BIG ALGEBRA IN TYPE A FOR THE COORDINATE RING OF THE MATRIX SPACE

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Abstract. In this note we consider the big algebra recently introduced by Hausel for the GL_n-action on the coordinate ring of the matrix space $\mathrm{Mat}(n, r)$. In particular, we obtain explicit formulas for the big algebra generators in terms of differential operators with polynomial coefficients. We show that big algebras in type A are commutative and relate them to the Bethe subalgebra in the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$. We apply these results to big algebras of symmetric powers of the standard representation of GL_n.

Contents

1. Introduction

1.1. Recently, Hausel introduced in [5] the notion of the *big algebra* associated to an irreducible representation $V(\lambda)$ of a complex semi-simple Lie group G. In fact, the corresponding big algebra $\mathscr{B}(\lambda)$ can be seen as a certain maximal commutative subalgebra of the algebra of all G-equivariant polynomial maps from $\mathfrak{g} = \mathrm{Lie}(G)$ to $\operatorname{End} V(\lambda)$.

From the geometric point of view, big algebras were motivated by the study of the geometry of affine Schubert varieties. Namely, big algebras are isomorphic to the equivariant intersection cohomology of affine Schubert varieties. In particular, this fact can be used to define a ring structure on equivariant intersection cohomology for such varieties, which is not possible in general. From the algebraic side, big algebras resemble objects that are well-studied in mathematical physics and representation theory. For example, Gaudin spin chain models and Mischenko–Fomenko integrable systems, see [21]. Moreover, big algebras give a novel approach to representations of complex semi-simple Lie groups. For instance, Hausel in [5] suggests how one can read off from the spectrum of the big algebra a lot of representation-theoretic data, such as the weight diagram, the crystal structure and Lusztig's q-weight multiplicity polynomial.

In this paper we work with big algebras from a purely algebraic point of view. We study the big algebras associated to polynomial representations of the general linear group $G = \mathrm{GL}_n$. We obtain certain explicit formulas for the generators of the big algebra in this case. Using these, we give an independent proof of the commutativity of big algebras by relating them to *Bethe subalgebras* of the *Yangian* $\mathrm{Y}(\mathfrak{gl}_n)$. As an application of our results, we use them to study the big algebras associated to symmetric powers of the vector representation of GL_n. Before we state our results more precisely, we first review the definition of big algebras.

1.2. **Big algebras.** Let G be a connected complex reductive group with Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$. Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where $\mathfrak{h}$ is a Cartan subalgebra and $\mathfrak{n}_\pm$ are nilpotent subalgebras spanned by positive and negative root spaces. In particular, we fix a choice of a set of positive roots of $\mathfrak{g}$ and this defines the set of dominant weights. For any dominant integral weight λ let $V(\lambda)$ be the irreducible representation of the highest weight λ and denote by $\pi_\lambda \colon \mathfrak{g} \to \operatorname{End} V(\lambda)$ be the corresponding Lie algebra homomorphism.

Although in [5] big algebras are defined only for irreducible representations, it will be useful for us to define them in a more general setting. Let $\pi \colon \mathfrak{g} \to \operatorname{End} V$ be a representation of $\mathfrak{g}$ which is isomorphic to a direct sum of finite-dimensional representations. Then, the *Kirillov algebra* [12] of the representation (π, V) is defined as $\mathscr{C}(V) = (S(\mathfrak{g}^*) \otimes \operatorname{End} V)^G$ (here we use the adjoint action of G on $S(\mathfrak{g}^*)$). Alternatively, we can view the

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elements of the Kirillov algebras as polynomial maps $F\colon \mathfrak{g} \to \operatorname{End} V$ which satisfy the following equivariance condition:

$$F(\mathrm{Ad}(g)(X)) = \pi(g)F(X)\pi(g)^{-1},\ X \in \mathfrak{g}, g \in G.$$

Note that $\mathscr{C}(V)$ is an $S(\mathfrak{g}^*)^G$-algebra since $S(\mathfrak{g}^*)^G$ embeds into $\mathscr{C}(V)$ as the subalgebra of scalar operators, i.e. operators of the form $P \cdot \mathrm{Id}_V$ with $P \in S(\mathfrak{g}^*)^G$.

In general, Kirillov algebras need not be commutative. However, one can construct a large commutative subalgebra inside $\mathscr{C}(V)$ and this is the *big algebra* $\mathscr{B}(V)$ which we define next. The subalgebra $\mathscr{B}(V)$ can be defined by means of a certain "differential-like" operator $\mathbf{D} = \mathbf{D}_V$ acting on $\mathscr{C}(V)$ which we call the *Kirillov–Wei operator* [12, Section 1.2] and [23, Eq. (42) and (57)].

Fix a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$ (e.g. the Killing form in the case of simple $\mathfrak{g}$) and let $\{X_i\}_{i=1}^{\dim \mathfrak{g}}$ and $\{X^i\}_{i=1}^{\dim \mathfrak{g}}$ be two bases of $\mathfrak{g}$ dual with respect to this form. Then, for any $F \in \mathscr{C}(V)$ define

$$(\mathbf{D}F)(X) = \sum_{i=1}^{\dim \mathfrak{g}} \frac{\partial F}{\partial X_i}(X) \cdot \pi(X^i),$$

where $\frac{\partial F}{\partial X_i}$ denotes the directional derivative of $F\colon \mathfrak{g} \to \operatorname{End} V$ along $X_i \in \mathfrak{g}$. It turns out that in types A, B, C, D and G there exist homogeneous generators $c_1, \dots, c_r$ of the ring $S(\mathfrak{g}^*)^G$ such that the subalgebra generated by elements $\mathbf{D}^p(c_q)$, is commutative (see [5, Section 2.1]). The resulting commutative subalgebra of $\mathscr{C}(V)$ is called the *big algebra* of V and denoted by $\mathscr{B}(V)$. More precisely, we set

$$\mathscr{B}(V) = \big\langle \mathbf{D}^p(c_q) : 1 \le q \le r, 0 \le p \le \deg(c_q) \big\rangle \subset \mathscr{C}(V).$$

For any dominant integral weight λ we define $\mathscr{B}(\lambda) \coloneqq \mathscr{B}(V(\lambda))$ and $\mathscr{C}(\lambda) \coloneqq \mathscr{C}(V(\lambda))$.

A different and more conceptual way to define the big algebra is through the notion of the *Feigin–Frenkel center* (see also [5, Section 2] and [21, Section 4]). Let $\hat{\mathfrak{g}}$ be the affine Kac–Moody algebra assoicated to $\mathfrak{g}$. As a Lie algebra, $\hat{\mathfrak{g}}$ is a central extension of the loop algebra $\mathfrak{g}[t, t^{-1}]$. Then, the Feigin–Frenkel center $\mathfrak{z}(\hat{\mathfrak{g}})$ can be regarded as a certain commutative subalgebra of the universal enveloping algebra $U(\hat{\mathfrak{g}}_-)$, where $\hat{\mathfrak{g}}_- = t^{-1}\mathfrak{g}[t^{-1}]$ is the negative part of $\hat{\mathfrak{g}}$. The big algebra $\mathscr{B}(V)$ now can be defined as the image of $\mathfrak{z}(\hat{\mathfrak{g}})$ under certain homomorphism $\varrho_z\colon U(\hat{\mathfrak{g}}_-) \to S(\mathfrak{g}) \otimes \operatorname{End} V \simeq S(\mathfrak{g}^*) \otimes \operatorname{End} V$ which depends on a non-zero complex parameter z. The homomorphism ϱ_z acts on $\hat{\mathfrak{g}}_-$ as follows:

$$\varrho_z(x \otimes t^{-k}) = z^{-k} \cdot 1 \otimes \pi(x) + \delta_{k,1} \cdot x \otimes 1,\ x \in \mathfrak{g},\ k = 1, 2, \dots$$

One can show that the image $\varrho_z(\mathfrak{z}(\hat{\mathfrak{g}}))$ does not depend on the choice of $z \in \mathbb{C}^\times$, see [21, Section 4, Corollary 3]. This approach, however, does not give a simple way to compute big algebras since the formulas for the generators of $\mathfrak{z}(\hat{\mathfrak{g}}_-)$ are rather complicated (see [16, Ch. 7–8]). Besides that, the commutativity of big algebras then relies on deep results of Feigin and Frenkel [4] on $\mathfrak{z}(\hat{\mathfrak{g}}_-)$. One of the motivations for our work was to obtain a more elementary proof of the commutativity of big algebras (in type A) using the definition involving the Kirillov–Wei operator.

1.3. **Main results.** In this paper we consider big algebras for $\mathfrak{g} = \mathfrak{gl}_n$. We fix a positive integer r and consider the coordinate ring $\mathcal{P}(n,r) = \mathbb{C}[\mathrm{Mat}(n,r)]$ of the affine space $\mathrm{Mat}(n,r)$ of complex $n \times r$ matrices. There is a natural action of $\mathrm{GL}_n \times \mathrm{GL}_r$ on $\mathrm{Mat}(n,r)$:

$$(g,h) \cdot A = (g^T)^{-1} A h^{-1}.$$

Using this action one can endow $\mathcal{P}(n,r)$ with the structure of $\mathrm{GL}_n \times \mathrm{GL}_r$-module. Moreover, one can give explicitly the decomposition of $\mathcal{P}(n,r)$ into the direct sum of irreducible $\mathrm{GL}_n \times \mathrm{GL}_r$-representations. Namely, by the *Howe duality* [8, Section 2.1.2] we have

$$\mathbb{C}[\mathrm{Mat}(n,r)] = \mathcal{P}(n,r) \simeq \bigoplus_{\lambda} V_{\mathrm{GL}_n}(\lambda) \otimes V_{\mathrm{GL}_r}(\lambda),$$

where λ runs over all partitions with length $\ell(\lambda) \le \min\{r,n\}$, while $V_{\mathrm{GL}_n}(\lambda)$ and $V_{\mathrm{GL}_n}(\lambda)$ denote irreducible representations of the highest weight λ of groups GL_n and GL_r, respectively. Note that if $r = n$, then $\mathcal{P}(n,r) = \mathcal{P}(n,n)$ contains all polynomial irreducible representations of GL_n.

In particular, we can regard $\mathcal{P}(n,r)$ as a $\mathfrak{gl}_n$-module and can consider the big algebra $\mathscr{B}(\mathcal{P}(n,r))$. It is not difficult to verify that the elements of $\mathscr{B}(\mathcal{P}(n,r))$ can be viewed as certain polynomial differential operators on $\mathrm{Mat}(n,r)$.

We find explicit formulas for the generators of $\mathscr{B}(\mathcal{P}(n,r))$ using direct calculations and the Kirillov–Wei operator $\mathbf{D}$ (Theorem 3.10 and Corollary 3.11). Using these formulas and the Capelli identities, we identify the generators of $\mathscr{B}(\mathcal{P}(n,r))$ with some elements of the Bethe subalgebra of the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$. As a consequence, we obtain that the big algebra $\mathscr{B}(\mathcal{P}(n,r))$ is commutative (Theorem 7.1). It follows from the big algebra construction that for any $V(\lambda)$ appearing in the decomposition of $\mathcal{P}(n,r)$ there is a surjective algebra homomorphism $\mathscr{B}(\mathcal{P}(n,r)) \twoheadrightarrow \mathscr{B}(\lambda)$. Therefore, the discussion above implies the following fact.

Theorem 1.1 (Corollary 7.2)**.** *The big algebra of any polynomial finite-dimensional irreducible representation of* GL_n *is commutative.*

The approach outlined above has an advantage of considering big algebras of all polynomial representations of $\mathfrak{gl}_n$ simultaneously. Besides that, our proof of commutativity relies only on direct calculations and thus, avoids the use of more complicated constructions such as the Feigin–Frenkel center.

Using the explicit formulas we can give a more explicit description of the big algebra $\mathscr{B}(m\varpi_1)$ for any positive integer m (Proposition 8.6). Here ϖ_1 is the first fundamental weight of $\mathfrak{g} = \mathfrak{gl}_n$ which corresponds to the n-dimensional vector representation $V(\varpi_1)$ of $\mathfrak{gl}_n$. In particular, $V(m\varpi_1)$ is isomorphic to the m-th symmetric power $S^m(V(\varpi_1))$. This description allows us to prove completely algebraically the following fact, which was first established geometrically by Hausel [5, Eq. (4.3)] using the technique of *fixed point schemes* from [6].

Theorem 1.2 (Theorem 8.9)**.** *There exists an isomorphism of* $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$*-algebras* $\mathscr{B}(m\varpi_1) \simeq S^m(\mathscr{B}(\varpi_1))$*.*

Let us also mention that the representations $V(m\varpi_1)$, $m = 1, 2, \ldots$, of $\mathfrak{g} = \mathfrak{gl}_n$ are particular examples of *weight multiplicity free representations.* Recall that a representation $V(\lambda)$ of $\mathfrak{g}$ is called weight multiplicity free, if all summands in its weight decomposition

$$V(\lambda) = \bigoplus_{\mu} V_\mu(\lambda), \text{ where } V_\mu(\lambda) = \{v \in V(\lambda) : \pi_\lambda(h)v = \lambda(h)v \text{ for all } h \in \mathfrak{h}\},$$

are at most one-dimensional. In these cases the Kirillov algebra was studied by several authors, see e.g. [12], [13], [19] and [20]. In particular, it is known that $\mathscr{C}(\lambda)$ is commutative (see also Subsection 8.1). Using the result announced by Hausel [5, Theorem 1.1], one can then deduce that in the weight multiplicity free case the big algebra $\mathscr{B}(\lambda)$ coincides with the Kirillov algebra $\mathscr{C}(\lambda)$.

The discussion above implies that $\mathscr{B}(m\varpi_1) = \mathscr{C}(m\varpi_1)$. Thus, the Kirillov-Wei operator $\mathbf{D}$ leaves the big algebra $\mathscr{B}(m\varpi_1)$ invariant (but this is not the case in general). We were able to derive a formula for $\mathbf{D}$ on $\mathscr{B}(m\varpi_1)$ in terms of the description from Proposition 8.6, see Subsection 8.6. The formula for $\mathbf{D}$ turns out to be related to some constructions from symmetric function theory and the ring of diagonal invariants. Although we do not yet have a conceptual explanation of these phenomena, we believe that these observations might be of independent interest.

1.4. **Contents.** Now let us briefly outline the contents of this paper.

In Section 2 we fix the notation and recall the necessary facts about the representation theory of $\mathfrak{gl}_n$ and its action on $\mathcal{P}(n, r) = \mathbb{C}[\mathrm{Mat}(n, r)]$.

In Section 3 we recall the notions of the Kirillov algebra and the big algebra. Then, we state the explicit formulas for the generators of big algebra of the coordinate ring of $\mathrm{Mat}(n, r)$ (see Theorem 3.10 and Corollary 3.11).

Section 4 is rather technical and is devoted to proofs of Theorem 3.10 and Corollary 3.11.

Sections 5–7 contain the proof of the commutativity of big algebras. In Section 5 we review the Capelli identity and its variants and in Section 6 we we recall the construction of a certain commutative subalgebra of $U(\mathfrak{gl}_n)$, called *Bethe subalgebra* following Molev [15, Section 1.14]. In Section 7 we prove the commutativity of the big algebra (in type A) using the explicit formulas obtained in Section 3 (Corollary 3.11) and the results from Sections 5 and 6.

In Section 8 we use the formulas obtained in Section 3 to prove several results about the big algebras $\mathscr{B}(m\varpi_1)$ including Proposition 8.6, which gives a description of $\mathscr{B}(m\varpi_1)$ in terms of certain functions on the weight diagram of $V(m\varpi_1)$, and Theorem 8.9 on the isomorphism between $\mathscr{B}(m\varpi_1)$ and $S^m(\mathscr{B}(\varpi_1))$. We obtain a presentation of $\mathscr{B}(m\varpi_1)$ in terms of generators and relations (see Theorem 8.11 and Corollary 8.13) and compare it to the previous computations of Hausel–Rychlewicz [6], Rozhkovskaya [20] and Hausel [5] (see Subsection 8.5). Besides that, we also prove a different formula for the Kirillov–Wei operator $\mathbf{D}$ on $\mathscr{B}(m\varpi_1)$ (see Proposition 8.18).

Finally, in Appendix A we prove Lemma 8.10 which was used in the proof of Theorem 8.9.

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2. Notation and preliminaries

Most of the proofs in this note involve many direct calculations. To simplify the formulas we introduce the following notations.

2.1. **Operations with tuples.** For every positive integer m we denote $[m] \coloneqq \{1, \ldots, m\}$ and let $\mathfrak{S}_m$ be the symmetric group of $[m]$. For any integer k such that $0 \le k \le m$ we define $\binom{[m]}{k}$ to be the set of all k-element subsets of $[m]$ and $[m]^{\underline{k}}$ to be the set of all k-tuples which consist of k distinct elements of m. Clearly,

$$\#\binom{[m]}{k} = \binom{m}{k}, \ \#[m]^{\underline{k}} = m^{\underline{k}},$$

where $m^{\underline{k}}$ is the so-called *falling factorial*:

$$m^{\underline{k}} = m(m-1)\dots(m-k+1) = k! \cdot \binom{m}{k}.$$

We also denote by $[m]^k = [m]^{[k]}$ the set of all k-tuples with entries in $[m]$. It is occasionally convenient to view a k-tuple $I = (i_1, \dots, i_k)$ as a function on the set $[k] = \{1, \dots, k\}$, namely, we set $I(l) = i_l$ for $l \in [k]$.

2.1.1. *Action of the symmetric group.* For any k-tuple $I = (i_1, \dots, i_k)$ and any permutation $\pi \in \mathfrak{S}_k$ define

$$\pi I = (i_{\pi^{-1}(1)}, \dots, i_{\pi^{-1}(k)}). \tag{2.1}$$

Regarding I as a funciton on $[k]$ we can alternatively write $\pi I = I \circ \pi^{-1}$.

Remark 2.1. Using this group action we can identify $\binom{[m]}{k}$ with quotient $\mathfrak{S}_k \backslash [m]^{\underline{k}}$. In particular, we will often regard a k-element subset as a k-tuple with arbitrarily chosen ordering (and in these situations the choice of ordering will be irrelevant).

2.1.2. *Sign functions for tuples.* For any $I, J \in [n]^{\underline{k}}$ define the generalized sign function as follows:

$$\operatorname{sgn}\begin{pmatrix} I \\ J \end{pmatrix} = \begin{cases} \operatorname{sgn}(\tau), & \text{if as sets } I = J, \\ 0, & \text{otherwise,} \end{cases}$$

where in the first case τ is the unique permutation in $\mathfrak{S}_k$ that maps the k-tuple J to I. In other words, τ satisfies $I = \tau \cdot J$.

Remark 2.2. Clearly, this agrees with the usual sign of permutation in the case when both I and J are permutations of $(1, 2, \dots, n)$. For instance, if $I = (1, \dots, n)$ and $J = (\tau(1), \dots, \tau(n))$ for $\tau \in \mathfrak{S}_n$, then

$$\operatorname{sgn}\begin{pmatrix} I \\ J \end{pmatrix} = \operatorname{sgn}\begin{pmatrix} 1 & 2 & \dots & n \\ \tau(1) & \tau(2) & \dots & \tau(n) \end{pmatrix} = \operatorname{sgn}(\tau).$$

Now assume that we have tuples $I_1, \dots, I_k$ and $J_1, \dots, J_k$ such that $I_l, J_l \in [n]^{\underline{p_l}}$ for each $l = 1, \dots, k$ and some positive integers $p_1, \dots, p_k$. We assume additionally that $I_1, \dots, I_k$ are disjoint as sets and similarly for $J_1, \dots, J_k$. Denote also $p = p_1 + \dots + p_k$. Then, we define

$$\operatorname{sgn}\begin{pmatrix} I_1 & \dots & I_k \\ J_1 & \dots & J_k \end{pmatrix} = \operatorname{sgn}\begin{pmatrix} I \\ J \end{pmatrix},$$

where I and J are p-tuples obtained by concatenating $I_1, \dots, I_k$ and $J_1, \dots, J_k$, respectively. For example, if $k = 2$, then

$$I(s) = \begin{cases} I_1(s), & s \in \{1, \dots, p_1\}, \\ I_2(s - p_1), & s \in \{p_1 + 1, \dots, p_1 + p_2\}, \end{cases} \quad \text{and } J_1(s) = \begin{cases} J_1(s), & s \in \{1, \dots, p_1\}, \\ J_2(s - p_1), & s \in \{p_1 + 1, \dots, p_1 + p_2\}. \end{cases}$$

In our calculations we also use another variant of the signature function. For any tuples $I_1, J_1 \in [m]^{\underline{p}}$ and $I_2, J_2 \in [m]^{\underline{q}}$ with $p \geq q$ define

$$\varepsilon(I_1, J_1, I_2, J_2) = \begin{cases} \operatorname{sgn}(\tau_1 \tau_2), & \text{if as sets } I_1 \setminus I_2 = J_1 \setminus J_2 \in \binom{[m]}{p-q}, \\ 0, & \text{otherwise.} \end{cases}$$

Here in the first case τ_1 and τ_2 are elements of $\mathfrak{S}_p$ that satisfy the following equalities:

$$\tau_1 I_1|_{[q]} = I_2, \;\; \tau_2 J_1|_{[q]} = J_2, \;\; \text{and } \tau_1 I_1|_{\{p+1,\dots,q\}} = \tau_2 J_1|_{\{p+1,\dots,q\}}.$$

Note that a pair (τ_1, τ_2) is not defined uniquely in general. However, for any other such pair (τ_1', τ_2') there exists an element $\sigma \in \mathfrak{S}_p$ which fixes each element of $[q]$ and such that $(\tau_1', \tau_2') = (\sigma\tau_1, \sigma\tau_2)$. In particular, $\operatorname{sgn}(\tau_1\tau_2) = \operatorname{sgn}(\tau_1'\tau_2')$ and, consequently, $\varepsilon(I_1, J_1, I_2, J_2)$ is well defined.

Observe that both generalized sign functions are skew-symmetric in the sense of the following lemma.

Lemma 2.1. *For any tuples $I, J \in [m]^{\underline{p}}$ and any permutations $\sigma, \tau \in \mathfrak{S}_p$ one has*

$$\operatorname{sgn}\begin{pmatrix} \sigma I \\ \tau J \end{pmatrix} = \operatorname{sgn}(\sigma\tau) \operatorname{sgn}\begin{pmatrix} I \\ J \end{pmatrix}.$$

Similarly, for any $I_1, J_1 \in [m]^{\underline{p}}$, $I_2, J_2 \in [m]^{\underline{q}}$ with $p \geq q$ and any permutations $\sigma_1, \tau_1 \in \mathfrak{S}_p$, $\sigma_2, \tau_2 \in \mathfrak{S}_q$ one has

$$\varepsilon(\sigma_1 I_1, \tau_1 J_1, \sigma_2 I_2, \tau_2 J_2) = \operatorname{sgn}(\sigma_1\tau_1) \operatorname{sgn}(\sigma_2\tau_2) \cdot \varepsilon(I_1, J_1, I_2, J_2).$$

2.2. **Non-commutative matrices.** We often work with matrices whose entries are elements of non-commutative (associative) algebras. In particular, many computations involve non-commutative versions of determinants.

Let $A = [a_{ij}]_{i,j=1}^N$ be an $N \times N$ matrix with entries in a certain non-commutative algebra. We define for matrix A its

- *row determinant*:
$$\mathrm{rdet}(A) = \sum_{\sigma \in \mathfrak{S}_N} \mathrm{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \dots a_{N,\sigma(N)},$$
- *column determinant*:
$$\mathrm{cdet}(A) = \sum_{\sigma \in \mathfrak{S}_N} \mathrm{sgn}(\sigma) a_{\sigma(1),1} a_{\sigma(2),2} \dots a_{\sigma(N),N},$$
- *symmetrized determinant*:
$$\mathrm{symdet}(A) = \sum_{\sigma,\tau \in \mathfrak{S}_N} \mathrm{sgn}(\sigma\tau) a_{\sigma(1),\tau(1)} a_{\sigma(2),\tau(2)} \dots a_{\sigma(N),\tau(N)}.$$

Observe that if the entries of A do commute, then the row and the column determinants coincide with the usual one while for the symmetrized version one has $\mathrm{symdet}(A) = N! \cdot \det(A)$. Finally, note that row (column) determinant is skew-symmetric with respect to columns (rows) while the symmetrized determinant is skew-symmetric with respect to both rows and columns.

For any $N \times N$ matrix M and any tuples $I = (i_1, \dots, i_k) \in [N]^k$ and $J = (j_1, \dots, j_l) \in [N]^l$ we denote by M_{IJ} the following $k \times l$ matrix:
$$M_{IJ} = [M_{i_\alpha, j_\beta}]_{\alpha \in [k], \beta \in [l]} = \begin{bmatrix} M_{i_1 j_1} & \dots & M_{i_1 j_l} \\ \vdots & \ddots & \vdots \\ M_{i_k j_1} & \dots & M_{i_k j_l} \end{bmatrix}.$$
In the case, when the entries of I and J are strictly increasing, M_{IJ} is a submatrix of M.

2.3. **The Lie algebra $\mathfrak{gl}_n$.** Let $\mathfrak{g} = \mathfrak{gl}_n$ be the general linear Lie algebra of complex $n \times n$ matrices. We denote by $\{E_{ij}\}_{i,j=1}^n$ the standard basis of $\mathfrak{gl}_n$ consisting of matrix units. They satsify the following commutation relations:
$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj}. \tag{2.2}$$
Denote also by $\{y_{ij}\}_{i,j=1}^n$ the corresponding coordinates on $\mathfrak{gl}_n$. In other words, $y_{ij} \in \mathfrak{gl}_n^*$ and for any $Y \in \mathfrak{gl}_n$ we have
$$Y = \sum_{i,j=1}^n y_{ij}(Y) \cdot E_{ij}.$$
In particular, we can view the algebra $\mathbb{C}[\mathfrak{gl}_n] = S(\mathfrak{gl}_n^*)$ as the polynomial ring $\mathbb{C}[y_{ij}, 1 \le i, j \le n]$.

Let $\mathfrak{h} = \mathrm{span}\{E_{ii}\}_{1 \le i \le n}$ be the Cartan subalgebra consisting of diagonal matrices. Let $\mathfrak{n}_+ = \mathrm{span}\{E_{ij}\}_{1 \le i < j \le n}$ and $\mathfrak{n}_- = \mathrm{span}\{E_{ij}\}_{1 \le j < i \le n}$ be the nilpotent subalgebras consisting of upper-triangular and lower-triangular matrices, respectively. Thus, we obtain a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$.

2.4. **The coordinate ring of the matrix space.** Consider the coordinate ring $\mathcal{P}(n,r) = \mathbb{C}[\mathrm{Mat}(n,r)]$ of the space $\mathrm{Mat}(n,r)$ of complex $n \times r$ matrices. Denote by $\{x_{ij} : 1 \le i \le n, 1 \le j \le r\}$ the standard coordinates on $\mathrm{Mat}(n,r)$. Then, $\mathcal{P}(n,r)$ is the polynomial ring $\mathbb{C}[x_{ij}, 1 \le i \le n, 1 \le j \le r]$ in rn variables. We denote by ∂_{ij} the partial derivative with respect to variable x_{ij}.

2.4.1. *The action of $\mathfrak{gl}_n$.* Note that the matrix space $\mathrm{Mat}(n,r)$ possesses the following GL_n-action:
$$(g, A) \mapsto (g^{-1})^T \cdot A, \ g \in \mathrm{GL}_n, A \in \mathrm{Mat}(n,r).$$
This action induces a GL_n-action on the coordinate ring $\mathcal{P}(n,r) = \mathbb{C}[\mathrm{Mat}(n,r)]$. Namely, for any $P \in \mathcal{P}(n,r)$ we have
$$(g \cdot P)(A) = P(g^T \cdot A), \ g \in \mathrm{GL}_n, A \in \mathrm{Mat}(n,r).$$
Denote by $\widetilde{L}(g)$ the corresponding linear operator in $\mathrm{End}\, \mathcal{P}(n,r)$. Differentiating this action along one-parameter subgroups in GL_n yields the infinitesimal $\mathfrak{gl}_n$-action on $\mathcal{P}(n,r)$, which we denote by L. Direct calculation shows that
$$L(E_{ij}) = \sum_{\alpha=1}^r x_{i\alpha} \partial_{j\alpha}, \ 1 \le i, j \le n. \tag{2.3}$$

2.4.2. *The algebra $\mathcal{PD}(n,r)$ of differential operators on $\mathcal{P}(n,r)$.* Since L is a representation of $\mathfrak{gl}_n$ it also gives rise to a representation of the universal enveloping algebra $U(\mathfrak{gl}_n)$ on $\mathcal{P}(n,r)$. In particular, the formula for $L(E_{ij})$ above implies that the elements of $U(\mathfrak{gl}_n)$ act on $\mathcal{P}(n,r)$ as differential operators with polynomial coefficients.

Let $\mathcal{PD}(n,r)$ be the (non-commutative) algebra of differential operators on $\mathcal{P}(n,r)$ with polynomial coefficients. In other words, $\mathcal{PD}(n,r)$ is the Weyl algebra generated by $\{x_{i\alpha}, \partial_{i\alpha} : 1 \le i \le n, 1 \le \alpha \le r\}$ and relations of the form

$$[\partial_{i\alpha}, x_{j\beta}] = \delta_{ij}\delta_{\alpha\beta},\ 1 \le i,j \le n, 1 \le \alpha,\beta \le r.$$

We will need the following fact in Section 5.

Proposition 2.2. *If $r = n$, then the map $L\colon U(\mathfrak{gl}_n) \to \mathcal{PD}(n,r)$ is injective.*

Remark 2.3. In fact, this follows from the fact that $U(\mathfrak{gl}_n)$ is isomorphic to the algebra of all left-invariant differential operators on GL_n. In general, the image of L coincides with the subalgebra of all differential operators in $\mathcal{PD}(n,r)$ which commute with the GL_r-action on $\mathbb{C}[\mathrm{Mat}(n,r)]$. See also [9, Section 1].

Denote by E, X and D the matrices, whose (i,j)-th entry equals E_{ij}, x_{ij} and ∂_{ij}, respectively. In particular, we have a formal identity

$$L(E) = X \cdot D^T.$$

3. Kirillov algebra, big algebra and medium algebra on $\mathcal{P}(n,r)$

3.1. **Invariant polynomials on $\mathfrak{gl}_n$.** For each $0 \le k \le n$ define the following element of $S(\mathfrak{gl}_n^*)$:

$$c_k(Y) = \mathrm{tr}(\Lambda^k Y),\ Y \in \mathfrak{gl}_n,$$

where $\Lambda^k Y$ is the operator in $\mathrm{End}\,\Lambda^k(\mathbb{C}^n)$ which is induced by the natural action of $Y \in \mathfrak{gl}_n$ on $\mathbb{C}^n$, i.e.

$$\Lambda^k Y \colon v_1 \wedge v_2 \wedge \ldots \wedge v_k \mapsto Yv_1 \wedge Yv_2 \wedge \ldots \wedge Yv_k,\ v_i \in \mathbb{C}^n.$$

We also set $c_0(Y) \equiv 1$.

Alternatively, one can define the elements c_k as the coefficients of the characteristic polynomial of Y:

$$\det(Y - z \cdot \mathrm{Id}_n) = \sum_{k=0}^{n} (-1)^{n-k} c_k(Y) \cdot z^{n-k}$$

Denote by y_{ij} the coordinates on $\mathfrak{gl}_n$ which correspond to standard matrix units $E_{ij} \in \mathfrak{gl}_n$. Then, $Y = [y_{ij}]_{i,j=1}^n$ and

$$c_k(Y) = \sum_{I \in \binom{[n]}{k}} \det Y_{II}. \tag{3.1}$$

It is well-known that the elements $c_1, \ldots, c_n$ generate the ring of GL_n-invariants of $S(\mathfrak{gl}_n^*)$.

Proposition 3.1. *The ring $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$ is a free polynomial ring in $c_1, \ldots, c_n$.*

3.2. **Construction of the Kirillov algebra.** Let $\widetilde{\pi}\colon \mathrm{GL}_n \to V$ be a finite-dimensional representation of the group GL_n and let $\pi\colon \mathfrak{gl}_n \to \mathrm{End}\,V$ be the associated representation of the Lie algebra $\mathfrak{gl}_n$.

Definition 3.2. The *Kirillov algebra*[1] of V is defined as the algebra $\mathscr{C}(V) = (S(\mathfrak{gl}_n^*) \otimes \mathrm{End}\,V)^{\mathrm{GL}_n}$.

In other words, the Kirillov algebra is the algebra of GL_n-equivariant polynomial maps from $\mathfrak{gl}_n$ to $\mathrm{End}\,V$, i.e. any $F \in \mathscr{C}(V)$ satisfies the following equivariance condition:

$$F(\mathrm{Ad}(g)(Y)) = \widetilde{\pi}(g)F(Y)\widetilde{\pi}(g)^{-1},\ g \in \mathrm{GL}_n, Y \in \mathfrak{gl}_n, \tag{3.2}$$

Note that $\mathscr{C}(V)$ is an algebra over the ring $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$. The elements of $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$ are realized inside $\mathscr{C}(V)$ as scalar operators.

Define the so-called *Kirillov–Wei operator* $\mathbf{D} = \mathbf{D}_V$ which acts on $\mathscr{C}(V)$ as follows: for any $F \in \mathscr{C}(V)$ set

$$(\mathbf{D}F)(Y) = \sum_{i,j=1}^{n} \frac{\partial F}{\partial y_{ji}}(Y) \cdot \pi(E_{ij}). \tag{3.3}$$

It follows from the definition that for any positive integer p we have

$$(\mathbf{D}^p F)(Y) = \sum_{i_1,\ldots,i_p=1}^{n} \sum_{j_1,\ldots,j_p=1}^{n} \frac{\partial^p F}{\partial y_{j_1 i_1} \ldots \partial y_{j_p i_p}}(Y) \cdot \pi(E_{i_1 j_1} \ldots E_{i_p j_p}). \tag{3.4}$$

In [12, Lemma 1] and [13, Section 1.4] Kirillov hinted at the following fact (see also [23, Proposition 5.2]):

Proposition 3.3. *The operator $\mathbf{D}$ maps $\mathscr{C}(V)$ to itself.*

[1] Also known as the *classical family algebra*, see Kirillov's original papers [12], [13].

