## 2.1

Ta tìm a và b sao cho

$$MSE(a,b) = E[(X_{n+h} - \hat{X}_{n+h})^2] = E[(X_{n+h} - aX_n - b)^2] \to \min$$

bằng cách giải hệ phương trình

$$\begin{cases} \frac{\partial}{\partial a} MSE(a,b) = 2E[X_n(aX_n + b - X_{n+h})] = 0 & (2.1.1) \\ \frac{\partial}{\partial b} MSE(a,b) = 2E[aX_n + b - X_{n+h}] = 0 & (2.1.2) \end{cases}$$

$$(2.1.1) \Leftrightarrow aE[X_n^2] + bE[X_n] - E[X_{n+h}X_n] = 0$$

$$\Leftrightarrow a(Var[X_n] + E[X_n]^2) + bE[X_n] - Cov(X_{n+h}, X_n) - E[X_{n+h}]E[X_n] = 0$$

$$\Leftrightarrow a(\gamma(0) + \mu^2) + b\mu - \gamma(h) - \mu^2 = 0 \quad (2.1.3)$$

$$(2.1.2) \Leftrightarrow aE[X_n] + b - E[X_{n+h}] = 0$$

$$\Leftrightarrow a\mu + b - \mu = 0 \quad (2.1.4)$$

$$(2.1.3), (2.1.4) \Rightarrow \begin{cases} a = \frac{\gamma(h)}{\gamma(0)} = \rho(h) \\ b = \mu\left(1 - \frac{\gamma(h)}{\gamma(0)}\right) = \mu(1 - \rho(h)) \end{cases}$$

## 2.2

$\{X_t\}$ là chuỗi dừng

$$X_t = A\cos(\omega t) + B\sin(\omega t), \quad t = 0, \pm 1, \dots$$

- $E[X_t] = E[A]\cos(\omega t) + E[B]\sin(\omega t) = 0$ không phụ thuộc $t$

- $\gamma_X(t+h,t) = Cov(X_{t+h}, X_t)$

$$= Cov\left(A\cos(\omega(t+h)) + B\sin(\omega(t+h)),\ A\cos(\omega t) + B\sin(\omega t)\right)$$

$$= \cos(\omega(t+h))\cos(\omega t)\, Cov(A,A) + \sin(\omega(t+h))\cos(\omega t)\, Cov(B,A) + \\ + \cos(\omega(t+h))\sin(\omega t)\, Cov(A,B) + \sin(\omega(t+h))\sin(\omega t)\, Cov(B,B)$$

$$= \cos(\omega(t+h))\cos(\omega t) + \sin(\omega(t+h))\sin(\omega t)$$

$$= \cos(\omega h) \qquad \text{không phụ thuộc } t$$

$\Rightarrow$ Chuỗi $\{X_t\}$ dừng

$\kappa(h) = \cos(\omega h)$ là hàm xác định không âm

Ta sẽ chứng minh $\sum_{i=1}^{n}\sum_{j=1}^{n} a_i \cos(\omega(i-j)) a_j \geq 0$, $\forall n \in \mathbb{N}^*$

- Với $n=1$: $\sum_{i=1}^{1}\sum_{j=1}^{1} a_i \cos(\omega(i-j)) a_j = a_1^2 \geq 0$, $\forall a_1 \in \mathbb{R}$

- Với $n \geq 2$, đặt $\underline{a} := [a_1\ a_2\ \dots\ a_n]' \in \mathbb{R}^{n\times 1}$, $K := [\cos(\omega(i-j))]_{1\leq i,j\leq n} \in \mathbb{R}^{n\times n}$

Ký hiệu $0_{m\times n} \in \mathbb{R}^{m\times n}$ là ma trận không, ta có

$$\cos(\omega(i-j)) = \cos(\omega i)\cos(\omega j) + \sin(\omega i)\sin(\omega j) = \begin{bmatrix} \cos(\omega i) \\ \sin(\omega i) \\ \hdashline 0_{(n-2)\times 1} \end{bmatrix}' \begin{bmatrix} \cos(\omega j) \\ \sin(\omega j) \\ \hdashline 0_{(n-2)\times 1} \end{bmatrix}$$

