

## Week2 monday

**Review:** Formal definition of DFA:  $M = (Q, \Sigma, \delta, q_0, F)$

- Finite set of states  $Q$
- Alphabet  $\Sigma$
- Transition function  $\delta$
- Start state  $q_0$
- Accept (final) states  $F$

In the state diagram of  $M$ , how many outgoing arrows are there from each state?

$M = (\{q, r, s\}, \{a, b\}, \delta, q, \{s\})$  where  $\delta$  is (rows labelled by states and columns labelled by symbols):

$\delta$	$a$	$b$
$q$	$r$	$q$
$r$	$r$	$s$
$s$	$s$	$s$

The state diagram for  $M$  is

Give two examples of strings that are accepted by  $M$  and two examples of strings that are rejected by  $M$ :

Add “labels” for states in the state diagram, e.g. “have not seen any of desired pattern yet” or “sink state”.

We can use the analysis of the roles of the states in the state diagram to describe the language recognized by the DFA.

$L(M) =$

A regular expression describing  $L(M)$  is

Let the alphabet be  $\Sigma_1 = \{0, 1\}$ .

A state diagram for a DFA that recognizes  $\{w \mid w \text{ contains at most two 1's}\}$  is

A state diagram for a DFA that recognizes  $\{w \mid w \text{ contains more than two 1's}\}$  is

*Extra example:* A state diagram for DFA recognizing

$$\{w \mid w \text{ is a string over } \{0,1\} \text{ whose length is not a multiple of } 3\}$$

Let  $n$  be an arbitrary positive integer. What is a formal definition for a DFA recognizing

$$\{w \mid w \text{ is a string over } \{0,1\} \text{ whose length is not a multiple of } n\}?$$

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Suppose  $A$  is a language over an alphabet  $\Sigma$ . By definition, this means  $A$  is a subset of  $\Sigma^*$ . **Claim:** if there is a DFA  $M$  such that  $L(M) = A$  then there is another DFA, let's call it  $M'$ , such that  $L(M') = \overline{A}$ , the complement of  $A$ , defined as  $\{w \in \Sigma^* \mid w \notin A\}$ .

**Proof idea:**

**Proof:**

A useful (optional) bit of terminology: the **iterated transition function** of a DFA  $M = (Q, \Sigma, \delta, q_0, F)$  is defined recursively by

$$\delta^*(q, w) = \begin{cases} q & \text{if } q \in Q, w = \varepsilon \\ \delta(q, a) & \text{if } q \in Q, w = a \in \Sigma \\ \delta(\delta^*(q, u), a) & \text{if } q \in Q, w = ua \text{ where } u \in \Sigma^* \text{ and } a \in \Sigma \end{cases}$$

Using this terminology,  $M$  accepts a string  $w$  over  $\Sigma$  if and only if  $\delta^*(q_0, w) \in F$ .

Fix  $\Sigma = \{a, b\}$ . A state diagram for a DFA that recognizes  $\{w \mid w \text{ has } ab \text{ as a substring and is of even length}\}$ :

Suppose  $A_1, A_2$  are languages over an alphabet  $\Sigma$ . **Claim:** if there is a DFA  $M_1$  such that  $L(M_1) = A_1$  and DFA  $M_2$  such that  $L(M_2) = A_2$ , then there is another DFA, let's call it  $M$ , such that  $L(M) = A_1 \cap A_2$ .

**Proof idea:**

**Formal construction:**

**Application:** When  $A_1 = \{w \mid w \text{ has } ab \text{ as a substring}\}$  and  $A_2 = \{w \mid w \text{ is of even length}\}$ .

Suppose  $A_1, A_2$  are languages over an alphabet  $\Sigma$ . **Claim:** if there is a DFA  $M_1$  such that  $L(M_1) = A_1$  and DFA  $M_2$  such that  $L(M_2) = A_2$ , then there is another DFA, let's call it  $M$ , such that  $L(M) = A_1 \cup A_2$ .  
*Sipser Theorem 1.25, page 45*

**Proof idea:**

**Formal construction:**

**Application:** A state diagram for a DFA that recognizes  $\{w \mid w \text{ has } ab \text{ as a substring or is of even length}\}$ :

