2.9

(a) $y_t = X_t + W_t = \phi X_{t-1} + Z_t + W_t$

Ta có $(1-\phi B)X_t = Z_t$

$$\Leftrightarrow \quad X_t = (1-\phi B)^{-1} Z_t = \sum_{j=0}^{+\infty} \phi^j B^j Z_t = \sum_{j=0}^{+\infty} \phi^j Z_{t-j}$$

$$\Rightarrow \quad Cov(X_t, W_s) = \sum_{j=0}^{+\infty} \phi^j Cov(Z_{t-j}, W_s) = 0 \quad , \forall t, s$$

+) $Ey_t = EX_t + EW_t = \phi \mu_X$ không phụ thuộc t

+) $$\begin{aligned}
\gamma_y(t+h, t) &= Cov(y_{t+h}, y_t) \\
&= Cov(X_{t+h} + W_{t+h}, X_t + W_t) \\
&= Cov(X_{t+h}, X_t) + Cov(W_{t+h}, X_t) + Cov(X_{t+h}, W_t) + Cov(W_{t+h}, W_t) \\
&= \phi^{|h|} \gamma_X(0) + \sigma_w^2 \chi_{\{0\}}(h)
\end{aligned}$$

Do đó $\{y_t\}$ là chuỗi dừng với hàm ACVF xác định bởi

$$\gamma_y(h) = \phi^{|h|} \gamma_X(0) + \sigma_w^2 \chi_{\{0\}}(h)$$

(b) $U_t = y_t - \phi y_{t-1}$

$$\begin{aligned}
\gamma_U(t+h, t) &= Cov(U_{t+h}, U_t) \\
&= Cov(y_{t+h} - \phi y_{t+h-1}, y_t - \phi y_{t-1}) \\
&= Cov(y_{t+h}, y_t) - \phi Cov(y_{t+h-1}, y_t) - \phi Cov(y_{t+h}, y_{t-1}) + \phi^2 Cov(y_{t+h-1}, y_{t-1}) \\
&= (\phi^2+1)\gamma_y(h) - \phi \gamma_y(h-1) - \phi \gamma_y(h+1) \\
&= (\phi^2+1)(\phi^{|h|}\gamma_X(0) + \sigma_w^2 \chi_{\{0\}}(h)) - \phi(\phi^{|h-1|}\gamma_X(0) + \sigma_w^2 \chi_{\{1\}}(h)) - \\
&\qquad - \phi(\phi^{|h+1|}\gamma_X(0) + \sigma_w^2 \chi_{\{-1\}}(h)) - \\
&= \ldots
\end{aligned}$$

$$= \begin{cases} \sigma_z^2 + (\phi^2+1)\sigma_w^2 & \text{nếu } h=0 \\ -\phi\sigma_w^2 & \text{nếu } |h|=1 \\ 0 & \text{nếu } |h| \geq 2 \end{cases} \qquad (2.9.1)$$

(c) Theo câu (b), U_t là 1-correlated nên $U_t \sim MA(1)$.
Đặt $U_t = \tilde{Z}_t + \theta \tilde{Z}_{t-1}$, $\tilde{Z}_t \sim WN(0, \sigma_{\tilde{Z}}^2)$. Khi đó

$$\gamma_U(h) = \begin{cases} \sigma_{\tilde{Z}}^2(1+\theta^2) & \text{nếu } h=0 \\ \sigma_{\tilde{Z}}^2\theta & \text{nếu } |h|=1 \\ 0 & \text{nếu } |h| \geq 2 \end{cases} \qquad (2.9.2)$$

Từ (2.9.1) và (2.9.2), ta được

$$\begin{cases} \sigma_{\tilde{Z}}^2(1+\theta^2) = -\sigma_Z^2 - (1+\phi^2)\sigma_w^2 \\ \sigma_Z^2\theta = -\sigma_w^2\phi \end{cases}$$

$$\Leftrightarrow \begin{cases} \theta = \frac{1}{2}\left(-A \pm \sqrt{A^2-4}\right) \\ \sigma_{\tilde{Z}}^2 = -\frac{1}{2}\phi\sigma_w^2\left(-A \mp \sqrt{A^2-4}\right) \end{cases}, \text{ trong đó } A = \frac{\sigma_Z^2}{\phi} + \frac{1+\phi^2}{\phi}$$

