1

$$\mu = [-3, 1, 4]^T; \quad \Sigma = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

a) $Cor(X_1, X_2) = \frac{-2}{\sqrt{1 \cdot 5}} \neq 0$

$\rightarrow X_1, X_2$ are <u>not</u> linearly independent

b) $Cor(X_2, X_3) = \frac{0}{\sqrt{5 \cdot 2}} = 0$

$\rightarrow X_2, X_3$ are linearly dependent ①

c) $Y = [X_1, X_2]^T$

$Cor(X_1, X_3) = \frac{0}{\sqrt{1 \cdot 2}} = 0 \rightarrow X_1, X_3$ are linearly independent ②

①, ② $\rightarrow Y, X_3$ are linearly independent

d) $Y = \frac{X_1 + X_2}{2} \sim N(\mu_Y = -1, \sigma_Y^2 = \frac{3}{2})$

$Cov(Y, X_3) = \frac{1}{2} Cov(X_1, X_3) + \frac{1}{2} Cov(X_2, X_3)$

$= 0$

$\rightarrow Cor(Y, X_3) = 0$

$\rightarrow Y, X_3$ linearly independent

e) $Cov(Y, X_2) = Cov(X_2, X_2) - \frac{5}{2} Cov(X_1, X_2) - Cov(X_1, X_3)$

$= 5 - \frac{5}{2}(-2) = 10 \neq 0$

$\rightarrow Y, X_2$ <u>not</u> linearly independent

2 Mahalanobis distance

$$d_M(y, \mu) = \sqrt{(y-\mu)^T \Sigma^{-1} (y-\mu)}$$

(a)

$$d_E(y, \mu_1) = \sqrt{1+4+4} = \boxed{3}$$

$$d_E(y, \mu_2) = \sqrt{4+4+2.25} \approx \boxed{3.20}$$

(b)

$$\Sigma = \begin{bmatrix} 9.0 & 8.1 & -3.6 \\ 8.1 & 9.0 & -4.8 \\ -3.6 & -4.8 & 4.0 \end{bmatrix} \rightarrow \Sigma^{-1} = \begin{bmatrix} 0.741 & -0.864 & -0.370 \\ -0.864 & 1.317 & 0.802 \\ -0.370 & 0.802 & 0.880 \end{bmatrix}$$

$$d_M(y, \mu_1) = \sqrt{(y-\mu)^T \Sigma^{-1} (y-\mu)} = \boxed{1.132}$$

$$d_M(y, \mu_2) = \boxed{0.703}$$

(c)

$$\bar{X} \mathrel{\dot\sim} MVN(\mu, \tfrac{1}{n}\Sigma)$$

$$(\bar{X}-\mu)^T \left(\tfrac{1}{n}\Sigma\right)^{-1} (\bar{X}-\mu) = (\bar{X}-\mu)^T \left(\tfrac{1}{n}\Sigma\right)^{-1/2} \left(\tfrac{1}{n}\Sigma\right)^{-1/2} (\bar{X}-\mu)$$

$$= \underbrace{\left[\left(\tfrac{1}{n}\Sigma\right)^{-1/2}(\bar{X}-\mu)\right]^T}_{\dot\sim MVN(\vec{0}, I_n)} \underbrace{\left[\left(\tfrac{1}{n}\Sigma\right)^{-1/2}(\bar{X}-\mu)\right]}_{\dot\sim MVN(\vec{0}, I_n)}$$

$$\rightarrow (\bar{X}-\mu)^T \left(\tfrac{1}{n}\Sigma\right)^{-1} (\bar{X}-\mu) \mathrel{\dot\sim} \chi^2_n$$

$$\parallel$$

$n \times$ Mahalanobis distance of $\bar{X}$ from μ

(d)

μ_2 is more plausible because it has smaller Mahalanobis distance from the sample mean.

$\boxed{3}$

(a)

$$E(Y_1) = \frac{1}{5}\sum_{i=1}^{5} E(X_i) = \boxed{\mu}$$

$$\Sigma_{Y_1} = Cov\left(\frac{1}{5}\sum_{i=1}^{5} X_i, \frac{1}{5}\sum_{j=1}^{5} X_j\right)$$

$$= \frac{1}{25}\sum_{i=1}^{5}\sum_{j=1}^{5} Cov(X_i, X_j)$$

$X_1, X_2, \dots, X_5$ iid $\Rightarrow Cov(X_i, X_j) = 0 \quad \forall i \neq j$

$$\Rightarrow \Sigma_{Y_1} = \boxed{\frac{1}{25}\left[\sum_{i=1}^{5} Cov(X_i, X_i)\right]}$$

