Propositional Logic

Besik Dundua

International Black Sea University
bdundua@gmail.com

Propositional Logic

Propositional Logic

What is a Propositional Logic?

- Propositional logic is a useful tool that allows us to reason about statements.
- Statements are propositions that is either true or false.
- Propositional logic has:
  - A syntax
  - A semantics
  - A proof theory.

Propositional Logic

Alphabet

The alphabet of propositional logic consists:

- Propositional letters: $p, q, r, \ldots$,
- Logical connectives,:
  - nullary connectives $\top$ and $\bot$,
  - unary connective $\neg$,
  - binary connectives $\wedge, \vee, \rightarrow$ and $\leftrightarrow$.
- The punctuation symbols "(" and ")".

Propositional Formulas

The set of propositional formulas (shortly, formulas) $\mathcal{P}$ is defined as follows:

- *Any propositional letter is in* $\mathcal{P}$*.*
- $\top$ *and* $\perp$ *are in* $\mathcal{P}$*.*
- *If* P *and* Q *are in* $\mathcal{P}$ *then* $\neg P, P \wedge Q, P \vee Q, P \rightarrow Q$ *and* $P \leftrightarrow Q$ *are also in* $\mathcal{P}$*. Formulas generated by this item we sometime call compound formulas.*
- *Nothing else is in* $\mathcal{P}$*.*

Examples of Propositional Formulas

Propositional letters together with logical connectives (connectives) are used to to form new expressions.

- p
- $\top$
- $p \vee \bot$
- $\top \vee (p \wedge q)$
- $\neg p \vee q$
- $p \wedge (q \rightarrow \neg r)$
- $(\neg p \vee q) \leftrightarrow (p \rightarrow q)$

Example of Problem Formulation

Suppose that stock prices go down if the prime interest rate goes up. Suppose also that most people are unhappy when stock prices go down. Assume that prime interest rate does go up. Are most people unhappy?

Example of Problem Formulation

Suppose that stock prices go down if the prime interest rate goes up. Suppose also that most people are unhappy when stock prices go down. Assume that prime interest rate does go up. Are most people unhappy?

Formalization:

- p : prime interest rate goes up.
- q : stock prices go down.
- r : most people are unhappy.
- $p \to q$: if the prime interest rate goes up, stock prices go down
- $q \to r$: if stock prices go down, most people are unhappy

Can we show that if $p \to q$, $q \to r$, and p hold, then r holds?

Propositional Logic

Interpretation of Propositional Letter and Constant

- An interpretation $\mathcal{I}$ assigns a truth value to each propositional letter
- The set of truth values is the set $\{true, false\}$

Example

- $\mathcal{I}(p) = true$
- $\mathcal{I}(q) = false$
- $\mathcal{I}(r) = true$

- The logical constant $\top$ is interpreted as true.
- The logical constant $\bot$ is interpreted as false.

Interpretation of Compound Formulas, Informally

- Interpretation of compound formulas is calculated by combining the truth values of propositional letters and constants according to the meaning of the connectives
- In the next slides we describe how connectives combine truth values to obtain a new truth value

Negation

- For any $P \in \mathcal{P}$, $\mathcal{I}(\neg P) = \mathit{true}$ if $\mathcal{I}(P) = \mathit{false}$.
- For any $P \in \mathcal{P}$, $\mathcal{I}(\neg P) = \mathit{false}$ if $\mathcal{I}(P) = \mathit{true}$.

P	$\neg P$
true	false
false	true

Conjunction

- For any $P, Q \in \mathcal{P}$, $\mathcal{I}(P \wedge Q) = \mathit{true}$ if $\mathcal{I}(P) = \mathit{true}$ and $\mathcal{I}(Q) = \mathit{true}$.
- For any $P, Q \in \mathcal{P}$, $\mathcal{I}(P \wedge Q) = \mathit{false}$ if $\mathcal{I}(P) = \mathit{false}$ or $\mathcal{I}(Q) = \mathit{false}$.

P	Q	$P \wedge Q$
true	*true*	*true*
false	*true*	*false*
true	*false*	*false*
false	*false*	*false*

Example

- $\mathcal{I}(p) = \mathit{true}$ and $\mathcal{I}(r) = \mathit{false}$ then $\mathcal{I}(p \wedge \neg r \wedge \top) =$
- $\mathcal{I}(p) = \mathit{true}$ and $\mathcal{I}(r) = \mathit{false}$ then $\mathcal{I}(\neg p \wedge \neg r) =$

Conjunction

- For any $P, Q \in \mathcal{P}$, $\mathcal{I}(P \wedge Q) = \mathit{true}$ if $\mathcal{I}(P) = \mathit{true}$ and $\mathcal{I}(Q) = \mathit{true}$.
- For any $P, Q \in \mathcal{P}$, $\mathcal{I}(P \wedge Q) = \mathit{false}$ if $\mathcal{I}(P) = \mathit{false}$ or $\mathcal{I}(Q) = \mathit{false}$.

