Modelling a way of cleaning Space Debris

Abstract

Whether it is natural or man-made, there has been a large increase of space debris over the years, that presents a detrimental threat to current satellites in space if collisions occur. The aim of this report is to find a method for cleaning this debris to prevent this from happening. To do this, we first investigate the impact of space debris and how long it takes to fall back to Earth by how it behaves in the atmosphere. We will then look at a method of cleaning debris that manoeuvres a spacecraft using force thrusters to capture the debris. We will model this by using Newtonian Mechanics to calculate the best distance, time and the minimum fuel used, which will help us find the best strategy for cleaning debris.

1 Introduction

A great deal of space debris has been built up over the years, whether it is natural debris from meteors or man-made debris. For example, as space missions have increased so has the formation of space debris from previous spacecrafts and broken satellites. Since 1957, there have been around 6170 successful rocket launches and to date, it is estimated that there have been more than 640 explosions or collisions resulting in debris. The debris can be micrometers to centimetres long, with 36500 objects having been recorded as being larger than 10cm[1]. Therefore, there is currently a large amount of space junk orbiting the Earth at a high speed which poses a threat of collision and damage to working satellites and any future launches into space. For example, in November 2021, the International Space Station (ISS) monitored passing a debris field every 90 minutes[2]. As a result, those aboard had to take shelter and for the first time due to space debris, a spacewalk was cancelled[3]. This was due to Russia destroying a satellite that resulted in around 1500 pieces of debris being dispersed. In the same month, the ISS had to drop the station by 310 metres for almost 3 minutes to avoid colliding with debris caused by a breakup in 1996 from a rocket launched by the US in 1994[4]. A similar situation occurred due to a satellite destroyed by China in 2007[5]. An example of when a collision occurred is in 2009, between satellites Cosmos-2251 and Iridium 33 [6], which created more debris. These examples, support a theoretical threat called Kessler Syndrome, which is a scenario where, if the density of the debris is very large due to its low orbit, then there will be detrimental collisions, forming more debris, which in turn will cause a domino effect for many more collisions. Therefore, we will attempt to model a scenario for cleaning debris by manoeuvring a spacecraft close enough to a piece of debris to catch it. To model this, we will first consider collisions in the atmosphere and its damage.

2 Period, Speed and Density of Orbital Debris

Q2

Here, we will start by analysing key characteristics of space debris such as how to calculate the period and hence the speed. Below, we have the period of circular orbit, T_0, and the angular speed,ω_0, in terms of G,M_E,R_E and r_0

$$T_0 = \frac{2\pi}{\omega_0} \qquad \omega_0 = \sqrt{\frac{GM_E}{r_0^3}} \tag{1}$$

where:

$$G = 6.763\times10^{-11}Nm^2kg^-2 \qquad M_E = 5.97\times10^{24}kg \qquad R_E = 6370km/6.37\times10^6m \qquad r_0 = R_E+h \tag{2}$$

G is the gravitational constant, M_E is the Earth's Mass and R_E is the Earth's radius. If we consider a polar orbit, instead of using very large values for r and very small values for θ that could lead to numerical inaccuracies, we have assumed that the debris is on a circular orbit hence why we have said the radius is the total of Earth's radius and altitude in the last equation from (2). From (1) substituting ω_0 into T_0 we have:

$$T_0 = 2\pi\sqrt{\frac{r_0^3}{GM_E}}$$

The speed in terms of T_0 is given by:

$$v = \frac{2\pi r_0}{T_0}$$

$$v = r_0\sqrt{\frac{GM_E}{r_0^3}}$$

$$v = \sqrt{\frac{GM_E}{r_0}} \tag{3}$$

To test what we have derived, we will calculate the time it takes for a piece of debris to orbit the Earth using two different heights corresponding to the lowest and highest orbits of the International Space Station:

Example 1:Height of 330km

$$r_0 = 6.37 \times 10^6 + 3.3 \times 10^5 = 6.7 \times 10^6$$

$$T_0 = 2\pi\sqrt{\frac{(6.7 \times 10^6)^3}{6.673 \times 10^{-11} \times 5.97 \times 10^{24}}}$$

$$T_0 = 5459 seconds = 91 minutes$$

Example 2:Height of 435km

$$r_0 = 6.37 \times 10^6 + 4.35 \times 10^5 = 6.805 \times 10^6$$

$$T_0 = 2\pi\sqrt{\frac{(6.805 \times 10^6)^3}{6.673 \times 10^{-11} \times 5.97 \times 10^{24}}}$$

$$T_0 = 5587 seconds = 93 minutes$$

If we compare these answers to the 92 minutes (on average) it takes the ISS to orbit[7], we can confirm that our times are close enough to this value (± 1 with respect to their heights), to consider them reasonable.

3 Atmospheric Density

First, we would like to consider how the density and frictional forces of the atmosphere affect orbital debris. Due to its fast speed, debris is slowed down by atmospheric drag until it is at a lower height enough to burn up and fall-back to Earth. In the following section, we will assess how to calculate the amount of collisions that can occur as well as how long the debris will take to fall respective to its frictional force, altitude and density. The amount of collisions per unit time between a satellite and piece of debris can be estimated if we know the density, cross-sectional area A and the speed v from the following equation:

$$k = \rho_d A v$$

For a speed in m/s the number of collisions per year is:

$$k = \rho_d A v \times 3.1536 \times 10^{-6}$$

Example 3: Number of collisions per year for the International Space Station

Using the following data from NASA[8][9], the ISS has: $A = 2500m^2$, $h = 400km$, $v = 7670m/s$,$v_{collision} = 10km/s$[10] and $\rho_d = 10^{-18}/m^{-3}$[11] we can calculate the number of collisions per year to be N = 0.0006, which is about once every 15 years. As a result, NASA uses radars to monitor debris.

