Indecomposable characters on direct limit of symmetric groups with diagonal embeddings

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Characters on a group

Let G be a (topological) group. Recall the notion of a character on G:

Definition

A (continuous) function χ on a group G is called a *(normalized) character* if

- χ is positive-definite, i.e. the matrix $[\chi(g_j^{-1} g_k)]_{j,k=1}^m$ is positive semi-definite for any $g_1, g_2, \dots, g_m \in G$,
- χ is central, i.e. the equality $\chi(gh) = \chi(hg)$ holds for all $g, h \in G$,
- $\chi(e) = 1$, where e is the identity element of G.

Definition

A character χ is called *indecomposable* if it is an extreme point of the simplex of all characters. In other words, χ is indecomposable if there are no distinct characters χ_1, χ_2 and $\alpha \in (0,1)$ such that $\chi = \alpha\chi_1 + (1-\alpha)\chi_2$.

The classical Thoma theorem

The natural inclusions $\{1, \dots, N\} \hookrightarrow \{1, \dots, N, N+1\}$ induce the following chain of embeddings of symmetric groups:

$$\mathfrak{S}_1 \to \mathfrak{S}_2 \to \dots \to \mathfrak{S}_n \to \mathfrak{S}_{n+1} \to \dots$$

The corresponding inductive limit $\mathfrak{S}_\infty = \varinjlim \mathfrak{S}_n$ is called the *infinite symmetric group*. The classical theorem of E. Thoma concerns the classification of all indecomposable characters of the infinite symmetric group.

Theorem (Thoma, 1964)

The indecomposable characters of the group $\mathfrak{S}_\infty$ are the functions of the form

$$\chi_{\alpha,\beta}(\sigma) = \prod_{k \in [\sigma]} \left(\sum_{i=1}^{\infty} \alpha_i^k + (-1)^{k-1} \sum_{i=1}^{\infty} \beta_i^k \right),$$

where $\alpha = \{\alpha_i\}_{i=1}^{\infty}$ and $\beta = \{\beta_i\}_{i=1}^{\infty}$ are sequences of non-negative real numbers (called *Thoma parameters*) such that

$$\alpha_1 \geqslant \alpha_2 \geqslant \dots \geqslant 0, \ \beta_1 \geqslant \beta_2 \geqslant \dots \geqslant 0, \ \text{and} \ \sum_{i=1}^{\infty} \alpha_i + \sum_{i=1}^{\infty} \beta_i \leqslant 1.$$

On the proof of Thoma theorem

Several proofs of Thoma theorem are known – see for instance works of Thoma, Vershik–Kerov and Kerov–Okounkov–Olshanski.

The original approach of Thoma involved the classification of the so called *totally positive sequences* and some non-trivial results of the theory of entire functions.

A different approach is based on the *ergodic method* due to A. Vershik and S. Kerov. Recall that the irreducible representations of the symmetric group $\mathfrak{S}_N$ are parameterized by the Young diagrams λ with N boxes.

Theorem (Vershik–Kerov, 1981)

(i) For any indecomposable character χ of $\mathfrak{S}_\infty$ there exist a sequence of the Young diagrams $\{\lambda(n)\}_{n=1}^{\infty}$ such that $\chi_{\lambda(n)}(g) \to \chi(g)$ as $n \to \infty$ for any fixed $g \in \mathfrak{S}_\infty$.

(ii) The sequence $\{\chi_{\lambda(n)}\}_{n=1}^{\infty}$ of irreducible characters converges pointwise iff the following limits exist:

$$\alpha_i := \lim_{n\to\infty} \frac{i\text{-th row of } \lambda(n)}{n} \text{ and } \beta_i := \lim_{n\to\infty} \frac{i\text{-th column of } \lambda(n)}{n}.$$

Moreover, the resulting limiting function in this case is precisely the indecomposable character $\chi_{\alpha,\beta}$ with Thoma parameters $\alpha = \{\alpha_i\}_{i=1}^{\infty}$ and $\beta = \{\beta_i\}_{i=1}^{\infty}$.