Proof. We first prove the following lemma.

Lemma 3.4. *For any $g \in \mathrm{GL}_n$ and any $1 \le k, l \le n$ we have the identity*

$$\sum_{i,j=1}^{n} [\mathrm{Ad}(g)(E_{ji})]_{lk} \cdot \mathrm{Ad}(g)(E_{ij}) = E_{kl}.$$

Proof. Recall that on $\mathfrak{gl}_n$ there is a non-degenerate GL_n-invariant pairing, namely the *trace form* $(A, B) \mapsto \mathrm{tr}(AB)$. Denote the element of $\mathfrak{gl}_n$ on the left-hand side by A. The equality $A = E_{kl}$ is equivalent to $\mathrm{Ad}(g)^{-1}(A) = \mathrm{Ad}(g)^{-1}(E_{kl})$. In order to prove the latter, it suffices to verify that

$$\mathrm{tr}(\mathrm{Ad}(g)^{-1}(A)E_{pq}) = \mathrm{tr}(\mathrm{Ad}(g^{-1})(E_{kl})E_{pq}) \text{ for all } 1 \le p, q \le n.$$

Indeed, $\mathrm{tr}(\mathrm{Ad}(g^{-1})(E_{kl})E_{pq}) = \mathrm{tr}(E_{kl}\,\mathrm{Ad}(g)(E_{pq})) = [\mathrm{Ad}(g)(E_{pq})]_{kl}$. On the other hand,

$$\mathrm{tr}(\mathrm{Ad}(g)^{-1}(A)E_{pq}) = \sum_{i,j=1}^{n} [\mathrm{Ad}(g)(E_{ji})]_{lk} \cdot \mathrm{tr}(E_{ij}E_{pq}) = \sum_{i,j=1}^{n} [\mathrm{Ad}(g)(E_{ji})]_{lk} \cdot \delta_{iq}\delta_{jp} = [\mathrm{Ad}(g)(E_{pq})]_{lk},$$

which concludes the proof of the lemma. □

Now let us return to the proof of Proposition 3.3. We know that for any $g \in \mathrm{GL}_n$ and $Y \in \mathfrak{gl}_n$ the identity $(F \circ \mathrm{Ad}(g^{-1}))(Y) = \widetilde{\pi}(g)F(Y)\widetilde{\pi}(g)^{-1}$ holds. Hence,

$$\frac{\partial}{\partial y_{ji}}\Big((F(\mathrm{Ad}(g)(Y)\Big) = \widetilde{\pi}(g) \cdot \frac{\partial F}{\partial y_{ji}}(Y) \cdot \widetilde{\pi}(g)^{-1}.$$

Then, we obtain

$$\begin{aligned}
\widetilde{\pi}(g) \cdot (\mathbf{D}F)(Y) \cdot \widetilde{\pi}(g)^{-1} &= \sum_{i,j=1}^{n} \widetilde{\pi}(g)\frac{\partial F}{\partial y_{ji}}(Y) \cdot \pi(E_{ij})\widetilde{\pi}(g)^{-1} = \sum_{i,j=1}^{n} \widetilde{\pi}(g)\frac{\partial F}{\partial y_{ji}}(Y)\widetilde{\pi}(g)^{-1} \cdot \pi(\mathrm{Ad}(g)(E_{ij})) = \\
&= \sum_{i,j=1}^{n} \frac{\partial}{\partial y_{ji}}\Big((F(\mathrm{Ad}(g)(Y)\Big) \cdot \pi(\mathrm{Ad}(g)(E_{ij})) = \sum_{i,j=1}^{n}\sum_{k,l=1}^{n} \frac{\partial F}{\partial y_{lk}}(\mathrm{Ad}(g)(Y)) \cdot [\mathrm{Ad}(g)(E_{ji})]_{lk} \cdot \pi(\mathrm{Ad}(g)(E_{ij})) = \\
&= \sum_{k,l=1}^{n} \frac{\partial F}{\partial y_{lk}}(\mathrm{Ad}(g)(Y)) \cdot \pi(E_{kl}) = (\mathbf{D}F)(\mathrm{Ad}(g)(Y)),
\end{aligned}$$

due to the lemma above. Thus, $\mathbf{D}F$ satisfies (3.2) and hence, belongs to $\mathscr{C}(V)$. □

Remark 3.1. One can generalize the construction of the Kirillov algebra, the operator $\mathbf{D}$ and the results of this subsection for any semi-simple Lie algebra $\mathfrak{g}$. The corresponding operator $\mathbf{D}$ can be described using a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$ (e.g. the Killing form in the case of simple $\mathfrak{g}$).

The explicit formulas for specific elements of $\mathscr{C}(V)$ are often quite complicated. However, these formulas can be simplified if one restrict elements of Kirillov algebra to Cartan subalgebra $\mathfrak{h}$. Moreover, these restrictions uniquely determine the elements of $\mathscr{C}(V)$:

Proposition 3.5. *The restriction map $\Phi = \Phi_V : F \mapsto F|_{\mathfrak{h}}$, $F \in \mathscr{C}(V)$ is an injective homomorphism of algebras over $S(\mathfrak{gl}_n^*)^{\mathrm{GL}_n} \simeq S(\mathfrak{h}^*)^W$. Moreover, the image of Φ is contained inside the algebra*

$$(S(\mathfrak{h}^*) \otimes \mathrm{End}_{\mathfrak{h}} V)^W = \left(\bigoplus_{\mu \in \mathrm{wt}(V)} S(\mathfrak{h}^*) \otimes \mathrm{End}\, V_\mu\right)^W.$$

Here $W \simeq \mathfrak{S}_n$ denotes the Weyl group of GL_n, $\mathrm{End}_{\mathfrak{h}} V$ is the space of all $\mathfrak{h}$-equivariant linear operators on V, and $\mathrm{wt}(V)$ is the set of weights of V.

Proof. We use the argument outlined in [12, Theorem 2]. It follows from the equality (3.2) that the restriction $F|_{\mathfrak{h}}$ completely determines the map $F : \mathfrak{gl}_n \to \mathrm{End}\, V$ on the set of all regular semi-simple elements of $\mathfrak{gl}_n$ (those are conjugate to elements of $\mathfrak{h}$). Since the latter set is Zariski dense in $\mathfrak{gl}_n$, the injectivity of Φ_V follows.

The second statement follows from the fact that the Weyl group W is the normalizer of $\mathfrak{h}$ in GL_n and the equivariance condition (3.2) applied for $g \in W$. □

The following statement shows that in order to study the Kirillov algebra of V one can study the Kirillov algebra of a "larger" representation.

Proposition 3.6. *Let $V = \bigoplus_\alpha V_\alpha$ be a direct sum of GL_n-modules. Then, the natural algebra homomorphism $\prod_\alpha \mathscr{C}(V_\alpha) \to \mathscr{C}(V)$ induced by the embeddings $\iota_\alpha : \mathrm{End}(V_\alpha) \hookrightarrow \mathrm{End}(V)$ is injective.*

Proof. The map in question sends a family of maps $F_\alpha : \mathfrak{gl}_n \to \mathrm{End}\, V_\alpha$ to the map $\sum_\alpha \iota_\alpha \circ F_\alpha : \mathfrak{gl}_n \to \mathscr{C}(V)$. Note that this expression is well defined since for every $Y \in \mathfrak{gl}_n$ the summand $(\iota_\alpha \circ F_\alpha)(Y)$ belongs to $\iota_\alpha(\mathrm{End}\, V_\alpha)$. In particular, this map is indeed an algebra homomorphism. The injectivity now follows from the injectivity of the maps ι_α. □

3.3. **Big and medium algebras.** First we recall the definitions of the big and the medium algebras in type A.

Definition 3.7. Let V be a finite-dimensional GL_n-module. The *big algebra* $\mathscr{B}(V)$ is the subalgebra of $\mathscr{C}(V)$ generated by the elements

$$M_{p,q} = \mathbf{D}^q(c_{p+q}),\ 0 \le p, q \le n,\ p+q \le n. \tag{3.5}$$

The *medium algebra* $\mathscr{M}(V)$ of $\mathcal{P}(n,r)$ is a subalgebra of $\mathscr{B}(V)$ defined as follows:

$$\mathscr{M}(V) = \langle F, \mathbf{D}(F) : F \in S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n} \rangle \subset \mathscr{B}(V). \tag{3.6}$$

Remark 3.2. In fact, the constructions of Kirillov, big and medium algebras described above also apply for arbitrary direct sums of finite-dimensional representations of GL_n. A similar approach can be used for complex semi-simple Lie groups of types B, C, D and G, see [5, Sections 1–2].

Now let us give a more explicit formula for the generators of $\mathscr{B}(V)$. Recall that $\pi\colon \mathfrak{gl}_n \to \operatorname{End} V$ is the representation of $\mathfrak{gl}_n$ which comes from the GL_n-action on V.

Proposition 3.8. *For any $Y \in \mathfrak{gl}_n$ we have*

$$M_{p,q}(Y) = \mathbf{D}^q(c_{p+q})(Y) = \sum_{\substack{I_1, J_1 \in \binom{[n]}{p} \\ I_2, J_2 \in \binom{[n]}{q} \\ I_1 \sqcup I_2 = J_1 \sqcup J_2}} \operatorname{sgn}\begin{pmatrix} I_1 & I_2 \\ J_1 & J_2 \end{pmatrix} \det Y_{I_1 J_1} \cdot \operatorname{symdet} \pi(E)_{J_2 I_2}. \tag{3.7}$$

Here $\pi(E)$ stands for the $n \times n$ matrix whose (i,j)-entry equals $\pi(E_{ij})$.

Proof. We have

$$c_{p+q}(Y) = \sum_{I \in \binom{[n]}{p+q}} \sum_{\sigma \in \mathfrak{S}_{p+q}} \operatorname{sgn}(\sigma) \prod_{s \in [p+q]} y_{i_s i_{\sigma(s)}},$$

where the first summation runs over all $(p+q)$-element subsets $I = \{i_1, \ldots, i_{p+q}\}$ of $[n]$. Here we identify $\binom{[n]}{p+q}$ with $\mathfrak{S}_{p+q} \backslash [n]^{\underline{p+q}}$ (see Remark 2.1). Applying the operator $\mathbf{D}^q$ (see also (3.4)) we obtain

$$\begin{aligned}
\mathbf{D}^q(c_{p+q})(Y) &= \sum_{I \in \binom{[n]}{p+q}} \sum_{\sigma \in \mathfrak{S}_{p+q}} \operatorname{sgn}(\sigma) \sum_{V \in \binom{[p+q]}{q}} \prod_{v \in V^c} y_{i_v i_{\sigma(v)}} \sum_{\tau \in \mathfrak{S}(V)} \prod_{v \in V} \pi(E_{i_{\sigma(\tau(v))} i_{\tau(v)}}) = \\
&= \sum_{I \in \binom{[n]}{p+q}} \sum_{V \in \binom{[p+q]}{q}} \sum_{[\sigma] \in \mathfrak{S}_{p+q}/\mathfrak{S}(V^c)} \sum_{\tau_1 \in \mathfrak{S}(V^c)} \operatorname{sgn}(\sigma\tau_1) \prod_{v \in V^c} y_{i_v i_{\sigma(\tau_1(v))}} \sum_{\tau_2 \in \mathfrak{S}(V)} \prod_{v \in V} \pi(E_{i_{\sigma(\tau_2(v))} i_{\tau_2(v)}}) = \\
&= \sum_{I \in \binom{[n]}{p+q}} \sum_{V \in \binom{[p+q]}{q}} \sum_{[\sigma] \in \mathfrak{S}_{p+q}/\mathfrak{S}(V^c)} \det(Y_{I(V^c), (\sigma^{-1}I)(V^c)}) \operatorname{sgn}(\sigma) \sum_{\tau_2 \in \mathfrak{S}(V)} \prod_{v \in V} \pi(E_{i_{\sigma(\tau_2(v))} i_{\tau_2(v)}}) = \\
&= \sum_{I \in \binom{[n]}{p+q}} \sum_{V \in \binom{[p+q]}{q}} \sum_{[\sigma] \in \mathfrak{S}_{p+q}/\mathfrak{S}(V) \times \mathfrak{S}(V^c)} \operatorname{sgn}(\sigma) \det(Y_{I(V^c), (\sigma^{-1}I)(V^c)}) \operatorname{symdet} \pi(E)_{(\sigma^{-1}I)(V), I(V)},
\end{aligned}$$

where we denoted $V^c = [p+q] \setminus V$. To conclude the proof it remains to note that the summation over $I \in \binom{[n]}{p+q}$, $V \in \binom{[p+q]}{p}$ and $[\sigma] \in \mathfrak{S}_{p+q}/\mathfrak{S}(V) \times \mathfrak{S}(V^c)$ is equivalent to the summation over I_1, I_2, J_1 and J_2 as in (3.7). Indeed, if we put

$$I_1 = I(V^c),\ J_1 = (\sigma^{-1}I)(V^c),\ I_2 = I(V),\ J_2 = (\sigma^{-1}I)(V),$$

we would get the right-hand side of (3.7) since $\operatorname{sgn}(\sigma) = \operatorname{sgn}\binom{I_1\ I_2}{J_1\ J_2}$. $\square$

The next proposition relates the big (resp. medium) algebras of direct sums with big (resp. medium) algebras of summands.

Proposition 3.9. *(i) The image of the algebra homomorphism $\prod_\alpha \mathscr{C}(V_\alpha) \to \mathscr{C}(V)$ from Proposition 3.6 contains the big algebra $\mathscr{B}(V)$.*
(ii) For each α there exist surjective homomorphisms $\mathscr{B}(V) \to \mathscr{B}(V_\alpha)$ and $\mathscr{M}(V) \to \mathscr{M}(V_\alpha)$. These homomorphisms are induced by the natural algebra homomorphisms $\prod_{\alpha'} \operatorname{End} V_{\alpha'} \to \operatorname{End} V_\alpha$.

Proof. The definition of the Kirillov–Wei operator implies the subalgebra $\prod_\alpha \mathscr{C}(V_\alpha)$ of $\mathscr{C}(V)$ is invariant under the action of $\mathbf{D}_V$. Moreover, $\mathbf{D}_V$ acts on $\prod_\alpha \mathscr{C}(V_\alpha)$ in the same way as the operator $\prod_\alpha \mathbf{D}_{V_\alpha}$. The first part now follows from the fact that $c_1, \ldots, c_n \in S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$ are realized inside $\mathscr{C}(V)$ as scalar operators and the identity $\mathbf{D}_V^p(c_k \otimes \mathrm{id}_V) = \prod_\alpha \mathbf{D}_{V_\alpha}^p(c_k \otimes \mathrm{id}_{V_\alpha})$ for all $p \ge 1$. The required homomorphisms $\mathscr{B}(V) \to \mathscr{B}(V_\alpha)$ and $\mathscr{M}(V) \to \mathscr{M}(V_\alpha)$ are induced by the projection homomorphisms $\prod_{\alpha'} \mathscr{C}(V_{\alpha'}) \to \mathscr{C}(V_\alpha)$. $\square$

3.4. **Kirillov algebra for $\mathcal{P}(n,r)$.** There are natural actions of the groups GL_n and GL_r on the matrix space $\mathrm{Mat}(n,r)$ that induce actions of these groups on $\mathcal{P}(n,r)$. Namely, for any $A \in \mathrm{Mat}(n,r)$, $g \in \mathrm{GL}_n$ and $h \in \mathrm{GL}_r$ we have

$$\widetilde{L}(g)(P)(A) = P(g^T \cdot A),\ \widetilde{R}(h)(P)(A) = P(A \cdot h).$$

Note that these two actions commute and give rise to an action of $\mathrm{GL}_n \times \mathrm{GL}_r$ on $\mathcal{P}(n,r)$. Then, *Howe duality* (see [8, Section 2.1.2]) implies that the $\mathrm{GL}_n \times \mathrm{GL}_r$-module $\mathcal{P}(n,r) = \mathbb{C}[\mathrm{Mat}(n,r)]$ decomposes as follows:

$$\mathcal{P}(n,r) \simeq \bigoplus_{\lambda\,:\,\ell(\lambda)\le r} V_{\mathrm{GL}_n}(\lambda) \otimes V_{\mathrm{GL}_r}(\lambda), \tag{3.8}$$

where the summation is over all partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ of length at most r (i.e. with $\lambda_i = 0$ for $i > r$). Here $V_{\mathrm{GL}_n}(\lambda)$ and $V_{\mathrm{GL}_r}(\lambda)$ denote the irreducible representations of highest weight λ of GL_n and GL_r, respectively. In particular, we conclude that $\mathcal{P}(n,r)$ contains as subrepresentations all irreducible representations of $\mathfrak{gl}_n$ that have highest weight λ with $\ell(\lambda) \le r$.

Remark 3.3. In fact, one can describe explicitly the highest weight vectors of the $\mathfrak{gl}_n$-module $\mathcal{P}(n,r) = \mathbb{C}[\mathrm{Mat}(n,r)]$. By [26, Ch. VII, §49] (see also [7, Proposition 3]) the highest weight vectors of $\mathcal{P}(n,r)$ are of the form

$$\begin{vmatrix} x_{i_1 1} \end{vmatrix}^{p_1} \cdot \begin{vmatrix} x_{i_1 1} & x_{i_1 2} \\ x_{i_2 1} & x_{i_2 2} \end{vmatrix}^{p_2} \cdot \ldots \cdot \begin{vmatrix} x_{i_1 1} & \ldots & x_{i_1 l} \\ \vdots & \ddots & \vdots \\ x_{i_l 1} & \ldots & x_{i_l l} \end{vmatrix}^{p_l},$$

where $0 \le l \le \min\{r,n\}$, $p_1, \ldots, p_l$ are arbitrary non-negative integers and $i_1, \ldots, i_l$ are arbitrary distinct elements of $\{1, \ldots, r\}$. The element above is of weight $(p_1 + \ldots + p_l, \ldots, p_{l-1} + p_l, p_l, 0, \ldots, 0)$.

3.5. **Generators of the big algebra of $\mathcal{P}(n,r)$.** From now on we consider the case $V = \mathcal{P}(n,r)$ and the corresponding Kirillov and big algebras. Proposition 3.9 implies that many properties of big algebras of irreducible representations can be read off from $\mathscr{B}(\mathcal{P}(n,r))$.

Observe that the big algebra $\mathscr{B}(\mathcal{P}(n,r))$ is contained in $S(\mathfrak{gl}_n^*) \otimes \mathcal{PD}(n,r)$. In other words, the elements of $\mathscr{B}(\mathcal{P}(n,r))$ are certain differential operators on $\mathcal{P}(n,r)$ whose coefficients are polynomials in variables $x_{i\alpha}$ and y_{jk}. One of the main results of this note is an explicit formula (the so-called *normal form*) for the operators $M_{p,q}$.

Theorem 3.10. *The normal form of the operator $M_{p,q}(Y)$ is as follows:*

$$\begin{aligned} M_{p,q}(Y) = &\sum_{\ell=0}^{q} (-1)^{q-\ell} (q-\ell)!\, \ell! \begin{Bmatrix} q \\ \ell \end{Bmatrix} \times \\ &\times \sum_{\substack{I_1, J_1 \in \binom{[n]}{p} \\ I_2, J_2 \in \binom{[n]}{q} \\ I_1 \sqcup I_2 = J_1 \sqcup J_2}} \operatorname{sgn} \begin{pmatrix} I_1 & I_2 \\ J_1 & J_2 \end{pmatrix} \det Y_{I_1 J_1} \sum_{R \in \binom{[r]}{\ell}} \sum_{V, W \in \binom{[q]}{\ell}} \varepsilon(J_2, I_2, J_2(V), I_2(W)) \det(X_{J_2(V),R}) \det(D_{I_2(W),R}). \end{aligned} \tag{3.9}$$

Here, for tuples I and J we denote by X_{IJ}, Y_{IJ} and D_{IJ} the corresponding submatrices of $X = [x_{ij}]_{i,j=1}^{n,r}$, $D = [\partial_{ij}]_{i,j=1}^{n,r}$ and $Y = [y_{ij}]_{i,j=1}^{n,n}$, respectively.

Remark 3.4. $\left\{ {k \atop \ell} \right\}$ stands for the *Stirling number of the second kind*, i.e. the number of ways to split a k-element set into ℓ non-empty subsets.

Corollary 3.11. *The big algebra $\mathscr{B}(\mathcal{P}(n,r))$ is generated by the operators $\{F_{p,q} : p, q \ge 0, p+q \le n\}$, where*

$$F_{p,q}(Y) = \sum_{\substack{I_1, J_1 \in \binom{[n]}{p} \\ I_2, J_2 \in \binom{[n]}{q} \\ I_1 \sqcup I_2 = J_1 \sqcup J_2}} \operatorname{sgn} \begin{pmatrix} I_1 & I_2 \\ J_1 & J_2 \end{pmatrix} \det Y_{I_1 J_1} \sum_{R \in \binom{[r]}{q}} \det(X_{J_2,R}) \det(D_{I_2,R}) \tag{3.10}$$

In particular, the operators $\{M_{p,q} : p, q \ge 0, p+q \le n\}$ and $\{F_{p,q} : p, q \ge 0, p+q \le n\}$ are related to each other in the following way:

$$M_{p,q} = \sum_{\ell=0}^{q} (-1)^{q-\ell} (q-\ell)!\, \ell! \begin{Bmatrix} q \\ \ell \end{Bmatrix} \binom{n-p-\ell}{q-\ell} F_{p,\ell}. \tag{3.11}$$

We prove these formulas for $M_{p,q}$ and $F_{p,q}$ in the next section.

Remark 3.5. Note that both sets $\{F_{p,0}\}_{p=1}^n$ and $\{M_{p,0}\}_{p=1}^n$ are generator sets for $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$.

The following fact is an immediate consequence of the explicit formulas from Corollary 3.11.

Corollary 3.12. *The big algebra $\mathscr{B}(\mathcal{P}(n,r))$ is generated by operators $\{F_{p,q} : 1 \le p \le n, 0 \le q \le r, p+q \le n\}$.*

Proof. Indeed, it follows from (3.10) that all operators $F_{p,q}$ with $q > r$ are zero. □

3.6. **Restriction to the Cartan subalgebra.** The formulas (3.9) and (3.10) become simpler if we restrict $Y \in \mathfrak{gl}_n$ to Cartan subalgebra $\mathfrak{h}$, i.e. to diagonal matrices.

Proposition 3.13. *For $Y = \operatorname{diag}(z_1, \dots, z_n) \in \mathfrak{h}$ we have*

$$F_{p,q}(Y) = \sum_{\substack{I \in \binom{[n]}{p} \\ J \in \binom{[n]}{q} \\ I \cap J = \varnothing}} \left(\prod_{i \in I} z_i \right) \left(\sum_{R \in \binom{[r]}{q}} \det(X_{JR}) \det(D_{JR}) \right).$$

Proof. This is a consequence of Corollary 3.11 and the observation that for $Y \in \mathfrak{h}$ the determinant $\det Y_{I_1 J_1}$ vanishes unless $I_1 = J_1$. □

We use this formula in the proof of commutativity of the big algebra (see Theorem 7.1).

4. Proofs of Theorem 3.10 and Corollary 3.11

In this section we prove Theorem 3.10 and Corollary 3.11, i.e. formulas (3.9) and (3.10). The proof is purely computational and reduces to some identities in the Weyl algebra.

4.1. **Computation of the symmetrized determinants.** In view of Proposition 3.8, to prove Theorem 3.10 we first need to get a formula for $\operatorname{symdet} L(E)_{IJ}$. Recall that for k-tuples $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_k)$ in $[n]^{\underline{k}}$ the symmetrized determinant $\operatorname{symdet} L(E_{IJ})$ is defined as (see (2.3))

$$\begin{aligned} \operatorname{symdet} L(E_{IJ}) &= \sum_{\sigma, \tau \in \mathfrak{S}_k} \operatorname{sgn}(\sigma\tau) L(E_{i_{\sigma(1)} i_{\tau(1)}}) \dots L(E_{i_{\sigma(k)} i_{\tau(k)}}) = \\ &= \sum_{a_1, \dots, a_k = 1}^{n} \sum_{\sigma, \tau \in \mathfrak{S}_k} \operatorname{sgn}(\sigma\tau) \cdot x_{i_{\sigma(1)} a_1} \partial_{j_{\tau(1)} a_1} \dots x_{i_{\sigma(k)} a_k} \partial_{j_{\tau(k)} a_k}. \end{aligned}$$

Our aim is to obtain a reduced expression (the normal form) for $\operatorname{symdet} L(E_{IJ})$ in the algebra $\mathcal{PD}(n, r)$ of differential operators on $\mathfrak{gl}_n$ with polynomial coefficients. The computation is divided into several steps.

We start with the following auxiliary identity.

Lemma 4.1. *Let N be a positive integer. For any permutations $\sigma, \tau \in \mathfrak{S}_N$ and any $s \in \{1, \dots, N\}$ define*

$$\delta_s(\sigma, \tau) = \begin{cases} 0, & \text{if } \sigma(s) \le \tau(s), \\ 1, & \text{if } \sigma(s) > \tau(s). \end{cases}$$

Then, for any $\alpha_1, \dots, \alpha_N \in \mathbb{C}$ the following identity holds:

$$\sum_{\sigma, \tau \in \mathfrak{S}_N} \operatorname{sgn}(\sigma\tau) \cdot (\alpha_1 + \delta_1(\sigma, \tau)) \dots (\alpha_N + \delta_N(\sigma, \tau)) = (-1)^{N-1} (N-1)! (\alpha_1 + \dots + \alpha_N). \tag{4.1}$$

Proof. The identity clearly holds for $N = 1$, so let us assume that $N \ge 2$. Observe that the left-hand side can be expressed as

$$\sum_{\ell=0}^{N} C_\ell \cdot e_\ell(\alpha_1, \dots, \alpha_N),$$

where $e_\ell(\alpha_1, \dots, \alpha_N)$ is the ℓ-th elementary symmetric polynomial in $\alpha_1, \dots, \alpha_N$ and $C_0, \dots, C_\ell$ are certain real numbers. Therefore, to check the identity (4.1) it sufices to calculate the coefficients C_ℓ for every $0 \le \ell \le N$. Note that C_ℓ is equal to the coefficient of the term $\alpha_1 \dots \alpha_\ell$ of the left-hand side. Then, it is not difiicult to see that

$$C_\ell = \sum_{\sigma, \tau \in \mathfrak{S}_N} \operatorname{sgn}(\sigma\tau) \prod_{s=\ell+1}^{N} \delta_s(\sigma, \tau).$$

In other words, C_ℓ is the sum of $\operatorname{sgn}(\sigma\tau)$, where (σ, τ) runs over all pairs of permutations in $\mathfrak{S}_N$ such that

$$\sigma(s) > \tau(s) \text{ for all } \ell + 1 \le s \le N.$$

Let $\Gamma_\ell \subset \mathfrak{S}_N \times \mathfrak{S}_N$ be the set of all such pairs. Now let us consider several cases:

- $\ell = 0$. In this case we have

  $$C_0 = \sum_{\sigma, \tau \in \mathfrak{S}_N} \operatorname{sgn}(\sigma\tau) \cdot \delta_1(\sigma, \tau) \dots \delta_N(\sigma, \tau) = 0$$

  because for any $\sigma, \tau \in \mathfrak{S}_N$ at least one of $\delta_s(\sigma, \tau)$ is zero (for example one can take $s = \sigma^{-1}(1)$).

- $\ell = 1$. We claim that in this case the set $\Gamma_\ell = \Gamma_1$ contains exactly $(N-1)!$ elements. Indeed, by definition a pair $(\sigma, \tau) \in \mathfrak{S}_N \times \mathfrak{S}_N$ belongs to Γ_1 if $\sigma(s) > \tau(s)$ for every $s \in \{2, \ldots, N\}$. One checks that this holds if and only if these permutations satisfy $\sigma(1) = 1$, $\tau(1) = N$ and $\sigma(s) = \tau(s) + 1$ for all $s = 2, \ldots, N$. In particular, $|\Gamma_1| = (N-1)!$ and for any $(\sigma, \tau) \in \Gamma_1$ the permutation $\sigma\tau^{-1}$ is the cycle $(1\ 2\ \ldots\ N)$. Hence,

$$C_1 = \sum_{(\sigma,\tau)\in\Gamma_1} \operatorname{sgn}(\sigma\tau) = |\Gamma_1| \cdot (-1)^{N-1} = (-1)^{N-1} \cdot (N-1)!.$$

- $2 \le \ell \le N$. Note that for any permutations $\sigma', \tau' \in \mathfrak{S}_N$ which fix each of $\ell+1, \ldots, N$ the pair (σ, τ) belongs to Γ_ℓ if and only if $(\sigma\sigma', \tau\tau') \in \Gamma_\ell$. This and the equality

$$\sum_{\sigma',\tau'\in\mathfrak{S}_\ell} \operatorname{sgn}(\sigma'\tau') = \left(\sum_{\sigma'\in\mathfrak{S}_\ell} \operatorname{sgn}\sigma'\right)\left(\sum_{\tau'\in\mathfrak{S}_\ell} \operatorname{sgn}\tau'\right) = 0$$

  imply that $C_\ell = 0$ for all $2 \le \ell \le N$.

Combining everything, we obtain

$$C_\ell = \begin{cases} 0, & \ell \neq 1, \\ (-1)^{N-1}(N-1)!, & \ell = 1, \end{cases}$$

which is equivalent to (4.1). $\square$

Corollary 4.2. *Let N be a positive integer and let $\ell \in \{0, 1, \ldots, N\}$. Then, in the notation of Lemma 4.1 we have*

$$\sum_{\sigma,\tau\in\mathfrak{S}_N} \operatorname{sgn}(\sigma\tau)(\alpha_1 + \delta_1(\sigma,\tau)) \ldots (\alpha_\ell + \delta_\ell(\sigma,\tau)) = \begin{cases} 0, & \ell \in \{0, 1, \ldots, N-2\}, \\ (-1)^{N-1}(N-1)!, & \ell = N-1, \\ (-1)^{N-1}(N-1)! \cdot (\alpha_1 + \ldots + \alpha_N), & \ell = N. \end{cases}$$

Remark 4.1. For $\ell = 0$ we define the left-hand side as $\sum_{\sigma,\tau\in\mathfrak{S}_N} \operatorname{sgn}(\sigma\tau)$.

Proof. This follows from (4.1) after taking derivatives of both sides with respect to $\alpha_{\ell+1}, \ldots, \alpha_N$. $\square$

Next we prove the following identity in the Weyl algebra which essentially computes the symmetrized determinant $\operatorname{symdet} L(E)_{IJ}$ in the simplest case $r = 1$.

Lemma 4.3. *Consider the Weyl algebra generated by variables $u_1, \ldots, u_n$ and the corresponding partial derivatives $\partial_1, \ldots, \partial_n$. Let $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$ be two k-tuples in $[n]^{\underline{k}}$. Assume that I and J have ℓ common elements, $0 \le \ell \le k$. Denote*

$$\Psi(I, J) = \operatorname{symdet}\left([u_{i_\alpha}\partial_{j_\beta}]_{\alpha,\beta=1}^k\right) = \sum_{\sigma,\tau\in\mathfrak{S}_k} \operatorname{sgn}(\sigma\tau) u_{i_{\sigma(1)}}\partial_{j_{\tau(1)}} \ldots u_{i_{\sigma(k)}}\partial_{j_{\tau(k)}}. \tag{4.2}$$

Then, $\Psi(I, J)$, as an element of the Weyl algebra, can be simplified as follows:

- *if $\ell = k$ and $\sigma \in \mathfrak{S}_k$ is such that $i_l = j_{\sigma(l)}$ for all $l \in \{1, \ldots, k\}$, then $\Psi(I, J) = (-1)^{k-1}(k-1)! \cdot \operatorname{sgn}(\sigma) \sum_{i\in I} u_i\partial_i$;*
- *if $\ell = k-1$ and $\pi, \sigma \in \mathfrak{S}_k$ are such that $i_{\pi(l)} = j_{\sigma(l)}$ for all $l \in \{1, \ldots, k-1\}$ and $i_{\pi(k)} \neq j_{\sigma(k)}$, then $\Psi(I, J) = (-1)^{k-1}(k-1)! \cdot \operatorname{sgn}(\pi\sigma) u_{i_{\pi(k)}}\partial_{j_{\sigma(k)}}$;*
- *if $\ell \le k-2$, then $\Psi(I, J) = 0$.*

Moreover, viewing I and J as functions on $[k] = \{1, \ldots, k\}$ one can express $\Psi(I, J)$ as follows:

$$\Psi(I, J) = (-1)^{k-1}(k-1)! \sum_{\pi\in\mathfrak{S}_k} \sum_{s=1}^{k} \mathbb{1}(I|_{[k]\setminus\{s\}} = \pi^{-1}J|_{[k]\setminus\{s\}}) \cdot \operatorname{sgn}(\pi) u_{i_s}\partial_{j_{\pi(s)}}.$$

Here, for an assertion A we put $\mathbb{1}(A) = 1$, if A holds, and $\mathbb{1}(A) = 0$ otherwise.