To make any assumptions about the speed of atmospheric molecules, we will compare the speed of these molecules with the orbital speed of the satellite. To calculate the speed, we use the following equations:

$$\frac{k_B T}{2} = \frac{mv^2}{2} \quad so \quad v = \sqrt{\frac{k_B T}{m}}$$

where $k_B = 1.38 \times 10^{23} J/K$ and T is the temperature in Kelvin. If we consider atmospheric particles at $h = 300km$, the temperature is approximately $1500K$ and at $h = 100km$, it is $300K$. At 300km the atmosphere is mostly Nitrogen, Helium and Oxygen[10] and as Nitrogen has the largest presence in the atmosphere at this height, we can use $m_N = 2.32 \times 10^{26} kg$ and therefore we have:

$$v_N = \sqrt{\frac{1.38 \times 10^{23} \times 1500}{2.32 \times 10^{26}}} = 945m/s$$

Using the equation in (3) the speed of the satellite at the same height is:

$$v_h = \sqrt{\frac{6.763 \times 10^{-11} \times 5.97 \times 10^{24}}{6.37 \times 10^6 + 0.3 \times 10^6}} = 7728m/s$$

Therefore, as the speed of the satellite is much faster than the average thermal speed of gas molecules, we can assume that the atmospheric molecules are at rest and that their speed is the same as the orbital speed of the piece of the debris being calculated. If the density is low in the atmosphere and is expressed in kg/m^3 the debris will collide with $v_h A \rho_{At}$ kg of molecules where ρ_{At} is the atmospheric density. When a molecule hits a piece of debris, it will rebound with the same speed. As a result, the momentum changes from v to 0 and then from 0 to -v. We can therefore say that the momentum is: $p = (mv) - (m * -v) = 2mv$. As the force on the debris is the same, the momentum gained by the molecules is the same as the resulting force. Furthermore, as we have said m = $v_h A \rho_{At}$ kg and v = v_h we have that:

$$F_{fr} = 2A\rho_{At} v_h^2$$

The power can be calculated using the following equation: $P = Fv$ where P is the power, F is the force, and v is the speed. Therefore, the power is the frictional force multiplied by the speed:

$$P = 2A\rho_{At} v_h^3$$

As power is measured in joules per second, we can also model this as the differentiation of energy/time [12]i.e energy lost

$$\frac{dE_h}{dt} = -2A\rho_{At}v_h^3 \tag{4}$$

where E_h is the total energy. We can also calculate this by adding the gravitational and kinetic energies. First we need to derive an equation for kinetic energy [13] by considering the equation of the motion of a satellite in circular orbit: **Q7**

$$\frac{mv^2}{r} = \frac{GM_E m}{r^2}$$

$$as \qquad K_h = \frac{1}{2}mv^2 \qquad then \qquad K_h = \frac{1}{2}\frac{GmM_E r}{r^2}$$

Therefore, we can now declare equations for gravitational, kinetic and total energy:

$$P_h = \frac{-GmM_E}{r_h} \qquad K_h = \frac{GmM_E}{2r_h} \qquad E_h = P_h + K_h = \frac{-GmM_E}{2r_h} \tag{5}$$

Considering a 2D motion, a negative total energy represents the amount of energy that the orbiting debris needs so that its trajectory is a circular orbit. Below, we will demonstrate the process of deriving an equation for the time of re entry of a piece of debris in the atmosphere as a function of its orbital altitude, and the ratio $\frac{mass}{Area}$ or $\frac{m}{A}$: First, we will express the velocity as a function of its total energy using substituting (3) into (5):

$$E_h = -v^2(\frac{m}{2}) \qquad v = \sqrt{\frac{-2E_h}{m}} \tag{6}$$

If we express $\frac{dE_h}{dt}$ as a function of $\frac{dE_h}{dr}$ we get:

$$\frac{dE_h}{dt} = \frac{dE_h}{dr} \times \frac{dr}{dt}$$

If we then substitute $\frac{dE_h}{dr}$ where $\frac{dE_h}{dr} = \frac{GM_E m}{2r_h^2}$ we get:

$$\frac{dE_h}{dt} = \frac{GMm}{2r_h^2} \times \frac{dr}{dt}$$

By also substituting (3) into (4) we get:

$$\frac{dE_h}{dt} = -2A\rho_{At}(\sqrt{\frac{GM_E}{r_h}})^3$$

As we have outlined above, the negative sign only represents its trajectory, so for our derivation we will neglect the sign from the equation above. Therefore, putting the above back into our original equation we get:

$$2A\rho_{At}(\sqrt{\frac{GM_E}{r_h}})^3 = \frac{GMm}{2r_h^2} \times \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{4A\rho_{At}\sqrt{GMr_h}}{m} \tag{7}$$

The average density of the high atmosphere can be found in [10] and can be modelled by:

$$\rho_{At}(h) = ae^{-hl} + B(\frac{h}{h_0})^{-\sigma}$$

where ρ and a are in kg/m^3,l is in km and $B = 8.82 \times 10^7 kg/m^3$. We can then use these values and our program to plot a curve of these values and fit it to find values of a,l and σ. Figure 1 outlines our graph as well as the values we have found:

