} \}$ 

# Review: Week 9 Friday

Please complete the review quiz questions on Gradescope about TBD

Pre class reading for next time: Skim Chapter 7.

In observance of Memorial Day, there will be no lecture or discussion section on Monday.