2.11

AR(1): $X_t - \mu = \phi(X_{t-1} - \mu) + Z_t$, $Z_t \sim WN(0, \sigma^2)$

$\gamma_X(h) = \dfrac{\phi^{|h|}\sigma^2}{1-\phi^2}$, thấy ngay $\sum_{h=-\infty}^{+\infty} \gamma(h) = \dfrac{\sigma^2}{(1-\phi)^2} < +\infty$. Khi đó

$$n E(\bar{X}_n - \mu)^2 \xrightarrow{n \to +\infty} \sum_{h=-\infty}^{+\infty} \gamma(h) = \frac{\sigma^2}{(1-\phi)^2}$$

$$\Rightarrow \bar{X}_n \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n(1-\phi)^2}\right)$$

Như vậy ta có ước lượng khoảng với độ tin cậy 95% như sau

$$\mu \in \left(\bar{x}_n - 1.96 \frac{\sigma}{\sqrt{n}(1-\Phi)} \, , \, \bar{x}_n + 1.96 \frac{\sigma}{\sqrt{n}(1-\Phi)} \right)$$

2.12.

MA(1) : $X_t = \mu + Z_t + \theta Z_{t-1}$, $Z_t \sim WN(0, \sigma^2)$

$$\gamma_X(h) = \begin{cases} \sigma^2(1+\theta^2) & \text{nếu } h = 0 \\ \sigma^2\theta & \text{nếu } |h| = 1 \\ 0 & \text{nếu } |h| \geq 2 \end{cases}$$

, thấy ngay $\sum_{h=-\infty}^{+\infty} \gamma(h) = \sigma^2(\theta+1)^2 < +\infty$.

Khi đó

$$nE(\bar{X}_n - \mu)^2 \xrightarrow{n \to +\infty} \sum_{|h| < \infty} \gamma(h) = \sigma^2(\theta+1)^2$$

$$\Rightarrow \quad \bar{X}_n \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}(\theta+1)^2\right)$$

Như vậy ta có ước lượng khoảng với độ tin cậy 95% như sau

$$\mu \in \left(\bar{x}_n - 1.96 \frac{\sigma}{\sqrt{n}}(\theta+1) \, , \, \bar{x}_n + 1.96 \frac{\sigma}{\sqrt{n}}(\theta+1) \right)$$

2.13

$\hat{\rho} \sim \mathcal{N}(\rho, n^{-1} W)$

trong đó $W = [w_{ij}]_{i,j=1}^{n}$,

$$w_{ij} = \sum_{k=1}^{+\infty} \left(\rho(k+i) + \rho(k-i) - 2\rho(i)\rho(k)\right)\left(\rho(k+j) + \rho(k-j) - 2\rho(j)\rho(k)\right)$$

Khoảng tin cậy 95% : $\hat{\rho}(h) \in \rho(h) \pm 1.96/\sqrt{n} \; w_{hh}$

$\Leftrightarrow \rho(h) \in \hat{\rho}(h) \pm 1.96/\sqrt{n} \; w_{hh}$

(a) AR(1) : $X_t - \phi X_{t-1} = Z_t$, $Z_t \sim WN(0, \sigma^2)$

$$\gamma_X(h) = \frac{\phi^{|h|}\sigma^2}{1-\phi} \quad , \quad \rho_X(h) = \phi^{|h|}$$

Ta có

$$w_{hh} = \sum_{k=1}^{+\infty} \left(\rho(k+h) + \rho(k-h) - 2\rho(k)\rho(h)\right)^2$$
$$= \sum_{k=1}^{+\infty} \left(\phi^{|k-h|} - \phi^{k+h}\right)^2$$