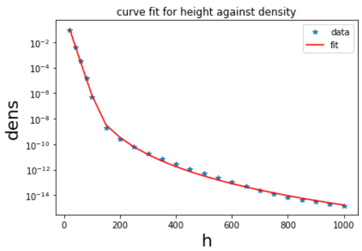


Figure 1: fitted curve of height in m against density in kg/m^3

$$a = 1.946 kg/m^3 \qquad l = 1.5 \times 10^{-4}/km \qquad \sigma = 7.57$$

If we consider $h_0 = 1m$ then we are able to say: $r_h \approx R_E$ *and* $\rho_{At}(h) \approx Bh^{-\sigma}$ We can then substitute this into (6) to compute $\frac{dr}{dt} = \frac{dr}{dt}$

$$\frac{dh}{dt} = \frac{4ABh^{-\sigma}\sqrt{GMR_E}}{m}$$

This gives a 1st order separable differential equation for h(t) which can be integrated from h = h_0 to h = 0

$$\int_0^{h_0} h^\sigma \, dh = \int \frac{4AB\sqrt{GMR_E}}{m} \, dt$$

$$\left[\frac{h^{\sigma+1}}{\sigma+1}\right]_0^{h_0} = \frac{4AB\sqrt{GMR_E}t}{m} + (c)$$

Though there is a constant time, (c), this represents the initial time hence we can assume this is 0, and neglect this in our final answer:

$$t = \frac{h_0^{\sigma+1}}{4B\sqrt{GMR_E}(\sigma+1)}(\frac{m}{A})$$

For clarity we have also factorised out the ratio of m/A. The ratio of m/A has been constantly increasing over the years which has been said to be directly cause many collisions[14]. To test this theory and our final answer above, we will now be using examples to plot graphs of t in years against h in metres, between 300m and 1km for 4 types of debris. We will convert the original values into metres for the examples:

$$a = 1.946 kg/m^3 \qquad l = 1.5 \times 10^{-4}/km \qquad B = 4.63 \times 10^{30} kg/m^3 \qquad \sigma = 7.57$$

The aim of the following examples is to find $\frac{m}{A}$ so that we can find t.

Example 4: Aluminium Bolt

An aluminium bolt (modelled as a plain cube) with cross sectional area $A = L^2$, $L = 1cm$ and $\rho = 2700kg/m^3$. From this information we would need to find the mass which can be done using $\rho = \frac{m}{V}$ where V in the volume. To calculate the volume, we use $(0.01)^3$. By multiplying this with the density we get a mass of $0.0027kg$. Therefore, as $A = L^2 = (0.01)^2, \frac{m}{A} = 27$. Figure 2 shows a graph representing this example.

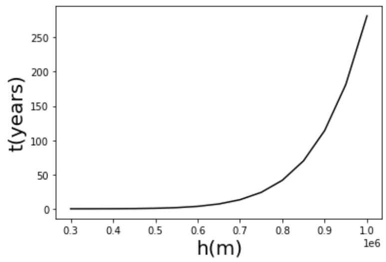


Figure 2: graph of height in m against t in years for an Aluminium Bolt. (Example 4)

Example 5: Aluminium Rod

An aluminium rod of length $L = 10cm$ and square cross section $l = 1cm$ and density $\rho = 2700kg/m^3$. The average area is $A \approx L\frac{l}{2}$, $m \approx l^2 L\rho$ and $\frac{m}{A} \approx 2\rho l$. So, $\frac{m}{A} = 54$. If we compare to Example 4, this demonstrates that as $\frac{m}{A}$ doubles so does t. Figure 3 shows a graph representing this example.

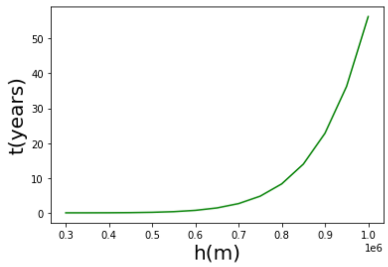


Figure 3: graph of height in m against t in years for an Aluminium Rod (Example 5)

Example 6: Aluminium Plate

A square aluminium plate of length L and thickness $l = 1mm$ has an average area $A \approx \frac{L^2}{2}$. If we use the same length and density as above A = 0.005. m = 2700 x (0.001x0.01), therefore $\frac{m}{A} = 5.4$. Similarly, if we compare to example 5, if $\frac{m}{A}$ is a tenth of the previous answer t is also a tenth. Figure 4 shows a graph representing this example.

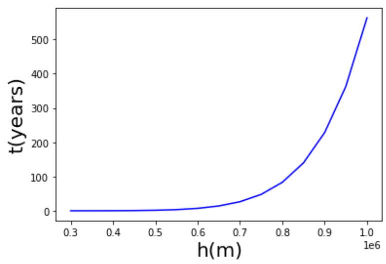


Figure 4: graph of height in m against t in years for an Aluminium Plate (Example 6)

Example 7: Gemini Spacecraft

Gemini spacecraft with mass m = 3850kg and a larger diameter of 3m. If we consider $A = \pi r^2 . \frac{m}{A} = \pi(1.5)^2 \approx 555$. Figure 5 shows a graph representing this example.

