1 Eigenvectors and Eigenvalues

Given a matrix A, an eigenvector of A is a non-zero vector v such that $Av = \lambda v$ for some scalar λ, called the eigenvalue corresponding to v. The eigenvectors and eigenvalues of a matrix provide important information about its behavior under transformation.

Consider the matrix $A = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$. To find the eigenvectors and eigenvalues of A, we can use the following steps:

1. Calculate the characteristic polynomial of the matrix. The characteristic polynomial is given by the formula:

   $$\det(A - \lambda I) = 0$$

   where I is the identity matrix.

   For the given matrix A, the characteristic polynomial is:

   $$\det\left(\begin{pmatrix} 3-\lambda & -1 \\ 2 & 0-\lambda \end{pmatrix}\right) = (3-\lambda)(0-\lambda) - 2(-1) = \lambda^2 - 3\lambda + 2 = 0$$

2. Solve the equation $\lambda^2 - 3\lambda + 2 = 0$ to find the eigenvalues. This equation can be factored as $(\lambda - 1)(\lambda - 2) = 0$, so the eigenvalues are $\lambda = 1$ and $\lambda = 3$.

3. For each eigenvalue, solve the equation $(A - \lambda I)x = 0$ to find the corresponding eigenvector.

   For $\lambda = 1$, the equation is $\begin{pmatrix} 3-1 & -1 \\ 2 & 0-1 \end{pmatrix} x = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} x = 0$. This equation has a non-trivial solution when the determinant of the matrix is zero, which occurs when $2(-1) - (-1) = 0$. Thus, the eigenvector corresponding to $\lambda = 1$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

   For $\lambda = 2$, the equation is $\begin{pmatrix} 3-2 & -1 \\ 2 & 0-2 \end{pmatrix} x = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} x = 0$. This equation has a non-trivial solution when the determinant of the matrix is zero, which occurs when $1(-2) - (-1)2 = 0$. Thus, the eigenvector corresponding to $\lambda = 2$ is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Therefore, the eigenvectors and eigenvalues of the matrix $A = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$ are:

- Eigenvalues: $\lambda = 1, \lambda = 2$

- Eigenvectors: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

It is worth noting that the eigenvectors of a matrix are not unique, since they can be multiplied by any non-zero scalar and still satisfy the equation $Av = \lambda v$. However, the eigenvalues of a matrix are unique.

In addition to providing insight into the behavior of a matrix under transformation, eigenvectors and eigenvalues also have applications in other areas of mathematics, such as differential equations and image compression.

2 Diagonalization

A matrix A is diagonalizable if and only if it has n linearly independent eigenvectors, where n is the size of the matrix. In other words, a matrix is diagonalizable if it is possible to find a basis consisting of its eigenvectors.

We have already determined that the matrix $A = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$ has two linearly independent eigenvectors: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Therefore, A is diagonalizable.

To find the matrix P that diagonalizes A, we can use the eigenvectors of A as the columns of P. Specifically, we can define P as follows:

$$P = \begin{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

Then, the diagonalized form of A can be found by computing $P^{-1}AP$.

$$P^{-1}AP = \left(\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \right)^{-1} \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

Thus, the results of the diagonalization of A are:

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

where D is the diagonal matrix containing the eigenvalues of A on the diagonal.