VIETNAM GENERAL CONFEDERATION OF LABOR
TON DUC THANG UNIVERSITY
FACULTY OF INFORMATION TECHNOLOGY

NGÔ CHÍ THUẬN – 523H0102

FINAL REPORT

APPLIED LINEAR ALGEBRA FOR IT

HO CHI MINH CITY, 2024

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Advised by
Master Phạm Kim Thủy

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Ho Chi Minh city, 17th May 2024.

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CHAPTER 1. THEORETICAL BASIS

1.1 Content 1

1.1.1 Cofactor Expansion

The determinant of A is defined as

$$\det(A) = \begin{cases} a_{11} & if\ n = 1 \\ a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n} & if n \geq 2 \end{cases}$$

Where $A_{ij} = (-1)^{i+j} det(M_{ij})$.

$A_{ij} = (-1)^{i+j} det(M_{ij})$ is called the (i, j)-cofactor of A.

This is known as cofactor expansion.

1.2 Content 2

1.2.1 Elementary Row Operations

The three operations

1) Multiply a row by a non-zero constant
2) Interchanging two rows
3) Adding a multiple of one row to another row

Performed on an augmented matrix are called elementary row operations

1.2.2 Row-echelon Form

An augmented matrix is said to be in row-echelon form if it has the following two properties.

1) If there are any rows consisting entirely of zeros, the they are grouped at the bottom of the matrix.
2) In any two successive rows that are not entirely zeros, the first non-zero number in the lower row occurs further to the right than the first non-zero number in the higher row.

The first non-zero number in every row is called the leading entry of that row.

1.2.3 Gaussian Elimination

Gaussian Elimination is an algorithm (systematic way of doing things) to reduce an augmented matrix to a row-echelon form using elementary row operations.

1.2.4 Gauss-Jordan Elimination

Gauss-Jordan Elimination is an algorithm to reduce an augmented matrix to the reduced row-echelon form using elementary row operations.

1.2.5 Linear Combination

Let $u_1, u_2, \dots, u_k$be vectors in $\mathbb{R}^n$.

For any real numbers $c_1, c_2, \dots, c_k$, the vectors c is a linear combination of $u_1, u_2, \dots, u_k$.

1.3 Content 3

1.3.1 Linear Independence

Let $S = \{u_1, u_2, \dots, u_k\} \subseteq \mathbb{R}^n$. Consider the solutions to the following equation (values of $c_1, c_2, \dots, c_k$)

$$c_1 u_1 + c_2 u_2 + \cdots + c_k u_k = 0 \ (*)$$

1) Clearly, $c_1 = 0, c_2 = 0, \dots, c_k = 0$ is a solution. This is called the trivial solution to (*)
2) S is called a linearly independent set if (*) has only the trivial solution. In this case, we say that $u_1, u_2, \dots, u_k$ are linearly independent vectors.
3) S is called a linearly dependent set if (*) has non- trivial solutions. In this case, we say that $u_1, u_2, \dots, u_k$ are linearly dependent vectors.

1.3.2 Linear Span

Let $S = \{u_1, u_2, \dots, u_k\}$ be a set of vectors in $\mathbb{R}^n$.

The set of all linear combinations of $u_1, u_2, \dots, u_k$, $\{c_1 u_1 + c_2 u_2 + \cdots + c_k u_k \mid c_1, c_2, \dots, c_k \in \mathbb{R}\}$ is called the linear span of S (or linear span of $u_1, u_2, \dots, u_k$).

This set is denoted by $span(S)$ or $span\{u_1, u_2, \dots, u_k\}$.

1.3.3 Basis

Let $S = \{u_1, u_2, \dots, u_k\}$ be a subset of a vector space V.

Then S is called a basis (plural bases) for V if

1. S is linearly independent and
2. $S\ spans\ V$.

1.4 Content 4

1.4.1 Eigenvalues and Eigenvectors

Let A be a square matrix of order n.

A non-zero column vector $u \in \mathbb{R}^n$ is called an eigenvector of A if $Au = \lambda u$ for some scalar λ.

The scalar λ is called an eigenvalue of A and u is said to be an eigenvector of A associated with the eigenvalue λ.

1.4.2 Characteristic Polynomial and Characteristic Equation

Let A be a square matrix of order n.

The polynomial $det(\lambda I - A)$ is called the characteristic polynomial of A

$$\mathrm{D}et(\lambda I - A) = c_n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0;$$

and the equation $det(\lambda I - A) = 0$ is called the characteristic equation of A.

$$c_n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0 = 0$$

1.4.3 Eigenspace

Let A be a square matrix of order n and λ an eigenvalue of A.