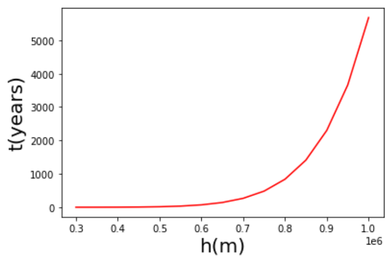


Figure 5: graph of height in m against t in years for a Gemini Spacecraft (Example 7)

Logarithmic graph

In [11] NASA quotes the following: *"Debris left in orbits below 600 km normally fall back to Earth within several years. At altitudes of 800 km, the time for orbital decay is often measured in decades. Above 1,000 km, orbital debris will normally continue circling the Earth for a century or more."*. All of the graphs above show that as $\frac{m}{A}$ increases, the time increases exponentially. Therefore, figure 6 shows a graph representing the logarithmic lines of these examples. We can now see that when $h < 600km$ the debris will fall in less than 100 years whereas when $h > 600km$ it can take 100s and eventually 1000s of years. Therefore, we can see how long pieces of debris can stay in the atmosphere, and as a result how much damage this can cause. Therefore, we would now like to investigate how we can clean this debris from the atmosphere:

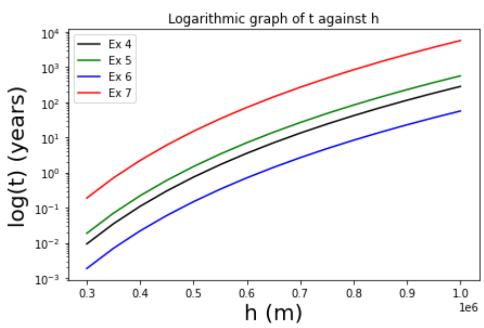


Figure 6: a logarithmic graph of height in m against t in years for examples 4,5,6 and 7

4 Orbital Dynamics

We will now be exploring a method for cleaning debris similar to one currently being developed called e.Deorbit,[15][16] which finds pieces of debris in a polar orbit. It attempts to collect debris using the "rendez vous" method. Just like in example 7, a Gemini Spacecraft was used in the first trial for the "rendez vous" method in 1965. We will now be modelling this using Newton's Laws and the Runge Kutta 4th order method, which solves first order differential equations by approximating a value of y for a given x. Using python, we will be iterating this for different times to find the closest distance we can get to the debris. We will start by deriving equations for the acceleration of the object in orbit in the x and y direction in terms of its force. Figure 6 shows a graph diagram representing the forces. Force balance against the x axis:[17] $m\ddot{x}$ = acceleration across x axis = $F_g cos(\theta) + F_x(t)$ and $cos(\theta) = \frac{x}{\sqrt{x^2+y^2}}$ and $F_g sin(\theta) = \frac{y}{\sqrt{x^2+y^2}}$

$$m\ddot{x} = \frac{-GM_E m}{x^2+y^2}\frac{x}{\sqrt{x^2+y^2}} + F_x(t) \tag{8}$$

$m\ddot{y}$ = acceleration across y axis = $F_g sin(\theta) + F_y(t)$

$$m\ddot{y} = \frac{-GM_E m}{x^2+y^2}\frac{x}{\sqrt{x^2+y^2}} + F_y(t) \tag{9}$$

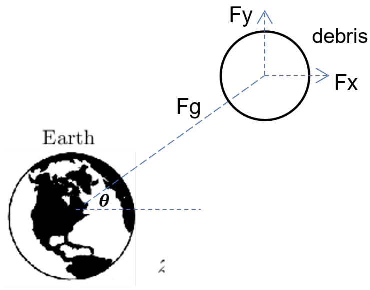


Diagram 1: Forces acting on a piece of debris when orbiting the Earth

Where and Fx is the thrust or frictional force exerted on the object along the x axis and Fy is the thrust or frictional force exerted on the object along the y axis. These thrusters are fired to manoeuvre the spacecraft within 1m of the target debris. As our satellite finds the pieces of debris in a polar orbit, and to make calculations easier in our modelling, we will change these equations into polar co-ordinates by using: **Q1**

$$x = rsin\theta \qquad y = rcos\theta \qquad r^2 = x^2+y^2 \tag{10}$$

Substituting these into (8) and (9) we have:

$$\ddot{x} = \frac{-GM_E}{r^2}sin\theta + \frac{F_x(t)}{m} \tag{11}$$

$$\ddot{y} = \frac{-GM_E}{r^2}cos\theta + \frac{F_y(t)}{m} \tag{12}$$

To change F_x and F_y in terms of F_r and F_θ we can say:

$$F_r = F_x + F_y \qquad F_\theta = F_x - F_y$$

We would now like to find F_r and F_θ in terms of F_x and F_y

$$F_r sin\theta = F_x sin^2(\theta) + F_y sin\theta cos\theta \qquad F_\theta cos\theta = F_x cos^2(\theta) - F_y sin\theta cos\theta$$

$$F_r sin\theta + F_\theta cos\theta = F_x \tag{13}$$

Similarly:

$$F_r cos\theta = F_x sin\theta cos\theta + F_y cos^2(\theta) \qquad F_\theta cos\theta = F_x sin\theta cos\theta - F_y sin^2(\theta)$$

$$F_r cos\theta - F_\theta sin\theta = F_y \tag{14}$$

If we put (13) and (14) into (11) and (12) we then have:

$$\ddot{x} = \frac{-GM_E}{r^2}sin\theta + \frac{F_r sin(\theta) + F_\theta cos\theta}{m} \tag{15}$$

$$\ddot{y} = \frac{-GM_E}{r^2} cos\theta + \frac{F_r cos\theta - F_\theta sin\theta}{m} \tag{16}$$