Block-diagonal embedding of symmetric groups

Fix arbitrary positive integers N and M. Then, define the block-diagonal embedding $\mathfrak{i} \colon \mathfrak{S}_N \hookrightarrow \mathfrak{S}_{MN}$ as follows: any $\sigma \in \mathfrak{S}_N$ is mapped to $\mathfrak{i}(\sigma) \in \mathfrak{S}_{MN}$ which is defined as

$$\mathfrak{i}(\sigma) = \begin{pmatrix} 1 & \dots & N & N+1 & \dots & 2N & \dots & MN \\ \sigma(1) & \dots & \sigma(N) & \sigma(1)+N & \dots & \sigma(N)+N & \dots & \sigma(N)+(M-1)N \end{pmatrix}.$$

To emphasize that $\mathfrak{i}$ depends on N and M we also denote it as $\mathfrak{i}_{N,MN}$.

The name "block-diagonal" is explained by the following illustration, which shows the permutation matrices of σ and $\mathfrak{i}_{N,MN}(\sigma)$:

$$[\sigma]_{N\times N} \xmapsto{\mathfrak{i}_{N,MN}} \begin{bmatrix} \sigma & & & \\ & \sigma & & \\ & & \ddots & \\ & & & \sigma \end{bmatrix}_{MN\times MN}$$

It is not difficult to see that these embeddings are compatible. This allows us to consider the inductive limit of symmetric groups with respect to embeddings $\{\mathfrak{i}_{N,MN}\}$.

The inductive limit associated to a sequence $\hat{\mathbf{n}}$

Now consider an arbitrary sequence $\hat{\mathbf{n}}$ of positive integers $\{n_j\}_{j=1}^{\infty}$ which are greater than 1. For such a sequence we define $N_k = n_1 n_2 \ldots n_k$.

Using the embeddings $\mathfrak{i}_{N,MN}$ we can associate to $\hat{\mathbf{n}}$ the corresponding inductive limit

$$\mathfrak{S}_{N_1} \xrightarrow{\mathfrak{i}_{N_1,N_2}} \mathfrak{S}_{N_2} \xrightarrow{\mathfrak{i}_{N_2,N_3}} \ldots \xrightarrow{\mathfrak{i}_{N_{k-1},N_k}} \mathfrak{S}_{N_k} \xrightarrow{\mathfrak{i}_{N_k,N_{k+1}}} \ldots$$

Denote the resulting group by $\mathfrak{S}_{\hat{\mathbf{n}}}$.

From now on we fix the sequence $\hat{\mathbf{n}}$ and for each i regard $\mathfrak{S}_{N_i}$ as a subgroup of $\mathfrak{S}_{\hat{\mathbf{n}}}$ using the embedding described above.

$\mathfrak{S}_{\hat{\mathfrak{n}}}$ as a subgroup of rational rearrangements of $[0,1)$

The group $\mathfrak{S}_{\hat{\mathfrak{n}}}$ can be naturally identified with a subgroup of certain measure--preserving transformations of the space $X = [0,1)$ equipped with the standard Lebesgue measure.

Indeed, let $\mathrm{Aut}_0(X,\nu)$ be the group of measure-preserving automorphisms of (X,ν). Then, to each element σ of $\mathfrak{S}_{N_i} \subset \mathfrak{S}_{\hat{\mathfrak{n}}}$ we can associate the following transformation $T(\sigma) \in \mathrm{Aut}_0(X,\nu)$:

$$T(\sigma)(x) = \left(x - \frac{k-1}{N_i}\right) + \frac{\sigma(k)-1}{N_i}, \text{ if } x \in \left[\frac{k-1}{N_i}, \frac{k}{N_i}\right), \quad k = 1\ldots, N_i.$$

In other words, $T(\sigma)$ permutes half-closed intervals $\left[\frac{k-1}{N_i}, \frac{k}{N_i}\right)$ via permutation σ.