P	Q	$P \wedge Q$
true	*true*	*true*
false	*true*	*false*
true	*false*	*false*
false	*false*	*false*

Example

- $\mathcal{I}(p) = \mathit{true}$ and $\mathcal{I}(r) = \mathit{false}$ then $\mathcal{I}(p \wedge \neg r \wedge \top) = \mathit{true}$.
- $\mathcal{I}(p) = \mathit{true}$ and $\mathcal{I}(r) = \mathit{false}$ then $\mathcal{I}(\neg p \wedge \neg r) = \mathit{false}$.

Disjunction

- For any $P, Q \in \mathcal{P}$, $\mathcal{I}(P \vee Q) = \mathit{true}$ if $\mathcal{I}(P) = \mathit{true}$ or $\mathcal{I}(Q) = \mathit{true}$
- For any $P, Q \in \mathcal{P}$, $\mathcal{I}(P \vee Q) = \mathit{false}$ if $\mathcal{I}(P) = \mathit{false}$ and $\mathcal{I}(Q) = \mathit{false}$

P	Q	$P \vee Q$
true	*true*	*true*
false	*true*	*true*
true	*false*	*true*
false	*false*	*false*

Example

- $\mathcal{I}(p) = \mathit{true}$ and $\mathcal{I}(r) = \mathit{true}$ then $\mathcal{I}((\neg p \vee r) \wedge (\neg p \vee \neg r)) =$
- $\mathcal{I}(p) = \mathit{true}$ and $\mathcal{I}(r) = \mathit{true}$ then $\mathcal{I}((p \vee \neg r) \wedge (\neg p \vee r)) =$

Disjunction

- For any $P, Q \in \mathcal{P}$, $\mathcal{I}(P \vee Q) = \mathit{true}$ if $\mathcal{I}(P) = \mathit{true}$ or $\mathcal{I}(Q) = \mathit{true}$
- For any $P, Q \in \mathcal{P}$, $\mathcal{I}(P \vee Q) = \mathit{false}$ if $\mathcal{I}(P) = \mathit{false}$ and $\mathcal{I}(Q) = \mathit{false}$

P	Q	$P \vee Q$
true	*true*	*true*
false	*true*	*true*
true	*false*	*true*
false	*false*	*false*

Example

- $\mathcal{I}(p) = \mathit{true}$ and $\mathcal{I}(r) = \mathit{true}$ then $\mathcal{I}((\neg p \vee r) \wedge (\neg p \vee \neg r)) = \mathit{false}$
- $\mathcal{I}(p) = \mathit{true}$ and $\mathcal{I}(r) = \mathit{true}$ then $\mathcal{I}((p \vee \neg r) \wedge (\neg p \vee r)) = \mathit{true}$

Implication

- For any $P, Q \in \mathcal{P}$, $\mathcal{I}(P \to Q) = \mathit{true}$ if $\mathcal{I}(P) = \mathit{false}$ or $\mathcal{I}(Q) = \mathit{true}$
- For any $P, Q \in \mathcal{P}$, $\mathcal{I}(P \to Q) = \mathit{false}$ if $\mathcal{I}(P) = \mathit{true}$ and $\mathcal{I}(Q) = \mathit{false}$

P	Q	$P \to Q$
true	*true*	*true*
false	*true*	*true*
true	*false*	*false*
false	*false*	*true*

Example

- $\mathcal{I}(p) = \mathit{true}$ and $\mathcal{I}(r) = \mathit{false}$ then $\mathcal{I}((\neg p \to r) \wedge (p \to \neg r)) =$
- $\mathcal{I}(p) = \mathit{true}$ and $\mathcal{I}(r) = \mathit{false}$ then $\mathcal{I}((\neg r \to \neg p) \vee (p \to r)) =$

Implication

- For any $P, Q \in \mathcal{P}$, $\mathcal{I}(P \rightarrow Q) = \mathit{true}$ if $\mathcal{I}(P) = \mathit{false}$ or $\mathcal{I}(Q) = \mathit{true}$
- For any $P, Q \in \mathcal{P}$, $\mathcal{I}(P \rightarrow Q) = \mathit{false}$ if $\mathcal{I}(P) = \mathit{true}$ and $\mathcal{I}(Q) = \mathit{false}$