Proof. We start with the following observation: for any permutation $\pi \in \mathfrak{S}_k$ we have $\Psi(\pi I, \sigma J) = \operatorname{sgn}(\pi\sigma) \cdot \Psi(I, J)$, where $\pi I = (i_{\pi^{-1}(1)}, \ldots, i_{\pi^{-1}(k)})$ and $\sigma J = (j_{\sigma^{-1}(1)}, \ldots, j_{\sigma^{-1}(k)})$. Therefore, it suffices to prove the statement in the case when $i_1 = j_1, \ldots, i_\ell = j_\ell$ and

$$\{i_1, \ldots, i_k\} \cap \{j_1, \ldots, j_k\} = \{i_1, \ldots, i_\ell\}.$$

Denote $K = (i_1, \ldots, i_\ell) = (j_1, \ldots, j_\ell)$. To simplify the notation let us also assume that $K = (1, \ldots, \ell)$. Now note that elements $u_{i_{\ell+1}}, \ldots, u_{i_k}$ and $\partial_{j_{\ell+1}}, \ldots, \partial_{j_k}$ commute with each other and also with u_i and ∂_i for $i \in K$. Therefore, we can rewrite $\Psi(I, J)$ as

$$\Psi(I, J) = u_{i_{\ell+1}} \ldots u_{i_k} \partial_{j_{\ell+1}} \ldots \partial_{j_k} \cdot \Phi(\ell; K), \tag{4.3}$$

where $\Phi(\ell;K)$ is a certain element in the Weyl algebra generated by u_i, ∂_i with $i \in K$. Namely, $\Phi(\ell;K)$ is obtained from the expression for $\Phi(I,J)$ by removing all u_i and ∂_i with $i \notin K$. One finds that the action of $\Phi(\ell;K)$ on a monomial $u_1^{\alpha_1}\dots u_\ell^{\alpha_\ell}$, where $\alpha_1,\dots,\alpha_\ell \in \mathbb{N}_0$, is given by

$$\Phi(\ell;K)(u_1^{\alpha_1}\dots u_\ell^{\alpha_\ell}) = \left(\sum_{\sigma,\tau\in\mathfrak{S}_k} \operatorname{sgn}(\sigma\tau)(\alpha_1+\delta_1(\sigma^{-1},\tau^{-1}))\dots(\alpha_\ell+\delta_\ell(\sigma^{-1},\tau^{-1}))\right)\cdot u_1^{\alpha_1}\dots u_\ell^{\alpha_\ell},$$

where the δ_i's are defined as in Lemma 4.1. Indeed, this follows from the identities

$$(u_i\partial_i)(u_1^{\alpha_1}\dots u_\ell^{\alpha_\ell}) = \alpha_i\cdot u_1^{\alpha_1}\dots u_\ell^{\alpha_\ell},\ (\partial_i u_i)(u_1^{\alpha_1}\dots u_\ell^{\alpha_\ell}) = (\alpha_i+1)\cdot u_1^{\alpha_1}\dots u_\ell^{\alpha_\ell},\ 1\le i\le \ell.$$

Now observe that

$$\sum_{\sigma,\tau\in\mathfrak{S}_k} \operatorname{sgn}(\sigma\tau)(\alpha_1+\delta_1(\sigma^{-1},\tau^{-1}))\dots(\alpha_\ell+\delta_\ell(\sigma^{-1},\tau^{-1})) = \sum_{\sigma,\tau\in\mathfrak{S}_k} \operatorname{sgn}(\sigma\tau)(\alpha_1+\delta_1(\sigma,\tau))\dots(\alpha_\ell+\delta_\ell(\sigma,\tau)),$$

and hence Corollary 4.2 implies that

- if $0\le\ell\le k-2$, then $\Phi(\ell;K)(u_1^{\alpha_1}\dots u_\ell^{\alpha_\ell}) = 0$;
- if $\ell = k-1$, then $\Phi(\ell;K)(u_1^{\alpha_1}\dots u_\ell^{\alpha_\ell}) = (-1)^{k-1}(k-1)!\cdot u_1^{\alpha_1}\dots u_\ell^{\alpha_\ell}$;
- if $\ell = k$, then $\Phi(\ell;K)(u_1^{\alpha_1}\dots u_\ell^{\alpha_\ell}) = (-1)^{k-1}(k-1)!\cdot(\alpha_1+\dots+\alpha_k)u_1^{\alpha_1}\dots u_\ell^{\alpha_\ell}$.

Since the elements of Weyl algebra are uniquely defined by their action on the corresponding polynomial ring $\mathbb{C}[u_1,\dots,u_n]$ we obtain

$$\Phi(\ell;K) = \begin{cases} 0, & 0\le\ell\le k-2,\\ (-1)^{k-1}(k-1)!, & \ell = k-1,\\ (-1)^{k-1}(k-1)!\sum_{i\in K} u_i\partial_i, & \ell = k. \end{cases}$$

Combining this with (4.3) concludes the proof. □

Lemma 4.4. *Fix $a_1,\dots,a_k \in \{1,\dots,r\}$. For any two k-tuples $I=(i_1,\dots,i_k)$ and $J=(j_1,\dots,j_k)$ of distinct elements of $\{1,\dots,n\}$ denote*

$$\Delta(I,J) = \sum_{\sigma,\tau\in\mathfrak{S}_k} \operatorname{sgn}(\sigma\tau)\cdot x_{i_{\sigma(1)}a_1}\partial_{j_{\tau(1)}a_1}\dots x_{i_{\sigma(k)}a_k}\partial_{j_{\tau(k)}a_k}.$$

Then, we have the following identity

$$\Delta(I,J) = (-1)^{k-\ell}(k-\ell)!\sum_{V,W\in\binom{[k]}{\ell}} \varepsilon(I,J,I(V),J(W))\det(X_{I(V),R})\cdot\det(D_{J(W),R}), \tag{4.4}$$

where $R=\{a_1,\dots,a_k\}$ and $\ell = |R|$.

Remark 4.2. Here $I(V)$ and $J(W)$ are the subtuples of I and J that correspond to V and W, respectively. The k-element subsets V and W are viewed as k-tuples with the ordering chosen in an arbitrary way (see Remark 2.1). Note that the summand does not depend on a choice of the ordering of V and W.

Remark 4.3. Observe that the formula for $\Psi(I,J)$ from Lemma 4.3 is a particular case of (4.4) when $\ell = |R| = 1$. Indeed, for any $I,J\in[n]^{\underline{k}}$ and any $l,s\in[k]$ we have

$$\sum_{\substack{\pi\in\mathfrak{S}_k\\ \pi(s)=l}} \mathbb{1}(I|_{[k]\setminus\{s\}} = \pi^{-1}J|_{[k]\setminus\{s\}})\cdot\operatorname{sgn}(\pi) = \varepsilon(I,J,i_s,j_l),$$

where in $\varepsilon(I,J,i_s,j_l)$ the elements i_s and j_l are viewed as 1-tuples.

Remark 4.4. In particular, when $I=J$, the formula (4.4) can be simplified as follows:

$$\begin{aligned}\Delta(I,I) &= (-1)^{k-\ell}(k-\ell)!\sum_{V\in\binom{[k]}{\ell}} \det([x_{I(v),l}]_{v\in V,l\in R})\cdot\det([\partial_{I(v),l}]_{v\in V,l\in R}) =\\ &= (-1)^{k-\ell}(k-\ell)!\sum_{V\in\binom{[k]}{\ell}} \det(X_{I(V),R})\cdot\det(D_{I(V),R}).\end{aligned}$$

Proof of Lemma 4.4. Define the decomposition $\{1,\dots,k\} = \bigsqcup_{l=1}^r K_l$ via

$$K_l = \{j\in\{1,\dots,k\} : a_j = l\}.$$

Let m_l be the cardinality of K_l for each l. Note that $R=\{l\in[r] : m_l>0\}$ and $\ell=|R|$. Denote by $\mathfrak{S}(K_l)$ the group of permutations of the set K_l viewed as a subgroup of $\mathfrak{S}_k$. In particular, $\mathfrak{S}(K_l)\simeq\mathfrak{S}_{m_l}$ for all l. We also denote by $\mathfrak{S}(K)$ the subgroup of $\mathfrak{S}_k$ which stabilizes each of the subsets R_l, i.e. $\mathfrak{S}(K) = \mathfrak{S}(K_1)\times\dots\times\mathfrak{S}(K_r)$.

Towards the end of the proof we will regard any sequence $\{c_j\}_{j\in K_l}$ indexed by elements of K_l as an m_l-tuple by considering the elements of K_l in ascending order. Since we are working in a non-commutative algebra let us make a convention that products over K_l are considered in ascending ordering as well.

For any permutation $\sigma \in \mathfrak{S}_k$ and any $l \in [r]$ define the m_l-tuples $\sigma^{-1}I|_{K_l}$ and $\sigma^{-1}J|_{K_l}$ as follows (cf. (2.1)):

$$\sigma^{-1}I|_{K_l} = \{i_{\sigma(s)}\}_{s\in K_l},\ \sigma^{-1}J|_{K_l} = \{j_{\sigma(s)}\}_{s\in K_l}.$$

For any two p-tuples $U = (u_1, \dots, u_p)$ and $V = (v_1, \dots, v_p)$ in $[k]^{\underline{p}}$ define

$$\Theta_l(U,V) = \prod_{i=1}^{p} x_{u_i l}\partial_{v_i l} = x_{u_1 l}\partial_{v_1 l} \dots x_{u_p l}\partial_{v_p l}.$$

We also denote (cf. Lemma 4.3)

$$\Psi_l(U,V) = \begin{cases} 1, & l \notin R, \\ \operatorname{symdet}\left([x_{u_\alpha l}\partial_{v_\beta l}]_{\alpha,\beta=1}^{p}\right), & l \in R. \end{cases}$$

Observe that for any $\pi_1, \pi_2 \in \mathfrak{S}(K)$ we have

$$\Psi_l((\sigma\pi_1)^{-1}I|_{K_l}, (\tau\pi_2)^{-1}J|_{K_l}) = \operatorname{sgn}(\pi_1|_{K_l})\operatorname{sgn}(\pi_2|_{K_l}) \cdot \Psi_l(\sigma^{-1}I|_{K_l}, \tau^{-1}J|_{K_l}), \tag{4.5}$$

where $\pi_i|_{K_l}$ is the restriction of permutation $\pi_i \in \mathfrak{S}(K)$ to the subset K_l (recall that $\pi_i(K_l) = K_l$). Finally, denote by $\mathfrak{S}_k/\mathfrak{S}(K)$ the set of all left cosets of $\mathfrak{S}(K)$ in $\mathfrak{S}_k$.

With all these notations we can now proceed to the proof of (4.4). Since x_{ia} and ∂_{jb} commute whenever $a \neq b$, we can rewrite $\Delta(I,J)$ as follows:

$$\begin{aligned}
\Delta(I,J) &= \sum_{\sigma,\tau\in\mathfrak{S}_k} \operatorname{sgn}(\sigma\tau) \cdot x_{i_{\sigma(1)}a_1}\partial_{j_{\tau(1)}a_1} \dots x_{i_{\sigma(k)}a_k}\partial_{j_{\tau(k)}a_k} = \\
&= \sum_{\sigma,\tau\in\mathfrak{S}_k} \operatorname{sgn}(\sigma\tau) \cdot \prod_{l=1}^{r} \Theta_l(\sigma^{-1}I|_{K_l}, \tau^{-1}J|_{K_l}) = \\
&= \sum_{[\sigma],[\tau]\in\mathfrak{S}_k/\mathfrak{S}(K)} \ \sum_{\pi_1,\pi_2\in\mathfrak{S}(K)} \operatorname{sgn}(\sigma\pi_1\tau\pi_2) \cdot \prod_{l=1}^{r} \Theta_l((\sigma\pi_1)^{-1}I|_{K_l}, (\tau\pi_2)^{-1}J|_{K_l}) = \\
&= \sum_{[\sigma],[\tau]\in\mathfrak{S}_k/\mathfrak{S}(K)} \operatorname{sgn}(\sigma\tau) \left(\sum_{\pi_1,\pi_2\in\mathfrak{S}(K)} \operatorname{sgn}(\pi_1\pi_2) \prod_{l=1}^{r} \Theta_l((\sigma\pi_1)^{-1}I|_{K_l}, (\tau\pi_2)^{-1}J|_{K_l}) \right).
\end{aligned}$$

Note that for any $\sigma \in \mathfrak{S}_k$ the tuples $\sigma^{-1}I|_{K_l}$ and $\sigma^{-1}J|_{K_l}$ depend only on how σ acts on K_l. Recall also that $\mathfrak{S}(K) = \mathfrak{S}(K_1) \times \ldots \times \mathfrak{S}(K_r)$. Using this for any $\sigma, \tau \in \mathfrak{S}_k$ we obtain

$$\begin{aligned}
&\sum_{\pi_1,\pi_2\in\mathfrak{S}(K)} \prod_{l=1}^{r} \left(\operatorname{sgn}(\pi_1\pi_2|_{K_l}) \Theta_l((\sigma\pi_1)^{-1}I|_{K_l}, (\tau\pi_2)^{-1}J|_{K_l}) \right) = \\
&\qquad = \prod_{l=1}^{r} \operatorname{symdet}\left([x_{i_{\sigma(s)}l}\partial_{j_{\tau(t)}l}]_{s,t\in K_l}\right) = \prod_{l=1}^{r} \Psi_l(\sigma^{-1}I|_{K_l}, \tau^{-1}J|_{K_l}).
\end{aligned}$$

Therefore, we get the following expression for $\Delta(I,J)$:

$$\Delta(I,J) = \sum_{[\sigma],[\tau]\in\mathfrak{S}_k/\mathfrak{S}(K)} \operatorname{sgn}(\sigma\tau) \prod_{l=1}^{r} \Psi_l(\sigma^{-1}I|_{K_l}, \tau^{-1}J|_{K_l}).$$

Remark 4.5. The summation here runs over all the left cosets of $\mathfrak{S}(K)$ in $\mathfrak{S}_k$. Note that the expression $\operatorname{sgn}(\sigma\tau)\prod_{l=1}^{r} \Psi_l(\sigma^{-1}I|_{K_l}, \tau^{-1}J|_{K_l})$ does not depend on the choice of representatives of $[\sigma], [\tau] \in \mathfrak{S}_k/\mathfrak{S}(K)$. Indeed, by (4.5) for any $\pi_1, \pi_2 \in \mathfrak{S}(K)$ we have

$$\begin{aligned}
\prod_{l=1}^{r} \Psi_l((\sigma\pi_1)^{-1}I|_{K_l}, (\tau\pi_2)^{-1}J|_{K_l}) &= \prod_{l=1}^{r} \operatorname{sgn}(\pi_1\pi_2|_{K_l}) \cdot \prod_{l=1}^{r} \Psi_l(\sigma^{-1}I|_{K_l}, \tau^{-1}J|_{K_l}) = \\
&= \operatorname{sgn}(\pi_1\pi_2) \cdot \prod_{l=1}^{r} \Psi_l(\sigma^{-1}I|_{K_l}, \tau^{-1}J|_{K_l}).
\end{aligned}$$

Now we apply Lemma 4.3 in order to compute for every $l \in R$ the symmetrized determinant $\Psi_l(\sigma^{-1}I|_{K_l}, \tau^{-1}J|_{K_l})$. Firstly, observe that $\Psi_l(I_l(\sigma), J_l(\tau)) = 0$ unless there exist a permutation $\pi_l \in \mathfrak{S}(K_l)$ and an element $s_l \in K_l$

such that $j_{\tau(\pi(s))} = i_{\sigma(s)}$ for all $s \in K_l \setminus \{s_l\}$. Moreover, in this case we have

$$\Psi_l(\sigma^{-1}I|_{K_l}, \tau^{-1}J|_{K_l}) = (-1)^{m_l-1}(m_l-1)!\times$$
$$\times \sum_{\pi_l\in\mathfrak{S}(K_l)} \sum_{s_l\in K_l} \mathbb{1}(\sigma^{-1}I|_{K_l\setminus\{s_l\}} = (\tau\pi_l)^{-1}J|_{K_l\setminus\{s_l\}}) \cdot \operatorname{sgn}(\pi_l) x_{(\sigma^{-1}I)(s_l),l}\partial_{(\tau^{-1}J)(\pi_l(s_l)),l}.$$

Combining everything we get

$$\Delta(I,J) = \sum_{[\sigma],[\tau]\in\mathfrak{S}_k/\mathfrak{S}(K)} \operatorname{sgn}(\sigma\tau)\prod_{l=1}^{r}\Psi_l(\sigma^{-1}I|_{K_l}, (\tau\pi_l)^{-1}J|_{K_l}) = \left(\prod_{l\in R}(-1)^{m_l-1}(m_l-1)!\right)\times$$
$$\times \sum_{[\sigma],[\tau]\in\mathfrak{S}_k/\mathfrak{S}(K)} \operatorname{sgn}(\sigma\tau) \sum_{\substack{s_l\in K_l\\ \pi_l\in\mathfrak{S}(K_l)\\ \text{for each } l\in R}} \prod_{l\in R}\Big(\mathbb{1}(\sigma^{-1}I|_{K_l\setminus\{s_l\}} = (\tau\pi_l)^{-1}J|_{K_l\setminus\{s_l\}}) \operatorname{sgn}(\pi_l)\, x_{(\sigma^{-1}I)(s_l),l)}\partial_{(\tau^{-1}J)(\pi_l(s_l)),l}\Big).$$

Now observe that $\mathfrak{S}(K) = \prod_{l\in R}\mathfrak{S}(K_l)$. Hence, when τ runs over $\mathfrak{S}_k/\mathfrak{S}(K)$ and π_l runs over $\mathfrak{S}(K_l)$ for each $l \in R$ the product $\tau \cdot \prod_{l\in R}\pi_l$ runs over $\mathfrak{S}_k$. Therefore, we can rewrite the formula above using the summation over $\tau \in \mathfrak{S}_k$ as follows:

$$\Delta(I,J) = \left(\prod_{l\in R}(-1)^{m_l-1}(m_l-1)!\right)\times$$
$$\times \sum_{[\sigma]\in\mathfrak{S}_k/\mathfrak{S}(K)} \operatorname{sgn}(\sigma) \sum_{\substack{s_l\in K_l\\ \text{for each } l\in R}} \sum_{\tau\in\mathfrak{S}_k} \mathbb{1}(\sigma^{-1}I|_{[k]\setminus\{s_l\colon l\in R\}} = \tau^{-1}J|_{[k]\setminus\{s_l\colon l\in R\}}) \cdot \operatorname{sgn}(\tau)\prod_{l\in R}\Big(x_{I(\sigma(s_l)),l}\partial_{J(\tau(s_l)),l}\Big).$$

Replacing τ with $\tau\sigma$ and introducing the summation over $\sigma \in \mathfrak{S}_k$ instead of the summation over $[\sigma] \in \mathfrak{S}_k/\mathfrak{S}(K)$ yields

$$\Delta(I,J) = (-1)^{k-\ell}\left(\prod_{l\in R} m_l^{-1}\right)\times$$
$$\times \sum_{\sigma\in\mathfrak{S}_k} \sum_{\substack{s_l\in K_l\\ \text{for each } l\in R}} \sum_{\tau\in\mathfrak{S}_k} \mathbb{1}(I|_{[k]\setminus\{\sigma(s_l)\colon l\in R\}} = \tau^{-1}J|_{[k]\setminus\{\sigma(s_l)\colon l\in R\}}) \cdot \operatorname{sgn}(\tau)\prod_{l\in R}\Big(x_{I(\sigma(s_l)),l}\partial_{J(\tau(\sigma(s_l))),l}\Big).$$

For a given ℓ-tuple $\{s_l\}_{l\in R}$, as σ runs over $\mathfrak{S}_k$ the tuple $\{\sigma(s_l)\}_{l\in R}$ runs over all ℓ-tuples in $[k]^{\underline{\ell}}$ and each of them occurs exactly $(k-\ell)!$ times. Since there are precisely $\prod_{l\in R} m_l$ tuples $\{s_l\}_{l\in R}$ in $\prod_{l\in R} R_l$ we have

$$\Delta(I,J) = (-1)^{k-\ell}(k-\ell)!\times$$
$$\times \sum_{U\in[k]^{\underline{R}}} \sum_{\tau\in\mathfrak{S}_k} \mathbb{1}(I|_{[k]\setminus\{u_l\colon l\in R\}} = \tau^{-1}J|_{[k]\setminus\{u_l\colon l\in R\}}) \cdot \operatorname{sgn}(\tau)\prod_{l\in R}\Big(x_{I(u_l),l}\partial_{J(\tau(u_l)),l}\Big).$$

Here by $[k]^{\underline{R}}$ we denote the set of all ℓ-tuples $U = \{u_l\}_{l\in R}$ of distinct elements in $[k]$. The next step is to replace the summation over ℓ-tuples $U \in [k]^{\underline{R}}$ by the summation over ℓ-element subsets of R. Namely, we collect the terms which correspond to the same ℓ-element subset of R and after some algebraic manipulations obtain the determinants in x_{ij} and ∂_{ij}. Let $\mathfrak{S}(U)$ be the permutation group of the ℓ-element set $\{u_l : l \in R\}$. Then, we can rewrite the last sum as follows:

$$\Delta(I,J) = \frac{(-1)^{k-\ell}(k-\ell)!}{\ell!}\times$$
$$\times \sum_{U\in[k]^{\underline{R}}} \sum_{\pi\in\mathfrak{S}(U)} \sum_{\tau\in\mathfrak{S}_k} \mathbb{1}(I|_{[k]\setminus\{u_l\colon l\in R\}} = \tau^{-1}J|_{[k]\setminus\{u_l\colon l\in R\}}) \cdot \operatorname{sgn}(\tau)\prod_{l\in R}\Big(x_{I(\pi(u_l)),l}\partial_{J(\tau(\pi(u_l))),l}\Big).$$

Clearly, for any $\pi \in \mathfrak{S}(U)$ we have

$$\mathbb{1}(I|_{[k]\setminus\{u_l\colon l\in R\}} = \tau^{-1}J|_{[k]\setminus\{u_l\colon l\in R\}}) = \mathbb{1}(I|_{[k]\setminus\{u_l\colon l\in R\}} = (\tau\pi)^{-1}J|_{[k]\setminus\{u_l\colon l\in R\}}). \tag{4.6}$$

Therefore, substituting $\tau \mapsto \tau\pi^{-1}$ we obtain

$$\Delta(I,J) = \frac{(-1)^{k-\ell}(k-\ell)!}{\ell!}\times$$
$$\times \sum_{U\in[k]^{\underline{R}}} \sum_{\pi\in\mathfrak{S}(U)} \sum_{\tau\in\mathfrak{S}_k} \mathbb{1}(I|_{[k]\setminus\{u_l\colon l\in R\}} = \tau^{-1}J|_{[k]\setminus\{u_l\colon l\in R\}}) \cdot \operatorname{sgn}(\tau\pi)\prod_{l\in R} x_{I(\pi(u_l)),l}\partial_{J(\tau(u_l)),l} =$$
$$= \frac{(-1)^{k-\ell}(k-\ell)!}{\ell!} \sum_{U\in[k]^{\underline{R}}} \sum_{\tau\in\mathfrak{S}_k} \mathbb{1}(I|_{[k]\setminus\{u_l\colon l\in R\}} = \tau^{-1}J|_{[k]\setminus\{u_l\colon l\in R\}}) \operatorname{sgn}(\tau)\det([x_{I(u_\alpha),\beta}]_{\alpha,\beta\in R})\prod_{l\in R}\partial_{J(\tau(u_l)),l}.$$

Similarly, note that

$$\begin{aligned}\sum_{U\in[k]^{\underline{R}}}\det([x_{I(u_\alpha),\beta}]_{\alpha,\beta\in R})\prod_{l\in R}\partial_{J(\tau(u_l)),l} &= \frac{1}{\ell!}\sum_{U\in[k]^{\underline{R}}}\sum_{\pi\in\mathfrak{S}(U)}\det([x_{I(\pi(u_\alpha)),\beta}]_{\alpha,\beta\in R})\prod_{l\in R}\partial_{J(\tau(\pi(u_l))),l} = \\ &= \frac{1}{\ell!}\sum_{U\in[k]^{\underline{R}}}\sum_{\pi\in\mathfrak{S}(U)}\det([x_{I(u_\alpha),\beta}]_{\alpha,\beta\in R})\operatorname{sgn}(\pi)\prod_{l\in R}\partial_{J(\tau(\pi(u_l))),l} = \\ &= \frac{1}{\ell!}\sum_{U\in[k]^{\underline{R}}}\det([x_{I(u_\alpha),\beta}]_{\alpha,\beta\in R})\cdot\det([\partial_{J(\tau(u_\alpha)),\beta}]_{\alpha,\beta\in R}) = \\ &= \sum_{V\in\binom{[k]}{\ell}}\det([x_{I(v),l}]_{v\in V,l\in R})\cdot\det([\partial_{J(\tau(v)),l}]_{v\in V,l\in R}).\end{aligned}$$

Plugging this into the previous formula and taking into account (4.6) gives

$$\Delta(I,J)=\frac{(-1)^{k-\ell}(k-\ell)!}{\ell!}\sum_{V\in\binom{[k]}{\ell}}\sum_{\tau\in\mathfrak{S}_k}\mathbb{1}(I|_{[k]\setminus V}=J\circ\tau|_{[k]\setminus V})\operatorname{sgn}(\tau)\det([x_{I(v),l}]_{v\in V,l\in R})\cdot\det([\partial_{J(\tau(v)),l}]_{v\in V,l\in R}). \tag{4.7}$$

Now observe that for any given subset $V\subset[k]$ with $|V|=\ell$ there exist either $\ell!$ permutations $\tau\in\mathfrak{S}_k$ such that

$$I|_{[k]\setminus V}=J\circ\tau|_{[k]\setminus V}, \tag{4.8}$$

or none. Indeed, suppose such a permutation τ_0 exists. Then, any other permutation τ that satisfies (4.8) is of the form $\tau=\tau_0\pi$, where $\pi\in\mathfrak{S}(V)$. In particular,

$$\operatorname{sgn}(\tau)\det([\partial_{J(\tau(v)),l}]_{v\in V,l\in R}=\operatorname{sgn}(\tau_0\pi)\operatorname{sgn}(\pi)\det([\partial_{J(\tau_0(v)),l}]_{v\in V,l\in R}=\operatorname{sgn}(\tau_0)\det([\partial_{J(\tau_0(v)),l}]_{v\in V,l\in R}$$

Hence, the summation over τ in (4.7) in fact contains $\ell!$ equal summands. Note that τ satisfying (4.8) exists if and only if all elements of $I|_{[k]\setminus V}$ are contained in J. This allows us to write the final formula for $\Delta(I,J)$:

$$\Delta(I,J)=(-1)^{k-\ell}(k-\ell)!\sum_{V,W\in\binom{[k]}{\ell}}\varepsilon(I,J,I(V),J(W))\det(X_{I(V),R})\cdot\det(D_{J(W),R}). \tag{4.9}$$

Indeed, to get (4.9) from (4.7) we just note that $\mathbb{1}(I|_{[k]\setminus V}=J\circ\tau|_{[k]\setminus V})\operatorname{sgn}(\tau)$ equals $\varepsilon(I,J,I(V),J(W))$ if $W=\tau(V)$ and τ satisfies (4.8), and is zero otherwise. ☐

Lemma 4.5. *For any $I,J\in[n]^{\underline{k}}$ we have*

$$\operatorname{symdet}L(E)_{IJ}=\sum_{\ell=0}^{k}(-1)^{k-\ell}(k-\ell)!\,\ell!\left\{{k \atop \ell}\right\}\sum_{R\in\binom{[r]}{\ell}}\sum_{V,W\in\binom{[k]}{\ell}}\varepsilon(I,J,I(V),J(W))\det(X_{I(V),R})\det(D_{J(W),R}).$$

Here by $\left\{{k \atop \ell}\right\}$ we denote the Stirling number of the second kind, i.e. the number of ways to split a k-element set into ℓ non-empty subsets.

Proof. We start with the identity

$$\operatorname{symdet}(L(E))_{IJ}=\sum_{a_1,\dots,a_k=1}^{n}\sum_{\sigma,\tau\in\mathfrak{S}_k}\operatorname{sgn}(\sigma\tau)\cdot x_{i_{\sigma(1)}a_1}\partial_{j_{\tau(1)}a_1}\dots x_{i_{\sigma(k)}a_k}\partial_{j_{\tau(k)}a_k}.$$

For each choice of $a_1,\dots,a_k\in\{1,\dots,r\}$ we rewrite the products using Lemma 4.4. It remains to notice that for any given $R\in\binom{r}{\ell}$ the number of sequences $(a_1,\dots,a_k)\in[r]^k$ such that $\{a_1,\dots,a_k\}=R$ equals $\left\{{k \atop \ell}\right\}\cdot\ell!$ ☐

4.2. **Proof of the main formulas.** Now we are ready to prove Theorem 3.10 and Corollary 3.11.

Proof of Theorem 3.10. By Proposition 3.8 we have

$$M_{p,q}(Y)=\sum_{\substack{I_1,J_1\in\binom{[n]}{p}\\ I_2,J_2\in\binom{[n]}{q}\\ I_1\sqcup I_2=J_1\sqcup J_2}}\operatorname{sgn}\begin{pmatrix}I_1 & I_2\\ J_1 & J_2\end{pmatrix}\det Y_{I_1J_1}\cdot\operatorname{symdet}L(E)_{J_2I_2}.$$

Applying Lemma 4.5 to $\operatorname{symdet}L(E)_{J_2I_2}$ gives the required identity (3.9). ☐

Proof of Corollary 3.11. Indeed, the identity (3.11) is a direct consequence of (3.9) and the fact that the expression $\operatorname{sgn}\binom{I_1\ I_2}{J_1\ J_2}\varepsilon(J_2,I_2,J_2(V),I_2(W))$, if non-zero, equals $\operatorname{sgn}\binom{I_1\ I_2(W)}{J_1\ J_2(V)}$. The coefficient $\binom{n-p-\ell}{q-\ell}$ appears as the number of ways to choose a $(q-\ell)$-element subset $I_2\setminus I_2(W)=J_2\setminus J_2(V)$ in the complement of the $(p+\ell)$-element subset $I_1\sqcup I_2(W)=J_1\sqcup J_2(V)$ of $[n]$.

Then, it follows from (3.11) that the difference $M_{p,q}-q!\cdot F_{p,q}$ is a linear combination of $F_{p,\ell}$ with $\ell<q$. Therefore, the sets $\{M_{p,q}\}_{p+q\le n}$ and $\{F_{p,q}\}_{p+q\le n}$ generate the same algebra inside $\mathscr{C}(\mathcal{P}(n,r))$. ☐

5. Capelli identities and their variations

In this section we give an account of Capelli identities and their variants. Since some of these generalizations are not widely known, we also include the proofs for the sake of completeness.

5.1. **Classical Capelli identity.** Assume that $r = n$, i.e. that $\mathcal{P}(n,n)$ is a polynomial ring $\mathbb{C}[\mathrm{Mat}(n,n)]$ in n^2 variables x_{ij}, $1 \le i,j \le n$. Recall that $\mathcal{PD}(n,n)$ is the algebra of differential operators on $\mathcal{P}(n,n)$ with polynomial coeficients. Define an element $\Pi \in U(\mathfrak{gl}_n)$ as

$$\Pi = \mathrm{rdet}(E_{ij} + (i-1)\delta_{ij})_{i,j=1}^n = \mathrm{rdet}\begin{bmatrix} E_{11}+0 & E_{12} & \dots & E_{1k} \\ E_{21} & E_{22}+1 & \dots & E_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ E_{k1} & E_{k2} & \dots & E_{kk}+k-1 \end{bmatrix}.$$

The next statement was observed by Alfredo Capelli [2] and was used by Hermann Weyl in his treatment of invariant theory for GL_n [24, Ch. II.4].

Proposition 5.1 (Capelli identity)**.** *The image of* Π *in* $\mathcal{PD}(n,n)$ *equals*

$$L(\Pi) = \det(X)\det(D). \tag{5.1}$$

In particular, the expression on the left is a GL_n*-invariant differential operator.*

Remark 5.1. Since $L(E) = XD^T$ one can rewrite this identity as $\mathrm{rdet}(XD^T + Q) = \det(X)\det(D^T)$, where Q is the diagonal matrix with entries $(0,1,\dots,n-1)$. In other words, the Capelli identity resembles the identity $\det(AB) = \det(A)\cdot\det(B)$ for matrices with commuting entries. However, since $U(\mathfrak{gl}_n)$ is non-commutative and one has to introduce the *quantum correction* Q to get a valid equality.

Example 5.1. For instance, if $n = r = 2$, then the identity is equivalent to

$$\mathrm{rdet}\begin{bmatrix} x_{11}\partial_{11} + x_{12}\partial_{12} + 0 & x_{11}\partial_{21} + x_{12}\partial_{22} \\ x_{21}\partial_{11} + x_{22}\partial_{12} & x_{21}\partial_{21} + x_{22}\partial_{22} + 1 \end{bmatrix} = \det\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22}\end{bmatrix}\cdot\det\begin{bmatrix}\partial_{11} & \partial_{12} \\ \partial_{21} & \partial_{22}\end{bmatrix}.$$

One can verify this by direct computations. Indeed, the left-hand side equals

$$\begin{aligned} \mathrm{LHS} &= (x_{11}\partial_{11} + x_{12}\partial_{12})(x_{21}\partial_{21} + x_{22}\partial_{22} + 1) - (x_{11}\partial_{21} + x_{12}\partial_{22})(x_{21}\partial_{11} + x_{22}\partial_{12}) = \\ &= x_{11}x_{21}\partial_{11}\partial_{21} + x_{11}x_{22}\partial_{11}\partial_{22} + x_{12}x_{21}\partial_{12}\partial_{21} + x_{12}x_{22}\partial_{12}\partial_{22} + x_{11}\partial_{11} + x_{12}\partial_{12} - \\ &- (x_{11}x_{21}\partial_{21}\partial_{11} + x_{11}\partial_{11} + x_{11}x_{22}\partial_{21}\partial_{12} + x_{12}x_{21}\partial_{22}\partial_{11} + x_{12}x_{22}\partial_{22}\partial_{12} + x_{12}\partial_{12}) = \\ &= x_{11}x_{22}\partial_{11}\partial_{22} + x_{12}x_{21}\partial_{12}\partial_{21} - x_{11}x_{22}\partial_{12}\partial_{21} - x_{12}x_{21}\partial_{11}\partial_{22} = \\ &= (x_{11}x_{22} - x_{12}x_{21})(\partial_{11}\partial_{22} - \partial_{12}\partial_{21}) = \det\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22}\end{bmatrix}\cdot\det\begin{bmatrix}\partial_{11} & \partial_{12} \\ \partial_{21} & \partial_{22}.\end{bmatrix}. \end{aligned}$$

5.2. **Capelli identities for rectangular matrices.** Now let r be an arbitrary positive integer. It turns out that one can generalize the Capelli identity for all minors of the matrix E.