Remark

This is a particular case of a group that is associated to path space of a Bratteli diagram. Our situation corresponds to the Bratteli diagram which has one vertex on each level and n_i edges between levels i and $i+1$.

Multiplicative characters of $\mathfrak{S}_{\hat{\mathbf{n}}}$

It is well known that symmetric groups $\mathfrak{S}_N$ admit only two one-dimensional representations (i.e. multiplicative characters), namely the trivial and the sign representation. There is an analogous statement for $\mathfrak{S}_{\hat{\mathbf{n}}}$.

Sign representation of $\mathfrak{S}_{\hat{\mathbf{n}}}$

For any $\sigma \in \mathfrak{S}_{N_l} \subset \mathfrak{S}_{\hat{\mathbf{n}}}$ define $\mathrm{sgn}_\infty(\sigma) = \lim_{k\to\infty} \mathrm{sgn}(\mathfrak{i}_{N_l,N_k}(\sigma))$.

Note that if the sequence $\hat{\mathbf{n}}$ contains infinitely many even integers, then $\mathrm{sgn}_\infty \equiv 1$.

Proposition

- If the sequence $\hat{\mathbf{n}}$ contains infinitely many even integers, then $\mathfrak{S}_{\hat{\mathbf{n}}}$ has no non-trivial multiplicative characters.
- If the sequence $\hat{\mathbf{n}}$ contains finitely many even integers, then the only non-trivial multiplicative character of $\mathfrak{S}_{\hat{\mathbf{n}}}$ is sgn_∞.

Natural character on $\mathfrak{S}_{\widehat{\mathfrak{n}}}$

The group $\mathfrak{S}_{\widehat{\mathfrak{n}}}$ possesses a natural character that comes from its action on X. Namely, for any $\sigma \in \mathfrak{S}_{N_i} \subset \mathfrak{S}_{\widehat{\mathfrak{n}}}$ define

$$\chi_{\mathrm{nat}}(\sigma) = \frac{\#\{k \in \{1, 2, \ldots, N_i\} : \sigma(k) = k\}}{N_i} = \nu(\mathrm{Fix}(T(\sigma))),$$

where $\mathrm{Fix}(Q) = \{x \in X : Qx = x\}$ is the set of fixed points of a transformation $Q \in \mathrm{Aut}_0(X, \nu)$.

One can give an explicit construction of a II_1-representation $(\pi, \mathcal{H})$ of $\mathfrak{S}_{\widehat{\mathfrak{n}}}$ with a cyclic unit vector $\xi \in \mathcal{H}$ such that

$$\chi_{\mathrm{nat}}(\sigma) = \langle \pi(\sigma)\xi, \xi \rangle_{\mathcal{H}}.$$

In particular, this implies that χ_{nat} is an indecomposable character of $\mathfrak{S}_{\widehat{\mathfrak{n}}}$.

Classification of indecomposable characters of $\mathfrak{S}_{\hat{\mathbf{n}}}$

Theorem 1 (Nessonov–N.)

The indecomposable characters of $\mathfrak{S}_{\hat{\mathbf{n}}}$ are the functions of the form $\chi = \chi_{\mathrm{nat}}^{p}$ or $\chi = \mathrm{sgn}_{\infty} \cdot \chi_{\mathrm{nat}}^{p}$ for $p \in \mathbb{N}_0 \cup \{\infty\}$.

Here, $\chi_{\mathrm{nat}}^{\infty} = \chi_{\mathrm{reg}}$ is the regular character, i.e. $(\chi_{\mathrm{nat}}^{\infty})(\sigma) = \begin{cases} 1, & \sigma = e, \\ 0, & \sigma \neq e. \end{cases}$

Previously known results for particular cases of the group $\mathfrak{S}_{\hat{\mathbf{n}}}$ or its analogues: see the works of Dudko (2011), Goryachko–Petrov (2011), Dudko–Medynets (2013*), and Leinen–Puglisi (2004).