We would now like to change all variables from x and y to r and θ. To do this, we would need to differentiate the first two equations in (10) twice using the product rule:

$$\dot{uv} = u\dot{v} + v\dot{u}$$

Differentiating x once we get:

$$\dot{x} = r\dot{sin\theta} + \dot{r}sin\theta = r\dot{\theta}cos\theta + \dot{r}sin\theta$$

Using the product rule again we get:

$$\ddot{x} = \ddot{r}sin\theta + 2\dot{r}\dot{\theta}cos\theta - r\theta^2 sin\theta + r\ddot{\theta}cos\theta \tag{17}$$

Similarly for y:

$$\dot{y} = r\dot{\theta}cos\theta - \dot{r}sin\theta$$

$$\ddot{y} = \ddot{r}cos\theta - 2\dot{r}\dot{\theta}sin\theta - r\theta^2 cos\theta + r\ddot{\theta}sin\theta \tag{18}$$

To get r and θ we substitute (17) into (15) and (18) into (16) and get:

$$\ddot{r}sin\theta + 2\dot{r}\dot{\theta}cos\theta - r\theta^2 sin\theta + r\ddot{\theta}cos\theta = \frac{-GM_E}{r^2} sin\theta + \frac{F_r sin(\theta) + F_\theta cos\theta}{m}$$

$$\ddot{r}cos\theta - 2\dot{r}\dot{\theta}sin\theta - r\theta^2 cos\theta + r\ddot{\theta}sin\theta = \frac{-GM_E}{r^2} cos\theta + \frac{F_r cos\theta - F_\theta sin\theta}{m}$$

multiplying both by $sin\theta$ we have:

$$\ddot{r}sin^2\theta + 2\dot{r}\dot{\theta}sin\theta cos\theta - r\theta^2 sin^2\theta + r\ddot{\theta}sin\theta cos\theta = \frac{-GM_E}{r^2} sin^2\theta + \frac{F_r sin^2(\theta) + F_\theta sin\theta cos\theta}{m} \tag{19}$$

$$\ddot{r}cos\theta sin\theta - 2\dot{r}\dot{\theta}sin^2\theta - r\theta^2 sin\theta cos\theta + r\ddot{\theta}sin^2\theta = \frac{-GM_E}{r^2} sin\theta cos\theta + \frac{F_r sin\theta cos\theta - F_\theta sin^2\theta}{m} \tag{20}$$

and multiplying both by $cos\theta$:

$$\ddot{r}sin\theta cos\theta + 2\dot{r}\dot{\theta}cos^2\theta - r\theta^2 sin\theta cos\theta + r\ddot{\theta}cos^2\theta = \frac{-GM_E}{r^2} sin\theta cos\theta + \frac{F_r sin(\theta)cos\theta + F_\theta cos^2\theta}{m} \tag{21}$$

$$\ddot{r}cos^2\theta - 2\dot{r}\dot{\theta}sin\theta cos\theta - r\theta^2 cos^2\theta + r\ddot{\theta}sin\theta cos\theta = \frac{-GM_E}{r^2} cos^2\theta + \frac{F_r cos^2\theta - F_\theta sin\theta cos\theta}{m} \tag{22}$$

Now using the 4 equations above and the identity $sin^2\theta + cos^2\theta = 1$ we can cancel these equations. (19) + (22):

$$\ddot{r} - r\theta^2 = \frac{-GM_E}{r^2} cos^2\theta + \frac{F_r}{m}$$

$$\ddot{r} = \frac{-GM_E}{r^2} cos^2\theta + r\theta^2 + \frac{F_r}{m} \tag{23}$$

(21) - (20)

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = \frac{F_\theta}{m}$$

$$\ddot{\theta} = \frac{F_\theta}{mr} - \frac{2\dot{\theta}\dot{r}}{r} \tag{24}$$

If we assume the spacecraft and the piece of orbit are confined to the same plane, we will use 4 small thrusters when manoeuvring the spacecraft. Therefore, using the polar co-ordinate equations found above: F_θ will represent two of the thrusters that push the spacecraft forward (positive value) and backwards (negative value), i.e parallel to its orbit and F_r pushes higher (positive value) and lower (negative value) i.e radially in orbit. We will also assume that the direction of the spacecraft is fixed and that the thrusters are always equal to or less than 100N. We will now use the radius formula from (2) and the angular speed as ω_0 given by (1) to find the co-ordinates for the reference trajectory: If we first consider setting $F_r = F_\theta = 0$, we can use the following equations for the co-ordinates of the trajectory of debris and the active spacecraft:

$$trajectory: \quad r = r_0 \qquad \theta = \omega_0 t$$

$$spacecraft: \quad r = r_0 + z(t) \qquad \theta = \omega_0 t + \phi(t)$$

therefore:

$$trajectory: \quad (r_0, \omega_0 t) \qquad spacecraft: \quad (r_0 + z(t), \omega_0 t + \phi(t))$$

where $z(t)$ and $\phi(t)$ are a function of their times. Using these, we can now find the distance for the polar co-ordinates between the spacecraft and the reference trajectory using the following equation:

Q4

$$S = \sqrt{r_A^2 + r_B^2 - 2r_A r_B cos(\theta_B - \theta_A)} \qquad co-ordinates: \quad (r_A, \theta_A), \ (r_B, \theta_B)$$