$(\lambda I - A)x = 0$: homogeneous linear system with coefficient matrix $(\lambda I - A)$

The solution space of $(\lambda I - A)x = 0$ is called the eigenspace of A associated with λ and is denoted by E_λ.

1.4.4 Column Space

Let A be a $m \times n$ matrix. Then the column space of A is

$$\left\{ A \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \middle| u_1, u_2, \dots, u_n \epsilon \mathbb{R} \right\} = \{Au \mid u \in \mathbb{R}^n\}$$

A system of linear equations $Ax = b$ is consistent if and only if b lies in the column space of A.

1.4.5 Diagonalization

Given a square matrix A, we wanted to know if it is possible to find an invertible matrix P such that

$$P^{-1}AP = D \text{ (a diagonal matrix)}$$

A square matrix A is called diagonalizable if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. In here, matrix P is said to diagonalize A.

1.5 Content 5

1.5.1 Coordinate vector

Let $S = \{u_1, u_2, u_3\}$ be a basis for a vector space V and v be a vector in V. If

$$v = c_1u_1 + c_2u_2 + \cdots + c_ku_k$$

Then the coefficients $c_1, c_2, \dots, c_k$ are called the coordinates of v relative to the basis S.

The vector

$$[v]_s = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} \text{(belonging to } \mathbb{R}^k\text{)}$$

Is called the coordinate vector of v relative to the basis S.

1.6 Content 6

1.6.1 Gram-Schmidt process

Let $\{u_1, u_2, \dots, u_k\}$ be a basis for a vector space V.

Let $v_1 = u_1$;

$$v_2 = u_2 - \frac{u_2 v_1}{\|v_1\|} v_1; \qquad v_3 = u_3 - \left(\frac{u_3 v_1}{\|v_1\|^2} v_1 + \frac{u_3 v_2}{\|v_2\|^2} v_2\right)$$

$$\vdots$$

$$v_k = u_k - \left(\frac{u_k v_1}{\|v_1\|^2} v_1 + \frac{u_k v_2}{\|v_2\|^2} v_2 + \cdots + \frac{u_k v_{k-1}}{\|v_{k-1}\|^2} v_{k-1}\right)$$

Then $\{v_1, v_2, \dots, v_k\}$ is an orthogonal basis for V. $\left\{\frac{1}{\|v_1\|} v_1, \frac{1}{\|v_2\|} v_2, \dots, \frac{1}{\|v_k\|} v_k\right\}$ is an orthonormal basis for V.

1.7 Content 7

1.7.1 Transition matrix

Let S and T be two bases of a vector space and let P be the transition matrix from S to T. Then

1) P is invertible; and
2) P^{-1} is the transition matrix from T to S.

CHAPTER 2. SOLUTIONS

2.1 Question 1

Given the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 1 \\ 1 & 2 & a \end{bmatrix}$. Find all values of a for which $det(A) = 0$.

Solve

Apply Cofactor expansion, we have:

$$det(A) = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 2 & 1 \\ 1 & 2 & a \end{vmatrix}$$

$$= 1(-1)^{1+1} \begin{vmatrix} 2 & 1 \\ 2 & a \end{vmatrix} + 2(-1)^{1+2} \begin{vmatrix} 2 & 1 \\ 1 & a \end{vmatrix} + (-1)(-1)^{1+3} \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix}$$

$$= 1 \times (2a - 2) + (-2) \times (2a - 1) + (-1) \times (4 - 2)$$

$$= 2a - 2 - 4a + 2 - 4 + 2 = -2a - 2$$

But $det(A) = 0 \Rightarrow -2a - 2 = 0 \Rightarrow -2a = 2 \Rightarrow a = -1$.

So, the value of a for which $det(A) = 0$ is $a = -1$

2.2 Question 2

Solve the following system of linear equations by using Gaussian Elimination method.

a) $\begin{cases} x + 5y - 2z = 4 \\ 3x - y + z = 3 \\ 5x + y - 2z = 4 \end{cases}$ b) $\begin{cases} x + 3y - z = 3 \\ x - 2y + 2z = 4 \\ 2x + y + z = 7 \end{cases}$

<u>Solve</u>

a) $\begin{cases} x + 5y - 2z = 4 \\ 3x - y + z = 3 \\ 5x + y - 2z = 4 \end{cases}$

