38/ We have: $(\lambda I_3 - A) = \cancel{A}\ \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{pmatrix}$

$$= \begin{pmatrix} \lambda + 1 & -4 & 2 \\ 3 & \lambda - 4 & 0 \\ 3 & -1 & \lambda - 3 \end{pmatrix}$$

Characteristic polynominal of A is:

$$\det(\lambda I_3 - A) = \begin{vmatrix} \lambda + 1 & -4 & 2 \\ 3 & \lambda - 4 & 0 \\ 3 & -1 & \lambda - 3 \end{vmatrix}$$

$$= (\lambda + 1).(\lambda - 4).(\lambda - 3) - 6 - 6.(\lambda - 4) + 12.(\lambda - 3)$$

$$= \lambda^3 - 6\lambda^2 + 11\lambda - 6$$

$$\Rightarrow \left[\begin{array}{l} x_1 = 1 \\ x_2 = 3 \\ x_3 = 2 \end{array}\right. \Rightarrow \lambda_1 = 1,\ \lambda_2 = 3,\ \lambda_3 = 2 \text{ are eigen values of } A$$

With $\lambda_1 = 1$, let $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$ is an eigen vector of A, that $(1I_3 - A)x = 0$

We have:

$$\begin{pmatrix} 2 & -4 & 2 \\ 3 & -3 & 0 \\ 3 & -1 & -2 \end{pmatrix} \xrightarrow[2R_3 - 3R_1 \to R_3]{2R_2 - 3R_1 \to R_2} \begin{pmatrix} 2 & -4 & 2 \\ 0 & 6 & -6 \\ 0 & 10 & -10 \end{pmatrix}$$

$$\xrightarrow{R_3 - \frac{10}{6}R_2 \to R_3} \begin{pmatrix} 2 & -4 & 2 \\ 0 & 6 & -6 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 2x_1 - 4x_2 + 2x_3 = 0 \\ 6x_2 - 6x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = a \\ x_2 = a \\ x_3 = a \end{cases}, a \in \mathbb{R} \setminus \{0\}$$

$$\Rightarrow x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow E_1 = \left\{ a\begin{pmatrix}1\\1\\1\end{pmatrix} \,\middle|\, a \in \mathbb{R}\setminus\{0\} \right\} = \operatorname{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$$

$\Rightarrow$ The basis of E_1 is $S_1 = \left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$

With $\lambda_2 = 3$, let $y = \begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix} \in \mathbb{R}^3 \setminus \{0\}$ is an eigen vector of A, that $(3I_3 - A)y = 0$

We have:

$$\begin{pmatrix}4 & -4 & 2\\ 3 & -1 & 0\\ 3 & -1 & 0\end{pmatrix} \xrightarrow[4R_3 - 3R_1 \to R_3]{4R_2 - 3R_1 \to R_2} \begin{pmatrix}4 & -4 & 2\\ 0 & 8 & -6\\ 0 & 8 & -6\end{pmatrix}$$

$$\xrightarrow{R_3 - R_2 \to R_3} \begin{pmatrix}4 & -4 & 2\\ 0 & 8 & -6\\ 0 & 0 & 0\end{pmatrix}$$

$$\Rightarrow \begin{cases}4y_1 - 4y_2 + 2y_3 = 0\\ 8y_2 - 6y_3 = 0\\ \cancel{2y_3 = 0}\end{cases} \Rightarrow \begin{cases}y_1 = \frac{1}{4}a\\ y_2 = \frac{3}{4}a\\ y_3 = a\end{cases},\ a \in \mathbb{R}\setminus\{0\}$$

$$\Rightarrow y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = a \begin{pmatrix} 1/4 \\ 3/4 \\ 1 \end{pmatrix}$$

$$\Rightarrow E_2 = \left\{ a \begin{pmatrix} 1/4 \\ 3/4 \\ 1 \end{pmatrix} \middle| a \in \mathbb{R} \setminus \{0\} \right\} = \text{Span}\left\{ \begin{pmatrix} 1/4 \\ 3/4 \\ 1 \end{pmatrix} \right.$$

$\Rightarrow$ The basis of E_2 is $S_2 = \left\{ \begin{pmatrix} 1/4 \\ 3/4 \\ 1 \end{pmatrix} \right\}$

With $\lambda_3 = 2$, let $z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \mathbb{R}^3$ is an eigen

vector of A, that $(2I_3 - A)z = 0$

We have:

$$\begin{pmatrix} 3 & -4 & 2 \\ 3 & -2 & 0 \\ 3 & -1 & -1 \end{pmatrix} \xrightarrow[R_3 - R_1 \to R_3]{R_2 - R_1 \to R_2} \begin{pmatrix} 3 & -4 & 2 \\ 0 & 2 & -2 \\ 0 & 3 & -3 \end{pmatrix}$$

$$\xrightarrow{2R_3 - 3R_2 \to R_3} \begin{pmatrix} 3 & -4 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 3z_1 - 4z_2 + 2z_3 = 0 \\ 2z_2 - 2z_3 = 0 \end{cases} \Rightarrow \begin{cases} z_1 = \frac{2}{3}a \\ z_2 = a \\ z_3 = a \end{cases}, a \in \mathbb{R} \setminus \{0\}$$

$$\Rightarrow z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = a \begin{pmatrix} 2/3 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow E_2 = \left\{ a \begin{pmatrix} 2/3 \\ 1 \\ 1 \end{pmatrix} \middle| a \in \mathbb{R} \setminus \{0\} \right\} = \text{Span}\left\{ \begin{pmatrix} 2/3 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$\Rightarrow$ The basis of E_2 is $S_2 = \left\{ \begin{pmatrix} 2/3 \\ 1 \\ 1 \end{pmatrix} \right\}$

Let $S = S_1 \cup S_3 \cup S_2$

$$= \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1/4 \\ 3/4 \\ 1 \end{pmatrix}, \begin{pmatrix} 2/3 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$\Rightarrow |S| = 3 = n \Rightarrow A$ is diagonalizable

let $P = \begin{pmatrix} 1 & 1/4 & 2/3 \\ 1 & 3/4 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ is said to diagonalizable A

We have:

$$P^{-1}.A.P = D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\Rightarrow A = P.D.P^{-1}$$

$$\Rightarrow A^{100} = P.D^{100}.P^{-1} = P.\begin{pmatrix} 1^{100} & 0 & 0 \\ 0 & 3^{100} & 0 \\ 0 & 0 & 2^{100} \end{pmatrix}.P^{-1}$$

$$= \begin{pmatrix} -2.10^{30} & -5.10^{47} & 5.10^{47} \\ -3.10^{30} & -1.10^{48} & 1.10^{48} \\ -3.10^{30} & -2.10^{48} & 2.10^{48} \end{pmatrix}$$