Substituting in the co-ordinates we have:

$$S = \sqrt{r_0^2 + (r_0 + z(t))^2 - 2r_0(r_0 + z(t))cos((\omega_0 t + \phi(t) - \omega_0 t)}$$

$$S = \sqrt{2r_0^2 + 2r_0 z(t) + z(t)^2 - (2r_0^2 + 2r_0 z(t))cos(\phi(t))}$$

Which gives the final expression:

$$S = \sqrt{(2r_0^2 + 2r_0 z(t))(1 - cos(\phi(t)) + z(t)^2}$$

To test the correctness of this, we would like to use the following special cases:

Table 1: Results of calculated and expected distances when varying z and phi

	z	ϕ	S	Expected
a)	0	0	0	As both z and ϕ are starting at 0 this means that they don't have any differences hence the distance is expected to be 0.
b)	a	0	a	As the difference in angle is 0, this means we . would only need to consider the 1 dimensional distance of z = a.
c)	0	a	$\sqrt{2r_0^2(1 - cos(a))}$	If we consider a to be very small we can use the Taylor series to say that $1 - cos(a) = \frac{a^2}{2}$ therefore we would expect the final result to be very small
d)	0	$\pi/2$	$\sqrt{2}r_0$	With this angle we can use trigonometry on a triangle of sides r_0 to find the distance would be $\sqrt{2}r_0$
e)	a	π	$\sqrt{4r_0(r_0 + a) + a^2}$	Similarly with d), we would expect to get this result with trigonometry

Now we would be able to add this to a program to be able to calculate the minimum distance between the spacecraft and the trajectory. Our next step is to substitute the co-ordinates into (3) and (4) to get:

Q3

$$\ddot{z} = \frac{-GM_E}{r_0 + z^2} + (r_0 + z)(\omega_0 + \dot{\phi})^2 + \frac{F_r}{m}$$

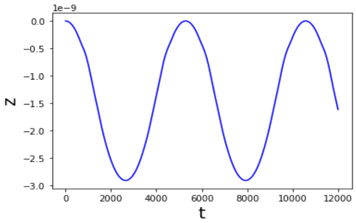
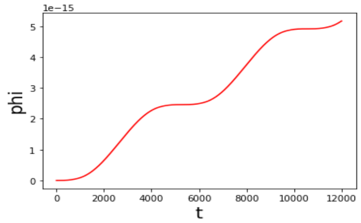
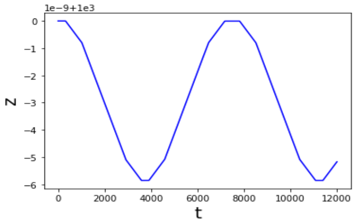
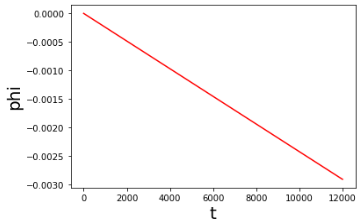
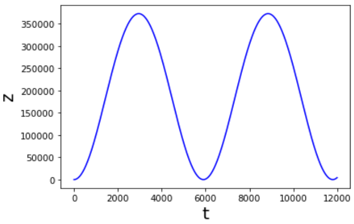
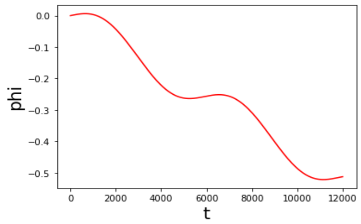
$$\ddot{\phi} = -2\frac{(\omega_0 + \dot{\phi})\dot{z}}{r_0 + z} + \frac{F_\theta}{m(r_0 + z)}$$

Finally, for our last step we would now like to covert the pair of second order ordinary differential equations into a system of 4 first order ordinary differential equations:

$$\dot{z} = v_z$$
$$\dot{\phi} = v_\phi$$
$$\dot{v_z} = \frac{-GM_E}{r_0 + z^2} + (r_0 + z)(\omega_0 + v_\phi)^2 + \frac{F_r}{m}$$
$$\dot{v_\phi} = -2\frac{(\omega_0 + v_\phi)v_z}{r_0 + z} + \frac{F_\theta}{m(r_0 + z)}$$

First, to test the derived equations above, we can now integrate these equations over 2 full periods to plot graphs of z(t) and phi(t) using the following three cases and h = 500km to check if they are reasonable:

Table 2: Graphs of z(t) and phi(t) when varying the values of the first order differential equations

z	ϕ	$\dot{z}$	$\dot{\phi}$	z(t)	phi(t)
0	0	0	0		
1000	0	0	$\sqrt{\frac{GM_E}{(r_0+z_0)^3}} - \omega_0$		
0	0	0	$\frac{100}{(r_0+z_0)}$		

If we pay attention to the axis' scale, we can note that some graphs can be modelled as very close to 0 or a straight line. If we now neglect and the assumption that $F_\theta = F_r = 0$ and assume that the thrusters will only be fired in the range of $t = 0$ to t_{thrust} and that both F_r and F_θ will be fixed (i.e. they will both start and stop at the same time) we can then use the equation below to work out the total amount of fuel used: $Fuel = (|F_r| + |F_\theta|)t_{thrust}$ in units kgm/s. Now, we would like to bring all the above factors together to integrate the equation for the spacecraft from $t = 0$ to $t = 4000$ using different values for F_r, F_θ and t_{thrust}. If we model a spacecraft that will initially be on an orbit 1km lower than the piece of debris and about 2km behind in our program, we will use the initial conditions of:

$$z = 1000 \qquad \dot{z} = 0 \qquad \phi = -2000 \qquad \dot{\phi} = \sqrt{\frac{GM_E}{(r_0 + z)^3}} - \omega_0 \tag{25}$$

Using this and varying F_θ, F_r and t_{thrust} we can then calculate the total fuel used, minimum distance and time between the spacecraft and the target debris, during its trajectory:

Q5

Table 3: Results for dmin, tmin and fuel when varying F_r,F_theta and t_thrust.