$h = 1$: $w_{11} = \sum_{k=1}^{+\infty} \left(\phi^{k-1} - \phi^{k+1}\right)^2 = 1 - \phi^2$

$h = 2$: $w_{22} = \sum_{k=1}^{+\infty} \left(\phi^{|k-2|} - \phi^{k+2}\right)^2 = (1-\phi^2)(1+3\phi^2)$

Như vậy

$$\rho(1) \in \hat{\rho}(1) \pm \frac{1.96}{\sqrt{n}}(1-\phi^2) = \dots$$
$$\rho(2) \in \hat{\rho}(2) \pm \frac{1.96}{\sqrt{n}}(1-\phi^2)(1+3\phi^2) = \dots$$

(b) MA(1) : $X_t = Z_t + \theta Z_{t-1}$, $Z_t \sim WN(0, \sigma^2)$

$$\gamma_X(h) = \begin{cases} \sigma^2(1+\theta^2) & \text{nếu } h = 0 \\ \sigma^2\theta & \text{nếu } |h| = 1 \\ 0 & \text{nếu } |h| \geq 2 \end{cases} \quad , \quad \rho_X(h) = \begin{cases} 1 & \text{nếu } h = 0 \\ \theta(1+\theta^2)^{-1} & \text{nếu } |h| = 1 \\ 0 & \text{nếu } |h| \geq 2 \end{cases}$$

Ta có

$$w_{hh} = \sum_{k=1}^{+\infty} \left(\rho(k+h) + \rho(k-h) - 2\rho(k)\rho(h)\right)^2$$

$h = 1$: $w_{11} = \sum_{k=1}^{+\infty} \left(\rho(k+1) + \rho(k-1) - 2\rho(k)\frac{\theta}{1+\theta^2}\right)^2 = \frac{5\theta^4 + \theta^2}{(1+\theta^2)^2}$

$h = 2$: $w_{22} = \sum_{k=1}^{+\infty} \left(\rho(k+2) + \rho(k-2)\right)^2 = \frac{\theta^4 + 3\theta^2 + 1}{(1+\theta^2)^2}$

Như vậy

$$\rho(1) \in \hat{\rho}(1) \pm \frac{1.96}{\sqrt{n}} \cdot \frac{5\theta^4+\theta^2}{(1+\theta^2)^2} = \dots$$

$$\rho(2) \in \hat{\rho}(2) \pm \frac{1.96}{\sqrt{n}} \cdot \frac{\theta^4+3\theta^2+1}{(1+\theta^2)^2} = \dots$$

2.14

$$X_t = A\cos(\omega t) + B\sin(\omega t), \quad \mu = 0$$

$$\gamma(h) = \cos(\omega h)$$

(1) $P_n X_{n+h} = \mu + \sum_{k=1}^{n} a_k (X_{n+1-k} - \mu) \xrightarrow{\mu=0} \underline{a}_n' \underline{X}_n = \underline{\gamma}_n(h)' \Gamma_n^{-1} \underline{X}_n$

(2) $E(X_{n+h} - P_n X_{n+h})^2 = \gamma(0) - \underline{a}_n' \underline{\gamma}_n(h)$

(3) $E[X_{n+h} - P_n X_{n+h}] = 0$

(4) $E[(X_{n+h} - P_n X_{n+h}) X_j] = 0, \quad \forall j = \overline{1,n}$

trong đó $\underline{a}_n = (a_1, a_2, \dots, a_n)^T$, $\underline{\gamma}_n(h) = (\gamma(h), \gamma(h+1), \dots, \gamma(h+n-1))^T$
$\Gamma_n = [\gamma(i-j)]_{i,j=1}^{n}$ và thoả mãn $\Gamma_n \underline{a}_n = \underline{\gamma}_n(h)$

(a) $a_1 = \gamma(0)^{-1}\gamma(1) = \cos(\omega)$

$$\Rightarrow \begin{cases} P_1 X_2 = a_1 X_1 = \cos(\omega) X_1 \\ E(X_2 - P_1 X_2)^2 = \gamma(0) - a_1 \gamma(1) = 1 - \cos^2(\omega) = \sin^2(\omega) \end{cases}$$