Xét ma trận
$$W = \begin{bmatrix} \cos(\omega) & \cos(2\omega) & \cdots & \cos(n\omega) \\ \sin(\omega) & \sin(2\omega) & \cdots & \sin(n\omega) \\ \hdashline 0_{(n-2)\times 1} & 0_{(n-2)\times 1} & & 0_{(n-2)\times 1} \end{bmatrix} \in \mathbb{R}^{n\times n}$$

Khi đó $K = W'W$ và từ đó suy ra

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i \cos(\omega(i-j)) a_j = \underline{a}' K \underline{a} = \underline{a}' W' W \underline{a} = \|W\underline{a}\|_2^2 \geq 0,\ \forall \underline{a} \in \mathbb{R}^{n\times 1}$$

## 2.3

Ta xét một kết quả tổng quát như sau:

$$MA(q): \quad X_t = Z_t + \sum_{j=0}^{q} \theta_j Z_{t-j}$$

$$\gamma_X(h) = Cov(X_{t+|h|}, X_t)$$

$$= Cov\left(\sum_{j=0}^{q} \theta_j Z_{t+|h|-j}, \sum_{j=0}^{q} \theta_j Z_{t-j}\right) \quad \text{(ở đây ta chọn } \theta_0 = 1\text{)}$$

$$= \sum_{i=0}^{q}\sum_{j=0}^{q} \theta_i \theta_j \, Cov(Z_{t+|h|-j}, Z_{t-j})$$

$$= \sum_{i=0}^{q}\sum_{j=0}^{q} \theta_i \theta_j \, \gamma_{\{i-j\}}(|h|) = \sigma^2 \sum_{j=0}^{q-|h|} \theta_j \theta_{j+|h|}$$

(a) $X_t = Z_t + 0.3 Z_{t-1} - 0.4 Z_{t-2}$, $Z_t \sim WN(0,1)$

- $h = 0$: $\gamma_X(0) = 1 \cdot (1^2 + 0.3^2 + (-0.4)^2) = 1.25$
- $h = \pm 1$: $\gamma_X(\pm 1) = 1 \cdot (1 \cdot 0.3 + 0.3 \cdot (-0.4)) = 0.18$
- $h = \pm 2$: $\gamma_X(\pm 2) = 1 \cdot (1 \cdot (-0.4)) = -0.4$
- $|h| \geq 3$: $\gamma_X(h) = 0$

(b) $y_t = \tilde{Z}_t - 1.2 \tilde{Z}_{t-1} - 1.6 \tilde{Z}_{t-2}$, $\tilde{Z}_t \sim WN(0, 0.25)$

- $h = 0$: $\gamma_X(0) = 0.25 \cdot (1^2 + 1.2^2 + 1.6^2) = 1.25$
- $h = \pm 1$: $\gamma_X(\pm 1) = 0.25 \cdot (1 \cdot (-1.2) + (-1.2)(-1.6)) = 0.18$
- $h = \pm 2$: $\gamma_X(\pm 2) = 0.25 \cdot (1 \cdot (-1.6)) = -0.4$
- $|h| \geq 3$: $\gamma_X(h) = 0$

## 2.4

(a) Chọn $X_t = (-1)^t y$, trong đó $y$ là bnn có $E[y] = 0$ và $Var[y] = 1$. Khi đó

$$\gamma_X(t+h, t) = Cov((-1)^{t+h} y, (-1)^t y) = (-1)^{2t+h} Var[y] = (-1)^h$$

(b) Chọn $X_t = y_1 \cos(\pi t/2) + y_2 \sin(\pi t/2) + y_3 \cos(\pi t/4) + y_4 \sin(\pi t/4) + y_5$,

trong đó $Y_k$ là bnn thoả mãn $E[Y_k]=0$ và $Var[Y_k]=1$.

NOTE: $X_t = Y_1\cos(\omega t) + Y_2\sin(\omega t) \longrightarrow \gamma_X(h) = \cos(\omega h)$

Khi đó

$$\gamma_X(t+h,t) = 1 + \cos\left(\frac{\pi h}{2}\right) + \cos\left(\frac{\pi h}{4}\right)$$

(c)

NOTE: Với AR(1): $X_t = Z_t + \theta Z_{t-1}$ thì $\gamma_X(h) = \begin{cases} \sigma^2(1+\theta)^2 & \text{nếu } h=0 \\ \sigma^2\theta & \text{nếu } h=\pm 1 \\ 0 & \text{nếu trái lại} \end{cases}$