Firstly, for arbitrary k-tuples $I = (i_1,\dots,i_k)$ and $J = (j_1,\dots,j_k)$ in $[n]^k$ introduce the following element $\Pi_{IJ} \in U(\mathfrak{gl}_n)$:

$$\Pi_{IJ} = \mathrm{rdet}[E_{i_\alpha j_\beta} + (\alpha-1)\delta_{i_\alpha j_\beta}]_{\alpha,\beta=1}^k.$$

For example, if $I = J = (i_1,\dots,i_k) \in [n]^{\underline{k}}$, then

$$\Pi_{II} = \mathrm{rdet}\begin{bmatrix} E_{i_1i_1}+0 & E_{i_1i_2} & \dots & E_{i_1i_k} \\ E_{i_2i_1} & E_{i_2i_2}+1 & \dots & E_{i_2i_k} \\ \vdots & \vdots & \ddots & \vdots \\ E_{i_ki_1} & E_{i_ki_2} & \dots & E_{i_ki_k}+k-1 \end{bmatrix},$$

which coincides with Π above when $I = (1,2,\dots,n)$.

Proposition 5.2. *The following holds in* $U(\mathfrak{gl}_n)$*:*

$$\mathrm{rdet}(E_{ij} + (i-1-z)\delta_{ij})_{i,j=1}^n = \sum_{k=0}^n (-1)^k z^{\underline{k}}\cdot C_{n-k}, \tag{5.2}$$

where $z^{\underline{k}} = z(z-1)\dots(z-k+1)$ *and*

$$C_k = \sum_{I\in\binom{[n]}{k}} \Pi_{II}.$$

Moreover, the elements C_k *belong to the center* $Z(\mathfrak{gl}_n)$ *of the universal enveloping algebra* $U(\mathfrak{gl}_n)$ *and their images in* $\mathcal{PD}(n,r)$ *are as follows:*

$$L(C_k) = \sum_{I\in\binom{[n]}{k}}\sum_{K\in\binom{[r]}{k}} \det(X_{IK})\det(D_{IK}) \tag{5.3}$$

In particular, if $r = n$, then plugging in $z = 0$ yields the classical Capelli identity (Proposition 5.1).

Remark 5.2. In formula (5.3) we regard $K \in \binom{[r]}{k}$ as an element of $\mathfrak{S}_k \backslash [r]^{\underline{k}}$, i.e. as k-tuple with arbitrarily chosen ordering of elements. Note that the term $\det(X_{IK})\det(D_{IK})$ does not depend on this choice, so the expression on the right is well defined.

Remark 5.3. The elements C_k are often called *Capelli generators*. Moreover, $Z(\mathfrak{gl}_n)$ is a free polynomial ring generated by $C_1, \dots, C_n$. The classical counterparts of the Capelli generators are the coefficients c_k (see (3.1)) of the characteristic polynomial which are the generators of $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$.

Instead of proving Proposition 5.2 directly, we prove more general versions of equalities (5.2) and (5.3), see Propositions 5.7 and 5.3 below. These generalizations are then verified by a straightforward inductive argument.

5.3. **Cauchy–Binet type identity.** We start with a formula for $L(\Pi_{IJ})$ for an arbitrary minor Π_{IJ}. It generalizes the classical Capelli identity and can be viewed as a non-commutative analogue of the Cauchy–Binet identity from linear algebra (see also [3, Corollary 1.3]).

Proposition 5.3 (Cauchy–Binet type identity)**.** *For any $I, J \in [n]^k$ the image of Π_{IJ} in $\mathcal{PD}(n,r)$ under the map L equals*

$$L(\Pi_{IJ}) = \sum_{K \in \binom{[r]}{k}} \det(X_{IK})\det(D_{JK}). \tag{5.4}$$

Remark 5.4. Note that the right-hand side of (5.4) is skew-symmetric in the entries of I and J and hence the same holds for $L(\Pi_{IJ})$. From the definition of Π_{IJ} it is clear that Π_{IJ} is skew-symmetric in J (as row determinant), but the skew-symmetricity in I is not immediate from this definition. Formula (5.4) in particular implies that $L(\Pi_{IJ})$ is zero whenever I or J contains equal elements.

Proof. Induct on k. For $k = 1$ the identity follows immediately from the definition of $L(E_{ij})$, see (2.3). Now assume that $k > 1$. Consider any k-tuples $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_k)$. Denote $I' = (i_1, \dots, i_{k-1})$ and $J^{(l)} = (j_1, \dots, \widehat{j_l}, \dots, j_k)$ for every $l \in [k]$. Expanding Π_{IJ} along the k-th row yields

$$\Pi_{IJ} = \sum_{l=1}^{k} (-1)^{k-l} \Pi_{I'J^{(l)}} \cdot (E_{i_k j_l} + (k-1)\delta_{i_k j_l}),$$

and hence by the inductive hypothesis

$$\begin{aligned} L(\Pi_{IJ}) &= \sum_{l=1}^{k} (-1)^{k-l} L(\Pi_{I'J^{(l)}}) \cdot \left(\sum_{\alpha \in [r]} x_{i_k \alpha} \partial_{j_l \alpha} + (k-1)\delta_{i_k j_l} \right) = \\ &= \sum_{l=1}^{k} (-1)^{k-l} \sum_{K' \in \binom{[r]}{k-1}} \det(X_{I'K'})\det(D_{J^{(l)}K'}) \cdot \left(\sum_{\alpha \in [r]} x_{i_k \alpha} \partial_{j_l \alpha} + (k-1)\delta_{i_k j_l} \right). \end{aligned}$$

Using the identity $\det(D_{J^{(l)}K'})x_{i_k\alpha} = x_{i_k\alpha}\det(D_{J^{(l)}K'}) + [\det(D_{J^{(l)}K'}), x_{i_k\alpha}]$ we can rewrite the last sum as follows:

$$\begin{aligned} L(\Pi_{IJ}) = &\sum_{l=1}^{k} \sum_{\alpha \in [r]} \sum_{K' \in \binom{[r]}{k-1}} (-1)^{k-l} \det(X_{I'K'})x_{i_k\alpha} \cdot \det(D_{J^{(l)}K'})\partial_{j_l\alpha} + \\ &+ (k-1) \sum_{l=1}^{k} \sum_{K' \in \binom{[r]}{k-1}} (-1)^{k-l} \det(X_{I'K'}) \cdot \det(D_{J^{(l)}K'})\delta_{i_k j_l} + \\ &+ \sum_{l=1}^{k} \sum_{\alpha \in [r]} \sum_{K' \in \binom{[r]}{k-1}} (-1)^{k-l} \det(X_{I'K'}) \cdot [\det(D_{J^{(l)}K'}), x_{i_k\alpha}]\partial_{j_l\alpha} \end{aligned} \tag{5.5}$$

Denote the three summands on the right-hand side of (5.5) by $S_{IJ}^{(1)}$, $S_{IJ}^{(2)}$ and $S_{IJ}^{(3)}$, respectively. Observe that $S_{IJ}^{(1)}$ equals

$$\begin{aligned} S_{IJ}^{(1)} &= \sum_{l=1}^{k} \sum_{\alpha \in [r]} \sum_{K' \in \binom{[r]}{k-1}} (-1)^{k-l} \det(X_{I'K'})x_{i_k\alpha} \cdot \det(D_{J^{(l)}K'})\partial_{j_l\alpha} = \\ &= \sum_{\alpha \in [r]} \sum_{K' \in \binom{[r]}{k-1}} \det(X_{I'K'})x_{i_k\alpha} \cdot \det(D_{JK'_{(\alpha)}}) = \sum_{K \in \binom{[r]}{k}} \det(X_{IK}) \cdot \det(D_{JK}). \end{aligned}$$

Here by $K'_{(\alpha)}$ we denote a k-tuple in $[n]^{\underline{k}}$ whose first $k-1$ entries coincide with those of K' and whose k-th entry equals α. (Recall that K' can be viewed as a $(k-1)$-tuple.) The last equality follows the cofactor expansion of $\det(X_{IK})$ along the last row. Indeed, for $K = (\beta_1, \dots, \beta_k)$ we have $\det(X_{IK}) = \sum_{l=1}^{l} (-1)^{k-l} \det(X_{I'K^{(l)}}) x_{i_k \beta_l}$, where $K^{(l)} = (\beta_1, \dots, \widehat{\beta_l}, \dots, \beta_k) \in [r]^{\underline{k-1}}$. If we denote $\widetilde{K}_{(l)} = (\beta_1, \dots, \beta_{l-1}, \beta_{l+1}, \dots, \beta_k, \beta_l) \in [r]^{\underline{k}}$ for $l = 1, \dots, k$, then we have the equality $\det(D_{JK}) = (-1)^{k-l} \det(D_{J\widetilde{K}_{(l)}})$ and the formula for $S^{(1)}_{IJ}$ above follows.

Thus, it remains to show that the sum $S^{(2)}_{IJ} + S^{(3)}_{IJ}$ is zero. We consider three cases depending on the number q of occurences of i_k in the k-tuple J.

Case 1. $q = 0$. Then, for any $l \in [k]$ and $\alpha \in [r]$ we have $\delta_{i_k j_l} = 0$ and $[\det(D_{J^{(l)}K'}), x_{i_k\alpha}] = 0$. Hence, both $S^{(2)}_{IJ}$ and $S^{(3)}_{IJ}$ are zero which concludes the proof.

In the remaining two cases we use the following lemma.

Lemma 5.4. *For any k-tuples $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_k)$ in $[n]^k$ and any $i, j \in [n]$, we have*

$$\sum_{\alpha \in [r]} [\det(D_{IJ}), x_{i\alpha}] \partial_{j\alpha} = \begin{cases} 0, & \text{if } I \notin [n]^{\underline{k}} \text{ or } i \notin I. \\ \det(D_{KJ}), & \text{if } i_p = i \text{ for } p \in [k], \end{cases}$$

where in the second case we put $K = (i_1, \dots, i_{p-1}, j, i_{p+1}, \dots, i_k)$.

Proof. If I contains equal entries or $i \notin I$, then it clear that $[\det(D_{IJ}), x_{i\alpha}] = 0$ for all $\alpha \in [r]$. Otherwise, if $p \in [k]$ is such that $i_p = i$, then $[\det(D_{IJ}), x_{i\alpha}]$ is zero if $\alpha \notin J$ and equals to the cofactor of the element $\partial_{j\alpha}$ of the matrix D_{KJ} otherwise. Thus, $\sum_{\alpha \in [r]} [\det(D_{IJ}), x_{i\alpha}] \partial_{j\alpha} = \det(D_{KJ})$, as claimed. ☐

Case 2. $q = 1$. Let $p \in [k]$ be such that $i_k = j_p$. Observe that both sides of (5.4) are skew-symmetric with respect to J (see also Remark 5.4). Thus, we may assume without loss of generality that $p = k$. In this case we can rewrite $S^{(2)}_{IJ}$ as follows:

$$S^{(2)}_{IJ} = (k-1) \sum_{l=1}^{k} \sum_{K' \in \binom{[r]}{k-1}} (-1)^{k-l} \det(X_{I'K'}) \cdot \det(D_{J^{(l)}K'}) \delta_{i_k j_l} = (k-1) \sum_{K' \in \binom{[r]}{k-1}} \det(X_{I'K'}) \cdot \det(D_{J^{(k)}K'}).$$

Applying Lemma 5.4 gives the following expression for $S^{(3)}_{IJ}$:

$$\begin{aligned} S^{(3)}_{IJ} &= \sum_{l=1}^{k} \sum_{\alpha \in [r]} \sum_{K' \in \binom{[r]}{k-1}} (-1)^{k-l} \det(X_{I'K'}) \cdot [\det(D_{J^{(l)}K'}), x_{i_k\alpha}] \partial_{j_l \alpha} = \\ &= \sum_{l \in [k-1]} \sum_{K' \in \binom{[r]}{k-1}} (-1)^{k-l} \det(X_{I'K'}) \cdot (-1)^{k-l-1} \det(D_{J^{(k)}K'}) = \\ &= -(k-1) \sum_{K' \in \binom{[r]}{k-1}} \det(X_{I'K'}) \cdot \det(D_{J^{(k)}K'}). \end{aligned}$$

Therefore, $S^{(2)}_{IJ}$ and $S^{(3)}_{IJ}$ add up to zero, as claimed.

Case 3. $q > 1$. Let us show that in this case both $S^{(2)}_{IJ}$ and $S^{(3)}_{IJ}$ vanish. If $q \geq 3$, then each $(k-1)$-tuple $J^{(l)}$, $l = 1, \dots, k$, contains at least two entries that are equal to i_k. Hence, all terms in sums $S^{(2)}_{IJ}$ and $S^{(3)}_{IJ}$ are zero since $\det(D_{J^lK'}) = 0$.

Now assume that $q = 2$. Similar to the second case we may assume that $j_{k-1} = j_k = i_k$ and $j_l \neq i_k$ for $l < k-1$. Then, for $S^{(2)}_{IJ}$ we have

$$\begin{aligned} S^{(2)}_{IJ} &= (k-1) \sum_{l=1}^{k} \sum_{K' \in \binom{[r]}{k-1}} (-1)^{k-l} \det(X_{I'K'}) \cdot \det(D_{J^{(l)}K'}) \delta_{i_k j_l} = \\ &= (k-1) \sum_{K' \in \binom{[r]}{k-1}} \Big(-\det(X_{I'K'}) \cdot \det(D_{J^{(k-1)}K'}) + \det(X_{I'K'}) \cdot \det(D_{J^{(k)}K'}) \Big) = 0 \end{aligned}$$

because $\det(D_{J^{(l)}K'}) = 0$ for $l < k-1$ while $J^{(k-1)} = J^{(k)}$. Lemma 5.4 implies that $S^{(3)}_{IJ}$ equals

$$S^{(3)}_{IJ} = \sum_{l=1}^{k} \sum_{\alpha\in[r]} \sum_{K'\in\binom{[r]}{k-1}} (-1)^{k-l} \det(X_{I'K'}) \cdot [\det(D_{J^{(l)}K'}), x_{i_k\alpha}]\partial_{j_l\alpha} =$$

$$= \sum_{K'\in\binom{[r]}{k-1}} \det(X_{I'K'}) \sum_{\alpha\in[r]} \Big(-[\det(D_{J^{(k-1)}K'}), x_{i_k\alpha}]\partial_{j_{k-1}\alpha} + [\det(D_{J^{(k)}K'}), x_{i_k\alpha}]\partial_{j_k\alpha}\Big) = 0$$

since $J^{(k-1)} = J^{(k)}$ and $j_{k-1} = j_k = i_k$.

Therefore, if $q > 1$, we have $S^{(2)}_{IJ} = S^{(3)}_{IJ} = 0$ which concludes the proof of the proposition. □

Corollary 5.5. *For any $I, J \in [n]^k$ and any $\sigma, \tau \in \mathfrak{S}_k$ we have $\Pi_{\sigma I,\tau J} = \operatorname{sgn}(\sigma\tau) \cdot \Pi_{IJ}$.*

Proof. By the previous proposition, we have the identity $L(\Pi_{\sigma I,\tau J}) = \operatorname{sgn}(\sigma\tau)\cdot L(\Pi_{IJ})$. By Proposition 2.2, for $r = n$ the map L faithfully maps $U(\mathfrak{gl}_n)$ to $\mathcal{PD}(n, r)$, hence the result follows. □

Corollary 5.6. *For any k-tuples $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_k)$ in $[n]^k$ we have*

$$\Pi_{IJ} = \operatorname{rdet}[E_{i_\alpha j_\beta} + (\alpha-1)\delta_{i_\alpha j_\beta}]^k_{\alpha,\beta=1} = \operatorname{cdet}[E_{i_\alpha j_\beta} + (k-\beta)\delta_{i_\alpha j_\beta}]^k_{\alpha,\beta=1}.$$

Proof. We prove that the images of Π_{IJ} and $\operatorname{cdet}[E_{i_\alpha j_\beta} + (k-\beta)\delta_{i_\alpha j_\beta}]^k_{\alpha,\beta=1}$ under the map L coincide. This is sufficient since by Proposition 2.2, for $r = n$ the map $L\colon U(\mathfrak{gl}_n) \to \mathcal{PD}(n, r)$ is injective.

In view of formula (5.4), the equality $L(\Pi_{IJ}) = \operatorname{cdet}[L(E_{i_\alpha j_\beta}) + (k-\beta)\delta_{i_\alpha j_\beta}]^k_{\alpha,\beta=1}$ is equivalent to

$$\operatorname{cdet}[L(E_{i_\alpha j_\beta}) + (k-\beta)\delta_{i_\alpha j_\beta}]^k_{\alpha,\beta=1} = \sum_{K\in\binom{[r]}{k}} \det(X_{IK})\det(D_{JK}).$$

This identity can be proved in essentially the same way as Proposition 5.3. □

Remark 5.5. One can prove this identity directly in $U(\mathfrak{gl}_n)$ using the ternary relations and the Yang–Baxter equation (see Lemma 6.10 in the next section).

5.4. **Characteristic polynomial of the Capelli matrix.** Now we state a generalization of the formula (5.2) for $k \times k$ minors. Define

$$C_k(z) = \sum_{I\in\binom{[n]}{k}} \operatorname{rdet}(E_{i_\alpha i_\beta} + (\alpha - 1 - z)\delta_{ij})_{\alpha,\beta\in[k]}.$$

Observe that $C_k(0) = C_k$ and $C_n(z)$ is the left-hand side of (5.2).

Proposition 5.7. *For any $0 \le k \le n$ we have*

$$C_k(z) = \sum_{m=0}^{k} (-1)^m \binom{n-k+m}{m} z^{\underline{m}} \cdot C_{k-m}. \tag{5.6}$$

In particular, for $k = n$ this identity is equivalent to (5.2).

Proof. We follow the proof from [9, Section 11.1]. Induct on k. The cases $k = 0$ and $k = 1$ are clear. Denote the right hand-side of (5.6) by $B_k(z)$. Let Δ be the difference operator defined as $(\Delta f)(z) = f(z+1) - f(z)$. Since both $B_k(z)$ and $C_k(z)$ are $U(\mathfrak{gl}_n)$-valued polynomials in z and $B_k(0) = C_k(0) = C_k$, it suffices to check that $(\Delta B_k)(z) = (\Delta C_k)(z)$. We have

$$(\Delta B_k)(z) = \sum_{m=0}^{k} (-1)^m \binom{n-k+m}{m} \Delta(z^{\underline{m}}) \cdot C_{k-m} = \sum_{m=1}^{k} (-1)^m m \binom{n-k+m}{m} z^{\underline{m-1}} \cdot C_{k-m}.$$

Note that $m\binom{n-k+m}{m} = (n-k+1)\binom{n-k+m}{m-1}$, and hence we obtain

$$(\Delta B_k)(z) = (n-k+1) \sum_{m=0}^{k-1} (-1)^{m+1} \binom{n-k+m+1}{m} z^{\underline{m}} \cdot C_{k-1-m} = -(n-k+1)B_{k-1}(z).$$

Now let us compute $(\Delta C_k)(z)$. For any k-element subset $I = \{i_1, \dots, i_k\}$ of $[n]$, where $i_1 < \ldots < i_k$, we set

$$E^\natural_{II} = \begin{bmatrix} E_{i_1i_1} + 0 & E_{i_1i_2} & \dots & E_{i_1i_k} \\ E_{i_2i_1} & E_{i_2i_2} + 1 & \dots & E_{i_2i_k} \\ \vdots & \vdots & \ddots & \vdots \\ E_{i_ki_1} & E_{i_ki_2} & \dots & E_{i_ki_k} + k - 1 \end{bmatrix} \quad \text{and } \Pi_{II}(z) = \operatorname{rdet}(E^\natural_{II} - z \cdot \mathrm{Id}_k).$$

Then,

$$(\Delta\Pi_{II})(z) = \Pi_{II}(z+1) - \Pi_{II}(z) = \sum_{p=1}^{k} \big(\operatorname{rdet}(E^{\natural}_{II} - (z+1)\cdot \mathrm{Id}_k + \Lambda_{p-1}) - \operatorname{rdet}(E^{\natural}_{II} - (z+1)\cdot \mathrm{Id}_k + \Lambda_p)\big),$$

where Λ_p is a diagonal matrix with the first p diagonal entries equal to 1 and the last $k-p$ entries equal to 0. Recall that the row determinant is an additive function of a fixed row. Applying this to the p-th row of an $n \times n$ matrix $E^{\natural} - (z+1)\cdot \mathrm{Id}_k + \Lambda_{p-1}$ gives

$$\operatorname{rdet}(E^{\natural}_{II} - (z+1)\cdot \mathrm{Id}_k + \Lambda_{p-1}) = \operatorname{rdet}(E^{\natural}_{II} - (z+1)\cdot \mathrm{Id}_k + \Lambda_p) - \operatorname{rdet}(E^{\natural}_{I^{(p)}I^{(p)}} - z\cdot \mathrm{Id}_{k-1}),$$

where $I^{(p)} = I \setminus \{i_p\}$. Therefore,

$$(\Delta C_k)(z) = \sum_{I \in \binom{[n]}{k}} (\Delta\Pi_{II})(z) = -\sum_{I \in \binom{[n]}{k}} \sum_{p=1}^{k} \Pi_{I^{(p)}I^{(p)}}(z) = -(n-k+1)C_{k-1}(z).$$

By the inductive hypothesis, $B_{k-1}(z) = C_{k-1}(z)$. Therefore, $(\Delta B_k)(z) = (\Delta C_k)(z)$ which concludes the inductive step due to the equality $B_k(0) = C_k(0) = C_k$. □

6. The Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ and its Bethe subalgebras

Here we recall the definitions and basic properties of the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ and the Bethe subalgebras. Our exposition follows the monograph by Molev (see [15, Section 1] for more details).

6.1. **Notation.** Let $\{e_{ij}\}_{i,j=1}^n$ be the standard matrix units of $\mathrm{Mat}(n,n)$ and let $\{E_{ij}\}_{i,j=1}^n$ be the corresponding generators of the universal enveloping algebra $U(\mathfrak{gl}_n)$. The matrix units $\{e_{ij}\}_{i,j=1}^n$ act on $\mathbb{C}^n$ spanned by $e_1, \dots, e_n$ in the usual way:

$$e_{ij}e_k = \delta_{jk}e_i.$$

We often work with algebras of the form $\mathcal{A} \otimes (\operatorname{End}\mathbb{C}^n)^{\otimes m}$, where $\mathcal{A}$ is an associative algebra. For any $C = \sum_{i,j=1}^n c_{ij} \otimes e_{ij} \in \mathcal{A} \otimes \operatorname{End}\mathbb{C}^n$ and any $D = \sum_{i,j,k,l=1}^n d_{ijkl}e_{ij} \otimes e_{kl} \in (\operatorname{End}\mathbb{C}^n)^{\otimes 2}$ (these operators might depend on some parameters) we define

$$C_a = \sum_{i,j=1}^n c_{ij} \otimes 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(m-a)} \in \mathcal{A} \otimes (\operatorname{End}\mathbb{C}^n)^{\otimes m}, \text{ and}$$

$$D_{ab} = \sum_{i,j=1}^n d_{ijkl} \cdot 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{kl} \otimes 1^{\otimes(m-b)} \in (\operatorname{End}\mathbb{C}^n)^{\otimes m}.$$

Usually we identify $(\operatorname{End}\mathbb{C}^n)^{\otimes m}$ with the subalgebra $1 \otimes (\operatorname{End}\mathbb{C}^n)^{\otimes m}$ inside $\mathcal{A} \otimes (\operatorname{End}\mathbb{C}^n)^{\otimes m}$.

Each element σ of the symmetric group $\mathfrak{S}_m$ defines an element of $(\operatorname{End}\mathbb{C}^n)^{\otimes m}$ whose action on $(\mathbb{C}^n)^{\otimes m}$ corresponds to permuting the tensors via σ. Namely, $\sigma \in \mathfrak{S}_m$ corresponds to

$$\sum_{i_1,\dots,i_m=1}^n e_{i_1 i_{\sigma(1)}} \otimes \dots \otimes e_{i_m i_{\sigma(m)}} \in (\operatorname{End}\mathbb{C}^n)^{\otimes m}.$$

Clearly, this gives rise to an embedding of the group algebra $\mathbb{C}[\mathfrak{S}_m]$ into $(\operatorname{End}\mathbb{C}^n)^{\otimes m}$. For any distinct $i,j \in \{1,\dots,m\}$ denote by P_{ij} the image of the transposition $(i\ j) \in \mathbb{C}[\mathfrak{S}_m]$ in $(\operatorname{End}\mathbb{C}^n)^{\otimes m}$. Let A_m be the antisymmetrization operator, i.e.

$$A_m = \sum_{\sigma \in \mathfrak{S}_m} \operatorname{sgn}(\sigma) \sum_{i_1,\dots,i_m=1}^n e_{i_1 i_{\sigma(1)}} \otimes \dots \otimes e_{i_m i_{\sigma(m)}} \in (\operatorname{End}\mathbb{C}^n)^{\otimes m}.$$

It is not difficult to see that $A_m^2 = m! \cdot A_m$ and $A_m P_{ij} = P_{ij} A_m = -A_m$ for any distinct $i,j \in \{1,\dots,m\}$.

6.2. **R-matrices.** Define the *Yang R-matrix* as an element $R(u) \in \mathrm{Y}(\mathfrak{gl}_n)[[u^{-1}]] \otimes \operatorname{End}(\mathbb{C}^n)^{\otimes 2}$ given by

$$R(u) = 1 - u^{-1}P, \text{ where } P = \sum_{i,j=1}^n e_{ij} \otimes e_{ji}.$$

Next, recall the *Yang–Baxter equation*:

Proposition 6.1 ([15, Proposition 1.2.1]). *For any commuting indeterminates u, v and w*

$$R_{12}(u-v)R_{13}(u-w)R_{23}(v-w) = R_{23}(v-w)R_{13}(u-w)R_{12}(u-v). \tag{6.1}$$

We also define for any $m \geq 2$ commuting indeterminates $u_1, \dots, u_m$ the following rational function:

$$R(u_1, u_2, \dots, u_m) = (R_{m-1,m})(R_{m-2,m}R_{m-2,m-1}) \dots (R_{1m} \dots R_{12}), \tag{6.2}$$

where we use shorthand notation $R_{ij} = R_{ij}(u_i - u_j)$. Note that for $m = 1$ this is just the Yang R-matrix: $R(u_1, u_2) = R_{12}(u_1 - u_2)$.

Using the Yang–Baxter equation, one can give an alternative definition of $R(u_1, u_2, \dots, u_m)$.

Lemma 6.2 ([15, Section 1.6])**.** *In the notation above, the following identity holds:*

$$R(u_1, u_2, \dots, u_m) = (R_{12} \dots R_{1m}) \dots (R_{m-2,m-1}R_{m-2,m})(R_{m-1,m})$$

In fact, the operator A_m is a certain specialization of $R(u_1, \dots, u_m)$.

Proposition 6.3 ([15, Proposition 1.6.2])**.** *If $u_i - u_{i+1} = 1$ for all $i = 1, \dots, m-1$, then*

$$R(u_1, u_2, \dots, u_m) = A_m.$$

6.3. **Yangian $\mathrm{Y}(\mathfrak{gl}_n)$.** The Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ is an associative algebra generated by elements $t_{ij}^{(r)}$, $1 \leq i, j \leq n$, $r \geq 1$, which satisfy certain relations. These relations can be succinctly written using the so-called *ternary (RTT) relation.* To state it, we need to introduce more notations.

For each $i, j \in \{1, \dots, n\}$ consider a formal Laurent series

$$t_{ij}(u) = \delta_{ij} + \sum_{r \geq 1} t_{ij}^{(r)} u^{-r} \in \mathrm{Y}(\mathfrak{gl}_n)[[u^{-1}]].$$

Let $T(u) = \sum_{i,j=1}^n t_{ij}(u) \otimes e_{ij} \in \mathrm{Y}(\mathfrak{gl}_n)[[u^{-1}]] \otimes \operatorname{End} \mathbb{C}^n$ be an $n \times n$ matrix whose (i,j)-entry equals $t_{ij}(u)$.

Now we can state the defining relations of $\mathrm{Y}(\mathfrak{gl}_n)$ (see [15, Proposition 1.2.2]):

$$R(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u - v). \tag{6.3}$$

Here both sides are viewed as elements of $\operatorname{End}(\mathbb{C}^n)^{\otimes 2} \otimes \mathrm{Y}(\mathfrak{gl}_n)[[u^{-1}, v^{-1}]]$. Expanding them in u^{-1} and v^{-1} gives infinitely many relations between generators $t_{ij}^{(r)}$.

We need the following generalization of the ternary relation (6.3):

Proposition 6.4. *For any $m \geq 2$ commuting indeterminates $u_1, \dots, u_m$*

$$R(u_1, \dots, u_m)T_1(u_1) \dots T_m(u_m) = T_m(u_m) \dots T_1(u_1)R(u_1, \dots, u_m).$$

6.4. **The evaluation homomorphism.** The relation between $\mathrm{Y}(\mathfrak{gl}_n)$ and $U(\mathfrak{gl}_n)$ is given by the so-called *evaluation homomorphism* $\mathrm{ev}\colon Y(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n)$. The map ev is defined by the formula $\mathrm{ev}(t_{ij}(u)) = \delta_{ij} + u^{-1}E_{ij}$, i.e. $\mathrm{ev}(t_{ij}^{(r)}) = \delta_{1r}E_{ij}$. One can verify that $\mathrm{ev}\colon Y(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n)$ is a surjective algebra homomorphism (see also [15, Proposition 1.1.3]). Note that ev maps the $n \times n$ matrix $T(u)$ to $1 + u^{-1}E$.

6.5. **Bethe subalgebras.** Define the *trace* on $(\operatorname{End} \mathbb{C}^n)^{\otimes m}$ as the linear map tr_m that acts on basis elements by

$$\mathrm{tr}_m \left(e_{i_1 j_1} \otimes \dots \otimes e_{i_m j_m}\right) = \delta_{i_1 j_1} \dots \delta_{i_m j_m}.$$

Observe that if we view $A \in (\operatorname{End} \mathbb{C}^n)^{\otimes m}$ as an element of $\operatorname{End}(\mathbb{C}^n)^{\otimes m}$, then $\mathrm{tr}_m(A)$ is indeed the trace of the operator A acting on $(\mathbb{C}^n)^{\otimes m}$. We extend tr_m to a map $\mathrm{tr}_m\colon \mathcal{A} \otimes (\operatorname{End} \mathbb{C}^n)^{\otimes m} \to \mathcal{A}$ where $\mathcal{A}$ is an arbitrary associative algebra:

$$\mathrm{tr}_m \left(a \otimes e_{i_1 j_1} \otimes \dots \otimes e_{i_m j_m}\right) = \delta_{i_1 j_1} \dots \delta_{i_m j_m} a, \ a \in \mathcal{A}.$$

Consider the elements $\tau_k(u)$ defined as follows:

$$\tau_k(u) = \frac{1}{k!} \mathrm{tr}_k\, A_k T_1(u) \dots T_k(u - k + 1).$$

Proposition 6.5 ([15, Proposition 1.14.1])**.** *The coefficients of the series $\tau_1(u), \dots, \tau_n(u) \in \mathrm{Y}(\mathfrak{gl}_n)[[u^{-1}]]$ generate a commutative subalgebra in $\mathrm{Y}(\mathfrak{gl}_n)$.*

Observe that for any complex $n \times n$ matrix C the matrix $C \cdot T(u)$ also satisfies the ternary relation (6.3). Indeed, this is a consequence of the fact that the element $C_1 C_2 \dots C_m$ commutes with the image of $\mathbb{C}[\mathfrak{S}_m]$ in $\operatorname{End}(\mathbb{C}^n)^{\otimes m}$. Thus, one can generalize Proposition 6.5 to construct commutative subalgebras of the Yangian which depend on a parameter in $\operatorname{End} \mathbb{C}^n$.

Corollary 6.6 ([15, Proposition 1.14.2])**.** *For any complex $n \times n$ matrix C the coefficients of the series*

$$\tau_k(u, C) = \frac{1}{k!} \mathrm{tr}_k\, A_k C_1 \dots C_k T_1(u) \dots T_k(u - k + 1)$$

generate a commutative subalgebra of $\mathrm{Y}(\mathfrak{gl}_n)$.

Corollary 6.7 ([15, Proposition 1.14.3]). *For any complex $n \times n$ matrix C the coefficients of the series*

$$\sigma_k(u, C) = \frac{1}{n!} \operatorname{tr}_n A_n T_1(u) \dots T_k(u-k+1) C_{k+1} \dots C_n$$

generate a commutative subalgebra of $\mathrm{Y}(\mathfrak{gl}_n)$.

Definition 6.8. The commutative subalgebras $\mathrm{Y}(\mathfrak{gl}_n)$ defined in Corollaries 6.6 and 6.7 are called the *Bethe subalgebras.* (See [15, Section 1.14] and also [17].)

In calculations we also use alternative formulas for the elements $\tau_k(u, C)$ and $\sigma_k(u, C)$, which are consequences of Proposition 6.4.

Lemma 6.9. *In the notation of Corollaries 6.6 and 6.7 we have*

$$\tau_k(u, C) = \frac{1}{k!} \operatorname{tr}_k A_k C_1 \dots C_k T_k(u-k+1) \dots T_1(u), \tag{6.4}$$

$$\sigma_k(u, C) = \frac{1}{n!} \operatorname{tr}_n A_n T_k(u-k+1) \dots T_1(u) C_{k+1} \dots C_n. \tag{6.5}$$

6.6. **Application to Capelli identities.** Let us show how the R-matrix formalism can be used in order to prove various identities related to the Capelli identity.

Lemma 6.10. *For any k-tuples $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_k)$ in $[n]^k$ we have the following identity in $U(\mathfrak{gl}_n)[z]$:*

$$\operatorname{rdet}[E_{i_\alpha j_\beta} + (z+\alpha-1)\delta_{i_\alpha j_\beta}]_{\alpha,\beta=1}^k = \operatorname{cdet}[E_{i_\alpha j_\beta} + (z+k-\beta)\delta_{i_\alpha j_\beta}]_{\alpha,\beta=1}^k.$$

In particular, these expressions are skew-symmetric in $i_1, \dots, i_k$ and $j_1, \dots, j_k$, respectively.