(b) $$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \Gamma_2^{-1} \underline{\gamma}_2(1) = \begin{bmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{bmatrix}^{-1} \begin{bmatrix} \gamma(1) \\ \gamma(2) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & \cos(\omega) \\ \cos(\omega) & 1 \end{bmatrix}^{-1} \begin{bmatrix} \cos(\omega) \\ \cos(2\omega) \end{bmatrix} = \begin{bmatrix} 2\cos(\omega) \\ -1 \end{bmatrix}$$

$\Rightarrow \dots$

$$\Rightarrow \begin{cases} P_2 X_3 = -X_1 + 2\cos(\omega) X_2 \\ E(X_3 - P_2 X_3)^2 = \gamma(0) - \underline{a}_2' \underline{\gamma}_2(1) \\ \qquad = 1 - \begin{bmatrix} 2\cos(\omega) \\ -1 \end{bmatrix}' \begin{bmatrix} \cos(\omega) \\ \cos(2\omega) \end{bmatrix} = 0 \end{cases}$$

(c) $\tilde{P}_n X_{n+1} = \sum_{j=1}^{+\infty} a_j X_{n+1-j}$

trong đó $a = (a_1, a_2, \dots)^T$ là nghiệm của hệ phương trình

$$\sum_{j=1}^{+\infty} a_j \gamma(i-j) = \gamma(h+i-1), \quad \forall i = 1, 2, \dots$$

$$\Leftrightarrow \quad \Gamma_\infty \underline{a}_\infty = \underline{\gamma}_\infty(h)$$

$$\xrightarrow{h=1} \quad \Gamma_\infty \underline{a}_\infty = \underline{\gamma}_\infty(1) \quad \Leftrightarrow \begin{cases} \gamma(0)a_1 + \gamma(1)a_2 + \gamma(2)a_3 + \dots = \gamma(1) \\ \gamma(1)a_1 + \gamma(0)a_2 + \gamma(1)a_3 + \dots = \gamma(2) \\ \gamma(2)a_1 + \gamma(1)a_2 + \gamma(0)a_3 + \dots = \gamma(3) \\ \vdots \end{cases}$$

2.15

$$X_{n+1} = Z_{n+1} + \sum_{j=1}^{n} \phi_j X_{n+1-j}$$

$$\Rightarrow P_n X_{n+1} = \underbrace{P_n Z_{n+1}}_{0} + \sum_{j=1}^{n} \phi_j P_n X_{n+1-j} = \sum_{j=1}^{n} \phi_j X_{n+1-j}$$

2.18

MA(1) : $\quad X_t = Z_t - \theta Z_{t-1}, \quad Z_t \sim WN(0, \sigma^2)$

MA(1) 2AR : $\quad Z_t = (1 - \theta B)^{-1} X_t$

$$= \sum_{j=0}^{+\infty} \theta^j B^j X_t = \sum_{j=0}^{+\infty} \theta^j X_{t-j}, \quad Z_t \sim WN(0, \sigma^2)$$

Tác động toán tử $\tilde{P}_n$ vào MA(1)2AR, ta có

$$\tilde{P}_n Z_{n+1} = \sum_{j=0}^{+\infty} \theta^j \tilde{P}_n X_{n+1-j}$$

$$\Leftrightarrow \quad 0 = \tilde{P}_n X_{n+1} + \sum_{j=1}^{+\infty} \theta^j X_{n+1-j}$$

$$\Leftrightarrow \quad \tilde{P}_n X_{n+1} = -\sum_{j=1}^{+\infty} \theta^j X_{n+1-j}$$

Trung bình bình phương sai số:

$$E(X_{n+1} - \tilde{P}_n X_{n+1})^2 = E\left(X_{n+1} + \sum_{j=1}^{+\infty} \theta^j X_{n+1-j}\right)^2 = EZ_t^2 = \sigma^2$$

