# Monday

Recap so far: In DFA, the only memory available is in the states. Automata can only "remember" finitely far in the past and finitely much information, because they can have only finitely many states. If a computation path of a DFA visits the same state more than once, the machine can't tell the difference between the first time and future times it visits this state. Thus, if a DFA accepts one long string, then it must accept (infinitely) many similar strings.

**Definition** A positive integer $p$ is a **pumping length** of a language $L$ over $\Sigma$ means that, for each string $s \in \Sigma^*$, if $|s| \geq p$ and $s \in L$, then there are strings $x, y, z$ such that

$$s = xyz$$

and

$$|y| > 0, \qquad \text{for each } i \geq 0,\ xy^iz \in L, \qquad \text{and} \qquad |xy| \leq p.$$

**Negation**: A positive integer $p$ is **not a pumping length** of a language $L$ over $\Sigma$ iff

$$\exists s \left( \ |s| \geq p \wedge s \in L \wedge \forall x \forall y \forall z \left( \ (s = xyz \wedge |y| > 0 \wedge |xy| \leq p \ ) \rightarrow \exists i (i \geq 0 \wedge xy^iz \notin L) \right) \ \right)$$

*Informally:*

Restating **Pumping Lemma**: If $L$ is a regular language, then it has a pumping length.

**Contrapositive**: If $L$ has no pumping length, then it is nonregular.

The Pumping Lemma *cannot* be used to prove that a language *is* regular.

The Pumping Lemma **can** be used to prove that a language *is not* regular.

*Extra practice*: Exercise 1.49 in the book.

**Proof strategy**: To prove that a language $L$ is **not** regular,

- Consider an arbitrary positive integer $p$
- Prove that $p$ is not a pumping length for $L$
- Conclude that $L$ does not have *any* pumping length, and therefore it is not regular.

**Example**: $\Sigma = \{0, 1\}$, $L = \{0^n 1^n \mid n \geq 0\}$.

Fix $p$ an arbitrary positive integer. List strings that are in $L$ and have length greater than or equal to $p$:

Pick $s =$

Suppose $s = xyz$ with $|xy| \leq p$ and $|y| > 0$.

Then when $i =$ , $xy^i z =$

**Example**: $\Sigma = \{0, 1\}$, $L = \{ww^{\mathcal{R}} \mid w \in \{0, 1\}^*\}$.

Fix $p$ an arbitrary positive integer. List strings that are in $L$ and have length greater than or equal to $p$:

Pick $s =$

Suppose $s = xyz$ with $|xy| \leq p$ and $|y| > 0$.

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**Example**: $\Sigma = \{0, 1\}$, $L = \{0^j1^k \mid j \geq k \geq 0\}$.

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Suppose $s = xyz$ with $|xy| \leq p$ and $|y| > 0$.

Then when $i =$ , $xy^iz =$

**Example**: $\Sigma = \{0, 1\}$, $L = \{0^n1^m0^n \mid m, n \geq 0\}$.

Fix $p$ an arbitrary positive integer. List strings that are in $L$ and have length greater than or equal to $p$:

Pick $s =$

Suppose $s = xyz$ with $|xy| \leq p$ and $|y| > 0$.

Then when $i =$ , $xy^iz =$

*Extra practice*:

| Language | $s \in L$ | $s \notin L$ | Is the language regular or nonregular? |
|---|---|---|---|
| $\{a^n b^n \mid 0 \leq n \leq 5\}$ | | | |
| $\{b^n a^n \mid n \geq 2\}$ | | | |
| $\{a^m b^n \mid 0 \leq m \leq n\}$ | | | |
| $\{a^m b^n \mid m \geq n+3, n \geq 0\}$ | | | |
| $\{b^m a^n \mid m \geq 1, n \geq 3\}$ | | | |
| $\{w \in \{a,b\}^* \mid w = w^{\mathcal{R}}\}$ | | | |
| $\{ww^{\mathcal{R}} \mid w \in \{a,b\}^*\}$ | | | |

## Review: Week 4 Monday

Recall: Review quizzes based on class material are assigned each day. These quizzes will help you track and confirm your understanding of the concepts and examples we work in class. Quizzes can be submitted on Gradescope as many times (with no penalty) as you like until the quiz deadline: the three quizzes each week are all due on Friday (with no penalty late submission open until Sunday).

Please complete the review quiz questions on Gradescope about pumping lemma and nonregular sets.

