Bài tập về nhà môn Numerical Linear Algebra - Đợt 2

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Ngày 20 tháng 10 năm 2023

b)

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Compute $\|A\| = \sup\limits_{\substack{x \in \mathbb{C}^2 \\ x \neq 0}} \dfrac{\|Ax\|_2}{\|x\|_2}$

<u>Giải</u>

Với $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{C}^2$, $Ax = \begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 \end{bmatrix}$, ta có: $\dfrac{\|Ax\|_2}{\|x\|_2} = \dfrac{\sqrt{|x_1 + x_2|^2 + |x_1 + x_2|^2}}{\sqrt{|x_1|^2 + |x_2|^2}}$

Xét:

$$\begin{aligned}
|x_1 + x_2|^2 + |x_1 + x_2|^2 &= 2(x_1 + x_2)\overline{(x_1 + x_2)} \\
&= 2\left[|x_1|^2 + |x_2|^2 + \overline{x_1}x_2 + x_1\overline{x_2}\right] \\
&= 2|x_1|^2 + 2|x_2|^2 + 2\left(\overline{x_1}x_2 + x_1\overline{x_2}\right) \\
&\leqslant 2|x_1|^2 + 2|x_2|^2 + 4|x_1||x_2| \\
&\leqslant 2|x_1|^2 + 2|x_2|^2 + \alpha|x_1|^2 + \beta|x_2|^2, \quad \alpha, \beta > 0 \\
&\leqslant (2 + \alpha)|x_1|^2 + (2 + \beta)|x_2|^2
\end{aligned}$$

Trong đó, ta chọn α, β thoả: $\begin{cases} \alpha, \beta > 0 \\ 2 + \alpha = 2 + \beta \\ \alpha\beta = 4 \end{cases} \Rightarrow \begin{cases} \alpha = 2 \\ \beta = 2 \end{cases} \Rightarrow |x_1 + x_2|^2 + |x_1 + x_2|^2 \leqslant 4\left(|x_1|^2 + |x_2|^2\right)$

Ta có:

$$\|A\| = \sup_{\substack{x \in \mathbb{C}^2 \\ x \neq 0}} \frac{\|Ax\|_2}{\|x\|_2} = \sup_{\substack{x \in \mathbb{C}^2 \\ x \neq 0}} \frac{\sqrt{|x_1 + x_2|^2 + |x_1 + x_2|^2}}{\sqrt{|x_1|^2 + |x_2|^2}} = \frac{\sqrt{4\left(|x_1|^2 + |x_2|^2\right)}}{\sqrt{|x_1|^2 + |x_2|^2}} = \frac{2\sqrt{|x_1|^2 + |x_2|^2}}{\sqrt{|x_1|^2 + |x_2|^2}} = 2$$

Vậy $\|A\| = 2$.

d)

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in \mathbb{C}^{2 \times 2}$$

Compute $\|A\| = \sup\limits_{\substack{x \in \mathbb{C}^2 \\ x \neq 0}} \dfrac{\|Ax\|_1}{\|x\|_1}$

<u>Giải</u>

Ta xét trường hợp tổng quát với $A = \left[\begin{array}{c|c|c} a_1 & \cdots & a_n \end{array}\right] \in \mathbb{C}^{m \times n}, a_j$ là vector m chiều. Xét quả cầu đơn vị chuẩn 1 trong $\mathbb{C}^n$, với $x \in \mathbb{C}^n : \sum\limits_{k=1}^{n} |x_j| \leqslant 1$, có:

$$\|Ax\|_1 = \left\| \sum_{j=1}^{n} x_j a_j \right\|_1 \leqslant \sum_{j=1}^{n} |x_j| \|a_j\|_1 \leqslant \max_{1\leqslant j\leqslant n} \|a_j\|_1$$

chọn $x = e_j$ với j cực đại hóa $\|a_j\|_1$, ta thu được:

$$\|A\|_1 = \max_{1\leqslant j\leqslant n} \|a_j\|_1$$

Với $n = 2$, $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in \mathbb{C}^{2\times 2}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{C}^2$, ta có:

$$\|A\|_1 = \max_{1\leqslant j\leqslant 2} \|a_j\|_1 = \max\{|a_1| + |a_3|, |a_2| + |a_4|\}$$

e)

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in \mathbb{C}^{2\times 2}$$

Compute $\|A\| = \sup\limits_{\substack{x\in\mathbb{C}^2 \\ x\neq 0}} \dfrac{\|Ax\|_\infty}{\|x\|_\infty}$

<u>Giải</u>

Ta có $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{C}^2, A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \left[\begin{array}{c|c} A_1 & A_2 \end{array}\right], Ax = \begin{bmatrix} a_1x_1 + a_2x_2 \\ a_3x_1 + a_4x_2 \end{bmatrix}$

Xét

$$\begin{aligned}
\frac{\|Ax\|_\infty}{\|x\|_\infty} &= \frac{\max\{|a_1x_1 + a_2x_2|, |a_3x_1 + a_4x_2|\}}{\|x\|_\infty} \\
&= \max\left\{\frac{|a_1x_1 + a_2x_2|}{\|x\|_\infty}, \frac{|a_3x_1 + a_4x_2|}{\|x\|_\infty}\right\} \\
&= \max\left\{\frac{|A_1^*x|}{\|x\|_\infty}, \frac{|A_2^*x|}{\|x\|_\infty}\right\} \quad (\text{Với } A_1^*, A_2^* \text{ là vector dòng của } A) \\
&\leqslant \max\left\{\frac{\|A_1^*\|_1\|x\|_\infty}{\|x\|_\infty}, \frac{\|A_2^*\|_1\|x\|_\infty}{\|x\|_\infty}\right\} \quad (\text{Bất đẳng thức Hölder}) \\
&= \max\left\{\|A_1^*\|_1, \|A_2^*\|_1\right\}
\end{aligned}$$

Vậy $\|A\| = \sup\limits_{\substack{x\in\mathbb{C}^2 \\ x\neq 0}} \dfrac{\|Ax\|_\infty}{\|x\|_\infty} = \max\{\|A_1^*\|_1, \|A_2^*\|_1\} = \max\{|a_1| + |a_2|, |a_3| + |a_4|\}$.

3.2: Let $\|\cdot\|$ denote any norm on $\mathbb{C}^m$ and also the induced matrix norm on $\mathbb{C}^{m\times m}$. Show that $\rho(A) \leqslant \|A\|$, where $\rho(A)$ is the *spectral radius* of A, i.e., the largest absolute value $|\lambda|$ of an eigenvalue λ of A.

<u>Giải</u>

Với eigenvalues λ tương ứng với eigenvectors $x \in \mathbb{C}^m \backslash \{0\}$ của A, ta có:

$$\begin{aligned}
&Ax = \lambda x \\
\Leftrightarrow &\|Ax\| = \|\lambda x\| \\
\Leftrightarrow &\|Ax\| = |\lambda|\|x\| \\
\Leftrightarrow &\frac{\|Ax\|}{\|x\|} = |\lambda|
\end{aligned}$$

$$\Rightarrow |\lambda| \leqslant \sup_{x\in\mathbb{C}^m\backslash\{0\}} \frac{\|Ax\|}{\|x\|} = \|A\|$$

Mà $\rho(A) = \max\{|\lambda| : \lambda \text{ là eigenvalue của } A\}$
Vậy $\rho(A) \leqslant \|A\|$.