2.19

$$X_t = Z_t - Z_{t-1}, \quad Z_t \sim WN(0, \sigma^2)$$

$$P_n X_{n+1} = P(Z_{n+1} - Z_n \mid X_n, \dots, X_1)$$

$$= \underbrace{P(Z_{n+1} \mid X_n, \dots, X_1)}_{0} - P(Z_n \mid X_n, \dots, X_1)$$

$$P(Z_n \mid X_n, \dots, X_1)$$

$$= EZ_n + \left(\Gamma_n^{-1} Cov(Z_n, [X_{n+1-j}]_{j=1}^n)\right)' [X_{n+1-j} - EX_{n+1-j}]_{j=1}^n$$

$$= 0 + [Cov(Z_n, Z_{n+1-j}) - Cov(Z_n, Z_{n-j})]_{j=1}^n \left([\gamma_X(i-j)]_{i,j=1}^n\right)^{-1} [X_{n+1-j}]_{j=1}^n$$

$$\overset{n \geq 2}{=} \begin{bmatrix} \sigma^2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}' \begin{bmatrix} 2\sigma^2 & -\sigma^2 & 0 & \cdots & 0 \\ -\sigma^2 & 2\sigma^2 & -\sigma^2 & \cdots & 0 \\ 0 & -\sigma^2 & 2\sigma^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2\sigma^2 \end{bmatrix}^{-1} \begin{bmatrix} X_n \\ X_{n-1} \\ X_{n-2} \\ \vdots \\ X_1 \end{bmatrix}$$

$$= \dots$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}' \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{bmatrix}^{-1} \begin{bmatrix} X_n \\ X_{n-1} \\ X_{n-2} \\ \vdots \\ X_1 \end{bmatrix}$$

$$= \frac{1}{n+1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}' \begin{bmatrix} n & n-1 & n-2 & \cdots & 1 \\ n-1 & 2(n-1) & 2(n-2) & \cdots & 2 \\ n-2 & 2(n-2) & 3(n-1) & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix} \begin{bmatrix} X_n \\ X_{n-1} \\ X_{n-2} \\ \vdots \\ X_1 \end{bmatrix}$$

$$= \frac{1}{n+1}\left(nX_n + (n-1)X_{n-1} + \dots + X_1\right)$$

Như vậy

$$P_n X_{n+1} = \frac{-1}{n+1}\left(nX_n + (n-1)X_{n-1} + \dots + X_1\right)$$

2.20

$$\mathrm{Cov}(X_n - \hat{X}_n, X_j) = \underbrace{E[X_j(X_n - \hat{X}_n)]}_{0} - E[X_j]\underbrace{E[X_n - \hat{X}_n]}_{0} = 0 \quad , \forall j = 1, \dots, n-1$$

$$\mathrm{Cov}(X_n - \hat{X}_n, X_j - \hat{X}_j) = E[(X_n - \hat{X}_n)(X_j - \hat{X}_j)] - \underbrace{E[X_n - \hat{X}_n]}_{0}\underbrace{E[X_j - \hat{X}_j]}_{0}$$

$$= \underbrace{E[X_j(X_n - \hat{X}_n)]}_{0} - \underbrace{E[\hat{X}_j(X_n - \hat{X}_n)]}_{0} = 0$$

2.21

(a) $P_2X_3 = P(X_3 | X_2, X_1)$

$$= \underbrace{P(Z_3 | X_2, X_1)}_{0} + \theta P(Z_2 | X_2, X_1)$$

$$P(Z_2 | X_2, X_1) = EZ_2 + Cov\left(Z_2, \begin{bmatrix} X_2 \\ X_1 \end{bmatrix}\right)' \Gamma_2^{-1} \begin{bmatrix} X_2 - EX_2 \\ X_1 - EX_1 \end{bmatrix}$$