Proof. We start with the ternary relation for $m = k$ (see Propositions 6.3 and 6.4):

$$A_k T_1(u) \dots T_k(u-k+1) = T_k(u-k+1) \dots T_1(u) A_k.$$

Now let us apply both sides as operators on $(\mathbb{C}^n)^{\otimes k}$ to the vector $e_{j_1} \otimes \dots e_{j_k}$. The left-hand side gives

$$(A_k T_1(u) \dots T_k(u-k+1))(e_{j_1} \otimes \dots e_{j_k}) = \sum_{l_1, \dots, l_k=1}^n t_{l_1 j_1}(u) \dots t_{l_k j_k}(u-k+1) \cdot A_k(e_{l_1} \otimes \dots e_{l_k}) =$$

$$= \sum_{l_1, \dots, l_k=1}^n \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}(\sigma) t_{l_1 j_1}(u) \dots t_{l_k j_k}(u-k+1) \cdot e_{l_{\sigma(1)}} \otimes \dots \otimes e_{l_{\sigma(k)}} =$$

$$= \sum_{l_1, \dots, l_k=1}^n \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}(\sigma) t_{l_{\sigma(1)} j_1}(u) \dots t_{l_{\sigma(k)} j_k}(u-k+1) \cdot e_{l_1} \otimes \dots e_{l_k},$$

while the right-hand side gives

$$(T_k(u-k+1) \dots T_1(u) A_k)(e_{j_1} \otimes \dots \otimes e_{j_k}) = \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}(\sigma) (T_k(u-k+1) \dots T_1(u))(e_{j_{\sigma(1)}} \otimes \dots \otimes e_{j_{\sigma(k)}}) =$$

$$= \sum_{\sigma \in \mathfrak{S}_k} \sum_{l_1, \dots, l_k=1}^n \operatorname{sgn}(\sigma) t_{l_1 j_{\sigma(1)}}(u-k+1) \dots t_{l_k j_{\sigma(k)}}(u) \cdot e_{l_1} \otimes \dots \otimes e_{l_k}.$$

Comparing the coefficients in front of $e_{i_1} \otimes \dots \otimes e_{i_k}$, we obtain

$$\sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}(\sigma) t_{i_{\sigma(1)} j_1}(u) \dots t_{i_{\sigma(k)} j_k}(u-k+1) = \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}(\sigma) t_{i_1 j_{\sigma(1)}}(u-k+1) \dots t_{i_k j_{\sigma(k)}}(u). \tag{6.6}$$

Apply the evaluation map to both sides and note that

$$\operatorname{ev}\left(\sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}(\sigma) t_{i_{\sigma(1)} j_1}(u) \dots t_{i_{\sigma(k)} j_k}(u-k+1) \right) = (u^{\underline{k}})^{-1} \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}(\sigma) \prod_{p=1,\dots,k} (E_{i_{\sigma(p)} j_p} + (u-p+1)\delta_{i_{\sigma(p)} j_p}) =$$
$$= (u^{\underline{k}})^{-1} \cdot \operatorname{cdet}[E_{i_\alpha j_\beta} + (u-\beta+1)\delta_{i_\alpha j_\beta}]_{\alpha,\beta=1}^k,$$

$$\operatorname{ev}\left(\sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}(\sigma) t_{i_1 j_{\sigma(1)}}(u-k+1) \dots t_{i_k j_{\sigma(k)}}(u) \right) = (u^{\underline{k}})^{-1} \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}(\sigma) \prod_{p=1,\dots,k} (E_{i_p j_{\sigma(p)}} + (u-k+p)\delta_{i_p j_{\sigma(p)}}) =$$
$$= (u^{\underline{k}})^{-1} \cdot \operatorname{rdet}[E_{i_\alpha j_\beta} + (u-k+\alpha)\delta_{i_\alpha j_\beta}]_{\alpha,\beta=1}^k.$$

Combining everything, we obtain

$$\operatorname{rdet}[E_{i_\alpha j_\beta} + (u-k+\alpha)\delta_{i_\alpha j_\beta}]_{\alpha,\beta=1}^k = \operatorname{cdet}[E_{i_\alpha j_\beta} + (u-\beta+1)\delta_{i_\alpha j_\beta}]_{\alpha,\beta=1}^k.$$

Substituting $u = z + k - 1$ finishes the proof of the lemma. $\square$

Remark 6.1. In particular, by plugging $z = 0$ into the lemma we obtain another proof of Corollary 5.6.

7. Commutativity of big algebras

The primary aim of this section is to prove the following crucial result.

Theorem 7.1. *The algebra $\mathscr{B}(\mathcal{P}(n,r))$ is commutative.*

Before we turn to the proof of the theorem, let us discuss its consequences for the big algebras of irreducible representations.

7.1. Big algebras of irreducible representations.

For any dominant integral weight $\lambda \in \mathfrak{h}^*$ we set $\mathscr{B}(\lambda) := \mathscr{B}(V(\lambda))$, i.e. the big algebra associated to the finite-dimensional irreducible GL_n-module $V(\lambda)$ of highest weight λ.

Recall from Proposition 3.9 that for any direct sum $V = \bigoplus_\alpha V_\alpha$ of GL_n-modules the big algebras $\mathscr{B}(V_\alpha)$ are homomorphic images of the big algebra $\mathscr{B}(V)$. Moreover, we have the inclusion $\mathscr{B}(V) \subset S(\mathfrak{gl}_n^*) \otimes \big(\prod_\alpha \operatorname{End} V_\alpha\big)$ and the map $\mathscr{B}(V) \to \mathscr{B}(V_\alpha)$ is induced by the projection $S(\mathfrak{gl}_n^*) \otimes \big(\prod_\alpha \operatorname{End} V_\alpha\big) \to S(\mathfrak{gl}_n^*) \otimes \operatorname{End} V_\alpha$. In particular, the image of an element $F \in \mathscr{B}(V)$ in $\mathscr{B}(V_\alpha)$ can be regarded as the "restriction of F to V_α". More precisely, for any $Y \in \mathfrak{gl}_n$ the linear operator $F(Y) \in \operatorname{End} V$ preserves each subspace V_α and its restriction to V_α is the image of F in $\mathscr{B}(V_\alpha)$ evaluated at Y.

Now let us apply these observations to the GL_n-module decomposition $\mathcal{P}(n,r) \simeq \bigoplus_{\lambda\,:\,\ell(\lambda)\le r} V(\lambda)^{\oplus d_\lambda}$, where $V(\lambda) = V_{\mathrm{GL}_n}(\lambda)$ and $d_\lambda = \dim V_{\mathrm{GL}_r}(\lambda)$ (see (3.8) for more details). Then, Proposition 3.9 implies that $\mathscr{B}(\lambda)$ is a quotient of $\mathscr{B}(\mathcal{P}(n,r))$ for any λ of the form $(\lambda_1,\dots,\lambda_r,0,\dots,0)$. In particular, the big algebra of any finite-dimensional irreducible polynomial GL_n-representation is a quotient of $\mathscr{B}(\mathcal{P}(n,n))$. The discussion above together with Theorem 7.1 gives the following corollary.

Corollary 7.2. *The big algebra $\mathscr{B}(\lambda)$ of any polynomial finite-dimensional irreducible GL_n-representation $V(\lambda)$ is commutative.*

Remark 7.1. The commutativity of big algebras defined by means of the Kirillov–Wei operator was first shown by Hausel and Zveryk, see [5, Theorem 2.3]. However, their proof involves quite non-trivial constructions such as the *Feigin–Frenkel center* and *Segal–Sugawara vectors*. Namely, they use explicit formulas for the Segal–Sugawara vectors for the Feigin–Frenkel center that were obtained by Yakimova in [25]. Our proof of the commutativity of big algebras in type A is different and essentially elementary. It relies on direct calculations and known facts about the Bethe subalgebras in Yangians and representation theory of $\mathfrak{gl}_n$.

7.2. Proof of Theorem 7.1.

Proof. In view of Corollary 3.11 it suffices to check that the operators $F_{p,q}$ commute. By (3.10) and (5.4)

$$F_{p,q}(Y) = \sum_{\substack{I_1,J_1\in\binom{[n]}{p}\\ I_2,J_2\in\binom{[n]}{q}\\ I_1\sqcup I_2=J_1\sqcup J_2}} \operatorname{sgn}\begin{pmatrix} I_1 & I_2\\ J_1 & J_2\end{pmatrix} \det Y_{I_1J_1}\cdot L(\Pi_{J_2I_2}).$$

Note that in view of Proposition 3.5, we only need to check the commutativity for $Y \in \mathfrak{h}$, i.e. for diagonal matrices. According to Proposition 3.13, for $Y = \operatorname{diag}(z_1,\dots,z_n)$ we have

$$F_{p,q}(Y) = \sum_{\substack{I\in\binom{[n]}{p}\\ J\in\binom{[n]}{q}\\ I\cap J=\varnothing}} \det Y_{II}\cdot L(\Pi_{JJ}) = \sum_{\substack{I\in\binom{[n]}{p}\\ J\in\binom{[n]}{q}\\ I\cap J=\varnothing}} \prod_{i\in I} z_i\cdot L(\Pi_{JJ}).$$

To show that the operators $F_{p,q}$ commute we will show that (see also (7.2) below)

$$L(\operatorname{ev}(\sigma_{n-p}(v,Y))) = \sum_{\ell=0}^{n-p} (v(v-1)\dots(v-\ell+1))^{-1}\cdot F_{p,\ell}(Y),\ Y\in\mathfrak{h} \tag{7.1}$$

and then use the commutativity of the Bethe subalgebra. (Here we use the notation from Corollary 6.7.)

Indeed, applying Lemma 6.9 to the diagonal matrix $C = Y = \operatorname{diag}(z_1,\dots,z_n)$ yields

$$\begin{aligned} n!\cdot\sigma_k(u,Y) &= \operatorname{tr}_n A_nT_k(u-k+1)\dots T_1(u)Y_{k+1}\dots Y_n = \\ &= \sum_{i_1,\dots,i_n}\sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma)t_{i_1i_{\sigma(1)}}(u-k+1)\dots t_{i_ki_{\sigma(k)}}(u)\prod_{j=k+1}^{n}\delta_{i_ji_{\sigma(j)}}z_{i_j} = \\ &= \sum_{(i_1,\dots,i_k)\in[n]^{\underline{k}}}\sum_{\sigma\in\mathfrak{S}_k} \operatorname{sgn}(\sigma)t_{i_1i_{\sigma(1)}}(u-k+1)\dots t_{i_ki_{\sigma(k)}}(u)\cdot\prod_{i\in I^c} z_i = \\ &= k!\sum_{i_1<\dots<i_k}\sum_{\sigma\in\mathfrak{S}_k} \operatorname{sgn}(\sigma)t_{i_1i_{\sigma(1)}}(u-k+1)\dots t_{i_ki_{\sigma(k)}}(u)\cdot\prod_{i\in I^c} z_i, \end{aligned}$$

where by I^c we denote the complement of $I = \{i_1, \dots, i_k\}$ in $[n]$. Therefore,

$$n! \cdot \mathrm{ev}(\sigma_k(u, Y)) = k! \cdot (u(u-1)\dots(u-k+1))^{-1} \sum_{I \in \binom{[n]}{k}} \Pi_{II}(k-1-u) \cdot \prod_{i \in I^c} z_i.$$

Here we used that the expression

$$\mathrm{ev}\left(\sum_{\sigma \in \mathfrak{S}_k} \mathrm{sgn}(\sigma) t_{i_1 i_{\sigma(1)}}(u-k+1)\dots t_{i_k i_{\sigma(k)}}(u)\right) = (u^{\underline{k}})^{-1} \cdot \mathrm{rdet}[E_{i_\alpha i_\beta} + (u-k+\alpha)\delta_{i_\alpha i_\beta}]_{\alpha,\beta=1}^{k}$$

is skew-symmetric in $i_1, \dots, i_k$ (see Lemma 6.10 and the formula (6.6)).

Thus,

$$n! \cdot (-1)^{n-p}\binom{u}{n-p} \cdot \mathrm{ev}(\sigma_{n-p}(n-p-1-u, Y)) = \sum_{I \in \binom{[n]}{n-p}} \Pi_{II}(u) \cdot \prod_{i \in I^c} z_i =$$

$$= \sum_{I \in \binom{[n]}{n-p}} \sum_{\ell=0}^{n-p} (-1)^{n-p-\ell} u^{\underline{n-p-\ell}} \sum_{K \in \binom{I}{\ell}} \Pi_{KK} \cdot \prod_{i \in I^c} z_i = \sum_{\ell=0}^{n-p} (-1)^{n-p-\ell} u^{\underline{n-p-\ell}} \sum_{\substack{J \in \binom{n}{p} \\ K \in \binom{[n]}{\ell} \\ J \cap K = \varnothing}} \Pi_{KK} \prod_{j \in J} z_j.$$

Applying the map L, we obtain

$$n! \cdot (-1)^{n-p}\binom{u}{n-p} \cdot L(\mathrm{ev}(\sigma_{n-p}(n-p-1-u, Y))) = \sum_{\ell=0}^{n-p} (-1)^{n-p-\ell} u^{\underline{n-p-\ell}} \cdot F_{p,\ell}(Y), \tag{7.2}$$

which is equivalent to (7.1).

Now Corollary 6.7 implies that for any $0 \le p_1, p_2 \le n$ we have the identity

$$\left[\sum_{\ell_1=0}^{n-p_1} (-1)^{n-p_1-\ell_1} u^{\underline{n-p_1-\ell_1}} \cdot F_{p_1,\ell_1}(Y), \sum_{\ell_2=0}^{n-p_2} (-1)^{n-p_2-\ell_2} v^{\underline{n-p_2-\ell_2}} \cdot F_{p_2,\ell_2}(Y)\right] = 0.$$

Finally, the polynomials $\{u^{\underline{k}} v^{\underline{l}}\}_{k,l \ge 0}$ are linearly independent and hence, $[F_{p_1,q_1}(Y), F_{p_2,q_2}(Y)] = 0$ for all p_1, p_2, q_1, q_2. $\square$

7.3. **Big algebras and Bethe subalgebras.** In fact, the images of elements of the Bethe subalgebra under the evaluation homomorphism can be regarded as elements of Kirillov algebra in the sense of the lemma below. This observation together with the formula (7.1) implies that big algebras are, roughly speaking, homomorphic images of the Bethe subalgebras of the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$.

Lemma 7.3. *Consider the Laurent expansion $\sigma_k(u, C) = \sum_{r=0}^{\infty} \sigma_k^{(r)}(C) u^{-r}$. Then, the maps $C \mapsto L(\mathrm{ev}(\sigma_k^{(r)}(C^T)))$ belong to the Kirillov algebra $\mathscr{C}(\mathcal{P}(n,r))$.*

Proof. Denote for brevity $\widetilde{T}(u) = \mathrm{ev}(T(u))$ and $\widetilde{t}_{ij}(u) = \mathrm{ev}(t_{ij}(u)) = \delta_{ij} + u^{-1}E_{ij}$. Observe that for any $g \in \mathrm{GL}_n$ the element A_n commutes with $g_1 g_2 \dots g_n$ in $\mathrm{End}(\mathbb{C}^n)^{\otimes n}$. Hence, by the cyclicity property of the trace,

$$\begin{aligned}
n! \cdot \sigma_k(u, \mathrm{Ad}(g)(C)) = n! \cdot \sigma_k(u, gCg^{-1}) &= \mathrm{tr}_n\, A_n T_1(u) \dots T_k(u-k+1) g_{k+1} C_{k+1} g_{k+1}^{-1} \dots g_n C_n g_n^{-1} = \\
&= \mathrm{tr}_n\, A_n g_{k+1} \dots g_n T_1(u) \dots T_k(u-k+1) C_{k+1} \dots C_n g_{k+1}^{-1} \dots g_n^{-1} = \\
&= \mathrm{tr}_n\, A_n (g_1 \dots g_n)(g_1^{-1} \dots g_k^{-1}) T_1(u) \dots T_k(u-k+1) C_{k+1} \dots C_n g_{k+1}^{-1} \dots g_n^{-1} = \\
&= \mathrm{tr}_n\, A_n g_1^{-1} \dots g_k^{-1} T_1(u) \dots T_k(u-k+1) C_{k+1} \dots C_n g_1 \dots g_k = \\
&= \mathrm{tr}_n\, A_n g_1^{-1} T_1(u) g_1 \dots g_k^{-1} T_k(u-k+1) g_k C_{k+1} \dots C_n.
\end{aligned}$$

On the other hand, for any $h \in \mathrm{GL}_n$ we have

$$\begin{aligned}
(\mathrm{id} \otimes h)\widetilde{T}(v)(\mathrm{id} \otimes h)^{-1} &= \sum_{i,j=1}^{n} \widetilde{t}_{ij}(v) \otimes \mathrm{Ad}(h)(e_{ij}) = \sum_{k,l=1}^{n} \sum_{i,j=1}^{n} [\mathrm{Ad}(h)(e_{ij})]_{kl} \cdot (\delta_{ij} + v^{-1}E_{ij}) \otimes e_{kl} = \\
&= \sum_{k,l=1}^{n} \left(\sum_{i,j=1}^{n} [\mathrm{Ad}(h)(e_{ij})]_{kl} \cdot \delta_{ij}\right) \otimes e_{kl} + \sum_{k,l=1}^{n} v^{-1} \left(\sum_{i,j=1}^{n} [\mathrm{Ad}(h)(e_{ij})]_{kl} \cdot E_{ij}\right) \otimes e_{kl} = \\
&= \sum_{k,l=1}^{n} (\delta_{kl} + v^{-1}\, \mathrm{Ad}(h^T)(E_{kl})) \otimes e_{kl} = \sum_{k,l=1}^{n} \mathrm{Ad}(h^T)(\widetilde{t}_{kl}(v)) \otimes e_{kl},
\end{aligned}$$

where the penultimate equality follows from

$$\sum_{i,j=1}^{n}[\mathrm{Ad}(h)(e_{ij})]_{kl}\cdot E_{ij}=\sum_{i,j=1}^{n}\mathrm{tr}(\mathrm{Ad}(h)(e_{ij})\,e_{lk})\cdot E_{ij}=\sum_{i,j=1}^{n}\mathrm{tr}(e_{ij}\,\mathrm{Ad}(h^{-1})(e_{lk}))\cdot E_{ij}=$$
$$=\sum_{i,j=1}^{n}\mathrm{tr}(\mathrm{Ad}(h^{T})(e_{kl})\,e_{ji})\cdot E_{ij}=\sum_{i,j=1}^{n}[\mathrm{Ad}(h^{T})(e_{kl})]_{ij}\cdot E_{ij}=\mathrm{Ad}(h^{T})(E_{kl}).$$

Therefore, $(\mathrm{id}\otimes h)\widetilde{T}(v)(\mathrm{id}\otimes h)^{-1}=(\mathrm{Ad}(h^{T})\otimes\mathrm{id})(\widetilde{T}(v))$. It follows that for any $g\in\mathrm{GL}_n$ we have

$$\mathrm{ev}\Big(\sigma_k(u,\mathrm{Ad}(g)(C))\Big)=\mathrm{Ad}\left((g^{T})^{-1}\right)\Big(\mathrm{ev}(\sigma_k(u,C))\Big).$$

If we denote $\Phi(u,C)=(L\circ\mathrm{ev})(\sigma_k(u,C^{T}))$, then for any $g\in\mathrm{GL}_n$ we obtain

$$\Phi(u,\mathrm{Ad}(g)(C))=L(\mathrm{Ad}(g)(\mathrm{ev}(\sigma_k(u,C))))=\widetilde{L}(g)\Phi(u,C)\widetilde{L}(g)^{-1},$$

which is precisely the condition (3.2) for the GL_n-representation $\widetilde{\pi}=\widetilde{L}$ on $\mathcal{P}(n,r)$. Hence, all coefficients of the Laurent series $\Phi(u,C)=(L\circ\mathrm{ev})(\sigma_k(u,C^{T}))$, viewed as functions in C, belong to $\mathscr{C}(\mathcal{P}(n,r))$. ☐

8. Big algebras of the symmetric powers of the vector representation

The goal of this section is to give a more explicit description of the big algebra for the case of symmetric powers $S^m(\mathbb{C}^n)$. These representations are also particular examples of weight multiplicity free representations.

8.1. **Kirillov algebras in the weight multiplicity free case.** Note that Kirillov algebra $\mathscr{C}(V)$ is not in general commutative. However, if V is weight multiplicity free, then it is the case (see [13, Corollary 1], [19, Introduction] and [20, Theorem 4.1]). Moreover, the following holds.

Proposition 8.1. *Let V be a weight multiplicity free representation of $\mathfrak{g}=\mathfrak{gl}_n$. Then, for any $A\in\mathscr{C}(V)$ and any $Y\in\mathfrak{h}$ the operator $A(Y)\in\mathrm{End}\,V$ is diagonal in the weight basis of V. In particular, the Kirillov algebra $\mathscr{C}(V)$ is commutative.*

Remark 8.1. Note that since V is weight multiplicity free, the elements of the weight basis of V are determined uniquely up to multiplication by a non-zero scalar.

Proof. This is an immediate consequence of Proposition 3.5. ☐

As was discussed in Subsection 7.1, we can treat the elements of $\mathscr{B}(\mathcal{P}(n,r))$ as elements of the big algebra $\mathscr{B}(\lambda)$ by restricting the corresponding linear operators to a subrepresentation V of $\mathcal{P}(n,r)$ which is isomorphic to $V(\lambda)$. For such a subrepresentation V and an element $F\in\mathscr{B}(n,r)$ we denote the image of F in $\mathscr{B}(\lambda)$ by $F|_{V(\lambda)}$ (or simply by F if the context is clear).

Corollary 8.2. *The big algebra $\mathscr{B}(\mathcal{P}(n,1))$ coincides with the medium algebra $\mathscr{M}(\mathcal{P}(n,1))$.*

Proof. By Corollary 3.12 applied to $r=1$, both algebras $\mathscr{M}(\mathcal{P}(n,1))$ and $\mathscr{B}(\mathcal{P}(n,1))$ are generated by the operators $\{F_{p,0}\}_{1\le p\le n}\cup\{F_{p,1}\}_{0\le p\le n-1}$. Hence, $\mathscr{M}(\mathcal{P}(n,1))=\mathscr{B}(\mathcal{P}(n,1))$. ☐

Recall that the medium algebra $\mathscr{M}(V)$ is the subalgebra of $\mathscr{C}(V)$ generated by $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$ and $\mathbf{D}(S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n})$ (see (3.6)). For a dominant integral weight λ we denote $\mathscr{M}(\lambda)\coloneqq\mathscr{M}(V(\lambda))$. We use the following result, announced by Hausel in [5].

Theorem 8.3 ([5, Theorem 2.1])**.** *For any dominant integral weight λ the medium algebra $\mathscr{M}(\lambda)$ is the center of the Kirillov algebra $\mathscr{C}(\lambda)$.*

Applying this theorem to the case of weight multiplicity free representations we obtain the following fact.

Proposition 8.4. *Let V be an irreducible weight multiplicity free representation. Then, $\mathscr{M}(V)=\mathscr{B}(V)=\mathscr{C}(V)$. In other words, the medium, big and Kirillov algebras of V all coincide.*

Remark 8.2. Since in the case $r=1$ the ring $\mathcal{P}(n,r)$ is isomorphic to $S(\mathbb{C}^n)$ as a $\mathfrak{gl}_n$-representation, its decomposition into irreducible subrepresentations is just $\bigoplus_{m\ge 0}S^m(\mathbb{C}^n)$. Note that all these summands are distinct and weight multiplicity free. Thus, taking into account Proposition 3.9, one can view Corollary 8.2 as a version of Proposition 8.4.

8.2. **Description of the algebra $\mathscr{B}(S^m(\mathbb{C}^n))$.** Now we apply our results to symmetric powers of the vector representation of $\mathfrak{gl}_n$. From now on we only consider $\mathcal{P}(n,1)$, i.e. we set $r=1$. Denote for brevity $x_j = x_{j1}$ and $\partial_j = \partial_{j1}$, $1 \le j \le n$. Then, $\mathcal{P}(n,r) = \mathcal{P}(n,1) = \mathbb{C}[x_1,\dots,x_n]$ is isomorphic to the symmetric algebra $S(\mathbb{C}^n) = \bigoplus_{m\ge 0} S^m(\mathbb{C}^n)$. Note that the latter decomposition is also compatible with the GL_n-action. Namely, the m-th summand $S^m(\mathbb{C}^n) \simeq \mathbb{C}[x_1,\dots,x_n]_m$ is isomorphic to $V(m\varpi_1)$. In particular, each irreducible representation of GL_n occurs in $\mathcal{P}(n,1)$ with the multiplicity at most 1.

Denote by $\{\varepsilon_1,\dots,\varepsilon_n\}$ the basis of the weight lattice in $\mathfrak{h}^*$ that is dual to $\{E_{11},\dots,E_{nn}\}$ in $\mathfrak{h}$. The fundamental weights $\varpi_1,\dots,\varpi_n$ are given by $\varpi_k = \varepsilon_1+\ldots+\varepsilon_k$, $1\le k\le n$. Recall that the Cartan subalgebra $\mathfrak{h}$ consists of diagonal matrices with n entries. We identify $S(\mathfrak{h}^*)$ with a polynomial ring $\mathbb{C}[t_1,\dots,t_n]$ using the natural coordinates on $\mathfrak{h}$. Then, by the Chevalley restriction theorem [10, Ch. VI, 23.1] $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n} \simeq S(\mathfrak{h}^*)^W = \mathbb{C}[t_1,\dots,t_n]^{\mathfrak{S}_n}$.

Proposition 8.5. *For any positive integer m the big algebra $\mathscr{B}(m\varpi_1)$ coincides with $\mathscr{C}(m\varpi_1)$ and is generated by the operators $\{F_{p,1}\}_{0\le p\le n-1}$ restricted to $V(m\varpi_1)$ over $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$.*

Proof. The first part follows from Proposition 8.4 and the fact that $V(m\varpi_1)\simeq S^m(\mathbb{C}^n)$ is weight multiplicity free. As for the second part, observe that $\mathcal{P}(n,1)\simeq S(\mathbb{C}^n)$ decomposes as a $\mathfrak{gl}_n$-module as a direct sum $\bigoplus_{m\ge 0} V(m\varpi_1)$. Thus, the second part of the proposition now follows from the proof of Corollary 8.2. $\square$

The following proposition gives a more explicit description of the big algebra for symmetric powers of the vector representation.

Proposition 8.6. *Let $\mathrm{Func}(\mathrm{wt}(m\varpi_1), S(\mathfrak{h}^*))$ be the algebra of $S(\mathfrak{h}^*)$-valued functions on $\mathrm{wt}(m\varpi_1)$. There exists an injective $S(\mathfrak{h}^*)^W$-algebra homomorphism $\mathfrak{i}_m\colon \mathscr{B}(m\varpi_1)\to \mathrm{Func}(\mathrm{wt}(m\varpi_1), S(\mathfrak{h}^*))$. Namely, the map $\mathfrak{i}_m$ sends an element $F\in\mathscr{B}(m\varpi_1)$ to a function $\hat{F}\in \mathrm{Func}(\mathrm{wt}(m\varpi_1), S(\mathfrak{h}^*))$ such that the operator $F(Y)$ acts as multiplication by $(\hat{F}(\mu))(Y)$ on $V_\mu(m\varpi_1)$ for all $\mu\in\mathrm{wt}(m\varpi_1)$ and $Y\in\mathfrak{h}$.*

The image $\mathfrak{i}_m(\mathscr{B}(m\varpi_1))\simeq\mathscr{B}(m\varpi_1)$ is the subalgebra of $\mathrm{Func}(\mathrm{wt}(m\varpi_1), S(\mathfrak{h}^*))$ generated by the following two subalgebras:

- *the subalgebra $\mathcal{F}_0^{(m)}$ of constant maps $\mu\mapsto f$ for $f\in S(\mathfrak{h}^*)^W\simeq \mathbb{C}[t_1,\dots,t_n]^{\mathfrak{S}_n}$, and*
- *the subalgebra $\mathcal{F}_1^{(m)}$ of maps $\mu\mapsto f(\underbrace{t_1,\dots,t_1}_{\mu_1},\dots,\underbrace{t_n,\dots,t_n}_{\mu_n})$ for $f\in\Lambda_m$.*

Here Λ_m stands for the ring of symmetric polynomials in m variables.

Remark 8.3. Using the language of symmetric functions, one can interpret $f(\underbrace{t_1,\dots,t_1}_{\mu_1},\dots,\underbrace{t_n,\dots,t_n}_{\mu_n})$ as the *plethystic substitution* $f[\mu_1 t_1+\ldots+\mu_n t_n]$, where f is a symmetric function (see [14, Ch. I.8] for definitions).

Proof. Using Proposition 3.5, we can view $\mathscr{B}(m\varpi_1)$ as a subalgebra of $S(\mathfrak{h}^*)\otimes \mathrm{End}_{\mathfrak{h}} S^m(\mathbb{C}^n)$. According to Proposition 3.13, for $Y=\mathrm{diag}(t_1,\dots,t_n)\in\mathfrak{h}$ we have

$$F_{p,0}(Y) = \sum_{I\in\binom{[n]}{p}}\prod_{i\in I} t_i = e_p(t_1,\dots,t_n), \text{ and}$$

$$F_{p,1}(Y) = \sum_{\substack{I\in\binom{[n]}{p}\\ j\in[n]\setminus I}}\left(\prod_{i\in I} t_i\right)\cdot x_j\partial_j = \sum_{j=1}^n e_p(t_1,\dots,\widehat{t_j},\dots,t_n)\cdot x_j\partial_j.$$

Now we restrict these operators to the subrepresentation $S^m(\mathbb{C}^n)\subset\mathcal{P}(n,1)\simeq\mathbb{C}[x_1,\dots,x_n]$. Consider the monomial basis $\{x^\mu\}_{\mu\in\mathrm{wt}(m\varpi_1)}$ of $S^m(\mathbb{C}^n)\simeq\mathbb{C}[x_1,\dots,x_n]_m$. Here we denote $x^\mu := x_1^{\mu_1}\dots x_n^{\mu_n}$ and $\mu=(\mu_1,\dots,\mu_n)$ runs over $\mathrm{wt}(m\varpi_1)$, that is, over all n-tuples of non-negative integers that add up to m. Clearly, the μ-eigenspace of $\mathbb{C}[x_1,\dots,x_n]_m$ is spanned by x^μ for all $\mu\in\mathrm{wt}(m\varpi_1)$.

The operator $x_j\partial_j$ acts on $\{x^\mu\}_{\mu\in\mathrm{wt}(m\varpi_1)}$ as $x_j\partial_j(x^\mu)=\mu_j x^\mu$. Thus, the operators $F_{p,0}$ and $F_{p,1}$ are diagonal in the basis $\{x^\mu\}_{\mu\in\mathrm{wt}(m\varpi_1)}$ and the corresponding matrices are as follows:

$$F_{p,0}(Y)\leftrightarrow\Big(e_p(t_1,\dots,t_n) : \mu\in\mathrm{wt}(m\varpi_1)\Big),\quad F_{p,1}(Y)\leftrightarrow\left(\sum_{j=1}^n \mu_j e_p(t_1,\dots,\widehat{t_j},\dots,t_n) : \mu\in\mathrm{wt}(m\varpi_1)\right).$$

Let $\hat{F}_{p,0}$ and $\hat{F}_{p,1}$ be the elements of $\mathrm{Func}(\mathrm{wt}(m\varpi_1), S(\mathfrak{h}^*))$ that correspond to $F_{p,0}$ and $F_{p,1}$, respectively. In other words, we have

$$\hat{F}_{p,0}(\mu) = e_p(t_1,\dots,t_n) \text{ and } \hat{F}_{p,1}(\mu)=\sum_{j=1}^n \mu_j e_p(t_1,\dots,\widehat{t_j},\dots,t_n).$$

Using this and the fact that the big algebra $\mathscr{B}(m\varpi_1)$ is generated by $\{F_{p,0}\}_{1\le p\le n}$ and $\{F_{p,1}\}_{0\le p\le n-1}$, we can identify $\mathscr{B}(m\varpi_1)$ with the subalgebra $\mathcal{A}$ of the algebra $\mathrm{Func}(\mathrm{wt}(m\varpi_1), S(\mathfrak{h}^*))$ generated by the elements $\{\hat{F}_{p,0}\}_{1\le p\le n}$ and $\{\hat{F}_{p,1}\}_{0\le p\le n-1}$. Our goal is to prove that $\mathcal{A}$ is generated by $\mathcal{F}_0^{(m)}$ and $\mathcal{F}_1^{(m)}$.

The elements $\{\hat{F}_{p,0}\}_{1\le p\le n}$ are constant functions on $\mathrm{wt}(m\varpi_1)$ and they generate the subalgebra $\mathcal{F}_0^{(m)} \subset \mathrm{Func}(\mathrm{wt}(m\varpi_1), S(\mathfrak{h}^*))$. Thus, $\mathcal{F}_0^{(m)} \subset \mathcal{A}$. Next, observe that

$$\hat{F}_{p,1}(\mu) = \sum_{j=1}^{n} \mu_j e_p(t_1,\dots,\hat{t_j},\dots,t_n) = \left(\mu_1\frac{\partial}{\partial t_1}+\ldots+\mu_n\frac{\partial}{\partial t_n}\right)(e_{p+1}(t_1,\dots,t_n)).$$

Recall that the elementary symmetric polynomials $e_q(t_1,\dots,t_n)$ generate the whole ring $\mathbb{C}[t_1,\dots,t_n]^{\mathfrak{S}_n}$. Then, the Leibniz rule implies that $\mathcal{A}$ contains all functions of the form

$$\mu \mapsto \left(\mu_1\frac{\partial}{\partial t_1}+\ldots+\mu_n\frac{\partial}{\partial t_n}\right)(f) \text{ for } f\in\mathbb{C}[t_1,\dots,t_n]^{\mathfrak{S}_n}.$$

In particular, by setting $f = p_r$, where $p_r(t_1,\dots,t_n) = t_1^r+\ldots+t_n^r$ is the r-th power sum, we obtain the function

$$\mu \mapsto \left(\mu_1\frac{\partial}{\partial t_1}+\ldots+\mu_n\frac{\partial}{\partial t_n}\right)(p_r(t_1,\dots,t_n)) = \sum_{j=1}^{n}\mu_j r t_j^{r-1} = r p_{r-1}(\underbrace{t_1,\dots,t_1}_{\mu_1},\dots,\underbrace{t_n,\dots,t_n}_{\mu_n}). \tag{8.1}$$

Since the power sums also generate the ring of symmetric functions, we conclude that $\mathcal{F}_1^{(m)} \subset \mathcal{A}$.

