Predicate Logic

Besik Dundua

International Black Sea University
bdundua@gmail.com

Contents

Why We Need Predicate Logic

- We studied propositional logic.
- Often we need more powerful logics.
- For instance, we need predicate logic to express this:
  - All men are mortal. Socrates is a man.
    Therefore Socrates is mortal.

Why We Need Predicate Logic

- We studied propositional logic.
- Often we need more powerful logics.
- For instance, we need predicate logic to express this:
  - All men are mortal. Socrates is a man.
    Therefore Socrates is mortal.
  - $\forall x.man(x) \Rightarrow mortal(x)$: All men are mortal.
  - $man(socrates)$: Socrates is a man.
  - $mortal(socrates)$: Socrates is mortal.

Alphabet

An alphabet consists of the following sets of symbols:

- A countable set of variables $\mathcal{V}$.
- For each $n \geq 0$, a set of n-ary function symbols $\mathcal{F}^n$.
- For each $n \geq 0$, a set of n-ary predicate symbols $\mathcal{P}^n$.
- Logical connectives $\bot$, $\top$, $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$.
- Quantifiers $\exists$, $\forall$.

Alphabet

An alphabet consists of the following sets of symbols:

- A countable set of variables $\mathcal{V}$.
- For each $n \geq 0$, a set of n-ary function symbols $\mathcal{F}^n$.
- For each $n \geq 0$, a set of n-ary predicate symbols $\mathcal{P}^n$.
- Logical connectives $\bot$, $\top$, $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$.
- Quantifiers $\exists$, $\forall$.

Notation:

- x, y, z for variables.
- f, g, a, b, c for function symbols.
- p, q for predicate symbols.

Terms

Definition

- A variable is a term.
- If $t_1, \ldots, t_n$ are terms and $f \in \mathcal{F}^n$, then $f(t_1, \ldots, t_n)$ is a term.

Terms

Definition

- A variable is a term.
- If $t_1, \ldots, t_n$ are terms and $f \in \mathcal{F}^n$, then $f(t_1, \ldots, t_n)$ is a term.

Notation:

- s, t, r for terms.

Formulas

The set of formulas $\mathcal{F}$ are constructed using the following rules:

- $\top$ and $\bot$ are in $\mathcal{F}$.
- If $t_1, \ldots, t_n$ are terms and $p \in \mathcal{P}^n$, then $p(t_1, \ldots, t_n)$ is a formula. It is called an atomic formula or an atom.
- If P is a formula, $\neg P$ is a formula.
- If P and Q are formulas, then $P \vee Q$, $P \wedge Q$, $P \rightarrow Q$, and $P \leftrightarrow Q$ are formulas.
- If P is a formula, then $\exists x.P$ and $\forall x.P$ are formulas.

Example

Translating English sentences into predicate logic formulas:

1. For each natural number there exists exactly one immediate successor natural number.

Assume:

- $zero$: nullary function symbol.
- $succ$, $pred$: unary function symbols.
- $=$: binary predicate symbol.

Example

Translating English sentences into predicate logic formulas:

1. For each natural number there exists exactly one immediate successor natural number.

$$\forall x.(\exists y.(y = succ(x) \land \forall z.(z = succ(x) \rightarrow y = z)))$$

Assume:

- $zero$: nullary function symbol.
- $succ$, $pred$: unary function symbols.
- $=$: binary predicate symbol.

Example

Translating English sentences into predicate logic formulas:

1. For each natural number there exists exactly one immediate successor natural number.

$$\forall x.(\exists y.(y = succ(x) \wedge \forall z.(z = succ(x) \rightarrow y = z)))$$

2. There is no natural number whose immediate successor is 0.

Assume:

- $zero$: nullary function symbol.
- $succ, pred$: unary function symbols.
- $=$: binary predicate symbol.

Example

Translating English sentences into predicate logic formulas:

1. For each natural number there exists exactly one immediate successor natural number.

$$\forall x.(\exists y.(y = succ(x) \wedge \forall z.(z = succ(x) \rightarrow y = z)))$$

2. There is no natural number whose immediate successor is 0.

$$\neg \exists x.zero = succ(x)$$

Assume:

- $zero$: nullary function symbol.
- $succ, pred$: unary function symbols.
- $=$: binary predicate symbol.

Example

Translating English sentences into predicate logic formulas:

1. For each natural number there exists exactly one immediate successor natural number.

$$\forall x.(\exists y.(y = succ(x) \land \forall z.(z = succ(x) \to y = z)))$$

2. There is no natural number whose immediate successor is 0.

$$\neg \exists x.zero = succ(x)$$

3. For each nonzero natural number there exists exactly one immediate predecessor natural number.

Assume:

- $zero$: nullary function symbol.
- $succ$, $pred$: unary function symbols.
- $=$: binary predicate symbol.