$$= 0 + \begin{bmatrix} Cov(Z_2, Z_2) + \theta(Z_2, Z_1) \\ Cov(Z_2, Z_1) + \theta(Z_1, Z_0) \end{bmatrix}' \begin{bmatrix} \sigma^2(1+\theta^2) & \sigma^2\theta \\ \sigma^2\theta & \sigma^2(1+\theta^2) \end{bmatrix}^{-1} \begin{bmatrix} X_2 \\ X_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix}' \begin{bmatrix} 1+\theta^2 & \theta \\ \theta & 1+\theta^2 \end{bmatrix}^{-1} \begin{bmatrix} X_2 \\ X_1 \end{bmatrix}$$

$$= \frac{1}{\theta^4+\theta^2+1}\left((1+\theta^2)X_2 + \theta X_1\right)$$

Như vậy

$$P_2X_3 = \frac{\theta}{\theta^4+\theta^2+1}\left((1+\theta^2)X_2 + \theta X_1\right)$$

(b) $P(X_3 | X_4, X_5) = P(Z_3 | X_4, X_5) + \theta P(Z_2 | X_4, X_5)$

$$P(Z_3 | X_4, X_5) = EZ_3 + Cov\left(Z_3, \begin{bmatrix} X_4 \\ X_5 \end{bmatrix}\right)' \Gamma_2^{-1} \begin{bmatrix} X_4 - EX_4 \\ X_5 - EX_5 \end{bmatrix}$$

$$= 0 + \begin{bmatrix} Cov(Z_3, Z_4 + \theta Z_3) \\ Cov(Z_3, Z_5 + \theta Z_4) \end{bmatrix}' \begin{bmatrix} \sigma^2(1+\theta^2) & \sigma^2\theta \\ \sigma^2\theta & \sigma^2(1+\theta^2) \end{bmatrix}^{-1} \begin{bmatrix} X_4 \\ X_5 \end{bmatrix}$$

$$= \begin{bmatrix} \theta \\ 0 \end{bmatrix}' \begin{bmatrix} 1+\theta^2 & \theta \\ \theta & 1+\theta^2 \end{bmatrix}^{-1} \begin{bmatrix} X_4 \\ X_5 \end{bmatrix}$$

$$= \frac{\theta}{\theta^4+\theta^2+1}\left(\theta(1+\theta^2)X_4 + \theta^2 X_5\right)$$

$$P(Z_2 \mid X_4, X_5) = 0 \quad \text{do} \quad Cov(Z_2, X_4) = Cov(Z_2, X_5) = 0$$

Như vậy

$$P(X_3 \mid X_4, X_5) = \theta(1+\theta^2) X_4 + \theta^2 X_5$$

(c) $W = (X_5, X_4, X_2, X_1)^T$

$$P(X_3 \mid W) = EX_3 + Cov(X_3, W)' Cov(W, W)^{-1} (W - EW)$$

$$= 0 + \begin{bmatrix} 0 \\ \sigma^2\theta \\ \sigma^2\theta \\ 0 \end{bmatrix}' \begin{bmatrix} \gamma(0) & \gamma(1) & \gamma(3) & \gamma(4) \\ \gamma(1) & \gamma(0) & \gamma(2) & \gamma(3) \\ \gamma(3) & \gamma(2) & \gamma(0) & \gamma(1) \\ \gamma(4) & \gamma(3) & \gamma(1) & \gamma(0) \end{bmatrix}^{-1} \begin{bmatrix} X_5 \\ X_4 \\ X_2 \\ X_1 \end{bmatrix}$$

$$= 0 + \begin{bmatrix} 0 \\ \theta \\ \theta \\ 0 \end{bmatrix}' \begin{bmatrix} \theta^2+1 & \theta & 0 & 0 \\ \theta & \theta^2+1 & 0 & 0 \\ 0 & 0 & \theta^2+1 & \theta \\ 0 & 0 & \theta & \theta^2+1 \end{bmatrix}^{-1} \begin{bmatrix} X_5 \\ X_4 \\ X_2 \\ X_1 \end{bmatrix}$$

$$= \frac{1}{\theta^4+\theta^2+1} \left(\theta^2 X_5 + \theta(\theta^2+1) X_4 + \theta(\theta^2+1) X_3 + \theta X_2 \right)$$

(d)