## Week2 friday

<b>Nondeterministic finite automaton</b> $M = (Q, \Sigma, \delta, q_0, F)$	
Finite set of states $Q$	Can be labelled by any collection of distinct names. Default: $q_0, q_1, \dots$
Alphabet $\Sigma$	Each input to the automaton is a string over $\Sigma$ .
Arrow labels $\Sigma_\varepsilon$	$\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$ .
Transition function $\delta$	Arrows in the state diagram are labelled either by symbols from $\Sigma$ or by $\varepsilon$ $\delta : Q \times \Sigma_\varepsilon \rightarrow \mathcal{P}(Q)$ gives the <b>set of possible next states</b> for a transition from the current state upon reading a symbol or spontaneously moving.
Start state $q_0$	Element of $Q$ . Each computation of the machine starts at the start state.
Accept (final) states $F$	$F \subseteq Q$ .
$M$ accepts the input string	if and only if <b>there is</b> a computation of $M$ on the input string that processes the whole string and ends in an accept state.
Page 53	

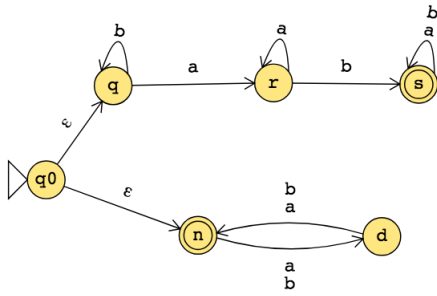
The formal definition of the NFA over  $\{0, 1\}$  given by this state diagram is:



The language over  $\{0, 1\}$  recognized by this NFA is:

Change the transition function to get a different NFA which accepts the empty string.

The state diagram of an NFA over  $\{a, b\}$  is below. The formal definition of this NFA is:



The language recognized by this NFA is:

# Week1 monday

We will use vocabulary that should be familiar from your discrete math and introduction to proofs classes. Some of the notation conventions may be a bit different: we will use the notation from this class' textbook<sup>1</sup>.

Write out in words the meaning of the symbols below:

$$\{a, b, c\}$$

$$|\{a, b, a\}| = 2$$

$$|aba| = 3$$

$$(a, 3, 2, b, b)$$

Term	Typical symbol	Meaning
Alphabet	$\Sigma, \Gamma$	A non-empty finite set
Symbol over $\Sigma$	$\sigma, b, x$	An element of the alphabet $\Sigma$
String over $\Sigma$	$u, v, w$	A finite list of symbols from $\Sigma$
The set of all strings over $\Sigma$	$\Sigma^*$	The collection of all possible strings formed from symbols from $\Sigma$
(Some) language over $\Sigma$	$L$	(Some) set of strings over $\Sigma$
Empty string	$\varepsilon$	The string of length 0
Empty set	$\emptyset$	The empty language
Natural numbers	$\mathcal{N}$	The set of positive integers
Finite set		The empty set or a set whose distinct elements can be counted by a natural number
Infinite set		A set that is not finite.
<i>Pages 3, 4, 13, 14</i>		

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<sup>1</sup>Page references are to the 3rd edition (International) of Sipser's Introduction to the Theory of Computation, available at the campus bookstore for under \$20. Copies of the book are also available for those who can't access the book to borrow from the course instructor, while supplies last (minnes@eng.ucsd.edu)



Term	Notation	Meaning
Reverse of a string $w$	$w^{\mathcal{R}}$	write $w$ in the opposite order, if $w = w_1 \cdots w_n$ then $w^{\mathcal{R}} = w_n \cdots w_1$ . Note: $\varepsilon^{\mathcal{R}} = \varepsilon$
Concatenating strings $x$ and $y$	$xy$	take $x = x_1 \cdots x_m$ , $y = y_1 \cdots y_n$ and form $xy = x_1 \cdots x_m y_1 \cdots y_n$
String $z$ is a substring of string $w$		there are strings $u, v$ such that $w = uzv$
String $x$ is a prefix of string $y$		there is a string $z$ such that $y = xz$
String $x$ is a proper prefix of string $y$		$x$ is a prefix of $y$ and $x \neq y$
Shortlex order, also known as string order over alphabet $\Sigma$		Order strings over $\Sigma$ first by length and then according to the dictionary order, assuming symbols in $\Sigma$ have an ordering.

Pages 13, 14

Circle the correct choice:

A **string** over an alphabet  $\Sigma$  is an element of  $\Sigma^*$  OR a subset of  $\Sigma^*$ .

A **language** over an alphabet  $\Sigma$  is an element of  $\Sigma^*$  OR a subset of  $\Sigma^*$ .