Example

Translating English sentences into predicate logic formulas:

1. For each natural number there exists exactly one immediate successor natural number.

$$\forall x.(\exists y.(y = succ(x) \land \forall z.(z = succ(x) \to y = z)))$$

2. There is no natural number whose immediate successor is 0.

$$\neg \exists x.zero = succ(x)$$

3. For each nonzero natural number there exists exactly one immediate predecessor natural number.

$$\forall x.(\neg(x = zero) \to \exists y.(y = pred(x) \land \forall z.(z = pred(x) \to y = z)))$$

Assume:

- $zero$: nullary function symbol.
- $succ$, $pred$: unary function symbols.
- $=$: binary predicate symbol.

Semantics

- Meaning of a predicate logic language consists of an universe and an appropriate meaning of each symbol.
- This pair is called structure.
- Structure fixes interpretation of function and predicate symbols.
- Meaning of variables is determined by a variable assignment.
- Interpretation of terms and formulas.

Structure

- Structure: a pair (D, I).
- D is a nonempty universe, the domain of interpretation.
- I is an interpretation function defined on D that fixes the meaning of each symbol associating
  - to each $f \in \mathcal{F}^n$ an n-ary function $f_I : D^n \to D$,
  - to each $p \in \mathcal{P}^n$ an n-ary relation p_I on D.

Variable Assignment

- A structure $\mathcal{S} = (D, I)$ is given.
- Variable assignment $\sigma_\mathcal{S}$ maps each $x \in \mathcal{V}$ into an element of D: $\sigma_\mathcal{S}(x) \in D$.
- Given a variable x, an assignment $\vartheta_\mathcal{S}$ is called an x-variant of $\sigma_\mathcal{S}$ iff $\vartheta_\mathcal{S}(y) = \sigma_\mathcal{S}(y)$ for all $y \neq x$.

Interpretation of Terms

- A structure $\mathcal{S} = (D, I)$ and a variable assignment $\sigma_{\mathcal{S}}$ are given.
- Value of a term t under $\mathcal{S}$ and $\sigma_{\mathcal{S}}$, $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(t)$:
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(x) = \sigma_{\mathcal{S}}(x)$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(f(t_1, \ldots, t_n)) = f_I(Val_{\mathcal{S},\sigma_{\mathcal{S}}}(t_1), \ldots, Val_{\mathcal{S},\sigma_{\mathcal{S}}}(t_n))$.

Interpretation of Terms

- A structure $\mathcal{S} = (D, I)$ and a variable assignment $\sigma_\mathcal{S}$ are given.
- Value of a term t under $\mathcal{S}$ and $\sigma_\mathcal{S}$, $Val_{\mathcal{S},\sigma_\mathcal{S}}(t)$:
  - $Val_{\mathcal{S},\sigma_\mathcal{S}}(x) = \sigma_\mathcal{S}(x)$.
  - $Val_{\mathcal{S},\sigma_\mathcal{S}}(f(t_1, \ldots, t_n)) = f_I(Val_{\mathcal{S},\sigma_\mathcal{S}}(t_1), \ldots, Val_{\mathcal{S},\sigma_\mathcal{S}}(t_n))$.

Interpretation of Formulas

- A structure $\mathcal{S} = (D, I)$ and a variable assignment $\sigma_{\mathcal{S}}$ are given.
- Value of an atomic formula under $\mathcal{S}$ and $\sigma_{\mathcal{S}}$ is one of true, false:
  - $\mathit{Val}_{\mathcal{S},\sigma_{\mathcal{S}}}(s \doteq t) = \mathit{true}$ iff $\mathit{Val}_{\mathcal{S},\sigma_{\mathcal{S}}}(s) = \mathit{Val}_{\mathcal{S},\sigma_{\mathcal{S}}}(t)$.
  - $\mathit{Val}_{\mathcal{S},\sigma_{\mathcal{S}}}(p(t_1, \ldots, t_n)) = \mathit{true}$ iff
    $(\mathit{Val}_{\mathcal{S},\sigma_{\mathcal{S}}}(t_1), \ldots, \mathit{Val}_{\mathcal{S},\sigma_{\mathcal{S}}}(t_n)) \in p_I$.

Interpretation of Formulas

- A structure $\mathcal{S} = (D, I)$ and a variable assignment $\sigma_{\mathcal{S}}$ are given.
- Value of an atomic formula under $\mathcal{S}$ and $\sigma_{\mathcal{S}}$ is one of true, false:
  - $\mathit{Val}_{\mathcal{S},\sigma_{\mathcal{S}}}(s \doteq t) = \mathit{true}$ iff $\mathit{Val}_{\mathcal{S},\sigma_{\mathcal{S}}}(s) = \mathit{Val}_{\mathcal{S},\sigma_{\mathcal{S}}}(t)$.
  - $\mathit{Val}_{\mathcal{S},\sigma_{\mathcal{S}}}(p(t_1, \ldots, t_n)) = \mathit{true}$ iff
    $(\mathit{Val}_{\mathcal{S},\sigma_{\mathcal{S}}}(t_1), \ldots, \mathit{Val}_{\mathcal{S},\sigma_{\mathcal{S}}}(t_n)) \in p_I$.

