# Monday

| Term                      | Typical symbol                 | Definition  |
|---------------------------|--------------------------------|---|
| Context-free grammar      | G                              | $G = (V, \Sigma, R, S)$   |
| (CFG)                     |                                |   |
| Variables                 | V                              | Finite set of symbols that represent phases in production                             |
|                           |                                | pattern   |
| Terminals                 | $\Sigma$                       | Alphabet of symbols of strings generated by CFG                                       |
|                           |                                | $V \cap \Sigma = \emptyset$   |
| Rules                     | R                              | Each rule is $A \to u$ with $A \in V$ and $u \in (V \cup \Sigma)^*$                   |
| Start variable            | S                              | Usually on LHS of first / topmost rule  |
| Derivation                |                                | Sequence of substitutions in a CFG  |
|                           | $S \implies \cdots \implies w$ | Start with start variable, apply one rule to one occurrence                           |
|                           |                                | of a variable at a time   |
| Language generated by the | L(G)                           | $\{w \in \Sigma^* \mid \text{ there is derivation in } G \text{ that ends in } w\} =$ |
| CFG G                     |                                | $\{w \in \Sigma^* \mid S \implies {}^*w\}$  |
| Context-free language     |                                | A language that is the language generated by some CFG                                 |
| Sipser pages 102-103      |                                |   |

Examples of context-free grammars, derivations in those grammars, and the languages generated by those grammars

$$G_1 = (\{S\}, \{0\}, R, S)$$
 with rules

$$S \to 0S$$
$$S \to 0$$

$$S$$
 -

In 
$$L(G_1)$$
 ...

Not in  $L(G_1)$  ...



 $S \to 0S \mid 1S \mid \varepsilon$ 

In  $L(G_2)$  ...

Not in  $L(G_2)$  ...

 $(\{S,T\},\{0,1\},R,S)$  with rules

$$\begin{split} S &\to T1T1T1T \\ T &\to 0T \mid 1T \mid \varepsilon \end{split}$$

In  $L(G_3)$  ...

Not in  $L(G_3)$  ...

$$G_4 = (\{A, B\}, \{0, 1\}, R, A)$$
 with rules

$$A \rightarrow 0A0 \mid 0A1 \mid 1A0 \mid 1A1 \mid 1$$

In  $L(G_4)$  ...

Not in  $L(G_4)$  ...

Extra practice: Is there a CFG G with  $L(G) = \emptyset$ ?

Three different CFGs that each generate the language  $\{abba\}$ 

$$(\{S,T,V,W\},\{a,b\},\{S\rightarrow aT,T\rightarrow bV,V\rightarrow bW,W\rightarrow a\},S)$$

$$(\{Q\},\{a,b\},\{Q\rightarrow abba\},Q)$$

$$(\{X,Y\},\{a,b\},\{X\to aYa,Y\to bb\},X)$$

Design a CFG to generate the language  $\{a^nb^n\mid n\geq 0\}$ 

| Sample derivation:   |  |  |  |  |  |  |  |  |
|--|--|--|--|--|--|--|--|--|
| Design a CFG to generate the language $\{a^ib^j\mid j\geq i\geq 0\}$ |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
| Sample derivation:   |  |  |  |  |  |  |  |  |
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**Theorem 2.20**: A language is generated by some context-free grammar if and only if it is recognized by some push-down automaton.

Definition: a language is called **context-free** if it is the language generated by a context-free grammar. The class of all context-free language over a given alphabet  $\Sigma$  is called **CFL**.

#### Consequences:

- Quick proof that every regular language is context free
- To prove closure of the class of context-free languages under a given operation, we can choose either of two modes of proof (via CFGs or PDAs) depending on which is easier
- To fully specify a PDA we could give its 6-tuple formal definition or we could give its input alphabet, stack alphabet, and state diagram. An informal description of a PDA is a step-by-step description of how its computations would process input strings; the reader should be able to reconstruct the state diagram or formal definition precisely from such a descripton. The informal description of a PDA can refer to some common modules or subroutines that are computable by PDAs:
  - PDAs can "test for emptiness of stack" without providing details. *How?* We can always push a special end-of-stack symbol, \$, at the start, before processing any input, and then use this symbol as a flag.
  - PDAs can "test for end of input" without providing details. How? We can transform a PDA to one where accepting states are only those reachable when there are no more input symbols.