It remains to show that the elements $\{\hat{F}_{p,0}\}_{1\le p\le n}$ and $\{\hat{F}_{p,1}\}_{0\le p\le n-1}$ also belong to the subalgebra of $\mathrm{Func}(\mathrm{wt}(m\varpi_1), S(\mathfrak{h}^*))$ generated by $\mathcal{F}_0^{(m)}$ and $\mathcal{F}_1^{(m)}$. Indeed, this is clear for $\hat{F}_{p,0}$ as it belongs to $\mathcal{F}_0^{(m)}$. As for the $\hat{F}_{p,1}$, note first that the function $\mu \mapsto \left(\sum_{i=1}^n \mu_i\frac{\partial}{\partial t_i}\right)(p_r(t_1,\dots,t_n))$ belongs to $\mathcal{F}_1^{(m)}$ due to the formula (8.1). Since the power sums $p_r(t_1,\dots,t_n)$ generate the ring $\mathbb{C}[t_1,\dots,t_n]^{\mathfrak{S}_n}$, the Leibniz rule implies that the function $\mu \mapsto \left(\sum_{i=1}^n \mu_i\frac{\partial}{\partial t_i}\right)(e_{p+1}(t_1,\dots,t_n))$ belongs to the subalgebra generated by $\mathcal{F}_0^{(m)}$ and $\mathcal{F}_1^{(m)}$. ☐

Remark 8.4. In fact, the proof of the proposition above can be easily modified to obtain a similar description of the medium algebra $\mathscr{M}(\lambda) = \mathscr{M}(V(\lambda))$ for any $\lambda = (\lambda_1,\dots,\lambda_n)$ with $\lambda_1\ge\ldots\ge\lambda_n\ge 0$.

As an application, we can explicitly describe the big algebra of the vector representation.

Corollary 8.7. *The $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$-algebra $\mathscr{B}(\varpi_1)$ is isomorphic to the $S(\mathfrak{h}^*)^W$-algebra $S(\mathfrak{h})^{W(\varpi_1)}$, where $W(\varpi_1)$ denotes the stabilizer of $\varpi_1\in\mathfrak{h}^*$ in the Weyl group $W\simeq\mathfrak{S}_n$. More explicitly, we have isomorphisms*

$$\mathscr{B}(\varpi_1) \simeq \mathbb{C}[t_1,\dots,t_n]^{\mathfrak{S}_1\times\mathfrak{S}_{n-1}} \simeq \mathbb{C}[t_1,\dots,t_n]^{\mathfrak{S}_n}[T]/(T^n - c_1T^{n-1}+\ldots+(-1)^n c_n).$$

Here, $c_1,\dots,c_n\in S(\mathfrak{h}^*)^W$ are defined as in Subsection 3.1 and T can be identified with t_1.

Remark 8.5. In fact, a similar result holds if one replaces ϖ_1 with ϖ_k, $1\le k\le n$ (the k-th exterior power of the vector representation). See also [19, Theorem 2.6].

Proof. For $m=1$ the set $\mathrm{wt}(m\varpi_1) = \mathrm{wt}(\varpi_1)$ consists of n weights which lie in the W-orbit of ϖ_1. Then, it is not difficult to see that the subalgebra $\mathcal{F}_1^{(1)}$ is isomorphic to $\mathbb{C}[t_1]$ in this case (namely, $F\mapsto F(\varpi_1)$ is an isomorphism). Therefore, $\mathscr{B}(\varpi_1)$ is isomorphic to the subalgebra of $\mathbb{C}[t_1,\dots,t_n]$ generated by $\mathbb{C}[t_1,\dots,t_n]^{\mathfrak{S}_n}$ and $\mathbb{C}[t_1]$, i.e. to $\mathbb{C}[t_1,\dots,t_n]^{\mathfrak{S}_1\times\mathfrak{S}_{n-1}}$.

The last assertion of the corollary follows from the fact that $\mathbb{C}[t_1,\dots,t_n]^{\mathfrak{S}_1\times\mathfrak{S}_{n-1}}$ is a free $\mathbb{C}[t_1,\dots,t_n]^{\mathfrak{S}_n}$-module generated by $1,t_1,\dots,t_1^{n-1}$ and the identity $t_1^n - c_1t_1^{n-1}+\ldots+(-1)^n c_n = 0$ in $\mathbb{C}[t_1,\dots,t_n]$. ☐

Remark 8.6. The element T can be described in terms of the isomorphism from Proposition 8.6. Namely, it corresponds to the function on $\mathrm{wt}(\varpi_1) = \{\varepsilon_1,\dots,\varepsilon_n\}$ that sends ε_i to t_i for $i=1,\dots,n$.

8.3. **An isomorphism between $\mathscr{B}(m\varpi_1)$ and $S^m(\mathscr{B}(\varpi_1))$.** For each positive integer α define an element $P_\alpha\in\mathscr{C}(\mathcal{P}(n,1))$ by the formula

$$P_\alpha(Y) = L(Y^\alpha), \tag{8.2}$$

where $Y\in\mathfrak{gl}_n$ is viewed as an $n\times n$ matrix. One checks using (3.2) that P_α is indeed an element of the Kirillov algebra $\mathscr{C}(\mathcal{P}(n,1))$.

Proposition 8.8. *The algebra $\mathscr{B}(m\varpi_1)$ is generated over the ring $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$ by the elements $P_1,\dots,P_m$ restricted to $V(m\varpi_1)$.*

Proof. Note that for $Y = \mathrm{diag}(t_1,\dots,t_n)\in\mathfrak{h}$ we have

$$P_\alpha(Y) = \sum_{i=1}^n t_i^\alpha\cdot x_i\partial_i.$$

It now follows from Proposition 3.5 and the proof of Proposition 8.6 that the restriction of P_α to $V(m\varpi_1)$ coincides with the preimage of the function $\mu\mapsto\sum_{i=1}^n\mu_i t_i^\alpha$ under the map $\mathfrak{i}_m$. In that proof we also showed that these functions generate the algebra $\mathfrak{i}_m(\mathscr{B}(m\varpi_1))$ over $S(\mathfrak{h}^*)^W$. Therefore, the elements $P_\alpha|_{V(m\varpi_1)}$ generate the algebra $\mathscr{B}(m\varpi_1)$ over $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$. ☐

Using the obtained facts, we can relate the big algebras of $\mathscr{B}(\varpi_1)$ and $\mathscr{B}(m\varpi_1)$.

Theorem 8.9. *The $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$-algebra $\mathscr{B}(m\varpi_1)$ is isomorphic to the subalgebra $S^m(\mathscr{B}(\varpi_1))$ of $\mathfrak{S}_m$-invariant elements of the tensor product $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$-algebra $\mathscr{B}(\varpi_1)^{\otimes m}$.*

Proof. Throughout this proof all tensor products are considered over the ring $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n} \simeq S(\mathfrak{h}^*)^W$. By Proposition 8.6, for any positive integer k we can view $\mathscr{B}(k\varpi_1)$ as a subalgebra $\operatorname{Func}(\operatorname{wt}(k\varpi_1), S(\mathfrak{h}^*))$. We use this identification for $k=1$ and $k=m$.

Consider the map $\Upsilon\colon \operatorname{Func}(\operatorname{wt}(\varpi_1), S(\mathfrak{h}^*))^{\otimes m} \to \operatorname{Func}(\operatorname{wt}(m\varpi_1), S(\mathfrak{h}^*))$ defined by the following formula:

$$\Upsilon(f_1 \otimes \ldots \otimes f_m)(\mu) = \prod_{j=1}^{n} \prod_{l=1}^{\mu_j} f_{\mu_1+\ldots+\mu_{j-1}+l}(\varepsilon_j) \text{ for } f_1, \ldots, f_m \in \operatorname{Func}(\operatorname{wt}(\varpi_1), S(\mathfrak{h}^*)), \mu \in \operatorname{wt}(m\varpi_1).$$

It is clear that Υ defined in this way is an $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$-algebra homomorphism.

Now let us show that Υ maps $S^m(\mathscr{B}(\varpi_1))$ to $\mathscr{B}(m\varpi_1)$ under the identification above. Using the notation of Corollary 8.7, any $f \in \mathscr{B}(\varpi_1)$ can be uniquely expressed as $a_0 + a_1 T + \ldots + a_{n-1}T^{n-1}$ with $a_i \in S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$. Define the following elements:

$$X_i = 1 \otimes \ldots \otimes \underset{i}{T} \otimes \ldots \otimes 1,\ i \in \{1, \ldots, m\}.$$

Claim. $S^m(\mathscr{B}(\varpi_1))$ is spanned by symmetric polynomials in $X_1, \ldots, X_m$ as a $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$-module.

Proof of the claim. As an $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$-module, $\mathscr{B}(\varpi_1)^{\otimes m}$ is a free module spanned by $X_1^{i_1} \ldots X_m^{i_m}$ with $i_1, \ldots, i_m \in \{0, 1, \ldots, n-1\}$. Since the action of $\mathfrak{S}_m$ permutes $X_1, \ldots, X_m$, the $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$-module $S^m(\mathscr{B}(\varpi_1)^{\otimes m})$ is free with generators $\sum_{\sigma \in \mathfrak{S}_m} X_1^{i_{\sigma(1)}} \ldots X_m^{i_{\sigma(m)}}$, where $(i_1, \ldots, i_m)$ runs over all m-tuples of integers such that $0 \le i_1 \le \ldots \le i_m \le n-1$. (In particular, the rank of this module equals $\binom{n+m-1}{m}$.) It is clear that these expressions are symmetric polynomials in $X_1, \ldots, X_m$ and generate all of them since $e_k(X_1, \ldots, X_m)$ are among the generators. □

Now let us prove that the image of the restriction of Υ to $S^m(\mathscr{B}(\varpi_1))$ coincides with $\mathscr{B}(m\varpi_1)$. To start with, observe that for any $\mu \in \operatorname{wt}(m\varpi_1)$ we have

$$(\Upsilon(X_1)(\mu), \Upsilon(X_2)(\mu), \ldots, \Upsilon(X_m)(\mu)) = (\underbrace{t_1, \ldots, t_1}_{\mu_1}, \ldots, \underbrace{t_n, \ldots, t_n}_{\mu_n}).$$

Therefore, for any polynomial f in m variables with coefficients in $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$ we have

$$\Upsilon(f(X_1, \ldots, X_m))(\mu) = f(\underbrace{t_1, \ldots, t_1}_{\mu_1}, \ldots, \underbrace{t_n, \ldots, t_n}_{\mu_n}). \tag{8.3}$$

In particular, if f is symmetric, then $\Upsilon(f(X_1, \ldots, X_m)) \in \mathcal{F}_1^{(m)} \subset \mathscr{B}(m\varpi_1)$. Hence, Υ maps elements of $S^m(\mathscr{B}(\varpi_1))$ onto $\mathscr{B}(m\varpi_1)$ (see also Proposition 8.6).

Finally, let us show that Υ is injective on $S^m(\mathscr{B}(\varpi_1))$. Suppose that f is a symmetric polynomial in m variables with coefficients in $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$ such that $\Upsilon(f(X_1, \ldots, X_m)) \equiv 0$ on $\operatorname{wt}(m\varpi_1)$. Since $S^m(\mathscr{B}(\varpi_1))$ is a free $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$-module generated by $\{\sum_{\sigma \in \mathfrak{S}_m} X_1^{i_{\sigma(1)}} \ldots X_m^{i_{\sigma(m)}}\}_{0 \le i_1 \le \ldots, \le i_m \le n-1}$, we may assume that f has degree at most $n-1$ in each of the variables. The symmetry and (8.3) now imply that $f(a_1, \ldots, a_m)$ vanishes for any $(a_1, \ldots, a_m) \in \{t_1, \ldots, t_n\}^m$. Now the following lemma, proved in Appendix A, implies that $f \equiv 0$:

Lemma 8.10. *Let f be a polynomial in m variables $u_1, \ldots, u_m$ with coefficients in $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$ which has degree at most $n-1$ in each of the variables. If $f(a_1, \ldots, a_m)$ vanishes for all $(a_1, \ldots, a_m) \in \{t_1, \ldots, t_n\}^m$, then $f \equiv 0$.*

Thus, the map $\Upsilon\colon S^m(\mathscr{B}(\varpi_1)) \to \mathscr{B}(m\varpi_1)$ is indeed an isomorphism of $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$-algebras. □

Remark 8.7. Let us explain why the isomorphism described in the proof of Theorem 8.9 is in fact "natural". Indeed, $S^m(\mathscr{B}(\varpi_1))$ and $\mathscr{B}(m\varpi_1)$ can be regarded as certain subalgebras of $S^m(S(\mathfrak{gl}_n^*) \otimes \operatorname{End} V(\varpi_1))$ and $S(\mathfrak{gl}_n^*) \otimes \operatorname{End} V(m\varpi_1)$, respectively. If we evaluate these algebras at an element $Y = \operatorname{diag}(t_1, \ldots, t_n) \in \mathfrak{h}$, we will obtain subalgebras of $S^m(\operatorname{End}_{\mathfrak{h}} V(\varpi_1))$ and $\operatorname{End}_{\mathfrak{h}} V(m\varpi_1)$, respectively, that depend on parameters $t_1, \ldots, t_n$. The formula for the isomorphism $\Upsilon\colon S^m(\mathscr{B}(\varpi_1)) \to \mathscr{B}(m\varpi_1)$ essentially comes from identifications $S^m(\operatorname{End}_{\mathfrak{h}} V(\varpi_1)) \simeq \operatorname{End}_{\mathfrak{h}} S^m(V(\varpi_1)) \simeq \operatorname{End}_{\mathfrak{h}} V(m\varpi_1)$.

8.4. **Generators and relations.** In this subsection we give a presentation of the big algebra $\mathscr{B}(m\varpi_1)$ in terms of generators and relations. Recall that we identify $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$ with $S(\mathfrak{h}^*)^W = \mathbb{C}[t_1, \ldots, t_n]^{\mathfrak{S}_n}$.

Theorem 8.11. *The $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$-algebra $\mathscr{B}(m\varpi_1)$ is isomorphic to*

$$\mathbb{C}[t_1, \ldots, t_n]^{\mathfrak{S}_n}[u_1, \ldots, u_m]^{\mathfrak{S}_m}/J, \tag{8.4}$$

where J is the ideal of $R[u_1, \ldots, u_m]^{\mathfrak{S}_m}$ given by

$$J = \langle f_1 + \ldots + f_m, u_1 f_1 + \ldots + u_m f_m, \ldots, u_1^{m-1} f_1 + \ldots + u_m^{m-1} f_m \rangle \tag{8.5}$$

and $f_i = (u_i - t_1) \ldots (u_i - t_n)$ for each $i = 1, \ldots, m$.

Remark 8.8. The group $\mathfrak{S}_m$ acts on $\mathbb{C}[t_1,\ldots,t_n]^{\mathfrak{S}_n}[u_1,\ldots,u_m]$ by permuting the generators $u_1,\ldots,u_m$.

Proof. Denote $R = S(\mathfrak{h}^*)^W = \mathbb{C}[t_1,\ldots,t_n]^{\mathfrak{S}_n}$. By Proposition 8.6, the big algebra $\mathscr{B}(m\varpi_1)$ is isomorphic to the subalgebra of $\operatorname{Func}(\operatorname{wt}(m\varpi_1), S(\mathfrak{h}^*))$ generated by the subalgebras $\mathcal{F}_0^{(m)}$ and $\mathcal{F}_1^{(m)}$. Define a homomorphism $\Psi\colon R[u_1,\ldots,u_m]^{\mathfrak{S}_m} \to \operatorname{Func}(\operatorname{wt}(m\varpi_1), S(\mathfrak{h}^*))$ which sends any symmetric polynomial $f(u_1,\ldots,u_m)$ with coefficients in R to the function[2]

$$\mu \mapsto f(\underbrace{t_1,\ldots,t_1}_{\mu_1},\ldots,\underbrace{t_n,\ldots,t_n}_{\mu_n}),\ \mu=(\mu_1,\ldots,\mu_n)\in \operatorname{wt}(m\varpi_1).$$

It follows from the definitions of $\mathcal{F}_0^{(m)}$ and $\mathcal{F}_1^{(m)}$ that Ψ defines a surjective homomorphism from $R[u_1,\ldots,u_m]^{\mathfrak{S}_m}$ to $\mathscr{B}(m\varpi_1) \subset \operatorname{Func}(\operatorname{wt}(m\varpi_1), S(\mathfrak{h}^*))$. Therefore, to finish the proof of theorem, it remains to show that $\ker\Psi = J$.

Indeed, Ψ sends a symmetric polynomial $f(u_1,\ldots,u_m)$ with coefficients in R to the zero function on $\operatorname{wt}(m\varpi_1)$ if and only if $f(\underbrace{t_1,\ldots,t_1}_{\mu_1},\ldots,\underbrace{t_n,\ldots,t_n}_{\mu_n}) \equiv 0$ for any $\mu=(\mu_1,\ldots,\mu_n)\in\operatorname{wt}(m\varpi_1)$. Since f is symmetric, the last condition is equivalent to the fact that f vanishes on the Cartesian product $\{t_1,\ldots,t_n\}^m$. Thus, $\ker\Psi$ is the ideal of those symmetric polynomials in $R[u_1,\ldots,u_m]^{\mathfrak{S}_m}$ which vanish whenever we substitute $(u_1,\ldots,u_m)\in\{t_1,\ldots,t_n\}^m$. This ideal coincides with J by Lemma 8.12 below. □

Lemma 8.12. *In the notation of the proof of Theorem 8.11 we have:*
(i) $J = \langle f_1,\ldots,f_m\rangle \cap R[u_1,\ldots,u_m]^{\mathfrak{S}_m}$*, where* $\langle f_1,\ldots,f_m\rangle$ *is the ideal of* $R[u_1,\ldots,u_m]$ *generated by* $\{f_i\}_{i=1}^m$*;*
(ii) $J = \left\{f(u_1,\ldots,u_m)\in R[u_1,\ldots,u_m]^{\mathfrak{S}_m} : f|_{\{t_1,\ldots,t_n\}^m}\equiv 0\right\}$.

Proof. (i) It is clear from (8.5) that $J\subset \langle f_1,\ldots,f_m\rangle\cap R[u_1,\ldots,u_m]^{\mathfrak{S}_m}$. Take any element $F\in\langle f_1,\ldots,f_m\rangle\cap R[u_1,\ldots,u_m]^{\mathfrak{S}_m}$. Then, we can write it as

$$F = a_1 f_1 + \ldots + a_m f_m,$$

where $a_1,\ldots,a_m\in R[u_1,\ldots,u_m]$. Averaging this expression over $\mathfrak{S}_m$ gives

$$F = \frac{1}{m!}\sum_{\sigma\in\mathfrak{S}_m}\sum_{i=1}^m(\sigma\cdot a_i)(\sigma\cdot f_i).$$

However, it is clear from the equality $f_i=(u_i-t_1)\ldots(u_i-t_n)$ that $\mathfrak{S}_m$ permutes $f_1,\ldots,f_m$, i.e. $\sigma\cdot f_i = f_{\sigma(i)}$ for all $i\in\{1,\ldots,m\}$. Therefore, we can rewrite the equality above as follows:

$$F = b_1f_1+\ldots+b_mf_m,\ \text{where } b_i = \frac{1}{m!}\sum_{j=1}^m\sum_{\substack{\sigma\in\mathfrak{S}_m\\ \sigma(j)=i}}(\sigma\cdot a_j) = \frac{1}{m!}\sum_{\tau\in\mathfrak{S}_m}(\tau^{-1}\cdot a_{\tau(i)}).$$

It is not difficult to see that for each $i\in\{1,\ldots,m\}$ the element b_i belongs to $R[u_1,\ldots,u_m]^{\mathfrak{S}_m(i)}$, where $\mathfrak{S}_m(i)=\operatorname{Stab}_{\mathfrak{S}_m}(i)$ is the stabilizer of $i\in\{1,\ldots,m\}$ in $\mathfrak{S}_m$. Indeed, it follows from a more general observation that $\sigma\cdot b_i = b_{\sigma(i)}$ for any $i\in\{1,\ldots,m\}$ and any $\sigma\in\mathfrak{S}_m$.

Note that $R[u_1,\ldots,u_m]^{\mathfrak{S}_m(1)}$ is in fact a free rank m module over $R[u_1,\ldots,u_m]^{\mathfrak{S}_m}$. Moreover, the elements $1,u_1,\ldots,u_1^{m-1}$ are free generators of this module. Therefore, there exist $g_0,\ldots,g_{m-1}\in R[u_1,\ldots,u_m]^{\mathfrak{S}_m}$ such that

$$b_1 = g_0+g_1u_1+\ldots+g_{m-1}u_1^{m-1}.$$

The discussion above implies that $g_i = g_0+g_1u_i+\ldots+g_{m-1}u_i^{m-1}$ for all i and thus,

$$F=\sum_{i=1}^m b_if_i = \sum_{i=1}^m(g_0+g_1u_i+\ldots+g_{m-1}u_i^{m-1})f_i = \sum_{j=0}^{m-1}g_j(u_1^jf_1+\ldots+u_m^jf_m)\in J.$$

Thus, $F\in J$ which concludes the proof of the first part of the lemma.

(ii) Observe that for each $i=1,\ldots,m$ we have $f_i|_{\{t_1,\ldots,t_n\}^m}\equiv 0$, where we regard $f_1,\ldots,f_m$ as polynomials in variables $u_1,\ldots,u_m$ with coefficients in R. Then, it follows from (i) that

$$J\subset\left\{f(u_1,\ldots,u_m)\in R[u_1,\ldots,u_m]^{\mathfrak{S}_m} : f|_{\{t_1,\ldots,t_n\}^m}\equiv 0\right\}.$$

To show the other inclusion, take any $f(u_1,\ldots,u_m)\in R[u_1,\ldots,u_m]^{\mathfrak{S}_m}$ that vanishes on $\{t_1,\ldots,t_n\}^m$. It is clear that there exists a polynomial $g(u_1,\ldots,u_m)\in R[u_1,\ldots,u_m]$ which has degree at most $n-1$ in each the variables $u_1,\ldots,u_m$ and such that $f\equiv g \pmod{\langle f_1,\ldots,f_m\rangle}$ in $R[u_1,\ldots,u_m]$. Then, $g(u_1,\ldots,u_m)$ vanishes on $\{t_1,\ldots,t_n\}^m$ as well and Lemma 8.10 implies that $g\equiv 0$. Thus, $f\in\langle f_1,\ldots,f_m\rangle$. As $f\in R[u_1,\ldots,u_m]^{\mathfrak{S}_m}$ by the assumption, we obtain $f\in\langle f_1,\ldots,f_m\rangle\cap R[u_1,\ldots,u_m]^{\mathfrak{S}_m}=J$, as needed. □

[2]Compare this to (8.3).

Remark 8.9. The ring (8.4) is related to the coordinate ring of a certain algebraic variety. Namely, take the product $X = \mathrm{Sym}^m(\mathbb{A}^1)\times\mathrm{Sym}^n(\mathbb{A}^1)$, i.e. the space of pairs $(\{a_1,\dots,a_m\},\{b_1,\dots,b_n\})$, where $\{a_1,\dots,a_m\}$ and $\{b_1,\dots,b_n\}$ are configurations on the affine line of m and n points, respectively. Then, consider the subvariety $Y\subset X$ which is given by the set-theoretic inclusion $\{a_1,\dots,a_m\}\subseteq\{b_1,\dots,b_n\}$ (in particular, we completely ignore multiplicities in configurations). One can verify that $\mathrm{Spec}(\mathscr{B}(m\varpi_1))$ coincides with Y if we identify $\mathrm{Sym}^m(\mathbb{A}^1)$ and $\mathrm{Sym}^n\mathbb{A}^1$ with $\mathrm{Spec}(\mathbb{C}[u_1,\dots,u_m]^{\mathfrak{S}_m})$ and $\mathrm{Spec}(R)=\mathrm{Spec}(\mathbb{C}[t_1,\dots,t_n]^{\mathfrak{S}_n})$, respectively. The inclusion $R\hookrightarrow\mathscr{B}(m\varpi_1)$ then corresponds to the projection $\mathrm{Sym}^m(\mathbb{A}^1)\times\mathrm{Sym}^n(\mathbb{A}^1)\to\mathrm{Sym}^n(\mathbb{A}^1)$ restricted to Y.

One can give a more explicit description of $\mathscr{B}(m\varpi_1)$.

Corollary 8.13. *We have the following isomorphism of $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$-algebras:*

$$\mathscr{B}(m\varpi_1)\simeq\mathbb{C}[c_1,\dots,c_n][P_1,\dots,P_m]/\langle P_{i+n}-c_1P_{i+n-1}+\ldots+(-1)^nc_nP_i : i=0,\dots,m-1\rangle, \tag{8.6}$$

where $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$ is identified with $\mathbb{C}[c_1,\dots,c_n]$ (see Subsection 3.1) and P_k is the element of $\mathscr{B}(m\varpi_1)$ given by the formula (8.2).

Remark 8.10. It follows from (8.2) and the proofs of Proposition 8.8 and Theorem 8.11 the image of the k-th power sum $p_k(u_1,\dots,u_m)=u_1^k+\ldots+u_m^k$ in $\mathscr{B}(m\varpi_1)$ coincides with P_k. In particular, any P_k with $k>m$ is a polynomial expression in $P_1,\dots,P_m$ (also note that $P_0\equiv m$). Therefore, to write down the explicit relations on $c_1,\dots,c_n,P_1,\dots,P_m$ one has to express $P_{m+1},\dots,P_{n+m-1}$ in terms of $P_1,\dots,P_m$. This can be done using the identities for power sum symmetric polynomials in m variables.

Proof. Denote as before $R=\mathbb{C}[c_1,\dots,c_n]$. We have $f_i=u_i^n-c_1u_i^{n-1}+\ldots+(-1)^nc_n$ and hence

$$u_1^jf_1+\ldots+u_m^jf_m=p_{j+n}(u_1,\dots,u_m)-c_1p_{j+n-1}(u_1,\dots,u_m)+\ldots+(-1)^nc_np_j(u_1,\dots,u_m).$$

Since $R[u_1,\dots,u_m]^{\mathfrak{S}_m}$ is freely generated over R by the power sums $p_k(u_1,\dots,u_m)=u_1^k+\ldots+u_m^k$, where $k=1,\dots,m$, the statement of the corollary follows from Theorem 8.11. □

Remark 8.11. The algebra $\mathscr{B}(m\varpi_1)$ coincides with the medium algebra $\mathscr{M}(m\varpi_1)$ (see (3.6) and Corollary 8.2). Jakub Löwit informed us that one can prove the isomorphism from Theorem 8.11 using the geometric interpretation of the medium algebra $\mathscr{M}(m\varpi_1)$. Namely, one can use the fact that for any dominant integral weight λ the algebra $\mathscr{M}(\lambda)$ is isomorphic to the GL_n-equivariant cohomology of the affine Schubert variety associated to λ.

8.5. **Examples.** In this subsection we use Corollary 8.13 to compute in several examples the explicit presentations of the big algebra $\mathscr{B}(m\varpi_1)$ for $\mathfrak{gl}_n$. Then, we deduce the presentations of the corresponding big algebras for $\mathfrak{sl}_n$ and compare them with the computations of Hausel–Rychlewicz [6, Example 4.6], Rozhkovskaya [20, Proposition 4.2] and Hausel [5, Section 4.2.3].

Recall that $V(m\varpi_1)$ is weight multiplicity free as a $\mathfrak{gl}_n$-module, hence $\mathscr{B}(m\varpi_1)$ coincides with the medium algebra $\mathscr{M}(m\varpi_1)$. The latter algebra is generated over $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$ by the operators $M_{0,1}|_{V(m\varpi_1)},\dots,M_{n-1,1}|_{V(m\varpi_1)}$ (see (3.5)). We denote $M_i\coloneqq M_{i,1}|_{V(m\varpi_1)}$ and call these elements *medium operators*. Note that $M_{0,1}|_{V(m\varpi_1)}\equiv m\cdot\mathrm{Id}_{V(m\varpi_1)}$ is a scalar operator and it can be discarded from the list of generators.

In all three examples below we present $\mathscr{B}(m\varpi_1)$ using as generators over $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}=\mathbb{C}[c_1,\dots,c_n]$ either $\{P_1,\dots,P_m\}$, or $\{M_1,\dots,M_{n-1}\}$. The transition between these presentations is done by means of the following identities for symmetric polynomials (the *Girard–Waring formula* and its inverse):

$$(-1)^ke_k(w_1,\dots,w_N)=\sum_{\substack{i_1,\dots,i_k\geq 0\\ i_1+2i_2+\ldots+ki_k=k}}\frac{(-1)^{i_1+i_2+i_3+\ldots}}{i_1!\cdot\ldots\cdot i_k!}\prod_{j=1}^k\left(\frac{1}{j}p_j(w_1,\dots,w_N)\right)^{i_j}, \tag{8.7}$$

$$\frac{1}{k}p_k(w_1,\dots,w_N)=\sum_{\substack{i_1,\dots,i_N\geq 0\\ i_1+2i_2+\ldots+Ni_N=k}}\frac{(-1)^{i_2+i_4+\cdots}(i_1+\ldots+i_N-1)!}{i_1!\cdot\ldots\cdot i_N!}\prod_{j=1}^N e_j(w_1,\dots,w_N)^{i_j}, \tag{8.8}$$

where k and N are arbitrary positive integers (recall that e_k and p_k are the elementary and the power sum symmetric polynomials, repsectively). These equalities are consequences of the following power series identities:

$$1-e_1(w_1,\dots,w_N)z+e_2(w_1,\dots,w_N)z^2-\ldots+(-1)^Ne_N(w_1,\dots,w_N)=(1-w_1z)\ldots(1-w_Nz)=$$
$$=\exp\left\{\sum_{i=1}^N\log(1-w_iz)\right\}=\exp\left\{-\sum_{i=1}^N\sum_{k=1}^\infty\frac{1}{k}w_i^kz^k\right\}=\exp\left\{-\sum_{k=1}^\infty\frac{1}{k}p_k(w_1,\dots,w_N)z^k\right\}.$$

For any positive integer N and any non-negative integer k define the polynomial $\Theta_k^{(N)}(v_1,\dots,v_N)$ in N variables as the polynomial satisfying the equality

$$\Theta_k^{(N)}(p_1(w_1,\dots,w_N),p_2(w_1,\dots,w_N),\dots,p_N(w_1,\dots,w_N))=p_k(w_1,\dots,w_N). \tag{8.9}$$

The existence of such polynomials follows from formulas (8.7) and (8.8) while the uniqueness follows from the fact that the ring of symmetric polynomial in $w_1, \ldots, w_N$ is freely generated by $\{p_i(w_1, \ldots, w_N)\}_{i=1}^N$. For instance, for any $k \in \{1, \ldots, N\}$, then $\Theta_k^{(N)}(v_1, \ldots, v_N) \equiv v_k$ and for $k = 0$ we have $\Theta_0^{(N)}(v_1, \ldots, v_N) \equiv N$.

The formula (8.7) allows us to express medium operators $M_1, \ldots, M_{n-1}$ as a $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$-linear combination of elements $P_1, \ldots, P_{n-1}$.

Lemma 8.14. *The k-th medium operator M_k in $\mathscr{B}(m\varpi_1)$ can be expressed as*

$$M_k = \sum_{\substack{i_1, \ldots, i_k, i_{k+1} \geq 0 \\ i_1 + 2i_2 + \ldots + (k+1)i_{k+1} = k+1}} \frac{(-1)^{(k+1)+(i_1+i_2+\ldots+i_{k+1})}}{i_1! \cdot \ldots \cdot i_{k+1}!} \left(\sum_{j=1}^{k+1} i_j P_{j-1} \Big(\frac{1}{j}\theta_j\Big)^{i_j - 1} \cdot \prod_{\substack{1 \leq l \leq k+1 \\ l \neq j}} \Big(\frac{1}{l}\theta_l\Big)^{i_l} \right), \tag{8.10}$$

where $k = 1, \ldots, n-1$ and θ_l is the element of $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n} \simeq \mathbb{C}[t_1, \ldots, t_n]^{\mathfrak{S}_n}$ given by the formula $\theta_l = t_1^l + \ldots + t_n^l$.