Interpretation of Formulas

- A structure $\mathcal{S} = (D, I)$ and a variable assignment $\sigma_{\mathcal{S}}$ are given.
- Value of an atomic formula under $\mathcal{S}$ and $\sigma_{\mathcal{S}}$ is one of true, false:
  - $\mathit{Val}_{\mathcal{S},\sigma_{\mathcal{S}}}(s \doteq t) = \mathit{true}$ iff $\mathit{Val}_{\mathcal{S},\sigma_{\mathcal{S}}}(s) = \mathit{Val}_{\mathcal{S},\sigma_{\mathcal{S}}}(t)$.
  - $\mathit{Val}_{\mathcal{S},\sigma_{\mathcal{S}}}(p(t_1, \ldots, t_n)) = \mathit{true}$ iff
    $(\mathit{Val}_{\mathcal{S},\sigma_{\mathcal{S}}}(t_1), \ldots, \mathit{Val}_{\mathcal{S},\sigma_{\mathcal{S}}}(t_n)) \in p_I$.

Interpretation of Formulas

- A structure $\mathcal{S} = (D, I)$ and a variable assignment $\sigma_{\mathcal{S}}$ are given.
- Value of an atomic formula under $\mathcal{S}$ and $\sigma_{\mathcal{S}}$ is one of *true*, *false*:
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(s \doteq t) = true$ iff $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(s) = Val_{\mathcal{S},\sigma_{\mathcal{S}}}(t)$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(p(t_1, \ldots, t_n)) = true$ iff
    $$(Val_{\mathcal{S},\sigma_{\mathcal{S}}}(t_1), \ldots, Val_{\mathcal{S},\sigma_{\mathcal{S}}}(t_n)) \in p_I.$$

Interpretation of Formulas

- A structure $\mathcal{S} = (D, I)$ and a variable assignment $\sigma_\mathcal{S}$ are given.
- Values of compound formulas under $\mathcal{S}$ and $\sigma_\mathcal{S}$ are also either *true* or *false*:
  - $Val_{\mathcal{S},\sigma_\mathcal{S}}(\neg A) = true$ iff $Val_{\mathcal{S},\sigma_\mathcal{S}}(A) = false$.
  - $Val_{\mathcal{S},\sigma_\mathcal{S}}(A \vee B) = true$ iff
    $Val_{\mathcal{S},\sigma_\mathcal{S}}(A) = true$ or $Val_{\mathcal{S},\sigma_\mathcal{S}}(B) = true$.
  - $Val_{\mathcal{S},\sigma_\mathcal{S}}(A \wedge B) = true$ iff
    $Val_{\mathcal{S},\sigma_\mathcal{S}}(A) = true$ and $Val_{\mathcal{S},\sigma_\mathcal{S}}(B) = true$.
  - $Val_{\mathcal{S},\sigma_\mathcal{S}}(A \rightarrow B) = true$ iff
    $Val_{\mathcal{S},\sigma_\mathcal{S}}(A) = false$ or $Val_{\mathcal{S},\sigma_\mathcal{S}}(B) = true$.
  - $Val_{\mathcal{S},\sigma_\mathcal{S}}(A \leftrightarrow B) = true$ iff $Val_{\mathcal{S},\sigma_\mathcal{S}}(A) = Val_{\mathcal{S},\sigma_\mathcal{S}}(B)$.
  - $Val_{\mathcal{S},\sigma_\mathcal{S}}(\exists x.A) = true$ iff
    $Val_{\mathcal{S},\vartheta_\mathcal{S}}(A) = true$ for some x-variant $\vartheta_\mathcal{S}$ of $\sigma_\mathcal{S}$.
  - $Val_{\mathcal{S},\sigma_\mathcal{S}}(\forall x.A) = true$ iff
    $Val_{\mathcal{S},\vartheta_\mathcal{S}}(A) = true$ for all x-variants $\vartheta_\mathcal{S}$ of $\sigma_\mathcal{S}$.

Interpretation of Formulas

- A structure $\mathcal{S} = (D, I)$ and a variable assignment $\sigma_{\mathcal{S}}$ are given.
- Values of compound formulas under $\mathcal{S}$ and $\sigma_{\mathcal{S}}$ are also either *true* or *false*:
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(\neg A) = \mathit{true}$ iff $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A) = \mathit{false}$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A \vee B) = \mathit{true}$ iff
    $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A) = \mathit{true}$ or $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(B) = \mathit{true}$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A \wedge B) = \mathit{true}$ iff
    $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A) = \mathit{true}$ and $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(B) = \mathit{true}$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A \rightarrow B) = \mathit{true}$ iff
    $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A) = \mathit{false}$ or $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(B) = \mathit{true}$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A \leftrightarrow B) = \mathit{true}$ iff $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A) = Val_{\mathcal{S},\sigma_{\mathcal{S}}}(B)$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(\exists x.A) = \mathit{true}$ iff
    $Val_{\mathcal{S},\vartheta_{\mathcal{S}}}(A) = \mathit{true}$ for some x-variant $\vartheta_{\mathcal{S}}$ of $\sigma_{\mathcal{S}}$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(\forall x.A) = \mathit{true}$ iff
    $Val_{\mathcal{S},\vartheta_{\mathcal{S}}}(A) = \mathit{true}$ for all x-variants $\vartheta_{\mathcal{S}}$ of $\sigma_{\mathcal{S}}$.

