1

$$\mu = [-3, 1, 4]^T; \quad \Sigma = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

a) $$Cor(X_1, X_2) = \frac{-2}{\sqrt{1 \cdot 5}} \neq 0$$

$\rightarrow X_1, X_2$ are <u>not</u> linearly independent

b) $$Cor(X_2, X_3) = \frac{0}{\sqrt{5 \cdot 2}} = 0$$

$\rightarrow X_2, X_3$ are linearly dependent 1

c) $Y = [X_1, X_2]^T$

$Cor(X_1, X_3) = \frac{0}{\sqrt{1 \cdot 2}} = 0 \rightarrow X_1, X_3$ are linearly independent 2

1, 2 $\rightarrow Y, X_3$ are linearly independent

d) $$Y = \frac{X_1 + X_2}{2} \sim N(\mu_Y = -1, \sigma_Y^2 = \tfrac{3}{2})$$

$$Cov(Y, X_3) = \frac{1}{2} Cov(X_1, X_3) + \frac{1}{2} Cov(X_2, X_3)$$
$$= 0$$

$\rightarrow Cor(Y, X_3) = 0$

$\rightarrow Y, X_3$ linearly independent

e) $$Cov(Y, X_2) = Cov(X_2, X_2) - \frac{5}{2} Cov(X_1, X_2) - Cov(X_1, X_3)$$
$$= 5 - \frac{5}{2}(-2) = 10 \neq 0$$

$\rightarrow Y, X_2$ <u>not</u> linearly independent

2 Mahalanobis distance

$$d_M(y,\mu) = \sqrt{(y-\mu)^T \Sigma^{-1} (y-\mu)}$$

(a)

$$d_E(y,\mu_1) = \sqrt{1+4+4} = \boxed{3}$$

$$d_E(y,\mu_2) = \sqrt{4+4+2.25} \approx \boxed{3.20}$$

(b)

$$\Sigma = \begin{bmatrix} 9.0 & 8.1 & -3.6 \\ 8.1 & 9.0 & -4.8 \\ -3.6 & -4.8 & 4.0 \end{bmatrix} \rightarrow \Sigma^{-1} = \begin{bmatrix} 0.741 & -0.864 & -0.370 \\ -0.864 & 1.317 & 0.802 \\ -0.370 & 0.802 & 0.880 \end{bmatrix}$$

$$d_M(y,\mu_1) = \sqrt{(y-\mu)^T \Sigma^{-1} (y-\mu)} = \boxed{1.132}$$

$$d_M(y,\mu_2) = \boxed{0.703}$$

(c) $\bar{X} \mathrel{\dot\sim} MVN(\mu, \frac{1}{n}\Sigma)$

$$(\bar{X}-\mu)^T \left(\tfrac{1}{n}\Sigma\right)^{-1} (\bar{X}-\mu) = (\bar{X}-\mu)^T \left(\tfrac{1}{n}\Sigma\right)^{-1/2} \left(\tfrac{1}{n}\Sigma\right)^{-1/2} (\bar{X}-\mu)$$

$$= \Big[\underbrace{\left(\tfrac{1}{n}\Sigma\right)^{-1/2} (\bar{X}-\mu)}_{\dot\sim MVN(\vec{0}, I_n)}\Big]^T \Big[\underbrace{\left(\tfrac{1}{n}\Sigma\right)^{-1/2} (\bar{X}-\mu)}_{\dot\sim MVN(\vec{0}, I_n)}\Big]$$

$$\rightarrow (\bar{X}-\mu)^T \left(\tfrac{1}{n}\Sigma\right)^{-1} (\bar{X}-\mu) \mathrel{\dot\sim} \chi^2_n$$

$\|$

$n \times$ Mahalanobis distance of $\bar{X}$ from μ

(d) μ_2 is more plausible because it has smaller Mahalanobis distance from the sample mean.

$\boxed{3}$

(a)
$$E(Y_1) = \frac{1}{5}\sum_{i=1}^{5} E(X_i) = \boxed{\mu}$$

$$\Sigma_{Y_1} = Cov\left(\frac{1}{5}\sum_{i=1}^{5} X_i, \frac{1}{5}\sum_{j=1}^{5} X_j\right)$$

$$= \frac{1}{25}\sum_{i=1}^{5}\sum_{j=1}^{5} Cov(X_i, X_j)$$

$X_1, X_2, \dots, X_5$ iid $\Rightarrow Cov(X_i, X_j) = 0 \quad \forall i \neq j$

$$\Rightarrow \Sigma_{Y_1} = \boxed{\frac{1}{25}\left[\sum_{i=1}^{5} Cov(X_i, X_i)\right]}$$

(b)
$$E(Y_2) = \sum_{i=1}^{5} (-1)^{i+1} E(X_i) = \boxed{\mu}$$

$$\Sigma_{Y_2} = \boxed{\sum_{i=1}^{5} Cov(X_i, X_i)}$$

$\boxed{4}$

$$X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 4 & 4 \\ 5 & 7 \\ 4 & 7 \end{bmatrix}$$

Likelihood: $L(\mu, \sigma^2 | X) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$

Log-likelihood: $LL(\mu, \sigma^2 | X) = \sum_{i=1}^{n}\left[-\frac{1}{2}\log(2\pi) - \frac{1}{2}\log(\sigma^2) - \frac{(x_i-\mu)^2}{2\sigma^2}\right]$

$$= \frac{-n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$\hat{\mu}_{MLE} = \arg\min \sum_{i=1}^{n}(x_i-\mu)^2 = \arg\min \sum_{i=1}^{n} x_i^2 - 2n\bar{X}\mu + n\mu^2) = \arg\min f(\mu)$

$f'(\mu) = 2n\mu - 2n\bar{X}$, $f'(\mu) = 0 \Leftrightarrow \boxed{\mu = \bar{X}}$; $f''(\bar{X}) = 2n > 0 \rightarrow$ minimum ✓

$$\rightarrow \hat{\mu}_{MLE} = \bar{X} = \boxed{\begin{bmatrix} 4 \\ 6 \end{bmatrix}}$$

4(cont)

Similarly, $\hat{\Sigma}_{MLE} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})(X_i - \bar{X})^T$

$$\rightarrow \hat{\Sigma}_{MLE} = \frac{1}{4}\left\{\begin{bmatrix}-1\\0\end{bmatrix}\begin{bmatrix}-1 & 0\end{bmatrix} + \begin{bmatrix}0\\-2\end{bmatrix}\begin{bmatrix}0 & -2\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}\right\}$$

$$= \frac{1}{4}\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 4\end{bmatrix} + \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$$

$$\hat{\Sigma}_{MLE} = \boxed{\begin{bmatrix}0.5 & 0.25\\0.25 & 1.5\end{bmatrix}}$$