We have augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 5 & -2 & 4 \\ 3 & -1 & 1 & 3 \\ 5 & 1 & -2 & 4 \end{array}\right] \xrightarrow[R_3 - 5R_1 \to R_3]{R_2 - 3R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 5 & -2 & 4 \\ 0 & -16 & 7 & -9 \\ 0 & -24 & 8 & -16 \end{array}\right] \xrightarrow{16R_3 - 24R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & 5 & -2 & 4 \\ 0 & -16 & 7 & -9 \\ 0 & 0 & -40 & -40 \end{array}\right]$$

We have system of equations:

$$\begin{cases} x + 5y - 2z = 4 \\ -16y + 7z = -9 \\ -40z = -40 \end{cases} \Rightarrow \begin{cases} x + 5y - 2 = 4 \\ -16y + 7 = -9 \\ z = 1 \end{cases} \Rightarrow \begin{cases} x + 5 - 2 = 4 \\ y = 1 \\ z = 1 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 1 \\ z = 1 \end{cases}$$

b) $\begin{cases} x + 3y - z = 3 \\ x - 2y + 2z = 4 \\ 2x + y + z = 7 \end{cases}$

We have augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 3 \\ 1 & -2 & 2 & 4 \\ 2 & 1 & 1 & 7 \end{array}\right] \xrightarrow[R_3 - 2R_1 \to R_3]{R_2 - R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 3 & -1 & 3 \\ 0 & -5 & 3 & 1 \\ 0 & -5 & 3 & 1 \end{array}\right] \xrightarrow{R_3 - R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & 3 & -1 & 3 \\ 0 & -5 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We have system of equations:

$$\begin{cases} x + 3y - z = 3 \\ -5y + 3z = 1 \end{cases} \Rightarrow \begin{cases} x + 3y - a = 3 \\ -5y + 3a = 1 \\ z = a \end{cases} \Rightarrow \begin{cases} x + 3\left(\dfrac{-1 + 3a}{5}\right) - a = 3 \\ y = \dfrac{-1 + 3a}{5} \\ z = a \end{cases}$$

$$\Rightarrow \begin{cases} x = 3 - 3\left(\dfrac{-1 + 3a}{5}\right) + a \\ y = \dfrac{-1 + 3a}{5} \\ z = a \end{cases} \quad (a \in \mathbb{R})$$

2.3 Question 3

Let $v_1 = (1;1;1), v_2 = (2;-5;1), v_3 = (3;0;5)$. Show that the set $B = \{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$.

<u>Solve</u>

We have: $|B| = dim(\mathbb{R}^3) = 3$

$\Rightarrow B\ spans\ \mathbb{R}^3$ (1)

Consider linear combination:

$$\sum_{i=1}^{3} C_i U_i = 0$$

We have augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & -5 & 0 & 0 \\ 1 & 1 & 5 & 0 \end{array}\right] \Rightarrow \begin{cases} C_1 = 0 \\ C_2 = 0 \\ C_3 = 0 \end{cases}$$

$\Rightarrow B$ is linear independence (2)

From (1) and (2) $\Rightarrow$ set B is a basis of $\mathbb{R}^3$.

2.4 Question 4

Find a matrix P that diagonalizes $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

<u>Solve</u>

Matrix $A \epsilon \mathcal{M}_3 (n = 3)$

We have:

$$(\lambda I_3 - A) = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -1 \\ 0 & 0 & \lambda - 1 \end{bmatrix}$$

Characteristic polynomial of A:

$$det(\lambda I_3 - A) = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -1 \\ 0 & 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^3 + 0 + 0 - 0 - 0 - 4(\lambda - 1)$$

$$= (\lambda - 1)^3 - 4(\lambda - 1)$$

And characteristic equation of A is $det(\lambda I_3 - A) = 0$

$\Rightarrow (\lambda-1)^3-4(\lambda-1)=0$

$$\Rightarrow \begin{cases} (\lambda_1-1)=2 \\ (\lambda_2-1)=-2 \\ (\lambda_3-1)=0 \end{cases} \Rightarrow \begin{cases} \lambda_1=3 \\ \lambda_2=-1 \\ \lambda_3=1 \end{cases}$$

With $\lambda_1=3$, we have $x=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \epsilon\mathbb{R}^3$is an eigen vector of A, that

$(3I_3-A)_x=0$

We have:

$$\begin{bmatrix} 2 & -2 & -2 \\ -2 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_2+R_1\to R_2} \begin{bmatrix} 2 & -2 & -2 \\ 0 & 0 & -3 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{3R_3+2R_2\to R_3} \begin{bmatrix} 2 & -2 & -2 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Longrightarrow \begin{cases} 2x_1-2x_2-2x_3=0 \\ -3x_3=0 \end{cases}$$

$$\Longrightarrow \begin{cases} x_1=a \\ x_2=a \\ x_3=0 \end{cases} (a\in\mathbb{R}) \Longrightarrow x=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} a \\ a \\ 0 \end{bmatrix}=a\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$E_3=\left\{a\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\middle| a\in\mathbb{R}\backslash\{0\}\right\} \Longrightarrow span\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$$