Interpretation of Formulas

- A structure $\mathcal{S} = (D, I)$ and a variable assignment $\sigma_{\mathcal{S}}$ are given.
- Values of compound formulas under $\mathcal{S}$ and $\sigma_{\mathcal{S}}$ are also either *true* or *false*:
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(\neg A) = true$ iff $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A) = false$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A \vee B) = true$ iff
    $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A) = true$ or $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(B) = true$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A \wedge B) = true$ iff
    $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A) = true$ and $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(B) = true$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A \rightarrow B) = true$ iff
    $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A) = false$ or $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(B) = true$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A \leftrightarrow B) = true$ iff $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A) = Val_{\mathcal{S},\sigma_{\mathcal{S}}}(B)$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(\exists x.A) = true$ iff
    $Val_{\mathcal{S},\vartheta_{\mathcal{S}}}(A) = true$ for some x-variant $\vartheta_{\mathcal{S}}$ of $\sigma_{\mathcal{S}}$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(\forall x.A) = true$ iff
    $Val_{\mathcal{S},\vartheta_{\mathcal{S}}}(A) = true$ for all x-variants $\vartheta_{\mathcal{S}}$ of $\sigma_{\mathcal{S}}$.

Interpretation of Formulas

- A structure $\mathcal{S} = (D, I)$ and a variable assignment $\sigma_\mathcal{S}$ are given.
- Values of compound formulas under $\mathcal{S}$ and $\sigma_\mathcal{S}$ are also either *true* or *false*:
  - $Val_{\mathcal{S},\sigma_\mathcal{S}}(\neg A) = true$ iff $Val_{\mathcal{S},\sigma_\mathcal{S}}(A) = false$.
  - $Val_{\mathcal{S},\sigma_\mathcal{S}}(A \vee B) = true$ iff
    $Val_{\mathcal{S},\sigma_\mathcal{S}}(A) = true$ or $Val_{\mathcal{S},\sigma_\mathcal{S}}(B) = true$.
  - $Val_{\mathcal{S},\sigma_\mathcal{S}}(A \wedge B) = true$ iff
    $Val_{\mathcal{S},\sigma_\mathcal{S}}(A) = true$ and $Val_{\mathcal{S},\sigma_\mathcal{S}}(B) = true$.
  - $Val_{\mathcal{S},\sigma_\mathcal{S}}(A \rightarrow B) = true$ iff
    $Val_{\mathcal{S},\sigma_\mathcal{S}}(A) = false$ or $Val_{\mathcal{S},\sigma_\mathcal{S}}(B) = true$.
  - $Val_{\mathcal{S},\sigma_\mathcal{S}}(A \leftrightarrow B) = true$ iff $Val_{\mathcal{S},\sigma_\mathcal{S}}(A) = Val_{\mathcal{S},\sigma_\mathcal{S}}(B)$.
  - $Val_{\mathcal{S},\sigma_\mathcal{S}}(\exists x.A) = true$ iff
    $Val_{\mathcal{S},\vartheta_\mathcal{S}}(A) = true$ for some x-variant $\vartheta_\mathcal{S}$ of $\sigma_\mathcal{S}$.
  - $Val_{\mathcal{S},\sigma_\mathcal{S}}(\forall x.A) = true$ iff
    $Val_{\mathcal{S},\vartheta_\mathcal{S}}(A) = true$ for all x-variants $\vartheta_\mathcal{S}$ of $\sigma_\mathcal{S}$.

Interpretation of Formulas

- A structure $\mathcal{S} = (D, I)$ and a variable assignment $\sigma_{\mathcal{S}}$ are given.
- Values of compound formulas under $\mathcal{S}$ and $\sigma_{\mathcal{S}}$ are also either *true* or *false*:
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(\neg A) = true$ iff $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A) = false$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A \vee B) = true$ iff
    $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A) = true$ or $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(B) = true$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A \wedge B) = true$ iff
    $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A) = true$ and $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(B) = true$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A \rightarrow B) = true$ iff
    $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A) = false$ or $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(B) = true$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A \leftrightarrow B) = true$ iff $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A) = Val_{\mathcal{S},\sigma_{\mathcal{S}}}(B)$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(\exists x.A) = true$ iff
    $Val_{\mathcal{S},\vartheta_{\mathcal{S}}}(A) = true$ for some x-variant $\vartheta_{\mathcal{S}}$ of $\sigma_{\mathcal{S}}$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(\forall x.A) = true$ iff
    $Val_{\mathcal{S},\vartheta_{\mathcal{S}}}(A) = true$ for all x-variants $\vartheta_{\mathcal{S}}$ of $\sigma_{\mathcal{S}}$.