(b)

$$E(Y_2) = \sum_{i=1}^{5}(-1)^{i+1} E(X_i) = \boxed{\mu}$$

$$\Sigma_{Y_2} = \boxed{\sum_{i=1}^{5} Cov(X_i, X_i)}$$

$\boxed{4}$

$$X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 4 & 4 \\ 5 & 7 \\ 4 & 7 \end{bmatrix}$$

Likelihood: $L(\mu, \sigma^2 | X) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$

Log-likelihood: $LL(\mu, \sigma^2 | X) = \sum_{i=1}^{n}\left[-\frac{1}{2}\log(2\pi) - \frac{1}{2}\log(\sigma^2) - \frac{(x_i-\mu)^2}{2\sigma^2}\right]$

$$= \frac{-n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\hat{\mu}_{MLE} = \operatorname{argmin} \sum_{i=1}^{n}(x_i-\mu)^2 = \operatorname{argmin} \left(\sum_{i=1}^{n} x_i^2 - 2n\bar{X}\mu + n\mu^2\right) = \operatorname{argmin} f(\mu)$$

$f'(\mu) = 2n\mu - 2n\bar{X}$, $f'(\mu) = 0 \Leftrightarrow \boxed{\mu = \bar{X}}$; $f''(\bar{X}) = 2n > 0 \rightarrow$ minimum ✓

$$\rightarrow \hat{\mu}_{MLE} = \bar{X} = \boxed{\begin{bmatrix} 4 \\ 6 \end{bmatrix}}$$

4(cont)

Similarly, $\hat{\Sigma}_{MLE} = \frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})(X_i-\bar{X})^T$

$$\rightarrow \hat{\Sigma}_{MLE} = \frac{1}{4}\left\{\begin{bmatrix}-1\\0\end{bmatrix}\begin{bmatrix}-1 & 0\end{bmatrix} + \begin{bmatrix}0\\-2\end{bmatrix}\begin{bmatrix}0 & -2\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}\right\}$$

$$= \frac{1}{4}\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 4\end{bmatrix} + \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$$

$$\hat{\Sigma}_{MLE} = \boxed{\begin{bmatrix}0.5 & 0.25\\0.25 & 1.5\end{bmatrix}}$$

Ngoc Ha

HW 2 - ST 557

Problem 5

$r_Q = \frac{\sum_{i=1}^{n}(x_{(j)}-\bar{x})(q_{(j)}-\bar{q})}{\sqrt{\sum_{i=1}^{n}(x_{(j)}-\bar{x})^2}\sqrt{\sum_{i=1}^{n}(q_{(j)}-\bar{q})^2}}$; where $q_{(j)} = \phi^{-1}\left(\frac{j-\frac{1}{2}}{n}\right)$

Prepare functions

In [1]:
```
inv_cdf <- function(i,n){
    return(qnorm((i-0.5)/n))
}
```

In [2]:
```
sampDiffVec <- function(sample){
    return(as.matrix(sort(sample)-mean(sample)))
}
```

In [3]:
```
theoDiffVec <- function(n){
    quantiles = rep(0,n)
    for (i in c(1:n)){
        quantiles[i] = inv_cdf(i,n)
    }
    return(as.matrix(quantiles-mean(quantiles)))
}
```

In [4]:
```
r_Q <- function(sampDiffVec, theoDiffVec){
    return((t(sampDiffVec)%*%theoDiffVec)/(norm(sampDiffVec, type='2')*norm(th
eoDiffVec, type='2')))
}
```

(5a)

In [5]:
```
rQVec_a <- rep(0,10000)
for (i in c(1:10000)){
    samp <- runif(10)
    rQVec_a[i] <- r_Q(sampDiffVec(samp),theoDiffVec(length(samp)))
}
```

In [6]:
```
cat("Part a's rejection rate:", length(rQVec_a[rQVec_a<0.9198])/length(rQVec_a
))
```

```
Part a's rejection rate: 0.056
```

(5b)

In [11]:
```
rQVec_b <- rep(0,10000)
for (i in c(1:10000)){
    samp <- rchisq(n=5,df=5)
    rQVec_b[i] <- r_Q(sampDiffVec(samp),theoDiffVec(length(samp)))
}
```

In [12]:
```
cat("Part b's rejection rate:", length(rQVec_b[rQVec_b<0.8788])/length(rQVec_b
))
```

```
Part b's rejection rate: 0.0847
```

(5c)

In [9]:
```
rQVec_c <- rep(0,10000)
for (i in c(1:10000)){
    samp <- rchisq(n=20,df=2)
    rQVec_c[i] <- r_Q(sampDiffVec(samp),theoDiffVec(length(samp)))
}
```

In [10]:
```
cat("Part c's rejection rate:", length(rQVec_c[rQVec_c<0.9508])/length(rQVec_c
))
```

```
Part c's rejection rate: 0.8067
```