In our work we obtain the description of all characters on discrete groups $\mathfrak{S}_{\hat{\mathbf{n}}}$ without any additional restrictions on the sequences $\{n_k\}_{k=1}^{\infty}$.

Isomorphism classes of groups $\mathfrak{S}_{\widehat{\mathbf{n}}}$

As an application of Theorem 1 one can give the complete classification of isomorphism classes of groups $\mathfrak{S}_{\widehat{\mathbf{n}}}$.

Corollary

Consider two sequences $\widehat{\mathbf{n}}' = \{n_j'\}_{j=1}^{\infty}$ and $\widehat{\mathbf{n}}'' = \{n_j''\}_{j=1}^{\infty}$ consisting of positive integers greater than 1. Then, the inductive limits $\mathfrak{S}_{\widehat{\mathbf{n}}'}$ and $\mathfrak{S}_{\widehat{\mathbf{n}}'}$ are isomorphic as groups if and only if for any prime p

$$\lim_{k\to\infty} v_p(N_k') = \lim_{k\to\infty} v_p(N_k''),$$

where $N_k' = n_1' n_2' \dots n_k'$, $N_k'' = n_1'' n_2'' \dots n_k''$ and v_p is the p-adic valuation.

Outline of the proof of Theorem 1

Proof of the main theorem has three parts:

- proving the approximation theorem: each indecomposable character of $\mathfrak{S}_{\widehat{\mathbf{n}}}$ is a weak limit of some sequence of irreducible characters of groups $\mathfrak{S}_{N_k}$;
- describing all weakly convergent sequences of characters of groups $\mathfrak{S}_{N_k}$ and computing the corresponding limits.
- checking that the resulting limiting functions are indeed indecomposable characters.

The first part can be considered as an application of the so-called *ergodic method* of Vershik and Kerov. Our proof of the approximation theorem relies on the connection between indecomposable characters and finite factor representations via the Gelfand–Naimark–Segal (GNS) construction.

In the second part we use the Okounkov–Vershik approach to representation theory of symmetric groups and also a deep result about the bounds for the characters of symmetric groups due to Roichmann.

On applications of the ergodic method

The idea of the ergodic method goes back to series of papers by Vershik and Kerov on asymptotic representation theory. It basically consists in approximating an infinite-dimensional object by its finite-dimensional analogues. Some examples:

- Vershik–Kerov: new proof of Thoma theorem on indecomposable characters of the infinite symmetric group $\mathfrak{S}_\infty = \varinjlim \mathfrak{S}_n$.
- Vershik–Kerov: description of indecomposable characters of the infinite unitary group $U_\infty = \varinjlim U_n$.
- Olshanski–Vershik: ergodic U_∞-invariant measures on the space of all infinite Hermitian matrices.

In fact, the problem of classification of indecomposable characters or spherical functions on a "big group" is often related to the description of ergodic invariant measures on certain spaces. Examples include the theorems of De Finetti and Schönberg.

The approximation theorem

In our work we prove directly a similar statement about the indecomposable characters of the group $\mathfrak{S}_{\widehat{\mathbf{n}}}$.

Proposition

For any indecomposable character χ of $\mathfrak{S}_{\widehat{\mathbf{n}}}$ there exists a sequence $\{\chi_{\lambda(k(l))}\}_{l=1}^{\infty}$ of irreducible characters of groups $\{\mathfrak{S}_{N_{k(l)}}\}_{l=1}^{\infty}$ that converges weakly (i.e. pointwise) to χ.

Remark

The irreducible characters of the symmetric group $\mathfrak{S}_N$ are parameterized by *partitions* $\lambda \vdash N$. As in Thoma's theorem, one can see how parameters (p in our case) for the indecomposable characters of $\mathfrak{S}_{\widehat{\mathbf{n}}}$ arise from the sequence $\lambda(k(l))$.