Interpretation of Formulas

- A structure $\mathcal{S} = (D, I)$ and a variable assignment $\sigma_{\mathcal{S}}$ are given.
- Values of compound formulas under $\mathcal{S}$ and $\sigma_{\mathcal{S}}$ are also either *true* or *false*:
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(\neg A) = true$ iff $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A) = false$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A \vee B) = true$ iff
    $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A) = true$ or $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(B) = true$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A \wedge B) = true$ iff
    $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A) = true$ and $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(B) = true$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A \rightarrow B) = true$ iff
    $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A) = false$ or $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(B) = true$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A \leftrightarrow B) = true$ iff $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A) = Val_{\mathcal{S},\sigma_{\mathcal{S}}}(B)$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(\exists x.A) = true$ iff
    $Val_{\mathcal{S},\vartheta_{\mathcal{S}}}(A) = true$ for some x-variant $\vartheta_{\mathcal{S}}$ of $\sigma_{\mathcal{S}}$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(\forall x.A) = true$ iff
    $Val_{\mathcal{S},\vartheta_{\mathcal{S}}}(A) = true$ for all x-variants $\vartheta_{\mathcal{S}}$ of $\sigma_{\mathcal{S}}$.

Interpretation of Formulas

- A structure $\mathcal{S} = (D, I)$ and a variable assignment $\sigma_\mathcal{S}$ are given.
- Values of compound formulas under $\mathcal{S}$ and $\sigma_\mathcal{S}$ are also either *true* or *false*:
  - $Val_{\mathcal{S},\sigma_\mathcal{S}}(\neg A) = true$ iff $Val_{\mathcal{S},\sigma_\mathcal{S}}(A) = false$.
  - $Val_{\mathcal{S},\sigma_\mathcal{S}}(A \vee B) = true$ iff
    $Val_{\mathcal{S},\sigma_\mathcal{S}}(A) = true$ or $Val_{\mathcal{S},\sigma_\mathcal{S}}(B) = true$.
  - $Val_{\mathcal{S},\sigma_\mathcal{S}}(A \wedge B) = true$ iff
    $Val_{\mathcal{S},\sigma_\mathcal{S}}(A) = true$ and $Val_{\mathcal{S},\sigma_\mathcal{S}}(B) = true$.
  - $Val_{\mathcal{S},\sigma_\mathcal{S}}(A \to B) = true$ iff
    $Val_{\mathcal{S},\sigma_\mathcal{S}}(A) = false$ or $Val_{\mathcal{S},\sigma_\mathcal{S}}(B) = true$.
  - $Val_{\mathcal{S},\sigma_\mathcal{S}}(A \leftrightarrow B) = true$ iff $Val_{\mathcal{S},\sigma_\mathcal{S}}(A) = Val_{\mathcal{S},\sigma_\mathcal{S}}(B)$.
  - $Val_{\mathcal{S},\sigma_\mathcal{S}}(\exists x.A) = true$ iff
    $Val_{\mathcal{S},\vartheta_\mathcal{S}}(A) = true$ for some x-variant $\vartheta_\mathcal{S}$ of $\sigma_\mathcal{S}$.
  - $Val_{\mathcal{S},\sigma_\mathcal{S}}(\forall x.A) = true$ iff
    $Val_{\mathcal{S},\vartheta_\mathcal{S}}(A) = true$ for all x-variants $\vartheta_\mathcal{S}$ of $\sigma_\mathcal{S}$.

Interpretation of Formulas

- A structure $\mathcal{S} = (D, I)$ and a variable assignment $\sigma_{\mathcal{S}}$ are given.
- Values of compound formulas under $\mathcal{S}$ and $\sigma_{\mathcal{S}}$ are also either *true* or *false*:
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(\neg A) = true$ iff $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A) = false$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A \vee B) = true$ iff
    $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A) = true$ or $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(B) = true$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A \wedge B) = true$ iff
    $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A) = true$ and $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(B) = true$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A \rightarrow B) = true$ iff
    $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A) = false$ or $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(B) = true$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A \leftrightarrow B) = true$ iff $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A) = Val_{\mathcal{S},\sigma_{\mathcal{S}}}(B)$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(\exists x.A) = true$ iff
    $Val_{\mathcal{S},\vartheta_{\mathcal{S}}}(A) = true$ for some x-variant $\vartheta_{\mathcal{S}}$ of $\sigma_{\mathcal{S}}$.
  - $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(\forall x.A) = true$ iff
    $Val_{\mathcal{S},\vartheta_{\mathcal{S}}}(A) = true$ for all x-variants $\vartheta_{\mathcal{S}}$ of $\sigma_{\mathcal{S}}$.