Over  $\Sigma = \{a, b\}$ , let  $L = \{a^n b^m \mid n \neq m\}$ . Goal: Prove L is context-free.

Suppose  $L_1$  and  $L_2$  are context-free languages over  $\Sigma$ . Goal:  $L_1 \cup L_2$  is also context-free.

Approach 1: with PDAs

Let  $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, F_1)$  and  $M_2 = (Q_2, \Sigma, \Gamma_2, \delta_2, q_2, F_2)$  be PDAs with  $L(M_1) = L_1$  and  $L(M_2) = L_2$ .

Define M =

 $Approach\ 2:\ with\ CFGs$ 

Let  $G_1 = (V_1, \Sigma, R_1, S_1)$  and  $G_2 = (V_2, \Sigma, R_2, S_2)$  be CFGs with  $L(G_1) = L_1$  and  $L(G_2) = L_2$ .

Define G =

Suppose  $L_1$  and  $L_2$  are context-free languages over  $\Sigma$ . Goal:  $L_1 \circ L_2$  is also context-free.

Approach 1: with PDAs

Let  $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, F_1)$  and  $M_2 = (Q_2, \Sigma, \Gamma_2, \delta_2, q_2, F_2)$  be PDAs with  $L(M_1) = L_1$  and  $L(M_2) = L_2$ .

Define M =

 $Approach\ 2:\ with\ CFGs$ 

Let  $G_1 = (V_1, \Sigma, R_1, S_1)$  and  $G_2 = (V_2, \Sigma, R_2, S_2)$  be CFGs with  $L(G_1) = L_1$  and  $L(G_2) = L_2$ .

Define G =

#### Summary

Over a fixed alphabet  $\Sigma$ , a language L is **regular** 

iff it is described by some regular expression iff it is recognized by some DFA iff it is recognized by some NFA

Over a fixed alphabet  $\Sigma$ , a language L is **context-free** 

iff it is generated by some CFG iff it is recognized by some PDA

**Fact**: Every regular language is a context-free language.

Fact: There are context-free languages that are not nonregular.

**Fact**: There are countably many regular languages.

Fact: There are countably inifnitely many context-free languages.

Consequence: Most languages are **not** context-free!

#### Examples of non-context-free languages

$$\begin{aligned} &\{a^nb^nc^n\mid 0\leq n, n\in\mathbb{Z}\}\\ &\{a^ib^jc^k\mid 0\leq i\leq j\leq k, i\in\mathbb{Z}, j\in\mathbb{Z}, k\in\mathbb{Z}\}\\ &\{ww\mid w\in\{0,1\}^*\} \end{aligned}$$

(Sipser Ex 2.36, Ex 2.37, 2.38)

There is a Pumping Lemma for CFL that can be used to prove a specific language is non-context-free: If A is a context-free language, there there is a number p where, if s is any string in A of length at least p, then s may be divided into five pieces s = uvxyz where (1) for each  $i \geq 0$ ,  $uv^ixy^iz \in A$ , (2) |uv| > 0, (3)  $|vxy| \leq p$ . We will not go into the details of the proof or application of Pumping Lemma for CFLs this quarter.

### Review: Week 5 Monday

Recall: Review quizzes based on class material are assigned each day. These quizzes will help you track and confirm your understanding of the concepts and examples we work in class. Quizzes can be submitted on Gradescope as many times (with no penalty) as you like until the quiz deadline: the three quizzes each week are all due on Friday (with no penalty late submission open until Sunday).