Partitions and Young diagrams

Definition

A *partition* of a positive integer N is a non-increasing sequence $\lambda = (\lambda_1, \dots, \lambda_\ell)$ of positive integers (parts) such that $\sum_i \lambda_i = N$. In this case $N = |\lambda|$ is called the *weight* of λ and $\ell = \ell(\lambda)$ is called the *length* of λ. If λ is a partition of N, we denote this by $\lambda \vdash N$.

The *Young diagram* associated to a partition $\lambda \vdash N$ is a collection of N boxes arranged in left-justified rows such that the i-th row consists of λ_i boxes (see examples below).

Young diagrams for $\lambda = (5,5,3,1,1)$ (left) and $\lambda' = (5,3,3,2,2)$ (right).

Flipping the Young diagram over its main diagonal gives another Young diagram. The corresponding partition is called the **conjugate** partition and is denoted by λ'.

A bound for the normalized characters of the symmetric group

Theorem (Roichmann, 1996)

There exist absolute constants $a \in (0,1)$ and $b > 0$ such that for any Young diagram λ with $N = |\lambda| \geqslant 4$ boxes and for any $\sigma \in \mathfrak{S}_N$ the following inequality holds:

$$\left|\chi_\lambda(\sigma)\right| \leq \left(\max\left\{\frac{\lambda_1}{N}, \frac{\lambda_1'}{N}, a\right\}\right)^{b|\operatorname{supp}_N(\sigma)|}.$$

Here λ_1 (λ_1') is the number of boxes in the first row (column) of the diagram λ and $\operatorname{supp}_N(\sigma)$ is the support of σ, i.e.

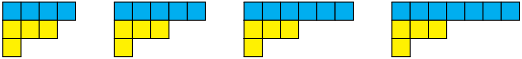
$$\operatorname{supp}_N(\sigma) = \{k \in \{1, 2, \ldots, N\} : \sigma(k) \neq k\}.$$

Calculation of the limits

Let χ be an indecomposable character of $\mathfrak{S}_{\widehat{\mathbf{n}}}$. The approximation theorem implies that there exists a sequence of normalized irreducible characters $\{\chi_{\lambda(k(l))}\}_{l=1}^{\infty}$ that converges weakly to χ. Essentially only three limiting behaviours are possible, as depicted below.

- (C_μ): the partition $\lambda(k_i)/(\lambda_1(k_i))$ is a fixed partition μ for all i.

$$\chi(\sigma) = \chi_{\text{nat}}(\sigma)^{|\mu|}$$



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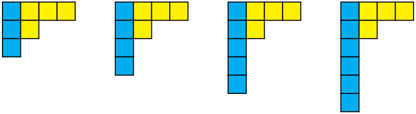
- $(\widehat{C}_\mu)$: the partition $\lambda'(k_i)/(1^{\lambda_1'(k_i)})$ is a fixed partition μ for all i.

$$\chi(\sigma) = \text{sgn}_\infty(\sigma)\chi_{\text{nat}}(\sigma)^{|\mu|}$$



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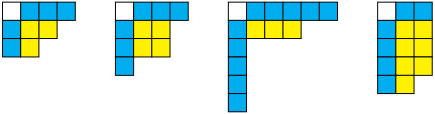
- (C_∞): $|\lambda(k_i)/\lambda_1(k_i)| \to \infty$ and $|\lambda'(k_i)/(1^{\lambda_1'(k_i)})| \to \infty$ as $i \to \infty$

$$\chi(\sigma) = \begin{cases} 1, & \sigma = e \\ 0, & \sigma \neq e \end{cases}$$



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Semidirect products $\mathbb{T}^N \rtimes \mathfrak{S}_N$

Denote $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

The symmetric group $\mathfrak{S}_N$ naturally acts on the torus $\mathbb{T}^N$ and hence, one can define the semidirect product $G_N = \mathbb{T}^N \rtimes \mathfrak{S}_N$. One view the elements of as permutation $N \times N$ matrices in which non-zero entires are replaced by arbitrary elements of $\mathbb{T}$.