Interpretation of Formulas

- A structure $\mathcal{S} = (D, I)$ and a variable assignment $\sigma_\mathcal{S}$ are given.
- Values of compound formulas under $\mathcal{S}$ and $\sigma_\mathcal{S}$ are also either *true* or *false*:
  - $Val_{\mathcal{S},\sigma_\mathcal{S}}(\neg A) = true$ iff $Val_{\mathcal{S},\sigma_\mathcal{S}}(A) = false$.
  - $Val_{\mathcal{S},\sigma_\mathcal{S}}(A \vee B) = true$ iff
    $Val_{\mathcal{S},\sigma_\mathcal{S}}(A) = true$ or $Val_{\mathcal{S},\sigma_\mathcal{S}}(B) = true$.
  - $Val_{\mathcal{S},\sigma_\mathcal{S}}(A \wedge B) = true$ iff
    $Val_{\mathcal{S},\sigma_\mathcal{S}}(A) = true$ and $Val_{\mathcal{S},\sigma_\mathcal{S}}(B) = true$.
  - $Val_{\mathcal{S},\sigma_\mathcal{S}}(A \rightarrow B) = true$ iff
    $Val_{\mathcal{S},\sigma_\mathcal{S}}(A) = false$ or $Val_{\mathcal{S},\sigma_\mathcal{S}}(B) = true$.
  - $Val_{\mathcal{S},\sigma_\mathcal{S}}(A \leftrightarrow B) = true$ iff $Val_{\mathcal{S},\sigma_\mathcal{S}}(A) = Val_{\mathcal{S},\sigma_\mathcal{S}}(B)$.
  - $Val_{\mathcal{S},\sigma_\mathcal{S}}(\exists x.A) = true$ iff
    $Val_{\mathcal{S},\vartheta_\mathcal{S}}(A) = true$ for some x-variant $\vartheta_\mathcal{S}$ of $\sigma_\mathcal{S}$.
  - $Val_{\mathcal{S},\sigma_\mathcal{S}}(\forall x.A) = true$ iff
    $Val_{\mathcal{S},\vartheta_\mathcal{S}}(A) = true$ for all x-variants $\vartheta_\mathcal{S}$ of $\sigma_\mathcal{S}$.

Interpretation of Formulas

- A structure $\mathcal{S} = (D, I)$ is given.
- The value of a formula A under $\mathcal{S}$ is either $true$ or $false$:
  - $Val_{\mathcal{S}}(A) = true$ iff $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A) = true$ for all $\sigma_{\mathcal{S}}$.
- $\mathcal{S}$ is called a model of A iff $Val_{\mathcal{S}}(A) = true$.
- Written $\models_{\mathcal{S}} A$.

Interpretation of Formulas

- A structure $\mathcal{S} = (D, I)$ is given.
- The value of a formula A under $\mathcal{S}$ is either *true* or *false*:
  - $Val_{\mathcal{S}}(A) = true$ iff $Val_{\mathcal{S}, \sigma_{\mathcal{S}}}(A) = true$ for all $\sigma_{\mathcal{S}}$.
- $\mathcal{S}$ is called a model of A iff $Val_{\mathcal{S}}(A) = true$.
- Written $\models_{\mathcal{S}} A$.

Interpretation of Formulas

- A structure $\mathcal{S} = (D, I)$ is given.
- The value of a formula A under $\mathcal{S}$ is either *true* or *false*:
  - $\mathit{Val}_{\mathcal{S}}(A) = \mathit{true}$ iff $\mathit{Val}_{\mathcal{S}}, \sigma_{\mathcal{S}}(A) = \mathit{true}$ for all $\sigma_{\mathcal{S}}$.
- $\mathcal{S}$ is called a model of A iff $\mathit{Val}_{\mathcal{S}}(A) = \mathit{true}$.
- Written $\models_{\mathcal{S}} A$.

Interpretation of Formulas

- A structure $\mathcal{S} = (D, I)$ is given.
- The value of a formula A under $\mathcal{S}$ is either *true* or *false*:
  - $Val_{\mathcal{S}}(A) = true$ iff $Val_{\mathcal{S}, \sigma_{\mathcal{S}}}(A) = true$ for all $\sigma_{\mathcal{S}}$.
- $\mathcal{S}$ is called a model of A iff $Val_{\mathcal{S}}(A) = true$.
- Written $\models_{\mathcal{S}} A$.

Example

- Formula: $\forall x.(p(x) \rightarrow q(f(x), a))$

Example

- Formula: $\forall x.(p(x) \rightarrow q(f(x), a))$
- Define $\mathcal{S} = (D, I)$ as
  - $D = \{1, 2\}$,
  - $a_I = 1$,
  - $f_I(1) = 2, f_I(2) = 1$,
  - $p_I = \{2\}$,
  - $q_I = \{(1, 1), (1, 2), (2, 2)\}$.

Example

- Formula: $\forall x.(p(x) \rightarrow q(f(x), a))$
- Define $\mathcal{S} = (D, I)$ as
  - $D = \{1, 2\}$,
  - $a_I = 1$,
  - $f_I(1) = 2, f_I(2) = 1$,
  - $p_I = \{2\}$,
  - $q_I = \{(1, 1), (1, 2), (2, 2)\}$.
- If $\sigma_{\mathcal{S}}(x) = 1$, then $Val_{\mathcal{S},\sigma_S}(\forall x.(p(x) \rightarrow q(f(x), a))) = true$.

Example

- Formula: $\forall x.(p(x) \rightarrow q(f(x), a))$
- Define $\mathcal{S} = (D, I)$ as
  - $D = \{1, 2\}$,
  - $a_I = 1$,
  - $f_I(1) = 2, f_I(2) = 1$,
  - $p_I = \{2\}$,
  - $q_I = \{(1, 1), (1, 2), (2, 2)\}$.
- If $\sigma_{\mathcal{S}}(x) = 1$, then $Val_{\mathcal{S},\sigma_S}(\forall x.(p(x) \rightarrow q(f(x), a))) = true$.
- If $\sigma_{\mathcal{S}}(x) = 2$, then $Val_{\mathcal{S},\sigma_S}(\forall x.(p(x) \rightarrow q(f(x), a))) = true$.