Please complete the review quiz questions on Gradescope about context-free grammars

Pre class reading for next time: Figure 3.1 (Pages 165-167)

## Wednesday

A set X is said to be **closed** under an operation OP if, for any elements in X, applying OP to them gives an element in X.

| True/False | Closure claim  |
|------------|--|
| True       | The set of integers is closed under multiplication.  |
|            | $\forall x \forall y  (\ (x \in \mathbb{Z} \land y \in \mathbb{Z}) \to xy \in \mathbb{Z}\ )$   |
| True       | For each set $A$ , the power set of $A$ is closed under intersection.  |
|            | $\forall A_1 \forall A_2 ( (A_1 \in \mathcal{P}(A) \land A_2 \in \mathcal{P}(A) \in \mathbb{Z}) \to A_1 \cap A_2 \in \mathcal{P}(A) )$ |
|            | The class of regular languages over $\Sigma$ is closed under complementation.  |
|            | The class of regular languages over $\Sigma$ is closed under union.  |
|            | The class of regular languages over $\Sigma$ is closed under intersection.   |
|            | The class of regular languages over $\Sigma$ is closed under concatenation.  |
|            | The class of regular languages over $\Sigma$ is closed under Kleene star.  |
|            | The class of context-free languages over $\Sigma$ is closed under complementation.   |
|            | The class of context-free languages over $\Sigma$ is closed under union.   |
|            | The class of context-free languages over $\Sigma$ is closed under intersection.  |
|            | The class of context-free languages over $\Sigma$ is closed under concatenation.   |
|            | The class of context-free languages over $\Sigma$ is closed under Kleene star.   |

Assume  $\Sigma = \{0, 1, \#\}$ 

**Turing machines**: unlimited read + write memory, unlimited time (computation can proceed without "consuming" input and can re-read symbols of input)

- Division between program (CPU, state diagram) and data
- Unbounded memory gives theoretical limit to what modern computation (including PCs, supercomputers, quantum computers) can achieve
- State diagram formulation is simple enough to reason about (and diagonalize against) while expressive enough to capture modern computation

For Turing machine  $M=(Q,\Sigma,\Gamma,\delta,q_0,q_{accept},q_{reject})$  the **computation** of M on a string w over  $\Sigma$  is:

- Read/write head starts at leftmost position on tape.
- Input string is written on |w|-many leftmost cells of tape, rest of the tape cells have the blank symbol. **Tape alphabet** is  $\Gamma$  with  $\bot \in \Gamma$  and  $\Sigma \subseteq \Gamma$ . The blank symbol  $\bot \notin \Sigma$ .
- Given current state of machine and current symbol being read at the tape head, the machine transitions to next state, writes a symbol to the current position of the tape head (overwriting existing symbol), and moves the tape head L or R (if possible). Formally, **transition function** is

$$\delta: Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$$

• Computation ends if and when machine enters either the accept or the reject state. This is called halting. Note:  $q_{accept} \neq q_{reject}$ .

The language recognized by the Turing machine M, is

 $\{w \in \Sigma^* \mid \text{computation of } M \text{ on } w \text{ halts after entering the accept state}\} = \{w \in \Sigma^* \mid w \text{ is accepted by } M\}$ 

An example Turing machine:  $\Sigma =$ 

$$,\Gamma =$$

$$\delta((q0,0)) =$$



Formal definition:

Sample computation:

| $q0\downarrow$ |   |   |   |   |   |   |
|----------------|---|---|---|---|---|---|
| 0              | 0 | 0 | J | J | J | J |
|                |   |   |   |   |   |   |
|                |   |   |   |   |   |   |
|                |   |   |   |   |   |   |
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|                |   |   |   |   |   |   |

The language recognized by this machine is ...

Extra practice:





Formal definition:

Sample computation:

| Review: Week 5 Wednesday  |    |
|---|----|
| Please complete the review quiz questions on Gradescope about context-free languages and Turing machine | s. |
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Friday: Midterm exam

The midterm exam policies are below