Extra examples for practice:

With  $\Sigma_1 = \{0, 1\}$  and  $\Sigma_2 = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$  and  $\Gamma = \{0, 1, x, y, z\}$

An example of a string of length 3 over  $\Sigma_1$  is \_\_\_\_\_

An example of a string of length 1 over  $\Sigma_2$  is \_\_\_\_\_

The number of distinct strings of length 2 over  $\Gamma$  is \_\_\_\_\_

An example of a language over  $\Sigma_1$  of size 1 is \_\_\_\_\_

An example of an infinite language over  $\Sigma_1$  is \_\_\_\_\_

An example of a finite language over  $\Gamma$  is \_\_\_\_\_

**True or False:**  $\varepsilon \in \Sigma_1$

**True or False:**  $\varepsilon$  is a string over  $\Sigma_1$

**True or False:**  $\varepsilon$  is a language over  $\Sigma_1$

**True or False:**  $\varepsilon$  is a prefix of some string over  $\Sigma_1$

**True or False:** There is a string over  $\Sigma_1$  that is a proper prefix of  $\varepsilon$

The first five strings over  $\Sigma_1$  in string order, using the ordering  $0 < 1$ :

The first five strings over  $\Sigma_2$  in string order, using the usual alphabetical ordering for single letters:

# Week1 wednesday

Our motivation in studying sets of strings is that they encode problems.

We need to describe the collection of all strings that match the pattern or property of a problem.

Let's start by thinking about how we can describe a language (a set of strings from a given alphabet).

**Definition 1.52:** A **regular expression** over alphabet  $\Sigma$  is a syntactic expression that can describe a language over  $\Sigma$ . The collection of all regular expressions is defined recursively:

*Basis steps of recursive definition*

$a$  is a regular expression, for  $a \in \Sigma$

$\varepsilon$  is a regular expression

$\emptyset$  is a regular expression

*Recursive steps of recursive definition*

$(R_1 \cup R_2)$  is a regular expression when  $R_1, R_2$  are regular expressions

$(R_1 \circ R_2)$  is a regular expression when  $R_1, R_2$  are regular expressions

$(R_1^*)$  is a regular expression when  $R_1$  is a regular expression

The *semantics* (or meaning) of the syntactic regular expression is the **language described by the regular expression**. The function that assigns a language to a regular expression over  $\Sigma$  is defined recursively, using familiar set operations:

*Basis steps of recursive definition*

The language described by  $a$ , for  $a \in \Sigma$ , is  $\{a\}$  and we write  $L(a) = \{a\}$

The language described by  $\varepsilon$  is  $\{\varepsilon\}$  and we write  $L(\varepsilon) = \{\varepsilon\}$

The language described by  $\emptyset$  is  $\{\}$  and we write  $L(\emptyset) = \emptyset$ .

*Recursive steps of recursive definition*

When  $R_1, R_2$  are regular expressions, the language described by the regular expression  $(R_1 \cup R_2)$  is the union of the languages described by  $R_1$  and  $R_2$ , and we write

$$L( (R_1 \cup R_2) ) = L(R_1) \cup L(R_2) = \{w \mid w \in L(R_1) \vee w \in L(R_2)\}$$

When  $R_1, R_2$  are regular expressions, the language described by the regular expression  $(R_1 \circ R_2)$  is the concatenation of the languages described by  $R_1$  and  $R_2$ , and we write

$$L( (R_1 \circ R_2) ) = L(R_1) \circ L(R_2) = \{uv \mid u \in L(R_1) \wedge v \in L(R_2)\}$$

When  $R_1$  is a regular expression, the language described by the regular expression  $(R_1^*)$  is the **Kleene star** of the language described by  $R_1$  and we write

$$L( (R_1^*) ) = ( L(R_1) )^* = \{w_1 \cdots w_k \mid k \geq 0 \text{ and each } w_i \in L(R_1)\}$$

For the following examples assume the alphabet is  $\Sigma_1 = \{0, 1\}$ :

The language described by the regular expression 0 is  $L(0) = \{0\}$

The language described by the regular expression 1 is  $L(1) = \{1\}$

The language described by the regular expression  $\varepsilon$  is  $L(\varepsilon) = \{\varepsilon\}$

The language described by the regular expression  $\emptyset$  is  $L(\emptyset) = \emptyset$

The language described by the regular expression  $((0 \cup 1) \cup 1)$  is  $L((0 \cup 1) \cup 1) =$

The language described by the regular expression  $1^+$  is  $L(1^+) =$

The language described by the regular expression  $\Sigma_1^*1$  is  $L(\Sigma_1^*1) =$