P	Q	$P \rightarrow Q$
true	*true*	*true*
false	*true*	*true*
true	*false*	*false*
false	*false*	*true*

Example

- $\mathcal{I}(p) = \mathit{true}$ and $\mathcal{I}(r) = \mathit{false}$ then $\mathcal{I}((\neg p \rightarrow r) \wedge (p \rightarrow \neg r)) = \mathit{true}$
- $\mathcal{I}(p) = \mathit{true}$ and $\mathcal{I}(r) = \mathit{false}$ then $\mathcal{I}((\neg r \rightarrow \neg p) \vee (p \rightarrow r)) = \mathit{false}$

Equivalence

- For any $P, Q \in \mathcal{P}$, $\mathcal{I}(P \leftrightarrow Q) = \mathit{true}$ if $\mathcal{I}(P) = \mathcal{I}(Q)$
- For any $P, Q \in \mathcal{P}$, $\mathcal{I}(P \leftrightarrow Q) = \mathit{false}$ if $\mathcal{I}(P) \neq \mathcal{I}(Q)$.

P	Q	$P \leftrightarrow Q$
true	*true*	*true*
false	*true*	*false*
true	*false*	*false*
false	*false*	*true*

Example

- $\mathcal{I}(p) = \mathit{true}$ and $\mathcal{I}(r) = \mathit{false}$ then $\mathcal{I}((p \leftrightarrow \neg r) \leftrightarrow (\neg p \vee r)) =$
- $\mathcal{I}(p) = \mathit{true}$ and $\mathcal{I}(r) = \mathit{false}$ then $\mathcal{I}((p \leftrightarrow r) \leftrightarrow (p \wedge r)) =$

Equivalence

- For any $P, Q \in \mathcal{P}$, $\mathcal{I}(P \leftrightarrow Q) = \mathit{true}$ if $\mathcal{I}(P) = \mathcal{I}(Q)$
- For any $P, Q \in \mathcal{P}$, $\mathcal{I}(P \leftrightarrow Q) = \mathit{false}$ if $\mathcal{I}(P) \neq \mathcal{I}(Q)$.

P	Q	$P \leftrightarrow Q$
true	*true*	*true*
false	*true*	*false*
true	*false*	*false*
false	*false*	*true*

Example

- $\mathcal{I}(p) = \mathit{true}$ and $\mathcal{I}(r) = \mathit{false}$ then $\mathcal{I}((p \leftrightarrow \neg r) \leftrightarrow (\neg p \vee r)) = \mathit{false}$
- $\mathcal{I}(p) = \mathit{true}$ and $\mathcal{I}(r) = \mathit{false}$ then $\mathcal{I}((p \leftrightarrow r) \leftrightarrow (p \wedge r)) = \mathit{true}$

A More Complex Truth Table

p	q	r	$p \to q$	$q \to r$	$((p \to q) \wedge (q \to r) \wedge p) \to r$
true	*true*	*true*	*true*	*true*	*true*
true	*true*	*false*	*true*	*false*	*true*
true	*false*	*true*	*false*	*true*	*true*
true	*false*	*false*	*false*	*true*	*true*
false	*true*	*true*	*true*	*true*	*true*
false	*false*	*true*	*true*	*true*	*true*
false	*true*	*false*	*true*	*false*	*true*
false	*false*	*false*	*true*	*true*	*true*

The table displays the truth values of a formula $((p \to q) \wedge (q \to r) \wedge p) \to r$ for all possible assignments of truth values to atoms occurring in it.

Interpretation of Formulas, Formally

- Lets S be a propositional formula and $p_1, \ldots, p_n$ be atoms occurring in S. Then an interpretation of S is an assignment of truth values to $p_1, \ldots, p_n$ in which every p_i is assigned either true or false, but not both.
- If there are n distinct propositional letters and constants in a formula, then there will be 2^n distinct interpretations for the formula.
- A formula S is said to be true under an interpretation if and only if S is evaluated to true in the interpretation. Otherwise, S is said to be false under the interpretation.