Proof. The formula (8.7) implies the following identity in $\mathbb{C}[t_1, \ldots, t_n]^{\mathfrak{S}_n}$:

$$e_{k+1}(t_1, \ldots, t_n) = \sum_{\substack{i_1, \ldots, i_k, i_{k+1} \geq 0 \\ i_1 + 2i_2 + \ldots + (k+1)i_{k+1} = k+1}} \frac{(-1)^{(k+1)+(i_1+i_2+\ldots+i_{k+1})}}{i_1! \cdot \ldots \cdot i_{k+1}!} \prod_{j=1}^{k+1} \Big(\frac{1}{l}(t_1^l + \ldots + t_n^l)\Big)^{i_l}, \tag{8.11}$$

which is equivalent to the equality

$$c_{k+1} = \sum_{\substack{i_1, \ldots, i_k, i_{k+1} \geq 0 \\ i_1 + 2i_2 + \ldots + (k+1)i_{k+1} = k+1}} \frac{(-1)^{(k+1)+(i_1+i_2+\ldots+i_{k+1})}}{i_1! \cdot \ldots \cdot i_{k+1}!} \prod_{j=1}^{k+1} \Big(\frac{1}{l}\theta_l\Big)^{i_l}$$

in $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$. By Corollary 3.11 we have $M_k = M_{k,1} = F_{k,1}$. The proof of Proposition 8.6 implies that the embedding of $\mathscr{B}(m\varpi_1)$ into $\mathrm{Func}(\mathrm{wt}(m\varpi_1), S(\mathfrak{h}^*))$ identifies c_{k+1} and $M_k = \mathbf{D}(c_{k+1})$ with functions $\mu \mapsto e_{k+1}(t_1, \ldots, t_n)$ and $\mu \mapsto (\sum_{i=1}^n \mu_i \frac{\partial}{\partial t_i})(e_{k+1}(t_1, \ldots, t_n))$, respectively. Note also that P_l corresponds to the function $\mu \mapsto \sum_{i=1}^n \mu_i t_i^l$ (see (8.2) and Proposition 8.6). Applying the differential operator $\sum_{i=1}^n \mu_i \frac{\partial}{\partial t_i}$ to both sides of (8.11) gives (8.10). □

Remark 8.12. It follows from (8.10) that $\{M_1, \ldots, M_{n-1}\}$ and $\{P_1, \ldots, P_{n-1}\}$ are related to each other by an invertible triangular matrix with entries in $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$. For instance, for any n we have $M_1 = -P_1 + mc_1$ and $M_2 = P_2 - c_1 P_1 + mc_2$.

8.5.1. *A general algorithm.* Now let us describe a procedure for computing a presentation of $\mathscr{B}(m\varpi_1)$.

(1) Using relations (8.7) for $N = m$ and $k > m$, express $P_{m+1}, \ldots, P_{m+n-1}$ via $P_1, \ldots, P_m$. Plugging these expressions into (8.6) gives relations for $c_1, \ldots, c_n$, $P_1, \ldots, P_m$ in $\mathscr{B}(m\varpi_1)$.

(2) Write $P_1, \ldots, P_m$ in terms of $\theta_1, \ldots, \theta_{m+1}$ and $M_1, \ldots, M_m$ using (8.10). Then, apply (8.8) for $N = n$ and $k = 1, \ldots, m+1$ to express $\theta_1, \ldots, \theta_{m+1}$ as polynomials in $c_1, \ldots, c_n$. This allows us to rewrite $P_1, \ldots, P_m$ as polynomial expressions in $c_1, \ldots, c_n$ and $M_1, \ldots, M_m$.

(3) Substituting expressions from **(2)** into relations which were obtained in **(1)** yields relations for $c_1, \ldots, c_n$, $M_1, \ldots, M_m$ in $\mathscr{B}(m\varpi_1)$.

In order to compare these results with computations of Hausel [5, Section 4.2] and Rozhkovskaya [20, Proposition 4.2], one has to calculate the $\mathfrak{sl}_n$-version[3] of $\mathscr{B}(m\varpi_1)$. It follows from a general construction that the big algebra $\mathscr{B}_{\mathfrak{sl}_n}(m\varpi_1)$ can be obtained from its $\mathfrak{gl}_n$-version $\mathscr{B}_{\mathfrak{gl}_n}(m\varpi_1)$ simply by quotienting out the *trace*, i.e. the element $c_1 \in S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$ (see [5, Section 2.1]). However, to match the presentations one has to take into account that the Kirillov–Wei operators for $\mathfrak{sl}_n$ and $\mathfrak{gl}_n$ are different. In particular, the medium operators $M_1, \ldots, M_{n-1}$ in $\mathscr{B}_{\mathfrak{gl}_n}(m\varpi_1)$ are replaced by the following operators in $\mathscr{B}_{\mathfrak{sl}_n}(m\varpi_1) \simeq \mathscr{B}_{\mathfrak{gl}_n}(m\varpi_1)/\langle c_1 \rangle$:

$$\widetilde{M}_i = M_i - \frac{m(n-i)}{n} c_i \pmod{c_1}, \ i = 1, \ldots, n-1. \tag{8.12}$$

In the examples below we give presentations of $\mathscr{B}_{\mathfrak{sl}_2}(m\varpi_1)$ in terms of $c_2, \ldots, c_n$ and $\widetilde{M}_1, \ldots, \widetilde{M}_{n-1}$.

8.5.2. *Explicit form of relations.* We apply the algorithm above to the following three cases: $(m, n) = (m, 2)$, where m is arbitrary, $(m, n) = (2, 3)$ and $(m, n) = (3, 3)$.

Example 8.1 ($n = 2$ and m is arbitrary)**.** This example was essentially computed by Hausel and Rychlewicz, see [6, Example 4.6] and [5, Section 4.2.2]. More precisely, they computed the SL_2-equivariant cohomology of the projective space $\mathbb{P}^m$ and the latter is known to be isomorphic to the medium algebra $\mathscr{M}_{\mathfrak{sl}_2}(m\varpi_1)$.

[3] To avoid a confusion we denote from now on in this subsection the two variants of big algebras by $\mathscr{B}_{\mathfrak{sl}_n}(m\varpi_1)$ and $\mathscr{B}_{\mathfrak{gl}_n}(m\varpi_1)$, respectively.

In this case the algebra $\mathscr{B}_{\mathfrak{gl}_2}(m\varpi_1)$ is isomorphic to the quotient of the ring $\mathbb{C}[c_1, c_2][P_1, \ldots, P_m]$ by the ideal with m generators

$$P_2 - c_1P_1 + c_2P_0, \ldots, P_m - c_1P_{m-1} + c_2P_{m-2}, P_{m+1} - c_1P_m + c_2P_{m-1},$$

where we set $P_0 = m$ and $P_{m+1} = \Theta_m^{(m+1)}(P_1, \ldots, P_m)$. Since $\mathscr{B}_{\mathfrak{sl}_2}(m\varpi_1) \simeq \mathscr{B}_{\mathfrak{gl}_2}(m\varpi_1)/\langle c_1\rangle$, we obtain

$$\mathscr{M}_{\mathfrak{sl}_2}(m\varpi_1) \simeq \mathbb{C}[c_2][P_1, \ldots, P_m]/\langle P_2 + mc_2, P_3 + c_2P_1 \ldots, P_m + c_2P_{m-2}, \Theta_{m+1}^{(m)}(P_1, \ldots, P_m) + c_2P_{m-1}\rangle.$$

It follows from the proof of Theorem 8.11 and Corollary 8.13 that the equality $P_{l+2} + c_2P_l = 0$ holds in the ring $\mathscr{M}_{\mathfrak{sl}_2}(m\varpi_1)$ for all l. Hence, $P_{2l} = (-1)^l mc_2^l$ and $P_{2l+1} = (-1)^l c_2^l P_1$, or equivalently,

$$P_k = \Theta_k^{(m)}(P_1, -c_2, -c_2P_1, c_2^2, \ldots) = \begin{cases} (-1)^l mc_2^l, & k = 2l, \text{ where } l \in \mathbb{Z}_{\geq 0}, \\ (-1)^l c_2^l P_1, & k = 2l+1, \text{ where } l \in \mathbb{Z}_{\geq 0}. \end{cases}$$

In particular, we can eliminate $P_2, \ldots, P_m$ from the list of generators for the $\mathbb{C}[c_2]$-algebra $\mathscr{M}_{\mathfrak{sl}_2}(m\varpi_1)$. We obtain that the medium algebra $\mathscr{M}_{\mathfrak{sl}_2}(m\varpi_1)$ is actually a quotient of $\mathbb{C}[c_2][P_1]$:

$$\mathscr{B}_{\mathfrak{sl}_2}(m\varpi_1) \simeq \begin{cases} \mathbb{C}[c_2][P_1] \Big/ \Big\langle \Theta_{m+1}^{(m)}(P_1, -c_2, -c_2P_1, c_2^2, \ldots) - (-1)^{m/2} c_2^{m/2} P_1 \Big\rangle, & \text{if } m \text{ is even}, \\ \mathbb{C}[c_2][P_1] \Big/ \Big\langle \Theta_{m+1}^{(m)}(P_1, -c_2, -c_2P_1, c_2^2, \ldots) - (-1)^{(m+1)/2} mc_2^{(m+1)/2} \Big\rangle, & \text{if } m \text{ is odd}. \end{cases} \tag{8.13}$$

Alternatively, consider the sequence $\{\psi_n(a, b)\}_{n=1}^\infty$ of polynomials in two variables defined by the recurrence relation $\psi_{n+2}(a, b) = -a\psi_n(a, b)$ with initial conditions $\psi_0(a, b) = m$ and $\psi_1(a, b) = b$. Then, $P_l = \psi_l(c_2, P_1)$ for all l and therefore,

$$\mathscr{B}_{\mathfrak{sl}_2}(m\varpi_1) \simeq \mathbb{C}[c_2, P_1] \Big/ \Big\langle \Theta_{m+1}^{(m)}(\psi_1(c_2, P_1), \ldots, \psi_m(c_2, P_1)) - \psi_{m+1}(c_2, P_1) \Big\rangle.$$

The recurrence relation imply the following explicit formula for $\psi_k(a, b)$:

$$\psi_k(a, b) = \left(\frac{m}{2} + \frac{b}{2i\sqrt{a}}\right) \left(i\sqrt{a}\right)^k + \left(\frac{m}{2} - \frac{b}{2i\sqrt{a}}\right) \left(-i\sqrt{a}\right)^k. \tag{8.14}$$

Now let us calculate explicitly the relation for P_1 and c_2 in $\mathscr{B}_{\mathfrak{sl}_2}(m\varpi_1)$. Consider the power series

$$\exp\left\{-p_1z - \frac{1}{2}p_2z^2 - \frac{1}{3}p_3z^3 - \ldots\right\} = \exp\left\{-\sum_{k=1}^\infty \frac{p_k}{k} z^k\right\}, \tag{8.15}$$

where $p_k = p_k(u_1, \ldots, u_m)$ is the k-th power sum in variables $u_1, \ldots, u_m$. Observe that the z^{m+1} term of this power series equals

$$\sum_{\substack{i_1, \ldots, i_{m+1} \geq 0 \\ i_1 + 2i_2 + \ldots + (m+1)i_{m+1} = m+1}} \frac{(-1)^{i_1 + i_2 + \ldots + i_{m+1}}}{i_1! \cdot \ldots \cdot i_{m+1}!} \prod_{j=1}^k \left(\frac{1}{j}p_j\right)^{i_j} = \Theta_{m+1}^{(m)}(p_1, \ldots, p_m) - p_{m+1},$$

where we used (8.7) for $N = m$ and $k = m+1$. This identity together with (8.13) implies that the relation for P_1 and c_2 in $\mathscr{B}_{\mathfrak{sl}_2}(m\varpi_1)$ is equal to the z^{m+1} term of the power series (8.15) after the substitution $(p_1, p_2, \ldots, p_m, \ldots) = (\psi_1(c_2, P_1), \psi_2(c_2, P_1), \ldots, \psi_m(c_2, P_1), \ldots)$. To compute the resulting power series, note that by (8.14) we have

$$\psi_k(c_2, P_1) = \alpha\left(i\sqrt{c_2}\right)^k + \beta\left(-i\sqrt{c_2}\right)^k, \text{ where } \alpha = \frac{m}{2} + \frac{P_1}{2i\sqrt{c_2}}, \beta = \frac{m}{2} - \frac{P_1}{2i\sqrt{c_2}}.$$

Then, the substitution gives

$$\exp\left\{-\sum_{k=1}^\infty \frac{p_k}{k} z^k\right\}\Bigg|_{p_j = \psi_j(c_2, P_1)} = \exp\left\{-\sum_{k=1}^\infty \frac{\alpha(i\sqrt{c_2})^k + \beta(-i\sqrt{c_2})^k}{k} z^k\right\}.$$

The discussion above implies that the remaining relation between P_1 and c_2 is given by the z^{m+1} term of the power series above. A direct calculation shows that

$$\exp\left\{-\sum_{k=1}^\infty \frac{\alpha(i\sqrt{c_2})^k + \beta(-i\sqrt{c_2})^k}{k} z^k\right\} = (1 - i\sqrt{c_2}z)^\alpha (1 + i\sqrt{c_2}z)^\beta =$$

$$= \sum_{k,l \geq 0} (-1)^k \binom{\alpha}{k}\binom{\beta}{l} (i\sqrt{c_2})^{k+l} z^{k+l} = \sum_{k,l \geq 0} (-1)^{k+l} \binom{\alpha}{k}\binom{-\beta + l - 1}{l} (i\sqrt{c_2})^{k+l} z^{k+l} =$$

$$= \sum_{k,l \geq 0} \frac{(-1)^{k+l} z^{k+l}}{2^{k+l} \cdot k! \cdot l!} \prod_{j=0}^{k-1} (P_1 + (m - 2j)i\sqrt{c_2}) \prod_{j=0}^{l-1} (P_1 - (m - 2j)i\sqrt{c_2}).$$

The z^{m+1} term of this expression equals

$$\frac{(-1)^{m+1}}{2^{m+1}}\sum_{k+l=m+1}\frac{1}{k!\cdot l!}\prod_{j=0}^{k-1}(P_1+(m-2j)i\sqrt{c_2})\prod_{j=0}^{l-1}(P_1-(m-2j)i\sqrt{c_2}).$$

However, for $k+l=m+1$ we have

$$\prod_{j=0}^{k-1}(P_1+(m-2j)i\sqrt{c_2})\prod_{j=0}^{l-1}(P_1-(m-2j)i\sqrt{c_2})=\prod_{j=0}^{m}(P_1+(m-2j)i\sqrt{c_2}),$$

so the sum above equals

$$\frac{(-1)^{m+1}}{2^{m+1}}\left(\sum_{k+l=m+1}\frac{1}{k!\cdot l!}\right)\prod_{j=0}^{l-1}(P_1-(m-2j)i\sqrt{c_2})=\frac{(-1)^{m+1}}{(m+1)!}\prod_{j=0}^{m}(P_1+(m-2j)i\sqrt{c_2}).$$

Therefore, the relation between P_1 and c_2 in $\mathscr{B}_{\mathfrak{sl}_2}(m\varpi_1)$ is just $\prod_{j=0}^{m}(P_1+(m-2j)i\sqrt{c_2})=0$. Finally, recall that $\widetilde{M}_1=M_1-\frac{m}{2}c_1=\frac{m}{2}c_1-P_1=-P_1 \pmod{c_1}$ in $\mathscr{B}_{\mathfrak{sl}_2}(m\varpi_1)$.

Our computations can be summarized in the following proposition.

Proposition 8.15. *We have the following presentations of the big algebra $\mathscr{B}_{\mathfrak{sl}_2}(m\varpi_1)$:*

$$\mathscr{B}_{\mathfrak{sl}_2}(m\varpi_1)\simeq\mathbb{C}[t_1,t_2]^{\mathfrak{S}_2}[u_1,\dots,u_m]^{\mathfrak{S}_m}\Big/\Big\langle t_1+t_2,\{u_1^r(u_1-t_1)(u_1-t_2)+\dots+u_m^r(u_m-t_1)(u_m-t_2)\}_{r=0}^{m-1}\Big\rangle,$$

$$\mathscr{B}_{\mathfrak{sl}_2}(m\varpi_1)\simeq\begin{cases}\mathbb{C}[c_2][\widetilde{M}_1]/\langle(\widetilde{M}_1^2+m^2c_2)(\widetilde{M}_1^2+(m-2)^2c_2)\dots(\widetilde{M}_1^2+4c_2)\widetilde{M}_1\rangle, & \textit{if } m \textit{ is even},\\ \mathbb{C}[c_2][\widetilde{M}_1]/\langle(\widetilde{M}_1^2+m^2c_2)(\widetilde{M}_1^2+(m-2)^2c_2)\dots(\widetilde{M}_1^2+c_2)\rangle, & \textit{if } m \textit{ is odd}.\end{cases}$$

These two presentations are identified via the following formulas:

$$\widetilde{M}_1=-(u_1+\dots+u_m),\ c_2=t_1t_2.$$

A similar calculation gives an analogous presentation of $\mathscr{B}_{\mathfrak{gl}_2}(m\varpi_1)$.

Proposition 8.16. *We have the following presentations of the big algebra $\mathscr{B}_{\mathfrak{gl}_2}(m\varpi_1)$:*

$$\mathscr{B}_{\mathfrak{gl}_2}(m\varpi_1)\simeq\mathbb{C}[t_1,t_2]^{\mathfrak{S}_2}[u_1,\dots,u_m]^{\mathfrak{S}_m}\Big/\Big\langle\{u_1^r(u_1-t_1)(u_1-t_2)+\dots+u_m^r(u_m-t_1)(u_m-t_2)\}_{r=0}^{m-1}\Big\rangle,$$

$$\mathscr{B}_{\mathfrak{sl}_2}(m\varpi_1)\simeq\begin{cases}\mathbb{C}[c_1,c_2][M_1]\Big/\Big\langle(M_1-\frac{m}{2}c_1)\prod_{j=0}^{m/2}(M_1^2-mc_1M_1+j(m-j)c_1^2+(m-2j)^2c_2)\Big\rangle, & \textit{if } m \textit{ is even},\\ \mathbb{C}[c_1,c_2][M_1]\Big/\Big\langle\prod_{j=0}^{(m-1)/2}(M_1^2-mc_1M_1+j(m-j)c_1^2+(m-2j)^2c_2)\Big\rangle, & \textit{if } m \textit{ is odd}.\end{cases}$$

These two presentations are identified via the following formulas:

$$M_1=\frac{m}{2}(t_1+t_2)-(u_1+\dots+u_m),\ c_1=t_1+t_2,\ c_2=t_1t_2.$$

Remark 8.13. Note that here $M_1=mc_1-P_1$. The relation between c_1, c_2 and P_1 in $\mathscr{B}_{\mathfrak{gl}_2}(m\varpi_1)$ can be also derived via the description of the medium algebra in terms of configurations of points on $\mathbb{A}^1$ (see Remark 8.9). Namely, the set-theoretic inclusion $\{u_1,\dots,u_m\}\subseteq\{t_1,t_2\}$ is equivalent to requiring that $u_j=\frac{1}{2}c_1\pm\frac{1}{2}\sqrt{c_1^2-4c_2}$ for each j (recall that $c_1=t_1+t_2$ and $c_2=t_1t_2$). Hence, the possible values of the sum $P_1=u_1+\dots+u_m$ are $\frac{m}{2}c_1+\frac{m}{2}\sqrt{c_1^2-4c_2}$, $\frac{m}{2}c_1+\frac{m-2}{2}\sqrt{c_1^2-4c_2},\dots,\frac{m}{2}c_1-\frac{m}{2}\sqrt{c_1^2-4c_2}$. One can check that this is confirmed by the relation which c_1, c_2 and P_1 satisfy in $\mathscr{B}_{\mathfrak{gl}_2}(m\varpi_1)$.

Example 8.2 ($n=3$ and $m=2$)**.** This example was computed[4] by Rozhkovskaya, see [20, Proposition 4.2]. Corollary 8.13 gives the following presentation of the $\mathscr{B}_{\mathfrak{gl}_3}(2\varpi_1)$:

$$\mathscr{B}_{\mathfrak{gl}_3}(2\varpi_1)\simeq\mathbb{C}[c_1,c_2,c_3][P_1,P_2]\Big/\left\langle\begin{matrix}P_3-c_1P_2+c_2P_1-c_3P_0,\\ P_4-c_1P_3+c_2P_2-c_3P_1\end{matrix}\right\rangle,$$

where we substitute $P_0=2$, $P_3=-\frac{1}{2}P_1^3+\frac{3}{2}P_1P_2$ and $P_4=-\frac{1}{2}P_1^4+P_1^2P_2$. This yields the following presentation:

$$\mathscr{B}_{\mathfrak{gl}_3}(2\varpi_1)\simeq\mathbb{C}[c_1,c_2,c_3][P_1,P_2]\Big/\left\langle\begin{matrix}P_1^3-3P_1P_2-2c_2P_1+2c_1P_2+4c_3,\\ P_1^4-c_1P_1^3-2P_1^2P_2+3c_1P_1P_2+2c_3P_1-P_2^2-2c_2P_2\end{matrix}\right\rangle. \tag{8.16}$$

[4]Strictly speaking, she computed a presentation of the *Kirillov algebra* $\mathscr{C}_{\mathfrak{sl}_3}(2\varpi_1)$, but in this case the Kirillov, medium and big algebras all coincide (cf. Proposition 8.4).

Using the fact that $M_1 = 2c_1 - P_1$ and $M_2 = P_2 - c_1P_1 + 2c_2$, one can obtain a presentation of $\mathscr{B}_{\mathfrak{gl}_2}(m\varpi_1)$ in terms of the medium operators:

(8.17)
$$\mathscr{B}_{\mathfrak{gl}_3}(2\varpi_1) \simeq \mathbb{C}[c_1,c_2,c_3][M_1,M_2]\Big/\left\langle \begin{matrix} M_1^3 - 3c_1M_1^2 - 3M_1M_2 + 2c_1^2M_1 + 4c_2M_1 + 4c_1M_2 - 4c_1c_2 - 4c_3, \\ M_1^4 - 5c_1M_1^3 - 2M_1^2M_2 + 8c_1^2M_1^2 + 4c_2M_1^2 + 7c_1M_1M_2 - 4c_1^3M_1 - \\ -12c_1c_2M_1 - 2c_3M_1 - M_2^2 - 6c_1^2M_2 + 2c_2M_2 + 8c_1^2c_2 + 4c_1c_3 \end{matrix}\right\rangle \simeq$$
$$\simeq \mathbb{C}[c_1,c_2,c_3][M_1,M_2]\Big/\left\langle \begin{matrix} M_1^3 - 3c_1M_1^2 - 3M_1M_2 + 2c_1^2M_1 + 4c_2M_1 + 4c_1M_2 - 4c_1c_2 - 4c_3, \\ M_1^2M_2 - 3c_1M_1M_2 + 2c_3M_1 - M_2^2 + 2c_1^2M_2 + 2c_2M_2 - 4c_1c_3 \end{matrix}\right\rangle.$$

In order to get a presentation of the $\mathfrak{sl}_2$-version, we use
$$\begin{cases} \widetilde{M}_1 \equiv M_1 - \frac{4}{3}c_1 \equiv -P_1 + \frac{2}{3}c_1 \equiv -P_1 \pmod{c_1}, \\ \widetilde{M}_2 \equiv M_2 - \frac{2}{3}c_2 \equiv P_2 - c_1P_1 + \frac{4}{3}c_2 \equiv P_2 + \frac{4}{3}c_2 \pmod{c_1}. \end{cases}$$

In particular, $P_1 \equiv -\widetilde{M}_1 \pmod{c_1}$ and $P_2 \equiv \widetilde{M}_2 - \frac{4}{3}c_2 \pmod{c_1}$. Plugging this into the presentation of $\mathscr{B}_{\mathfrak{gl}_3}(2\varpi_1)$ above and using the fact that $\mathscr{B}_{\mathfrak{sl}_3}(2\varpi_1) \simeq \mathscr{B}_{\mathfrak{sl}_3}(2\varpi_1)/\langle c_1\rangle$, we obtain

(8.18)
$$\mathscr{B}_{\mathfrak{sl}_3}(2\varpi_1) \simeq \mathbb{C}[c_2,c_3][\widetilde{M}_1,\widetilde{M}_2]\Big/\left\langle \begin{matrix} \widetilde{M}_1^3 - 3\widetilde{M}_1\widetilde{M}_2 + 2c_2\widetilde{M}_1 - 4c_3, \\ 9\widetilde{M}_1^4 - 18\widetilde{M}_1^2\widetilde{M}_2 + 24c_2\widetilde{M}_1^2 - 18c_3\widetilde{M}_1 - 9\widetilde{M}_2^2 + 6c_2\widetilde{M}_2 + 8c_2^2 \end{matrix}\right\rangle \simeq$$
$$\simeq \mathbb{C}[c_2,c_3][\widetilde{M}_1,\widetilde{M}_2]\Big/\left\langle \begin{matrix} \widetilde{M}_1^3 - 3\widetilde{M}_1\widetilde{M}_2 + 2c_2\widetilde{M}_1 - 4c_3, \\ 9\widetilde{M}_1^2\widetilde{M}_2 + 6c_2\widetilde{M}_1^2 + 18c_3\widetilde{M}_1 - 9\widetilde{M}_2^2 + 6c_2\widetilde{M}_2 + 8c_2^2 \end{matrix}\right\rangle.$$

To obtain Rozhkovskaya's presentation, we introduce the following elements:
$$M = -\widetilde{M}_1, \quad N = -\frac{1}{12}\widetilde{M}_1^2 + \frac{1}{4}\widetilde{M}_2 - \frac{1}{6}c_2, \quad C_2 = -\frac{1}{3}c_2, \quad C_3 = \frac{1}{3}c_3,$$
which is equivalent to
$$\widetilde{M}_1 = -M, \quad \widetilde{M}_2 = \frac{1}{3}M^2 + 4N - 2C_2, \quad c_2 = -3C_2, \quad c_3 = 3C_3.$$

Substituting this into (8.18) gives the following presentation:
$$\mathscr{B}_{\mathfrak{sl}_3}(2\varpi_1) \simeq \mathbb{C}[C_2,C_3][M,N]\Big/\left\langle \begin{matrix} MN - C_3, \\ M^4 + 6M^2N - 15C_2M^2 - 27C_3M - 72N^2 + 36C_2N + 36C_2^2 \end{matrix}\right\rangle \simeq$$
$$\simeq \mathbb{C}[C_2,C_3][M,N]\Big/\left\langle \begin{matrix} MN - C_3, \\ M^4 - 15C_2M^2 - 21C_3M - 72N^2 + 36C_2N + 36C_2^2 \end{matrix}\right\rangle.$$

Then, we obtain the generators and relations given in [20, Proposition 4.2].

Example 8.3 ($n = 3$ and $m = 3$)**.** This example was computed by Hausel, see [5, Section 4.2.3]. In this case Corollary 8.13 gives the following of $\mathscr{B}_{\mathfrak{gl}_3}(3\varpi_1)$:

(8.19)
$$\mathscr{B}_{\mathfrak{gl}_3}(3\varpi_1) \simeq \mathbb{C}[c_1,c_2,c_3][P_1,P_2,P_3]\Big/\left\langle \begin{matrix} P_3 - c_1P_2 + c_2P_1 - c_3P_0, \\ P_4 - c_1P_3 + c_2P_2 - c_3P_1, \\ P_5 - c_1P_4 + c_2P_3 - c_3P_2 \end{matrix}\right\rangle,$$

where we substitute $P_0 = 3$, $P_4 = \frac{1}{6}P_1^4 - P_1^2P_2 + \frac{1}{2}P_2^2 + \frac{4}{3}P_1P_3$ and $P_5 = \frac{1}{6}P_1^5 - \frac{5}{6}P_1^3P_2 + \frac{5}{6}P_1^2P_3 + \frac{5}{6}P_2P_3$. We obtain the following presentation:
$$\mathscr{B}_{\mathfrak{gl}_3}(3\varpi_1) \simeq \mathbb{C}[c_1,c_2,c_3][P_1,P_2,P_3]\Bigg/\left\langle \begin{matrix} c_2P_1 - c_1P_2 + P_3 - 3c_3, \\ P_1^4 - 6P_1^2P_2 + 8P_1P_3 - 6c_3P_1 + 3P_2^2 + 6c_2P_2 - 6c_1P_3, \\ P_1^5 - c_1P_1^4 - 5P_1^3P_2 + 6c_1P_1^2P_2 + 5P_1^2P_3 - \\ -8c_1P_1P_3 - 3c_1P_2^2 + 5P_2P_3 - 6c_3P_2 + 6c_2P_3 \end{matrix}\right\rangle \simeq$$
$$\simeq \mathbb{C}[c_1,c_2,c_3][P_1,P_2]\Bigg/\left\langle \begin{matrix} P_1^4 - 6P_1^2P_2 - 8c_2P_1^2 + 8c_1P_1P_2 + 6c_1c_2P_1 + 18c_3P_1 + 3P_2^2 - 6c_1^2P_2 + \\ +6c_2P_2 - 18c_1c_3, \\ P_1^5 - c_1P_1^4 - 5P_1^3P_2 - 5c_2P_1^3 + 11c_1P_1^2P_2 + 8c_1c_2P_1^2 + 15c_3P_1^2 - 8c_1^2P_1P_2 + \\ +2c_1P_2^2 - 5c_2P_1P_2 - 6c_2^2P_1 - 24c_1c_3P_1 + 6c_1c_2P_2 + 9c_3P_2 + 18c_2c_3 \end{matrix}\right\rangle.$$

As $M_1 = 3c_1 - P_1$ and $M_2 = P_2 - c_1p_1 + 3c_2$, we have
$$\mathscr{B}_{\mathfrak{gl}_3}(3\varpi_1) \simeq \mathbb{C}[c_1,c_2,c_3][M_1,M_2]\Bigg/\left\langle \begin{matrix} M_1^4 - 6c_1M_1^3 - 6M_1^2M_2 + 11c_1^2M_1^2 + 10c_2M_1^2 + 22c_1M_1M_2 - 6c_1^3M_1 - 30c_1c_2M_1 - \\ -18c_3M_1 + 3M_2^2 - 18c_1^2M_2 - 12c_2M_2 + 18c_1^2c_2 + 36c_1c_3 + 9c_2^2, \\ M_1^5 - 9c_1M_1^4 - 5M_1^3M_2 + 29c_1^2M_1^3 + 10c_2M_1^3 + 34c_1M_1^2M_2 - 39c_1^3M_1^2 - 60c_1c_2M_1^2 - \\ -15c_3M_1^2 - 73c_1^2M_1M_2 - 5c_2M_1M_2 + 18c_1^4M_1 + 108c_1^2c_2M_1 + 75c_1c_3M_1 + 9c_2^2M_1 - \\ -2c_1M_2^2 + 48c_1^3M_2 + 21c_1c_2M_2 - 9c_3M_2 - 54c_1^3c_2 - 90c_1^2c_3 - 27c_1c_2^2 + 9c_2c_3 \end{matrix}\right\rangle.$$

The medium operators for the $\mathscr{B}_{\mathfrak{sl}_3}(3\varpi_1)$ are given by the following formulas:

$$\begin{cases} \widetilde{M}_1 \equiv M_1 - 2c_1 \equiv -P_1 + c_1 = -P_1 \pmod{c_1}, \\ \widetilde{M}_2 \equiv M_2 - c_2 \equiv P_2 - c_1 P_1 + 2c_2 \equiv P_2 + 2c_2 \pmod{c_1}. \end{cases}$$

Hence, $P_1 = -\widetilde{M}_1 \pmod{c_1}$ and $P_2 = \widetilde{M}_2 - 2c_2 \pmod{c_1}$. Plugging this into the presentation above and using the isomorphism $\mathscr{B}_{\mathfrak{sl}_3}(3\varpi_1) \simeq \mathscr{B}_{\mathfrak{gl}_3}(3\varpi_1)/\langle c_1 \rangle$, we get

$$\mathscr{B}_{\mathfrak{sl}_3}(3\varpi_1) = \mathbb{C}[c_2, c_3][\widetilde{M}_1, \widetilde{M}_2] \Big/ \left\langle \begin{matrix} \widetilde{M}_1^4 - 6\widetilde{M}_1^2\widetilde{M}_2 + 4c_2\widetilde{M}_1^2 - 18c_3\widetilde{M}_1 + 3\widetilde{M}_2^2 - 6c_2\widetilde{M}_2, \\ \widetilde{M}_1^5 - 5\widetilde{M}_1^3\widetilde{M}_2 + 5c_2\widetilde{M}_1^3 - 15c_3\widetilde{M}_1^2 - 5c_2\widetilde{M}_1\widetilde{M}_2 + 4c_2^2\widetilde{M}_1 - 9c_3\widetilde{M}_2 \end{matrix} \right\rangle.$$

Subtracting from the second relation the first one multiplied by $\widetilde{M}_1$, we obtain

$$\mathscr{B}_{\mathfrak{sl}_3}(3\varpi_1) = \mathbb{C}[c_2, c_3][\widetilde{M}_1, \widetilde{M}_2] \Big/ \left\langle \begin{matrix} \widetilde{M}_1^4 - 6\widetilde{M}_1^2\widetilde{M}_2 + 4c_2\widetilde{M}_1^2 - 18c_3\widetilde{M}_1 + 3\widetilde{M}_2^2 - 6c_2\widetilde{M}_2, \\ \widetilde{M}_1^3\widetilde{M}_2 + c_2\widetilde{M}_1^3 + 3c_3\widetilde{M}_1^2 - 3\widetilde{M}_1\widetilde{M}_2^2 + c_2\widetilde{M}_1\widetilde{M}_2 + 4c_2^2\widetilde{M}_1 - 9c_3\widetilde{M}_2 \end{matrix} \right\rangle, \tag{8.20}$$

which coincides with the presentation given in [5, (4.6)].

8.6. **Formula for the Kirillov–Wei operator.** The aim of this subsection is to derive a more explicit formula for the Kirillov–Wei operator on the Kirillov algebra of $V(m\varpi_1) \simeq S^m(\mathbb{C}^n)$.

Lemma 8.17. *For any $A \in S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$ and any element B of the Kirillov algebra $\mathscr{C}(V)$ we have the identity*

$$\mathbf{D}(A \cdot B) = \mathbf{D}(A) \cdot B + A \cdot \mathbf{D}(B).$$

Proof. This is a direct consequence of the definition of $\mathbf{D}$ and the usual Leibniz rule, see (3.3). ☐

Remark 8.14. Note however that $\mathbf{D}$ does **not** satisfy the Leibniz rule for arbitrary $A, B \in \mathscr{C}(V)$. It is important here that A is a scalar matrix.

We use the map $\mathfrak{i}_m$ in order to obtain a different interpretation for the Kirillov–Wei operator on[5] $\mathscr{B}(m\varpi_1)$. Namely, define the operator $\widehat{\mathbf{D}}\colon \mathfrak{i}_m(\mathscr{B}(m\varpi_1)) \to \mathfrak{i}_m(\mathscr{B}(m\varpi_1))$ so that the diagram

$$\begin{tikzcd} \mathscr{B}(m\varpi_1) \arrow[r, "\mathbf{D}"] \arrow[d, "\mathfrak{i}_m"'] & \mathscr{B}(m\varpi_1) \arrow[d, "\mathfrak{i}_m"] \\ \mathfrak{i}_m(\mathscr{B}(m\varpi_1)) \arrow[r, "\widehat{\mathbf{D}}"] & \mathfrak{i}_m(\mathscr{B}(m\varpi_1)) \end{tikzcd}$$

commutes. It turns out that $\widehat{\mathbf{D}}$ can be described using some constructions from symmetric function theory.