The language described by the regular expression  $(\Sigma_1\Sigma_1\Sigma_1\Sigma_1\Sigma_1)^*$  is  $L((\Sigma_1\Sigma_1\Sigma_1\Sigma_1\Sigma_1)^*) =$

A regular expression that describes the language  $\{00, 01, 10, 11\}$  is

A regular expression that describes the language  $\{0^n1 \mid n \text{ is even}\}$  is

### *Shorthand and conventions*

Assuming $\Sigma$ is the alphabet, we use the following conventions	
$\Sigma$	regular expression describing language consisting of all strings of length 1 over $\Sigma$
$*$ then $\circ$ then $\cup$	precedence order, unless parentheses are used to change it
$R_1R_2$	shorthand for $R_1 \circ R_2$ (concatenation symbol is implicit)
$R^+$	shorthand for $R^* \circ R$
$R^k$	shorthand for $R$ concatenated with itself $k$ times, where $k$ is a natural number
Pages 63 - 65	

**Caution:** many programming languages that support regular expressions build in functionality that is more powerful than the “pure” definition of regular expressions given here.

Regular expressions are everywhere (once you start looking for them).

Software tools and languages often have built-in support for regular expressions to describe **patterns** that we want to match (e.g. Excel/ Sheets, grep, Perl, python, Java, Ruby).

Under the hood, the first phase of **compilers** is to transform the strings we write in code to tokens (keywords, operators, identifiers, literals). Compilers use regular expressions to describe the sets of strings that can be used for each token type.

Next time: we’ll start to see how to build machines that decide whether strings match the pattern described by a regular expression.

*Extra examples for practice:*

Which regular expression(s) below describe a language that includes the string  $a$  as an element?

$a^*b^*$

$a(ba)^*b$

$a^* \cup b^*$

$(aaa)^*$

$(\varepsilon \cup a)b$

## Week1 friday

**Review:** Determine whether each statement below about regular expressions over the alphabet  $\{a, b, c\}$  is true or false:

True or False:  $a \in L((a \cup b) \cup c)$

True or False:  $ab \in L((a \cup b)^*)$

True or False:  $ba \in L(a^*b^*)$

True or False:  $\varepsilon \in L(a \cup b \cup c)$

True or False:  $\varepsilon \in L((a \cup b)^*)$

True or False:  $\varepsilon \in L(a^*b^*)$

**From the pre-class reading, pages 34-36:** A deterministic finite automaton (DFA) is specified by  $M = (Q, \Sigma, \delta, q_0, F)$ . This 5-tuple is called the **formal definition** of the DFA. The DFA can also be represented by its state diagram: with nodes for the state, labelled edges specifying the transition function, and decorations on nodes denoting the start and accept states.

Finite set of states  $Q$  can be labelled by any collection of distinct names. Often we use default state labels  $q_0, q_1, \dots$

The alphabet  $\Sigma$  determines the possible inputs to the automaton. Each input to the automaton is a string over  $\Sigma$ , and the automaton “processes” the input one symbol (or character) at a time.

The transition function  $\delta$  gives the next state of the DFA based on the current state of the machine and on the next input symbol.

The start state  $q_0$  is an element of  $Q$ . Each computation of the machine starts at the start state.

The accept (final) states  $F$  form a subset of the states of the DFA,  $F \subseteq Q$ . These states are used to flag if the machine accepts or rejects an input string.

The computation of a machine on an input string is a sequence of states in the machine, starting with the start state, determined by transitions of the machine as it reads successive input symbols.

The DFA  $M$  accepts the given input string exactly when the computation of  $M$  on the input string ends in an accept state.  $M$  rejects the given input string exactly when the computation of  $M$  on the input string ends in a nonaccept state, that is, a state that is not in  $F$ .

The language of  $M$ ,  $L(M)$ , is defined as the set of all strings that are each accepted by the machine  $M$ . Each string that is rejected by  $M$  is not in  $L(M)$ . The language of  $M$  is also called the language recognized by  $M$ .

What is **finite** about all deterministic finite automata? (Select all that apply)

- ☐ The size of the machine (number of states, number of arrows)
- ☐ The number of strings that are accepted by the machine
- ☐ The length of each computation of the machine



The formal definition of this DFA is

Classify each string  $a, aa, ab, ba, bb, \varepsilon$  as accepted by the DFA or rejected by the DFA.

*Why are these the only two options?*

The language recognized by this DFA is



The language recognized by this DFA is



The language recognized by this DFA is