Valid Formulas

A formula is said to be valid if and only if it is true under all its interpretations.

p	q	$p \to q$	$(p \to q) \land p$	$((p \to q) \land p) \to q$
true	*true*	*true*	*true*	*true*
true	*false*	*false*	*false*	*true*
false	*true*	*true*	*false*	*true*
false	*false*	*true*	*false*	*true*

Inconsistent Formulas

A formula is said to be inconsistent if and only if it is false under all its interpretations.

p	q	$\neg q$	$p \to q$	$p \wedge \neg q$	$(p \to q) \wedge (p \wedge \neg q)$
true	*true*	*false*	*true*	*false*	*false*
true	*false*	*true*	*false*	*true*	*false*
false	*true*	*false*	*true*	*false*	*false*
false	*false*	*true*	*true*	*false*	*false*

Invalid and Consistent Formulas

- A formula is invalid if and only if there is at least one interpretation under which the formula is false.
- A formula is consistent if and only if there is at least one interpretation under which the formula is true.

p	$p \to \neg p$
true	*false*
false	*true*

Logical Equivalences

- Two formulas P and Q are said to be logically equivalent denoted $P \Leftrightarrow Q$, if and only if the truth values of P and Q are the same under every interpretation of P and Q.
- Notice that, $P \Leftrightarrow Q$ if and only if $P \leftrightarrow Q$ is valid.

Example

We show $p \to q \Leftrightarrow \neg p \vee q$:

p	q	$p \to q$	$\neg p \vee q$
true	*true*	*true*	*true*
true	*false*	*false*	*false*
false	*true*	*true*	*true*
false	*false*	*true*	*true*

Some Useful Logical Equivalences

$$P \vee \top \Leftrightarrow \top \tag{1}$$
$$P \vee \bot \Leftrightarrow P \tag{2}$$
$$P \wedge \top \Leftrightarrow P \tag{3}$$
$$P \wedge \bot \Leftrightarrow \bot \tag{4}$$
$$P \wedge P \Leftrightarrow P \tag{5}$$
$$P \vee \neg P \Leftrightarrow \top \tag{6}$$
$$P \wedge \neg P \Leftrightarrow \bot \tag{7}$$
$$\neg(\neg P) \Leftrightarrow P \tag{8}$$
$$\neg \top \Leftrightarrow \bot \tag{9}$$
$$\neg \bot \Leftrightarrow \top \tag{10}$$

Some Useful Logical Equivalences. Cont.

$$P \leftrightarrow Q \Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P) \tag{11}$$

$$P \rightarrow Q \Leftrightarrow \neg P \vee Q \tag{12}$$

$$(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R) \tag{13}$$

$$(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R) \tag{14}$$

$$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R) \tag{15}$$

$$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R) \tag{16}$$

$$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q \tag{17}$$

$$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q \tag{18}$$

$$P \vee Q \Leftrightarrow Q \vee P \tag{19}$$

$$P \wedge Q \Leftrightarrow Q \wedge P \tag{20}$$

Logical Consequence

Definition

A formula P is a logical consequence of formulas $S_1, \ldots, S_n$ if and only if for any interpretation $\mathcal{I}$ in which $S_1 \wedge \cdots \wedge S_n$ is *true*, P is also *true*.

Theorem

A formula P is a logical consequence of formulas $S_1, \ldots, S_n$ if and only if the formula $(S_1 \wedge \cdots \wedge S_n) \rightarrow P$ is valid.

Example of Logical Consequence

> *Suppose that stock prices go down if the prime interest rate goes up. Suppose also that most people are unhappy when stock prices go down. Assume that prime interest rate does go up. Are most people unhappy?*

- p : prime interest rate goes up.
- q : stock prices go down.
- r : most people are unhappy.
- $p \to q$: if the prime interest rate goes up, stock prices go down
- $q \to r$: if stock prices go down, most people are unhappy

We need to show r is logical consequence of $p \to q \wedge q \to r \wedge p$. That means, we have to prove $((p \to q) \wedge (q \to r) \wedge p) \to r$ is valid.

Propositional Logic

Sequent

Definition

A sequent is a pair (Γ, Δ) of finite (possibly empty) sequences $\Gamma = P_1, \ldots, P_n$ and $\Delta = R_1, \ldots, R_m$ of propositional formulas written as $\Gamma \vdash \Delta$ where Γ is called antecedent and Δ succedent or consequent.

Semantics of Sequent

- Comma to the left of the turnstile of a sequent $P_1, \ldots, P_n \vdash R_1, \ldots, R_m$ is thought as "and", and a comma to the right of the turnstile is thought as "or".
- The semantics of a sequent $P_1, \ldots, P_n \vdash R_1, \ldots, R_m$ is an assertion that says whenever $P_1 \wedge \ldots \wedge P_n$ is true then $R_1 \vee \ldots \vee R_m$ is also true.