Example

- Formula: $\forall x.(p(x) \to q(f(x), a))$
- Define $\mathcal{S} = (D, I)$ as
  - $D = \{1, 2\}$,
  - $a_I = 1$,
  - $f_I(1) = 2, f_I(2) = 1$,
  - $p_I = \{2\}$,
  - $q_I = \{(1, 1), (1, 2), (2, 2)\}$.
- If $\sigma_{\mathcal{S}}(x) = 1$, then $Val_{\mathcal{S},\sigma_S}(\forall x.(p(x) \to q(f(x), a))) = true$.
- If $\sigma_{\mathcal{S}}(x) = 2$, then $Val_{\mathcal{S},\sigma_S}(\forall x.(p(x) \to q(f(x), a))) = true$.
- Hence, $\models_{\mathcal{S}} A$.

Validity, Unsatisfiability

- A formula A is valid, if $\models_{\mathcal{S}} A$ for all $\mathcal{S}$.
- Written $\models A$.
- A formula A is unsatisfiable, if $\models_{\mathcal{S}} A$ for no $\mathcal{S}$.
- If A is valid, then $\neg A$ is unsatisfiable and vice versa.

Validity, Unsatisfiability

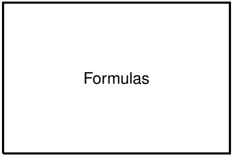
- A formula A is valid, if $\models_{\mathcal{S}} A$ for all $\mathcal{S}$.
- Written $\models A$.
- A formula A is unsatisfiable, if $\models_{\mathcal{S}} A$ for no $\mathcal{S}$.
- If A is valid, then $\neg A$ is unsatisfiable and vice versa.

Validity, Unsatisfiability

- A formula A is valid, if $\models_{\mathcal{S}} A$ for all $\mathcal{S}$.
- Written $\models A$.
- A formula A is unsatisfiable, if $\models_{\mathcal{S}} A$ for no $\mathcal{S}$.
- If A is valid, then $\neg A$ is unsatisfiable and vice versa.

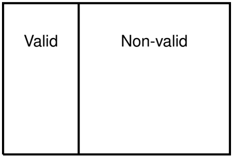
Validity, Unsatisfiability

- A formula A is valid, if $\models_{\mathcal{S}} A$ for all $\mathcal{S}$.
- Written $\models A$.
- A formula A is unsatisfiable, if $\models_{\mathcal{S}} A$ for no $\mathcal{S}$.
- If A is valid, then $\neg A$ is unsatisfiable and vice versa.



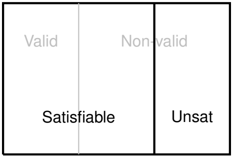
Validity, Unsatisfiability

- A formula A is valid, if $\models_{\mathcal{S}} A$ for all $\mathcal{S}$.
- Written $\models A$.
- A formula A is unsatisfiable, if $\models_{\mathcal{S}} A$ for no $\mathcal{S}$.
- If A is valid, then $\neg A$ is unsatisfiable and vice versa.



Validity, Unsatisfiability

- A formula A is valid, if $\models_{\mathcal{S}} A$ for all $\mathcal{S}$.
- Written $\models A$.
- A formula A is unsatisfiable, if $\models_{\mathcal{S}} A$ for no $\mathcal{S}$.
- If A is valid, then $\neg A$ is unsatisfiable and vice versa.



Validity, Unsatisfiability

- A formula A is valid, if $\models_{\mathcal{S}} A$ for all $\mathcal{S}$.
- Written $\models A$.
- A formula A is unsatisfiable, if $\models_{\mathcal{S}} A$ for no $\mathcal{S}$.
- If A is valid, then $\neg A$ is unsatisfiable and vice versa.

Valid	Non-valid sat	Unsat

Sequent

Definition

A sequent is a pair (Γ, Δ) of finite (possibly empty) sequences $\Gamma = P_1, \ldots, P_n$ and $\Delta = R_1, \ldots, R_m$ of propositional formulas written as $\Gamma \vdash \Delta$ where Γ is called antecedent and Δ succedent or consequent.

Semantics of Sequent

- Comma to the left of the turnstile of a sequent $P_1, \ldots, P_n \vdash R_1, \ldots, R_m$ is thought as "and", and a comma to the right of the turnstile is thought as "or".
- The semantics of a sequent $P_1, \ldots, P_n \vdash R_1, \ldots, R_m$ is an assertion that says whenever $P_1 \wedge \ldots \wedge P_n$ is true then $R_1 \vee \ldots \vee R_m$ is also true.