A basis of E_3 is $S_3=\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$

With $\lambda_2=-1$, we have $y=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \epsilon\mathbb{R}^3$is an eigen vector of A, that

$(-I_3-A)_y=0$

We have:

$$\begin{bmatrix} -2 & -2 & -2 \\ -2 & -2 & -1 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{R_2-R_1\longrightarrow R_2} \begin{bmatrix} -2 & -2 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{R_3-2R_2\longrightarrow R_3} \begin{bmatrix} -2 & -2 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Longrightarrow \begin{cases} -2y_1-2y_2-2y_3=0 \\ -y_3=0 \end{cases} \Longrightarrow \begin{cases} y_1=-a \\ y_2=a\ (a\in\mathbb{R}) \\ y_3=0 \end{cases}$$

$$\Longrightarrow y=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} -a \\ a \\ 0 \end{bmatrix}=a\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$E_{-1} = \left\{ a \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \middle| a \in \mathbb{R} \setminus \{0\} \right\} \Longrightarrow span \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

A basis of E_{-1} is $S_{-1} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

With $\lambda_3 = 1$, let $z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ is an eigen vector of A that $(I_3 - A)_z = 0$

We have:

$$\begin{bmatrix} 0 & -2 & -2 \\ -2 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{cases} -2z_2 - 2z_3 = 0 \\ -2z_1 - z_3 = 0 \end{cases} \Rightarrow \begin{cases} z_1 = -\dfrac{a}{2} \\ z_2 = -a \\ z_3 = a \end{cases} (a \in \mathbb{R})$$

$$\Rightarrow z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} -\frac{a}{2} \\ -a \\ a \end{bmatrix} = a \begin{bmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{bmatrix}$$

$$E_1 = \left\{ a \begin{bmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{bmatrix} \middle| a \in \mathbb{R} \setminus \{0\} \right\} \Longrightarrow span \left\{ \begin{bmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{bmatrix} \right\}$$

A basis of E_1 is $S_1 = \left\{ \begin{bmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{bmatrix} \right\}$

Let $S = S_3 \cup S_1 \cup S_{-1} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{bmatrix} \right\}$

$\Rightarrow |S| = n = 3 \Longrightarrow$ Matrix A is diagonalizable and $P = \begin{bmatrix} 1 & -1 & -\frac{1}{2} \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ is a matrix that diagonalizes matrix A.

2.5 Question 5

Let $S = \{v_1 = (2; 4; 3), v_2 = (2; 4; 2), v_3 = (-6; 4; 2)\}$. Find the coordinate vector of $v = (54, 12, 9)$ relative to S.

<u>Solve</u>

We have $|S| = \dim(\mathbb{R}^3) = 3$

$\Rightarrow S\ spans\ \mathbb{R}^3$ (1)

Consider linear combination:

$$\sum_{i=1}^{3} C_i U_i = 0$$

We have augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & 2 & -6 & 0 \\ 4 & 4 & 4 & 0 \\ 3 & 2 & 2 & 0 \end{array}\right] \Rightarrow \begin{cases} C_1 = 0 \\ C_2 = 0 \\ C_3 = 0 \end{cases}$$

$\Rightarrow S$ is linear independence (2)

From (1) and (2) $\Rightarrow$ set S is a basis of $\mathbb{R}^3$

$$(54, 12, 9) = a_1(2,4,3) + a_2(2,4,2) + a_3(-6,4,2)$$
$$= (2a_1, 4a_1, 3a_1) + (2a_2, 4a_2, 2a_2) + (-6a_3, 4a_3, 2a_3)$$

We have system of equations:

$$\begin{cases} 2a_1 + 2a_2 - 6a_3 = 54 \\ 4a_1 + 4a_2 + 4a_3 = 12 \\ 3a_1 + 2a_2 + 2a_3 = 9 \end{cases} \Longrightarrow \begin{cases} a_1 = 3 \\ a_2 = 6 \\ a_3 = -6 \end{cases}$$

$$\Rightarrow [v]_S = \begin{bmatrix} 3 \\ 6 \\ -6 \end{bmatrix}.$$

2.6 Question 6

Use the Gram-Schmidt orthonormalization process to transform the basis $S = \{u_1 = (4;-8;8), u_2 = (8;8;4), u_3 = (8;-4;-8)\}$ for the $\mathbb{R}^3$ into an orthonormal basis.