	$F_r(N)$	$F_\theta(N)$	$t_{thrust}(s)$	$dmin(metres)$	$tmin(minutes)$	$Fuel(kgm/s)$
a)	50	100	100	499.59	8.73	15000
b)	25	50	200	432.73	9.47	15000
c)	10	20	500	279.70	11.56	15000

Figure 7 below demonstrates a graph of these trajectories:

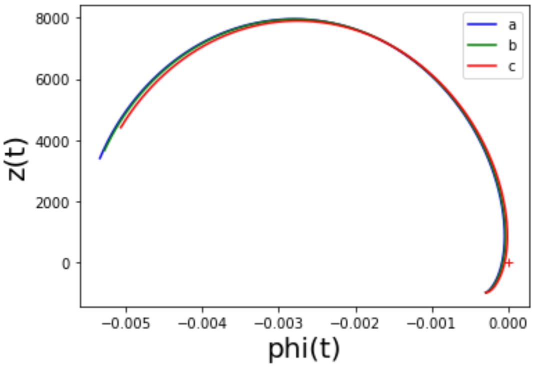


Figure 7: Graph, phi(t) against z(t) showing trajectories of Table 3

With the cases above, we can see that the time taken to get the minimum distance to the target is between 8 and 12 minutes.

When we vary t_{thrust}, we find that as we increase t_{thrust}, the minimum distance decreases. However, once we reach $t_{thrust} = 346$ the minimum time and distance no longer changes past 1 decimal place and tends to 220.6m, where 220.627m is the absolute closest distance we can get to the piece of debris by setting $t_{thrust} = 500$.

We would think that we could simply aim towards the debris in a straight line, however, if we factor in speed and altitude this poses a problem. If we aim in a straight line, as the speed of the spacecraft increases, so does its altitude and therefore would miss its target. At high altitudes, the spacecraft's velocity would decrease and so would be too slow to catch the debris. Therefore, we would have to strategically adjust F_r, t_{thrust} and F_θ to reach the debris.

If we take t_{max}=4000, we will try to find a trajectory that reaches the debris whilst also using the least amount of fuel by adjusting F_r and t_{thrust} but keeping $F_\theta = 100$.

We will now create a program that uses a brute force attack for finding a minimum distance of less than 1m, that uses the least amount of fuel, by varying F_r and t_{thrust} and fixing $F_\theta = 100$.

Table 4: Fuel used for when dmin < 1m, setting F_theta = 100N and varying F_r and t_thrust

	$F_r(N)$	$t_{thrust}(s)$	$F_{theta}(N)$	$dmin(m)$	$tmin(s)$	$Fuel(kgm/s)$
a)	0	67	100	0.905	781.2	6700
b)	2	74	100	0.994	737.4	7548
c)	3	22	100	0.226	1411.2	2266
d)	4	81	100	0.826	699.1	8424
e)	6	88	100	0.332	665.3	9328
f)	7	20	100	0.143	1483.9	2140
g)	8	12	100	0.636	3171.2	1296
h)	11	108	100	0.414	588.6	11988
i)	12	112	100	0.675	575.8	12544
j)	14	121	100	0.859	549.8	13794
k)	15	126	100	0.179	536.9	14490
l)	16	17	100	0.77	1621.6	1972
m)	17	136	100	0.58	513.5	15912
n)	19	147	100	0.56	491.1	17493
o)	20	16	100	0.415	1682.0	1920
p)	21	159	100	0.679	470.2	19239
q)	22	166	100	0.493	459.5	20252
r)	23	173	100	0.707	449.7	21279
s)	24	180	100	0.288	440.7	22320
t)	25	15	100	0.257	1751.3	1875
u)	26	196	100	0.319	422.9	24696
v)	27	205	100	0.374	414.3	26035
w)	28	214	100	0.694	406.7	27392
x)	29	225	100	0.343	398.6	29025
y)	30	236	100	0.812	391.4	30680
z)	31	249	100	0.857	384.4	32619
α)	32	264	100	0.966	377.7	34848
β)	33	283	100	0.548	371.2	37639
γ)	34	308	100	0.770	365.4	41272

From the table above, we can see that the 6 best scenarios that use the least amount of fuel whilst having a minimum distance of less than 1m are c), g), f), l), o) and t). All of these use fuel between 1800kg m/s and 2300 kg m/s where g) uses the least amount of fuel and f) reaches the closest to the piece of debris. Figure 9 below is an illustration of their trajectories:

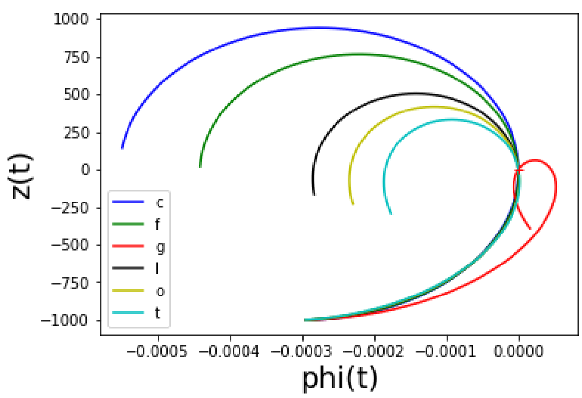


Figure 8: Graph, phi(t) against z(t) showing 6 best trajectories from Table 4

Q6

5 Analysing our model

Though scenario f) gets us the closest to the debris, g) is not far behind with a miniscule difference and uses only 1296 kg m/s. Therefore, we have found that if we set the parallel force to 8N, the radial force to 100N and the thrust force to 12s we use the least amount of fuel. As outlined by our graph we can see that the trajectory of g) is different from the others therefore, could this give us an indication of the best strategy to use when catching the debris? To outline this we will aim to get within 1m of our debris whilst using less fuel. We will investigate this, using our program, by neglecting the assumption that the spacecraft activates its thrusters when it is 2km behind the debris and vary the distance as well as F_θ,F_r and t_{thrust}. A starting point for this would be to try numbers close to those we found to get the result from g).

Table 5: Fuel used by varying dstart

	$F_r(N)$	$F_{theta}(N)$	$t_{thrust}(s)$	$dstart(m)$	$dmin(metres)$	$Fuel(kgm/s)$
a)	0	95	12	2392	0.377	1140
b)	0	95	12	2393	0.0367	1140
c)	0	95	12	2394	0.267	1140
d)	0	95	12	2395	0.609	1140
e)	0	95	12	2396	0.609	1140

Using trial and error we have found that, we can use a small amount of fuel (1140 kg m/s) whilst also getting a minimum distance of less than 1m by starting from a distance between 2.392km and 2.396km. This result was found by setting F_θ to 95N, F_r to 0N and t_{thrust} to 12s, which is not far off from our values for g). As shown from the table above, all distances tested will give us a minimum distance of less than 1m, 2393m will give us a minimum distance of 0.0367m. Using this data we can model a graph of our best result:

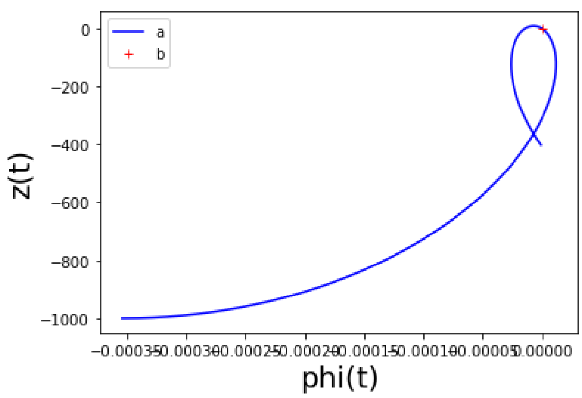


Figure 9: Graph, phi(t) against z(t) showing best trajectory from Table 5

As we can see from Figure 9 and Figure 10, the best trajectories found have been ones that have caught the debris from travelling below and around. Therefore, we can then conclude that the best strategy for cleaning debris would be to approach our target from behind.

However, there are also problems with our model that we need to consider. Firstly, if we consider that the ejection speed of the thruster's gas is 1km/s and that we need to capture a debris on an orbit 1km above the spacecraft, we will need 1kg of fuel for every km travelled. Therefore, theoretically, if the spacecraft has 500kg of fuel we could capture a piece of debris in the range 300-500km. Furthermore, if we include in our model locations for several pieces of debris that are close to each other, we can have a starting distance from the co-ordinate of the last piece of debris rather than using the starting distance we have used in our model. This way we may be able to collect several pieces of debris at once, which saves fuel and time. However, we cannot assume that we only need fuel to reach the debris, we also need to consider how much we can use, as the spacecraft would need to land back safely after the mission. Furthermore, as we have now considered the mass and loss of the fuel, we can see that in reality, the mass of the rocket will change and will not stay constant as our model has assumed, which is another limitation. Furthermore, we have not considered the atmospheric drag, not modelling the debris and spacecraft on the same plane and their exact dimensions/surface areas.

6 Conclusion

We first modelled how our debris behaves in the atmosphere using height and density as factors and worked out how long it can take for a piece of debris to fall back to Earth. Using curve fitting, we found values to fit our equation for density, allowing us to model the expression we derived for time in terms of density, height, mass and area. When we investigated this, we found that the time taken was largely dependent on the ratio of m/A and that as this increased, so did the time it took to fall. When taking this into account, we could see how much damage debris can cause and how long it can stay in the atmosphere for. Therefore, we investigated how we could clean this debris by manoeuvring a spacecraft to catch the debris. We then used Newtons Laws to derive equations for acceleration and used this equation to derive expressions for r and θ. Next, we used a program to model and solve these equations using the Runge-Kutta 4th order method. From our equations we found that our model was strongly dependent on factors such as force thrusters and fuel. Therefore, we tested different values for these to find the best distance between the spacecraft and the piece of debris so that it would use the least amount of fuel. The results showed us that at a distance of 2km away when setting F_r to 8N, F_θ to 100N and t_{thrust} to 12s we can get within 1m of the debris only using 1296 kg/m^3 of fuel. We then found that if we set our distance to start using the forces to around 2.4km away we could use only 1140 kg/m^3 of fuel by setting the F_r to 0N, F_θ to 95N and t_{thrust} to 12s. When plotting the graphs of these trajectories we discovered the best strategy for catching the debris and found that we can take a trajectory that goes below the debris and then catches it from behind.

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