**Pre class reading for next time**: Page 112

## Wednesday

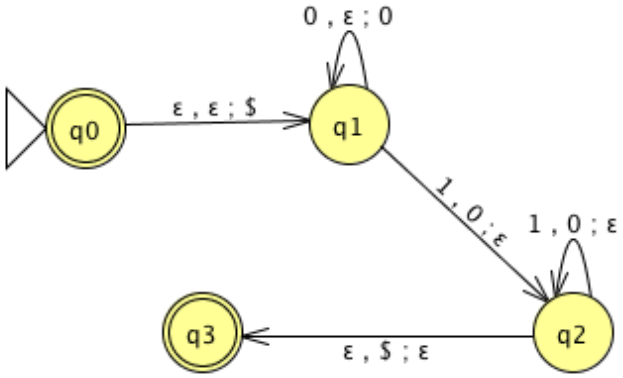
Regular sets are not the end of the story

- Many nice / simple / important sets are not regular
- Limitation of the finite-state automaton model: Can't "count", Can only remember finitely far into the past, Can't backtrack, Must make decisions in "real-time"
- We know actual computers are more powerful than this model...

The **next** model of computation. Idea: allow some memory of unbounded size. How?

- To generalize regular expressions: **context-free grammars**
- To generalize NFA: **Pushdown automata**, which is like an NFA with access to a stack: Number of states is fixed, number of entries in stack is unbounded. At each step (1) Transition to new state based on current state, letter read, and top letter of stack, then (2) (Possibly) push or pop a letter to (or from) top of stack. Accept a string iff there is some sequence of states and some sequence of stack contents which helps the PDA processes the entire input string and ends in an accepting state.

Is there a PDA that recognizes the nonregular language $\{0^n 1^n \mid n \geq 0\}$?




The PDA with state diagram above can be informally described as:

> Read symbols from the input. As each 0 is read, push it onto the stack. As soon as 1s are seen, pop a 0 off the stack for each 1 read. If the stack becomes empty and we are at the end of the input string, accept the input. If the stack becomes empty and there are 1s left to read, or if 1s are finished while the stack still contains 0s, or if any 0s appear in the string following 1s, reject the input.

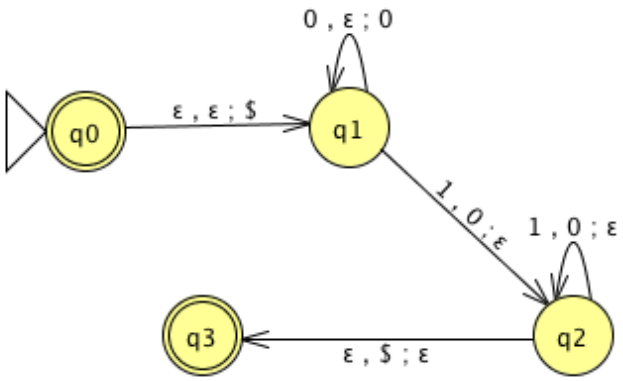
Trace the computation of this PDA on the input string 01.

Trace the computation of this PDA on the input string 011.

A PDA recognizing the set { } can be informally described as:

> Read symbols from the input. As each 0 is read, push it onto the stack. As soon as 1s are seen, pop a 0 off the stack for each 1 read. If the stack becomes empty and there is exactly one 1 left to read, read that 1 and accept the input. If the stack becomes empty and there are either zero or more than one 1s left to read, or if the 1s are finished while the stack still contains 0s, or if any 0s appear in the input following 1s, reject the input.

Modify the state diagram below to get a PDA that implements this description:



**Definition** A **pushdown automaton** (PDA) is specified by a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where $Q$ is the finite set of states, $\Sigma$ is the input alphabet, $\Gamma$ is the stack alphabet,

$$\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \to \mathcal{P}(Q \times \Gamma_\varepsilon)$$

is the transition function, $q_0 \in Q$ is the start state, $F \subseteq Q$ is the set of accept states.

Draw the state diagram and give the formal definition of a PDA with $\Sigma = \Gamma$.

Draw the state diagram and give the formal definition of a PDA with $\Sigma \cap \Gamma = \emptyset$.

*Extra practice*: Consider the state diagram of a PDA with input alphabet $\Sigma$ and stack alphabet $\Gamma$.

| Label | means |
|---|---|
| $a, b; c$ when $a \in \Sigma$, $b \in \Gamma$, $c \in \Gamma$ | |
| $a, \varepsilon; c$ when $a \in \Sigma$, $c \in \Gamma$ | |
| $a, b; \varepsilon$ when $a \in \Sigma$, $b \in \Gamma$ | |
| $a, \varepsilon; \varepsilon$ when $a \in \Sigma$ | |

How does the meaning change if $a$ is replaced by $\varepsilon$?