Proposition 8.18. *The Kirillov–Wei operator $\mathbf{D}$ on $\mathscr{B}(m\varpi_1)$ induces on $\mathfrak{i}_m(\mathscr{B}(m\varpi_1)) \subset \mathrm{Func}(\mathrm{wt}(m\varpi_1), S(\mathfrak{h}^*))$ the operator $\widehat{\mathbf{D}}$ which acts as follows: for any symmetric functions f and g it maps the function $\mu \mapsto f[t_1 + \ldots + t_n]g[\mu_1 t_1 + \ldots + \mu_n t_n]$ to the function*

$$\begin{aligned} \mu \mapsto & \left(\mu_1 \frac{\partial}{\partial t_1} + \ldots + \mu_n \frac{\partial}{\partial t_n}\right) \big(f[t_1 + \ldots + t_n]g[\mu_1 t_1 + \ldots + \mu_n t_n]\big) + \\ & + f[t_1 + \ldots + t_n] \sum_{\substack{1 \le i,j \le n \\ i \ne j}} \frac{\mu_j(\mu_i + 1)}{t_i - t_j} \big(g[\mu_1 t_1 + \ldots + \mu_n t_n + (t_i - t_j)] - g[\mu_1 t_1 + \ldots \mu_n t_n]\big). \end{aligned} \tag{8.21}$$

Here, $F[-]$ for a symmetric function F stands for the plethystic substitution (see Remark 8.3 and [14, Ch. I.8]*).*

Remark 8.15. It is suggestive to regard the "shifts" $t_i - t_j$ as corresponding to the elements of the A_{n-1} root system.

Proof. Throughout the proof we perform computations in the big algebra $\mathscr{B}(\mathcal{P}(n,1))$. To obtain formulas for $\mathscr{B}(m\varpi_1)$ one simply restricts all operators to the subspace $V(m\varpi_1) \simeq \mathbb{C}[x_1, \ldots, x_n]_m \subset \mathcal{P}(n,1)$.

Since the operators $\mathbf{D}$ and $\widehat{\mathbf{D}}$ are additive, it suffices to verify the statement from the proposition only for homogeneous f and g. Applying Lemma 8.17 we can reduce[6] this to the two cases: $f \equiv 1$ and $g \equiv 1$.

Case 1. Assume that $g \equiv 1$. It is well known that the ring $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$ is a free polynomial ring in the elements $\theta_1, \ldots, \theta_n \in S(\mathfrak{gl}_n^*)$, where $\theta_k(Y) = \mathrm{tr}(Y^k)$, $Y \in \mathfrak{gl}_n$. Now let us show that $\mathbf{D}(\theta_\alpha) = \alpha P_{\alpha-1}$ for all $\alpha = 1, \ldots, n$.

[5]Recall that $\mathscr{B}(m\varpi_1) = \mathscr{C}(m\varpi_1)$ and hence, $\mathbf{D}$ indeed acts on $\mathscr{B}(m\varpi_1)$.

[6]Note that if we denote the right-hand side of (8.21) by $D_\mu(f,g)$, then $D_\mu(f,g)$ satisfies an analogue of Lemma 8.17. Namely, $D_\mu(f,g) = f[t_1 + \ldots + t_n]D_\mu(1,g) + D_\mu(f,1)g[\mu_1 t_1 + \ldots + \mu_n t_n]$.

Indeed, we have

$$\mathbf{D}(\theta_\alpha)(Y) = \sum_{i,j=1}^{n} \frac{\partial(\mathrm{tr}(Y^\alpha))}{\partial y_{ji}} \cdot L(E_{ij}) = \sum_{i,j=1}^{n}\sum_{l=1}^{n}\sum_{\beta=1}^{\alpha}[Y^{\beta-1}]_{lj}[Y^{\alpha-\beta}]_{il} \cdot L(E_{ij}) =$$
$$= \sum_{i,j=1}^{n}\sum_{\beta=1}^{\alpha}[Y^{\beta-1}\cdot Y^{\alpha-\beta}]_{ij} \cdot L(E_{ij}) = \alpha \sum_{i,j=1}^{n}[Y^{\alpha-1}]\cdot L(E_{ij}) = \alpha L(Y^{\alpha-1}) = \alpha P_{\alpha-1}(Y).$$

Now note that for $Y = \mathrm{diag}(t_1,\dots,t_n) \in \mathfrak{h}$ we have $\theta_\alpha(Y) = t_1^\alpha + \ldots + t_n^\alpha$. Therefore, $\widehat{\mathbf{D}}$ maps a constant function $\mu \mapsto \sum_{i=1}^n t_i^\alpha$ to the function $\mu \mapsto \sum_{i=1}^n \alpha\mu_i t_i^{\alpha-1} = (\sum_i \mu_i \frac{\partial}{\partial t_i})(\sum_{i=1}^n t_i^\alpha)$. This together with Lemma 8.17 implies that for any $f \in \mathbb{C}[t_1,\dots,t_n]^{\mathfrak{S}_n}$ the operator $\widehat{\mathbf{D}}$ maps any constant function $\mu \mapsto f$ to $\mu \mapsto (\sum_i \mu_i \frac{\partial}{\partial t_i})(f)$.

Case 2. Assume that $f \equiv 1$. Since $\widehat{\mathbf{D}}$ is $\mathbb{C}$-linear we may assume without loss of generality that $g = \prod_{k=1}^l p_{\alpha_k}$ for some positive integers $\alpha_1,\dots,\alpha_l$, where p_α is the α-th power sum. In other words, we consider the function

$$\mu \mapsto \prod_{k=1}^{l}\left(\sum_{i=1}^{n}\mu_i t_i^{\alpha_k}\right) = \prod_{k=1}^{l} p_{\alpha_k}[\mu_1 t_1 + \ldots + \mu_n t_n].$$

Note that this element of $\mathfrak{i}_m(\mathscr{B}(m\varpi_1))$ corresponds to the element $P_{\alpha_1}P_{\alpha_2}\dots P_{\alpha_l}$ in $\mathscr{B}(m\varpi_1)$. Then, we can compute the action of the Kirillov–Wei operator on this element:

$$\mathbf{D}(P_{\alpha_1}\dots P_{\alpha_l})(Y) = \sum_{i_1,j_1,\dots,i_l,j_l=1}^{n}\sum_{i,j=1}^{n}\frac{\partial}{\partial y_{ji}}\big([Y^{\alpha_1}]_{i_1j_1}\dots[Y^{\alpha_l}]_{i_lj_l}\big)\cdot L(E_{i_1j_1})\dots L(E_{i_lj_l})L(E_{ij}) =$$
$$= \sum_{i_1,j_1,\dots,i_l,j_l=1}^{n}\sum_{i,j=1}^{n}\left(\sum_{k=1}^{l}\sum_{\alpha=0}^{\alpha_k-1}[Y^{\alpha_1}]_{i_1j_1}\dots[Y^{\alpha}]_{i_kj}[Y^{\alpha_k-\alpha-1}]_{ij_k}\dots[Y^{\alpha_l}]_{i_lj_l}\right)\cdot L(E_{i_1j_1})\dots L(E_{i_lj_l})L(E_{ij}) =$$
$$= \sum_{k=1}^{l} L(Y^{\alpha_1})\dots L(Y^{\alpha_{k-1}})\sum_{i_k,j_k,\dots,i_l,j_l=1}^{n}\sum_{i,j=1}^{n}\left(\sum_{\alpha=0}^{\alpha_k-1}[Y^\alpha]_{i_kj}[Y^{\alpha_k-\alpha-1}]_{ij_k}\right)[Y^{\alpha_{k+1}}]_{i_{k+1}j_{k+1}}\dots[Y^{\alpha_l}]_{i_lj_l}\times$$
$$\times L(E_{i_kj_k})L(E_{i_{k+1}j_{k+1}})\dots L(E_{i_lj_l})L(E_{ij}).$$

For $Y = \mathrm{diag}(t_1,\dots,t_n) \in \mathfrak{h}$ this simplifies as follows:

$$\mathbf{D}(P_{\alpha_1}\dots P_{\alpha_l})(Y) =$$
$$= \sum_{k=1}^{l} P_{\alpha_1}(Y)\dots P_{\alpha_{k-1}}(Y)\sum_{i_{k+1},\dots,i_l=1}^{n}\sum_{i,j=1}^{n} h_{\alpha_k-1}(t_i,t_j)t_{i_{k+1}}^{\alpha_{k+1}}\dots t_{i_l}^{\alpha_l}\cdot x_j\partial_i x_{i_{k+1}}\partial_{i_{k+1}}\dots x_{i_l}\partial_{i_l}x_i\partial_j.$$

where h_α denotes the α-th *complete homogeneous symmetric polynomial*. This formula can be simplified if we restrict the operator on the right to $V(m\varpi_1) \simeq \mathbb{C}[x_1,\dots,x_n]_m$. Note that the differential operator $x_j\partial_i x_{i_{k+1}}\partial_{i_{k+1}}\dots x_{i_l}\partial_{i_l}x_i\partial_j$ acts on the monomial $x^\mu = x_1^{\mu_1}\dots x_n^{\mu_n}$ as follows:

$$\big(x_j\partial_i x_{i_{k+1}}\partial_{i_{k+1}}\dots x_{i_l}\partial_{i_l}x_i\partial_j\big)(x^\mu) = \mu_j(\mu_i+1-\delta_{ij})\prod_{s=1}^{l}(\mu_s+\delta_{i,i_s}-\delta_{j,i_s})\cdot x^\mu.$$

The last equality follows from the identity $x_i\partial_i(x^\mu) = \mu_i\cdot x^\mu$. Therefore, the map $\mathfrak{i}_m$ sends the element $\mathbf{D}(P_{\alpha_1}\dots P_{\alpha_k})$ of $\mathscr{B}(m\varpi_1)$ to the function in $\mathrm{Func}(\mathrm{wt}(m\varpi_1), S(\mathfrak{h}^*))$ whose value at $\mu \in \mathrm{wt}(m\varpi_1)$ is equal to

$$\sum_{k=1}^{l}\prod_{s=1}^{k-1}(\mu_1 t_1^{\alpha_s}+\ldots+\mu_n t_n^{\alpha_s})\sum_{i_{k+1},\dots,i_l=1}^{n}\sum_{i,j=1}^{n}h_{\alpha_k-1}(t_i,t_j)t_{i_{k+1}}^{\alpha_{k+1}}\dots t_{i_l}^{\alpha_l}\cdot\mu_j(\mu_i+1-\delta_{ij})\prod_{s=k+1}^{l}(\mu_s+\delta_{i,i_s}-\delta_{j,i_s}) =$$
$$= \sum_{k=1}^{l}\prod_{s=1}^{k-1}(\mu_1 t_1^{\alpha_s}+\ldots+\mu_n t_n^{\alpha_s})\times\left(\sum_{i,j=1}^{n}\mu_j(\mu_i+1-\delta_{ij})h_{\alpha_k-1}(t_i,t_j)\prod_{s=k+1}^{l}(\mu_1 t_1^{\alpha_s}+\ldots+\mu_n t_n^{\alpha_s}+(t_i^{\alpha_s}-t_j^{\alpha_s}))\right),$$

where we used that for any $s \in \{k+1,\dots,l\}$ we have

$$\sum_{i_s=1}^{n}(\mu_s+\delta_{i,i_s}-\delta_{j,i_s})t_{i_s}^{\alpha_s} = (\mu_1 t_1^{\alpha_s}+\mu_2 t_2^{\alpha_s}+\ldots+\mu_n t_n^{\alpha_s})+(t_i^{\alpha_s}-t_j^{\alpha_s}).$$

Using the fact that $h_{\alpha_k-1}(t_i,t_j)$ equals $\alpha_k t_i^{\alpha_k-1}$ if $i=j$ and $\frac{1}{t_i-t_j}(t_i^{\alpha_k}-t_j^{\alpha_k})$ otherwise, we obtain that the value of $\mathfrak{i}_m(\mathbf{D}(P_{\alpha_1}\dots P_{\alpha_k}))$ at μ equals

$$\sum_{k=1}^{l}\left(\sum_{i=1}^{n}\mu_i^2 t_i^{\alpha_k-1}\right)\prod_{\substack{1\le s\le l\\ s\neq k}}(\mu_1 t_1^{\alpha_s}+\ldots+\mu_n t_n^{\alpha_s})+$$

$$+\sum_{i\neq j}\frac{\mu_j(\mu_i+1)}{t_i-t_j}\sum_{k=1}^{l}\left(\prod_{s=1}^{k-1}(\mu_1 t_1^{\alpha_s}+\ldots+\mu_n t_n^{\alpha_s})\right)(t_i^{\alpha_k}-t_j^{\alpha_k})\left(\prod_{s=k+1}^{l}(\mu_1 t_1^{\alpha_s}+\ldots+\mu_n t_n^{\alpha_s}+(t_i^{\alpha_s}-t_j^{\alpha_s}))\right)=$$

$$=\left(\sum_{i=1}^{n}\mu_i\frac{\partial}{\partial t_i}\right)\left(\prod_{s=1}^{l}(\mu_1 t_1^{\alpha_s}+\ldots+\mu_n t_n^{\alpha_s})\right)+$$

$$+\sum_{i\neq j}\frac{\mu_j(\mu_i+1)}{t_i-t_j}\left[\prod_{s=1}^{l}(\mu_1 t_1^{\alpha_s}+\ldots+\mu_n t_n^{\alpha_s}+(t_i^{\alpha_s}-t_j^{\alpha_s}))-\prod_{s=1}^{l}(\mu_1 t_1^{\alpha_s}+\ldots+\mu_n t_n^{\alpha_s})\right].$$

Combining everything, we obtain that for $g=\prod_{k=1}^{l}p_{\alpha_k}$ the operator $\widehat{\mathbf{D}}$ maps the function $\mu\mapsto g[\mu_1 t_1+\ldots+\mu_n t_n]$ to the function

$$\mu\mapsto\left(\mu_1\frac{\partial}{\partial t_1}+\ldots+\mu_n\frac{\partial}{\partial t_n}\right)(g[\mu_1t_1+\ldots+\mu_nt_n])+$$
$$+\sum_{\substack{1\le i,j\le n\\ i\neq j}}\frac{\mu_j(\mu_i+1)}{t_i-t_j}\big(g[\mu_1t_1+\ldots+\mu_nt_n+(t_i-t_j)]-g[\mu_1t_1+\ldots\mu_nt_n]\big),$$

as claimed. □

Remark 8.16. In fact, one can modify the argument above in order to show directly that $\mathbf{D}$ leaves $\mathscr{B}(m\varpi_1)$ invariant. Namely, one simply needs to verify that for any symmetric functions f and g the expression in (8.21) defines an element of the subalgebra generated by $\mathcal{F}_0^{(m)}$ and $\mathcal{F}_1^{(m)}$ inside $\mathrm{Func}(\mathrm{wt}(m\varpi_1),S(\mathfrak{h}^*))$. This is essentially done in the proof of Proposition 8.21.

8.6.1. *Discussion of the formula.* The expression in (8.21) can be put in a broader context. Consider the ring of *diagonal invariants* $DI_n=\mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]^{\mathfrak{S}_n}$, where $\mathfrak{S}_n$ acts diagonally on variables x_i,y_i. It is known that this ring is generated by the so-called *polarized power sums* (see [24, Ch. II, §A.3]):

$$p_{a,b}=\sum_{i=1}^{n}x_i^a y_i^b.$$

Note that the generators $\{p_{a,b}\}_{a,b\ge0}$ are not algebraically independent.

Let DI_n^1 be the subring of DI_n generated by $p_{a,0}$ and $p_{a,1}$ for all non-negative integers a. It turns out that DI_n^1 is isomorphic to a polynomial ring in $2n$ variables.

Proposition 8.19. *DI_n^1 is a free polynomial ring in the $2n$ generators $p_{1,0},\ldots,p_{n,0}$ and $p_{0,1},\ldots,p_{n-1,1}$.*

Proof. Indeed, these $2n$ polynomials are algebraically independent since the corresponding Jacobian

$$\frac{\partial(p_{1,0},\ldots,p_{n,0},p_{0,1},\ldots,p_{n-1,1})}{\partial(x_1,\ldots,x_n,y_1,\ldots,y_n)}=n!\cdot\prod_{i<j}(x_i-x_j)^2$$

is non-zero. The elements $p_{1,0},\ldots,p_{n,0}$ generate all symmetric polynomials in the variables $x_1,\ldots,x_n$ and in particular all $p_{a,0}$ for $a>n$. Finally, for any $a\ge n$ we have the identity

$$p_{a,1}-e_1p_{a-1,1}+e_2p_{a-1,1}-\ldots+(-1)^ne_np_{a-n,1}=0,$$

where $e_i=e_i(x_1,\ldots,x_n)$ is the i-th elementary symmetric polynomial in $x_1,\ldots,x_n$. It follows that all elements $p_{a,1}$ belong to the algebra generated by $p_{1,0},\ldots,p_{n,0}$ and $p_{0,1},\ldots,p_{n-1,1}$. Thus, these $2n$ polynomials freely generate DI_n^1. □

We can now restate the results of the previous subsection.

Proposition 8.20. *The big algebra $\mathscr{B}(S^m(\mathbb{C}^n))$ is a homomorphic image of the ring DI_n^1. Namely, the map*

$$\mathfrak{i}_m^{-1}\circ\left(\prod_{\mu\in\mathrm{wt}(m\varpi_1)}\mathrm{ev}_\mu\right)\colon DI_n^1\to\mathscr{B}(S^m(\mathbb{C}^n)),$$

where the ring homomorphism $\mathrm{ev}_\mu\colon DI_n^1\to\mathbb{C}[t_1,\ldots,t_n]$ sends the generator $p_{a,b}$ to $\sum_{i=1}^n\mu_i^bt_i^a$, is a surjective $\mathbb{C}[t_1,\ldots,t_n]^{\mathfrak{S}_n}$-algebra homomorphism.

Recall that Λ is the ring of symmetric functions in infinitely many variables. For $f \in \Lambda$ we use the notation $f[A]$ for the *plethystic substitution*, where A is a certain expression in x_i, y_i. In plethystic substitutions we treat x_i as variables and y_i as constants. It turns out that the expression on the right-hand side of (8.21) can be lifted to an operator on DI_n^1.

Proposition 8.21. *There exists a $\mathbb{C}$-linear operator $\widehat{\mathbf{D}}_{x,y} \colon DI_n^1 \to DI_n^1$ such that for any $f, g \in \Lambda$ we have*

(8.22)
$$\widehat{\mathbf{D}}_{x,y}(f[x_1+\ldots+x_n]g[x_1y_1+\ldots+x_ny_n]) = \left(y_1\frac{\partial}{\partial x_1}+\ldots+y_n\frac{\partial}{\partial x_n}\right)(f[x_1+\ldots+x_n]g[x_1y_1+\ldots+x_ny_n])+$$
$$+f[x_1+\ldots+x_n]\cdot\sum_{\substack{1\le i,j\le n\\ i\neq j}} y_j(y_i+1)\frac{g[x_1y_1+\ldots+x_ny_n+(x_i-x_j)]-g[x_1y_1+\ldots+x_ny_n)]}{x_i-x_j}.$$

The map $\widehat{\mathbf{D}}_{x,y}$ induces the Kirillov–Wei operator $\mathbf{D}$ on $\mathscr{B}(S^m(\mathbb{C}^n))$.

Proof. Denote the expression on the right-hand side of (8.22) by $D(f,g)$. It is clear that $D(f,g) \in DI_n$ for any $f, g \in \Lambda$.

Note that the subalgebras $\{f[x_1+\ldots+x_n] : f \in \Lambda\}$ and $\{g[x_1y_1+\ldots+x_ny_n] : g \in \Lambda\}$ of DI_n^1 coincide with $\mathbb{C}[p_{1,0},\ldots,p_{n,0}]$ and $\mathbb{C}[p_{0,1},\ldots,p_{n-1,1}]$, respectively. It follows now from Proposition 8.19 that one can define (uniquely) a $\mathbb{C}$-linear map $\widehat{\mathbf{D}}_{x,y} \colon DI_n^1 \to DI_n$ by the formula
$$\widehat{\mathbf{D}}_{x,y}(f[x_1+\ldots+x_n]g[x_1y_1+\ldots+x_ny_n]) = D(f,g), \text{ where } f,g\in\Lambda.$$
Observe that the right-hand side depends only on $f[x_1+\ldots+x_n]$ and $g[x_1y_1+\ldots+x_ny_n]$ rather than on the choice of f and g. Indeed, it follows from the fact that if for $f_0, g_0 \in \Lambda$ we have $f_0[x_1+\ldots+x_n] \equiv 0$ or $g_0[x_1y_1+\ldots+x_ny_n] \equiv 0$, then $D(f_0,g_0) \equiv 0$ as well. Therefore, we obtain a well-defined $\mathbb{C}$-linear map $\widehat{\mathbf{D}}_{x,y} \colon DI_n^1 \to DI_n$.

To show that $\widehat{\mathbf{D}}_{x,y}$ is an operator acting on DI_n^1 it remains to verify that $D(f,g)$ is indeed an element of DI_n^1 for any $f,g\in\Lambda$. Since
$$D(f,g) = D(f,1)\cdot g[x_1y_1+\ldots+x_ny_n] + f[x_1+\ldots+x_n]\cdot D(1,g),$$
it suffices to check that $D(f,1)$ and $D(1,g)$ belong to DI_n^1 for any $f,g\in\Lambda$.

To compute $D(f,1)$ note that
$$D(f,1) = \left(y_1\frac{\partial}{\partial x_1}+\ldots+y_n\frac{\partial}{\partial x_n}\right)(f[x_1+\ldots+x_n]),$$
Without loss of generality, we may assume that $f = p_{\alpha_1}\ldots p_{\alpha_l}$ for $\alpha_i \in \{1,\ldots,n\}$, i.e. that $f[x_1+\ldots+x_n] = p_{\alpha_1,0}\ldots p_{\alpha_l,0}$. Then,
$$D(f,1) = \left(y_1\frac{\partial}{\partial x_1}+\ldots+y_n\frac{\partial}{\partial x_n}\right)(p_{\alpha_1,0}\ldots p_{\alpha_l,0}) = \sum_{k=1}^{l}\alpha_k p_{\alpha_k-1,1}\prod_{s\neq k}p_{\alpha_s,0} \in DI_n^1.$$

Similarly, to check that $D(1,g)\in DI_n^1$ we may assume that $g = p_{\alpha_1}\ldots p_{\alpha_l}$ for $\alpha_i \in \{0,1,\ldots,n-1\}$, i.e. that $g[x_1y_1+\ldots+x_ny_n] = p_{\alpha_1,1}\ldots p_{\alpha_l,1}$. We obtain
$$D(1,g) = \left(\sum_{i=1}^{n}y_i\frac{\partial}{\partial x_i}\right)(p_{\alpha_1,1}\ldots p_{\alpha_l,1}) + \sum_{\substack{1\le i,j\le n\\ i\neq j}}\frac{y_j(y_i+1)}{x_i-x_j}\left(\prod_{k=1}^{l}(p_{\alpha_k,1}+(x_i^{\alpha_k}-x_j^{\alpha_k})) - \prod_{k=1}^{l}p_{\alpha_k,1}\right) =$$
$$= \sum_{k=1}^{l}\alpha_k\left(\sum_{i=1}^{n}y_i^2x_i^{\alpha_k-1}\right)\prod_{s\neq k}p_{\alpha_s,1} + \sum_{\substack{1\le i,j\le n\\ i\neq j}}\frac{y_j(y_i+1)}{x_i-x_j}\sum_{\substack{I\subset[l]\\ I\neq\varnothing}}\prod_{s\in I}(x_i^{\alpha_s}-x_j^{\alpha_s})\prod_{s\notin I}p_{\alpha_s,1}.$$
Now observe that for any non-empty subset I of $[l]=\{1,\ldots,l\}$ the expression $\frac{1}{x_i-x_j}\prod_{s\in I}(x_i^{\alpha_s}-x_j^{\alpha_s})$ is in fact a polynomial in x_i and x_j. Moreover, if I is not a singleton, then this expression vanishes when $x_i = x_j$. Thus, we can rewrite the formula above as follows:
$$\widehat{\mathbf{D}}_{x,y}(p_{\alpha_1,1}\ldots p_{\alpha_l,1}) = -\sum_{k=1}^{l}\alpha_k\left(\sum_{i=1}^{n}y_ix_i^{\alpha_k-1}\right)\prod_{s\neq k}p_{\alpha_s,1} + \sum_{i,j=1}^{n}y_j(y_i+1)\sum_{\substack{I\subset[l]\\ I\neq\varnothing}}\left(\frac{\prod_{s\in I}(x_i^{\alpha_s}-x_j^{\alpha_s})}{x_i-x_j}\right)\prod_{s\notin I}p_{\alpha_s,1} =$$
$$= -\sum_{k=1}^{l}\alpha_k p_{\alpha_k-1,1}\prod_{s\neq k}p_{\alpha_s,1} + \sum_{i,j=1}^{n}y_j(y_i+1)\sum_{\substack{I\subset[l]\\ I\neq\varnothing}}Q_I(x_i,x_j)\prod_{s\notin I}p_{\alpha_s,1}.$$

where $Q_I(x_i, x_j) = \frac{1}{x_i - x_j} \prod_{s\in I}(x_i^{\alpha_s} - x_j^{\alpha_s})$ is a certain polynomial in x_i and x_j whose coefficients depend on I. Finally, note that for any polynomial $Q(u, v) = \sum_{a,b} Q_{a,b} u^a v^b$ we have

$$\sum_{i,j=1}^{n} y_j(y_i + 1)Q(x_i, x_j) = \sum_{a,b} Q_{a,b} \left(\sum_{i=1}^{n}(y_i + 1)x_i^a\right)\left(\sum_{j=1}^{n} y_j x_j^b\right) = \sum_{a,b} Q_{a,b}(p_{a,0} + p_{a,1})p_{b,1} \in DI_n^1.$$

This completes the proof of the fact that $D(1, g) \in DI_n^1$. Thus, $\widehat{\mathbf{D}}_{x,y}$ is an operator on DI_n^1 satisfying (8.22). □

Remark 8.17. Note that a priori it is not clear that $\widehat{\mathbf{D}}_{x,y}$ descends to a well defined map on $\mathscr{B}(S^m(\mathbb{C}^n))$. It is only a consequence of the computations from Proposition 8.18.

Appendix A. Proof of Lemma 8.10

In this appendix we prove Lemma 8.10. In fact, it is a direct consequence of Corollary A.3. The latter fact is one of the variants of the celebrated Combinatorial Nullstellensatz by Alon [1, Theorems 1.1 and 1.2] (see also [11, Theorem 4]). For the sake of completeness, we present here a short proof of this result following [11].

Lemma A.1. *Let A any finite subset of a field $\mathbb{F}$. Then, for any $k \in \{0, 1, \ldots, |A| - 1\}$ we have*

$$\sum_{a\in A} \frac{a^k}{\prod_{b\in A\setminus\{a\}}(b - a)} = \begin{cases} 0, & k \in \{0, 1, \ldots, |A| - 2\}, \\ 1, & k = |A| - 1. \end{cases}$$

Proof. Applying the Lagrange interpolation formula to the polynomial x^k and the elements of A (note that $k < |A|$) gives the identity

$$\sum_{a\in A} a^k \cdot \frac{\prod_{b\in A\setminus\{a\}}(x - a)}{\prod_{b\in A\setminus\{a\}}(b - a)} = x^k.$$

Equating the coefficients in front of $x^{|A|-1}$ of both sides, we obtain the required equality. □

Proposition A.2. *Let $g \in \mathbb{F}[X_1, \ldots, X_m]$ be a polynomial in m variables over a field $\mathbb{F}$ of total degree d. Let $d_1, \ldots, d_m$ be non-negative integers such that $d_1 + \ldots + d_m = d$. Then, for any subsets $A_1, \ldots, A_m$ of $\mathbb{F}$ such that $|A_i| = d_i + 1$ for all i, the $X_1^{d_1} \ldots X_m^{d_m}$ term of g equals*

$$[g]_{X_1^{d_1}\ldots X_m^{d_m}} = \sum_{\substack{(a_1,\ldots,a_m)\in\\ A_1\times\ldots\times A_m}} g(a_1, \ldots, a_m) \prod_{i=1}^{m} \frac{1}{\prod_{a\in A_i\setminus\{a_i\}}(a_i - a)}.$$

Proof. Because of linearity it suffices to verify the identity in the case when $g = X_1^{i_1} \ldots X_m^{i_m}$ is a monomial such that $i_1 + \ldots + i_m \le d$. Then, either $(i_1, \ldots, i_m) = (d_1, \ldots, d_m)$, or $i_j < d_j$ for some j. Applying Lemma A.1 m times now yields the required identity. □

Corollary A.3. *Let $g \in \mathbb{F}[X_1, \ldots, X_m]$ be a non-zero polynomial in m variables over a field $\mathbb{F}$. Let $d_1, \ldots, d_m$ be non-negative integers such that for each $i \in \{1, \ldots, m\}$ the degree of g in the variable x_i is less than d_i. Let $A_1, \ldots, A_m$ be subsets of $\mathbb{F}$ such that $|A_i| = d_i + 1$ for all i. Then, g cannot vanish on $A_1 \times \ldots \times A_m$.*

Proof. The corollary is a direct consequence of the proposition above. Indeed, assume that g vanishes on $A_1 \times \ldots \times A_m$ and choose a monomial $X_1^{i_1} \ldots X_m^{i_m}$ of the largest degree which occurs in g. Then, by the assumption, we have $0 \le i_j \le d_j - 1$ for all j. Choose arbitrary subsets $B_j \subset A_j$ such that $|B_j| = i_j + 1$. Applying Proposition A.2 to g, the subsets $B_1, \ldots, B_m$ and the monomial $X_1^{i_1} \ldots X_m^{i_m}$ yields a contradiction. □

Now Lemma 8.10 follows as a consequence of Corollary A.3 applied to the field $\mathbb{F} = \mathrm{Frac}(S(\mathfrak{gl}_n^*))$.

References

[1] N. Alon, *Combinatorial Nullstellensatz*, Combin. Prob. Comput. **8** (1999), 7–29.

[2] A. Capelli, *Über die Zurückführung der Cayley'schen Operation Ω auf gewöhnlichen Polar-Operationen*, Math. Ann. **29** (1887), 331–338.

[3] S. Caracciolo, A. D. Sokal, A. Sportiello, *Noncommutative determinants, Cauchy–Binet formulae, and Capelli-type identities I. Generalizations of the Capelli and Turnbull identities*, Electron. J. Combin. **16** (2009), #R103.

[4] B. Feigin, E. Frenkel, *Affine Kac–Moody algebras at the critical level and Gelfand-Dikii algebras*, Int. Journ. Mod. Phys. **A7**, Supplement 1A (1992), 197–215.

[5] T. Hausel, *Commutative avatars of representations of semisimple Lie groups*, Proc. Natl. Acad. Sci. USA **121** (2024), no. 38.

[6] T. Hausel, K. Rychlewicz, *Spectrum of equivariant cohomology as fixed point scheme*, to appear in Épijournal de Géométrie Algébrique (2024).

[7] R. Howe, *Remarks on classical invariant theory*, Trans. Amer. Math. Soc. **313** (1989), 539–570.

[8] R. Howe, *Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond*, Israel Math. Conference Proceedings (1995), 1–182.

[9] R. Howe and T. Umeda, *The Capelli identity, the double commutant theorem and multiplicity-free actions*, Math. Ann. **290** (1991), 569–619.

[10] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics **9**, Springer-Verlag, New York, 1972.
[11] R. N. Karasev, F. V. Petrov, *Partitions of nonzero elements of a finite field into pairs*, Israel J. Math. **192** (2012), 143–156.
[12] A. Kirillov, *Family algebras*, Electron. Res. Announc. Amer. Math. Soc. **6** (2000), 7–20.
[13] A. Kirillov, *Introduction to family algebras*, Mosc. Math. J. **1** (2001), 49–63.
[14] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Second edition, with contributions by A. Zelevinsky, The Clarendon Press, Oxford University Press, New York (1995).
[15] A. Molev, *Yangians and classical Lie algebras*, Mathematical Surveys and Monographs **143**, American Mathematical Society, Providence, RI, 2007.
[16] A. Molev, *Sugawara operators for classical Lie algebras*, Mathematical Surveys and Monographs **229**, American Mathematical Society, Providence, RI, 2018.
[17] M. Nazarov, G. Olshanski, *Bethe subalgebras in twisted Yangians*, Comm. Math. Phys. **178** (1996), 483–506.
[18] A. Okounkov, *Quantum immanants and higher Capelli identities*, Transformation Groups **1** (1996), 99–126.
[19] D. Panyushev, *Weight multiplicity free representations, $\mathfrak{g}$-endomorphism algebras, and Dynkin polynomials*, J. London Math. Soc. (2) 69, **2** (2004), 273–290.
[20] N. Rozhkovskaya, *Commutativity of quantum family algebras*, Lett. Math. Phys. **63** (2003), 87–103.
[21] L. G. Rybnikov, *The argument shift method and the Gaudin model*, Funct. Anal. Appl. **40** (2006), 188–199.
[22] T. Umeda, *On the proof of the Capelli identities*, Funkcial. Ekvac. **51**, (2008), 1–15.
[23] Z. Wei, *The noncommutative Poisson bracket and the deformation of the family algebras*, J. Math. Phys. **56** (2015), no. 7.
[24] H. Weyl, *The classical groups, their invariants and representations*, Second Edition, Princeton University Press, Princeton, N.J. (1946).
[25] O. Yakimova, *Symmetrisation and the Feigin-Frenkel centre*, Compos. Math. **158** (2022), no. 3, 585–622.
[26] D. Zhelobenko, *Compact Lie groups and their representations*, Translations of Mathematical Monographs **40** (1973).