<u>Solve</u>

Let $v_1 = u_1 = (4, -8{,}8)$

We have:

$u_2 v_1 = 8(4) + 8(-8) + 4(8) = 0$

$\|v_1\| = \sqrt{4^2 + (-8)^2 + 8^2} = 12$

Let $v_2 = (8, 8,4) - \frac{0}{12^2}(4, -8,8) = (8,8,4)$

We have:

$u_3 v_1 = 8(4) + (-4)(-8) + (-8)(8) = 0$

$u_3 v_2 = 8(8) + (-4)(8) + (-8)(4) = 0$

$\|v_2\| = \sqrt{8^2 + 8^2 + 4^2} = 12$

Let $v_3 = (8, -4, -8) - \frac{0}{12^2}(4, -8,8) - \frac{0}{12^2}(8,8,4) = (8, -4, -8)$

$\Rightarrow$ An orthogonal basis for $\mathbb{R}^3$ is $V = \{v_1, v_2, v_3\}$

We have:

$\|v_3\| = \sqrt{8^2 + (-4)^2 + (-8)^2} = 12$

Let:

$w_1 = \frac{1}{12}(4, -8,8)$

$w_2 = \frac{1}{12}(8,8,4)$

$w_3 = \frac{1}{12}(8, -4, -8)$

$\Rightarrow$ An orthonormal basis for $\mathbb{R}^3$ is $W = \{w_1, w_2, w_3\}$.

2.7 Question 7

Consider the vector space $\mathbb{R}^3$ with two bases:

$\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$ in which $\mathcal{E}_1 = (1, 0, 0)$, $\mathcal{E}_2 = (0, 1, 0)$, $\mathcal{E}_3 = (0, 0, 1)$

$\theta = \{\theta_1, \theta_2, \theta_3\}$ in which $\theta_1 = (1, 1, 0)$, $\theta_2 = (0, 1, 1)$, $\theta_3 = (1, 0, 1)$

a) Find the transition matrix from the basis $\mathcal{E}$ to the basis θ.

b) Find the transition matrix from the basis θ to the basis $\mathcal{E}$.

<u>Solve</u>

a) Consider linear combination:

$$\varepsilon_1 = \sum_{i=1}^{3} a_i u_i\,; \qquad \varepsilon_2 = \sum_{i=1}^{3} b_i u_i\,;$$

$$\varepsilon_3 = \sum_{i=1}^{3} c_i u_i$$

We have augmented matrix:

$$\left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$\Rightarrow \begin{cases} a_1 = 1 \\ a_2 = 1 \\ a_3 = 0 \end{cases}; \qquad \begin{cases} b_1 = 0 \\ b_2 = 1 \\ b_3 = 1 \end{cases}; \qquad \begin{cases} c_1 = 1 \\ c_2 = 0 \\ c_3 = 1 \end{cases};$$

$$\Rightarrow \begin{cases} [\theta_1]_\varepsilon = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ [\theta_2]_\varepsilon = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\ [\theta_3]_\varepsilon = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \end{cases}$$

$\Rightarrow$ the transition matrix from the basis $\mathcal{E}$ to the basis θ:

$$P_{\mathcal{E}\to\theta} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

b) Consider linear combination:

$$\varepsilon_1 = \sum_{i=1}^{3} a_i u_i ; \qquad \varepsilon_2 = \sum_{i=1}^{3} b_i u_i ;$$

$$\varepsilon_3 = \sum_{i=1}^{3} c_i u_i$$

We have augmented matrix:

$$\left[\begin{array}{ccc|c|c|c} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \begin{cases} a_1 = \dfrac{1}{2} \\ a_2 = -\dfrac{1}{2} \\ a_3 = \dfrac{1}{2} \end{cases}; \qquad \begin{cases} b_1 = \dfrac{1}{2} \\ b_2 = \dfrac{1}{2} \\ b_3 = -\dfrac{1}{2} \end{cases}; \qquad \begin{cases} c_1 = -\dfrac{1}{2} \\ c_2 = \dfrac{1}{2} \\ c_3 = \dfrac{1}{2} \end{cases};$$

$$\Rightarrow \begin{cases} [\epsilon_1]_\theta = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \\ [\varepsilon_2]_\theta = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \\ [\varepsilon_3]_\theta = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \end{cases}$$

$\Rightarrow$ the transition matrix from the basis θ to the basis Ɛ:

$$P_{\mathcal{E}\to\theta} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

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