Rules for Sequent Calculus

$$\frac{}{\Gamma_1, P, \Gamma_2 \vdash \Delta, P}(A) \qquad \frac{}{\Gamma \vdash \Delta, \top}(A\top) \qquad \frac{}{\Gamma_1, \bot, \Gamma_2 \vdash \Delta}(A\bot)$$

$$\frac{\Gamma_1, \Gamma_2 \vdash P, \Delta}{\Gamma_1, \neg P, \Gamma_2 \vdash \Delta}(\neg L) \qquad \frac{\Gamma, P \vdash \Delta_1, \Delta_2}{\Gamma \vdash \Delta_1, \neg P, \Delta_2}(\neg R)$$

Rules for Sequent Calculus

$$\frac{\Gamma_1, P, Q, \Gamma_2 \vdash \Delta}{\Gamma_1, P \wedge Q, \Gamma_2 \vdash \Delta}(\wedge L) \qquad \frac{\Gamma \vdash \Delta_1, P, \Delta_2 \quad \Gamma \vdash \Delta_1, Q, \Delta_2}{\Gamma \vdash \Delta_1, P \wedge Q, \Delta_2}(\wedge R)$$

$$\frac{\Gamma_1, P, \Gamma_2 \vdash \Delta \quad \Gamma_1, Q, \Gamma_2 \vdash \Delta}{\Gamma_1, P \vee Q, \Gamma_2 \vdash \Delta}(\vee L) \qquad \frac{\Gamma \vdash \Delta_1, P, Q, \Delta_2}{\Gamma \vdash \Delta_1, P \vee Q, \Delta_2}(\vee R)$$

$$\frac{\Gamma_1, \Gamma_2 \vdash P, \Delta \quad \Gamma_1, Q, \Gamma_2 \vdash \Delta}{\Gamma_1, P \rightarrow Q, \Gamma_2 \vdash \Delta}(\rightarrow L) \qquad \frac{\Gamma, P \vdash \Delta_1, Q, \Delta_2}{\Gamma \vdash \Delta_1, P \rightarrow Q, \Delta_2}(\rightarrow R)$$

Deduction in the Sequent Calculus

- The set of inference rules generate deduction trees.
- The set of deduction trees is defined inductively as the least set of trees containing all one-node trees.
- The set of proof trees is the set of deduction trees where all one-node trees are labeled with an axiom.
- A proof of a formula P is a proof of the sequent $\vdash P$,

Conjunction Introduction

$$\frac{P \qquad Q}{P \wedge Q}$$

This rule says that if we know P and we know Q then we can conclude $P \wedge Q$.

Conjunction Elimination

$$\frac{P \wedge Q}{P} \qquad \frac{P \wedge Q}{Q}$$

This is the inverse operation of the previous one and says that if we know $P \wedge Q$ we know P and we know Q.

Implication Introduction

$$\frac{\begin{array}{c} P \\ \vdots \\ Q \end{array}}{P \to Q}$$

If supposing P made Q to hold, then we can assure $P \to Q$ to holds. Note, that P is supposed and then removed so that $P \to Q$ does not depend on P.

Implication Elimination

$$\frac{P \qquad P \to Q}{Q}$$

If we know P and $P \to Q$ to hold, then we can assure Q to holds.

Disjunction Introduction

$$\frac{P}{P \vee Q} \qquad \frac{Q}{P \vee Q}$$

The introduction rules are straightforward. If we know "today is a sunny day" we also know "today is a sunny or rainy day".

Disjunction Elimination

$$\frac{P \vee Q \qquad \begin{array}{c} P \\ \vdots \\ R \end{array} \qquad \begin{array}{c} Q \\ \vdots \\ R \end{array}}{R}$$

The elimination rule is a little more complex, because if we know "today is a sunny or rainy day" what can we deduce from it? That it is sunny day? That it is rainy day? We need more information to eliminate $\vee$. In particular, if we manage to show $P \to R$ and $Q \to R$, then we can derive R from $P \vee Q$.

Negation Introduction

$$\frac{\begin{array}{cc} P & P \\ \vdots & \vdots \\ Q & \neg Q \end{array}}{\neg P}$$

When both Q and $\neg Q$ can be achieved, we have contradiction and conclude P does not holds, which means $\neg P$ holds.

Negation Elimination

$$\frac{\neg\neg P}{P}$$

When we have negation of negation of a propositional formula, we can eliminate these two negations.

Literature

Chin-Liang Chang and Richard C. T. Lee. Symbolic logic and mechanical theorem proving. Computer science classics. Academic Press, 1973.

Melvin Fitting. First-order logic and automated theorem proving. Springer, 2nd edition, 1996.

Gallier, J.H.: Logic for Computer Science: Foundations of Automatic Theorem Proving. Wiley, 1987.