*Note: alternate notation is to replace* ; *with* $\rightarrow$

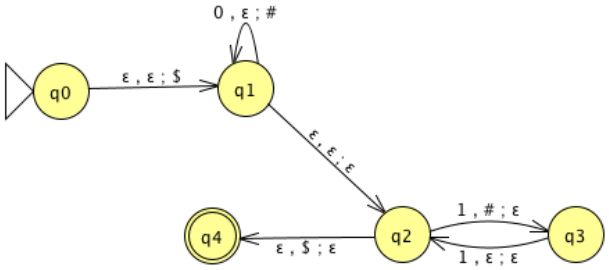
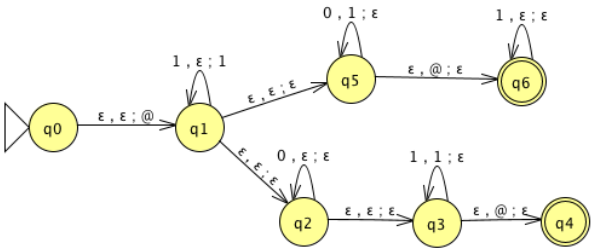
## Review: Week 4 Wednesday

Please complete the review quiz questions on Gradescope about PDA definitions.

**Pre class reading for next time**: Page 102

## Friday

For the PDA state diagrams below, $\Sigma = \{0, 1\}$.

| Mathematical description of language | State diagram of PDA recognizing language |
|---|---|
| | $\Gamma = \{\$, \#\}$   |
| | $\Gamma = \{@, 1\}$   |

$\{0^i 1^j 0^k \mid i, j, k \geq 0\}$

*Big picture*: PDAs were motivated by wanting to add some memory of unbounded size to NFA. How do we accomplish a similar enhancement of regular expressions to get a syntactic model that is more expressive?

DFA, NFA, PDA: Machines process one input string at a time; the computation of a machine on its input string reads the input from left to right.

Regular expressions: Syntactic descriptions of all strings that match a particular pattern; the language described by a regular expression is built up recursively according to the expression's syntax

**Context-free grammars**: Rules to produce one string at a time, adding characters from the middle, beginning, or end of the final string as the derivation proceeds.

| Term | Typical symbol | Definition |
|---|---|---|
| **Context-free grammar** (CFG) | $G$ | $G = (V, \Sigma, R, S)$ |
| **Variables** | $V$ | Finite set of symbols that represent phases in production pattern |
| **Terminals** | $\Sigma$ | Alphabet of symbols of strings generated by CFG<br>$V \cap \Sigma = \emptyset$ |
| **Rules** | $R$ | Each rule is $A \to u$ with $A \in V$ and $u \in (V \cup \Sigma)^*$ |
| Start variable | $S$ | Usually on LHS of first / topmost rule |
| **Derivation** | | Sequence of substitutions in a CFG |
| | $S \implies \cdots \implies w$ | Start with start variable, apply one rule to one occurrence of a variable at a time |
| **Language** generated by the CFG $G$ | $L(G)$ | $\{w \in \Sigma^* \mid \text{there is derivation in } G \text{ that ends in } w\} = \{w \in \Sigma^* \mid S \implies {}^*w\}$ |
| **Context-free language** | | A language that is the language generated by some CFG |
| Sipser pages 102-103 | | |

**Examples of context-free grammars, derivations in those grammars, and the languages generated by those grammars**

$G_1 = (\{S\}, \{0\}, R, S)$ with rules

$$S \to 0S$$
$$S \to 0$$

In $L(G_1)$ ...

Not in $L(G_1)$ ...

$G_2 = (\{S\}, \{0, 1\}, R, S)$

$$S \to 0S \mid 1S \mid \varepsilon$$

In $L(G_2)$ ...

Not in $L(G_2)$ ...

$(\{S, T\}, \{0, 1\}, R, S)$ with rules

$$S \to T1T1T1T$$
$$T \to 0T \mid 1T \mid \varepsilon$$

In $L(G_3)$ ...

Not in $L(G_3)$ ...

$G_4 = (\{A, B\}, \{0, 1\}, R, A)$ with rules

$$A \to 0A0 \mid 0A1 \mid 1A0 \mid 1A1 \mid 1$$

In $L(G_4)$ ...

Not in $L(G_4)$ ...

*Extra practice*: Is there a CFG $G$ with $L(G) = \emptyset$?

## Review: Week 4 Friday

Please complete the review quiz questions on Gradescope about PDA construction.