Rules for Sequent Calculus

$$\frac{}{\Gamma_1, P, \Gamma_2 \vdash \Delta, P}(A) \qquad \frac{}{\Gamma \vdash \Delta, \top}(A\top) \qquad \frac{}{\Gamma_1, \bot, \Gamma_2 \vdash \Delta}(A\bot)$$

$$\frac{\Gamma_1, \Gamma_2 \vdash P, \Delta}{\Gamma_1, \neg P, \Gamma_2 \vdash \Delta}(\neg L) \qquad \frac{\Gamma, P \vdash \Delta_1, \Delta_2}{\Gamma \vdash \Delta_1, \neg P, \Delta_2}(\neg R)$$

Rules for Sequent Calculus

$$\frac{\Gamma_1, P, Q, \Gamma_2 \vdash \Delta}{\Gamma_1, P \wedge Q, \Gamma_2 \vdash \Delta}(\wedge L) \qquad \frac{\Gamma \vdash \Delta_1, P, \Delta_2 \quad \Gamma \vdash \Delta_1, Q, \Delta_2}{\Gamma \vdash \Delta_1, P \wedge Q, \Delta_2}(\wedge R)$$

$$\frac{\Gamma_1, P, \Gamma_2 \vdash \Delta \quad \Gamma_1, Q, \Gamma_2 \vdash \Delta}{\Gamma_1, P \vee Q, \Gamma_2 \vdash \Delta}(\vee L) \qquad \frac{\Gamma \vdash \Delta_1, P, Q, \Delta_2}{\Gamma \vdash \Delta_1, P \vee Q, \Delta_2}(\vee R)$$

$$\frac{\Gamma_1, \Gamma_2 \vdash P, \Delta \quad \Gamma_1, Q, \Gamma_2 \vdash \Delta}{\Gamma_1, P \rightarrow Q, \Gamma_2 \vdash \Delta}(\rightarrow L) \qquad \frac{\Gamma, P \vdash \Delta_1, Q, \Delta_2}{\Gamma \vdash \Delta_1, P \rightarrow Q, \Delta_2}(\rightarrow R)$$

Rules for Sequent Calculus

$$\frac{\Gamma_1, P[t/x], \Gamma_2 \vdash \Delta}{\Gamma_1, \forall x P, \Gamma_2 \vdash \Delta}(\forall L) \qquad \frac{\Gamma \vdash \Delta_1, P[y/x], \Delta_2}{\Gamma \vdash \Delta_1, \forall x P, \Delta_2}(\forall R)$$

$$\frac{\Gamma_1, P[y/x], \Gamma_2 \vdash \Delta}{\Gamma_1, \exists x P, \Gamma_2 \vdash \Delta}(\exists L) \qquad \frac{\Gamma \vdash \Delta_1, P[t/x], \Delta_2}{\Gamma \vdash \Delta_1, \exists x P, \Delta_2}(\exists R)$$

Deduction in the Sequent Calculus

- The set of inference rules generate deduction trees.
- The set of deduction trees is defined inductively as the least set of trees containing all one-node trees.
- The set of proof trees is the set of deduction trees where all one-node trees are labeled with an axiom.
- A proof of a formula P is a proof of the sequent $\vdash P$,

What is Unification

- Unification is an algorithmic process to identify two symbolic expressions by replacing certain subexpressions (variables) by other expressions.
- $f(a,x)$ and $f(y,b)$ terms are identified by replacing the variable x by the term b and the variables y by the term a.
- A unification problem for two terms t and s is represented by $t \doteq s$.

Substitution and Unifier

- Substitution is a mapping from variables, where all but finitely many variables are mapped to themselves. For example, $\sigma = \{x \mapsto b, y \mapsto a\}$ is a substitution.
- A substitution σ is a is a unifier of the unification problem $s \doteq t$ if $t\sigma = s\sigma$. For example, $\sigma = \{x \mapsto b, y \mapsto a\}$ is a matcher of the matching problem $f(a, x) \doteq f(y, b)$.

SLD Resolution Step

- Clause: $P \leftarrow P_1 \wedge \cdots \wedge P_n$
- Goal: $\leftarrow Q_1 \wedge \cdots \wedge Q_m$

Resolvent:
$\leftarrow Q_1\theta \wedge \cdots \wedge Q_{i-1}\theta \wedge P_1\theta \wedge \cdots \wedge P_n\theta \wedge Q_{i+1}\theta \wedge \cdots \wedge Q_m\theta$ where θ is a unifier of $P \doteq Q_i$.

Genealogy Database

$$parent(gia, nika).$$
$$parent(gia, lika).$$
$$parent(dato, gia).$$
$$parent(nato, gia).$$
$$male(gia).$$
$$male(nika).$$
$$male(dato).$$
$$male(lika).$$
$$male(nato).$$
$$father(x, y) \leftarrow parent(x, y) \wedge male(x).$$
$$son(x, y) \leftarrow man(x) \wedge parent(y, x).$$
$$grandfather(x, y) \leftarrow father(x, z) \wedge parent(z, y).$$

Literature

Chin-Liang Chang and Richard C. T. Lee. Symbolic logic and mechanical theorem proving. Computer science classics. Academic Press, 1973.

Melvin Fitting. First-order logic and automated theorem proving. Springer, 2nd edition, 1996.

Gallier, J.H.: Logic for Computer Science: Foundations of Automatic Theorem Proving. Wiley, 1987.