Using the block-diagonal embeddings, one can define the inductive limit $G_{\widehat{\mathfrak{n}}} = \varinjlim G_{N_k}$. It contains $\mathfrak{S}_{\widehat{\mathfrak{n}}} = \varinjlim \mathfrak{S}_{N_k}$ and $\mathbb{T}^{\widehat{\mathfrak{n}}} = \varinjlim \mathbb{T}^{N_k}$ as subgroups.

The group $\mathbb{T}^{\widehat{\mathfrak{n}}}$ can be naturally identified with a certain subgroup of the group of measurable functions from $X = [0,1)$ to $\mathbb{T}$. Namely, an element $\mathfrak{t} = (t_1, \ldots, t_{N_i}) \in \mathbb{T}^{N_i}$ is identified with the following piecewise constant function:

$$\mathfrak{t}(x) = \begin{cases} t_1, & x \in \left[0, \frac{1}{N_i}\right), \\ \ldots & \ldots \\ t_{N_i}, & x \in \left[\frac{N_i-1}{N_i}, 1\right). \end{cases}$$

Classification of indecomposable characters on $G_{\hat{\mathfrak{n}}}$

The resulting group is not compact and one cannot use the conventional representation-theoretic tools. Nevertheless, the indecomposable characters on $G_{\hat{\mathfrak{n}}}$ can be still classified:

Theorem 2 (Nessonov–N.)

For any $M \in \mathbb{Z}$ define $\phi_M \colon G_{\hat{\mathfrak{n}}} \to \mathbb{C}$ by the formula

$$\phi_M(g) = \int_{\mathrm{Fix}(T(\sigma))} \mathfrak{t}(x)^M d\nu(x), \;\; g = (\mathfrak{t}, \sigma) \in G_{\hat{\mathfrak{n}}}.$$

Then, the indecomposable characters of the group $G_{\hat{\mathfrak{n}}}$ are the functions of the form χ_{reg}, or

$$\chi_{p,\mathbf{m}}(g) = \mathrm{sgn}_{\infty}^{p}(\sigma)\phi_{M_1}(g)\ldots\phi_{M_l}(g), \;\; g = (\mathfrak{t}, \sigma) \in G_{\hat{\mathfrak{n}}},$$

where $p \in \{0, 1\}$ and $\mathbf{m} = (M_1, \ldots, M_l)$ is an arbitrary finite non-increasing sequence of integers.

Some ideas of the proof of Theorem 2

The proof strategy is essentially the same: we use the ergodic method and compute the limits of irreducible characters.

The irreducible representations of the group G_N are all of the form

$$\pi_{\mathfrak{m},\rho} = \mathrm{Ind}_{\mathbb{T}^N \rtimes \mathrm{Stab}(\mathfrak{m})}^{G_N} \mathfrak{m} \otimes \rho,$$

where $\mathfrak{m} \in \mathbb{Z}^N$ is a one-dimensional multiplicative character of $\mathbb{T}^N$ and ρ is an irreducible representation of the stabilizer subgroup $\mathrm{Stab}(\mathfrak{m}) \subset \mathfrak{S}_N$.

Since the stabilizer $\mathrm{Stab}(\mathfrak{m})$ is the direct product of symmetric groups, one can use the Frobenius formula to express the character $\chi_{\mathfrak{m},\rho}$ of $\pi_{\mathfrak{m},\rho}$ in terms of characters of the symmetric groups.

Any indecomposable character χ on $G_{\widehat{\mathfrak{n}}}$ is a weak limit of a certain sequence of characters $\chi_{\mathfrak{m}_k,\rho_k}$. Analysis of the convergence shows that $\mathfrak{m}_k$ must of the form $(M_1, \dots, M_l, 0, \dots, 0)$ for all k. Computing limits in this case yields the formula for $\chi_{p,\mathbf{m}}$.

Related work in progress: classification of indecomposable characters of the unitary group $U_{\widehat{\mathfrak{n}}}$ (using the fact that G_N is the normalizer of $\mathbb{T}^N$ in U_N).

Thank you!