Chọn $X_t = Z_t + \theta Z_{t-1}$, trong đó $Z_t \sim WN(0,\sigma^2)$

Chọn tiếp $(\sigma,\theta)$ thoả mãn $\begin{cases} \sigma^2(1+\theta)^2 = 1 \\ \sigma^2\theta = 0.4 \end{cases} \Leftrightarrow (\sigma,\theta) = (\ldots,\ldots)$

## 2.5

Đặt $S = \sum_{j=1}^{+\infty} \theta^j X_{n-j}$, $E[X_t] = \mu_X$ và $E[X_t^2] = \sigma_X^2 < +\infty$. Ta sẽ chứng minh

$$\lim_{m\to+\infty} E[|S_m - S|] = 0 \quad (1) \quad \text{và} \quad \lim_{m\to+\infty} E[(S_m-S)^2] = 0 \quad (2)$$

Ta chỉ cần chứng minh (2), từ đó suy trực tiếp ra (1) nhờ bất đẳng thức $E[|X|]^2 \leq E[X^2]$. Với $m,n \in \mathbb{N}$ bất kỳ $(m<n)$, ta có

$$\begin{aligned}
E[(S_m-S_n)^2] &= E\left[\left(\sum_{j=m+1}^{n} \theta^j X_{t-j}\right)^2\right] \\
&= \sum_{j=m+1}^{n} \theta^{2j} E[X_{t-j}^2] + 2\sum_{m+1\leq k<l\leq n} \theta^{k+l} E[X_{t-k}X_{t-l}] \\
&= \sigma_X^2 \sum_{j=m+1}^{n} \theta^{2j} + 2\mu_X^2 \sum_{m+1\leq k<l\leq n} \theta^{k+l} \\
&= \sigma_X^2\left(\theta^{2(n+1)} - \theta^{2(m+1)}\right)(1-\theta^2)^{-1} + 2\mu_X^2 \sum_{m+1\leq k<l\leq n} \theta^{k+l} \xrightarrow[n\to+\infty]{m\to+\infty} 0
\end{aligned}$$

Do đó $\lim_{m\to+\infty} E[(S_m - S)^2] = 0$ theo tiêu chuẩn Cauchy

2.6. CMR $\sum_{k=-\infty}^{+\infty} \alpha_k \beta_{j-k} = \sum_{k=-\infty}^{+\infty} \beta_k \alpha_{j-k}$

$$\sum_{k=-\infty}^{+\infty} \alpha_k \beta_{j-k} \overset{(k=j-m)}{=} \sum_{j-m=-\infty}^{+\infty} \alpha_{j-m} \beta_{j-(j-m)} = \sum_{m=-\infty}^{+\infty} \alpha_{j-m} \beta_m = \sum_{k=-\infty}^{+\infty} \alpha_{j-k} \beta_k$$

2.7.

Chọn $x = (\phi z)^{-1}$, do $|\phi| > 1$ và $|z| \geq 1$ nên $|x| = |\phi|^{-1}|z|^{-1} < 1$. Khi đó

$$\frac{1}{1-x} = \sum_{j=0}^{+\infty} x^j \overset{x=(\phi z)^{-1}}{\Longrightarrow} \frac{1}{1-(\phi z)^{-1}} = \sum_{j=0}^{+\infty} (\phi z)^{-j}$$

$$\Leftrightarrow \frac{\phi z}{1-\phi z} = -\sum_{j=0}^{+\infty} \phi^{-j} z^{-j}$$

2.8

Giả sử quá trình AR(1): $X_t = \phi X_{t-1} + Z_t$ với $|\phi| = 1$ có nghiệm dừng. Ta có

$$X_t = \phi X_{t-1} + Z_t = \phi^2 X_{t-2} + \phi Z_{t-1} + Z_t$$
$$= \dots$$
$$= \phi^{n+1} X_{t-n-1} + \sum_{j=0}^{n} \phi^j Z_{t-j}$$

$$\Leftrightarrow \sum_{j=0}^{n} \phi^j Z_{t-j} = X_t - \phi^{n+1} X_{t-n-1}, \forall n$$

Lại có

$$Var\left[\sum_{j=0}^{n} \phi^j Z_{t-j}\right] = \sum_{j=0}^{n} \underbrace{\phi^{2j}}_{1} Var[Z_{t-j}] + 2\sum_{1\leq i<j\leq n} \phi^{i+j} \underbrace{Cov(Z_{t-i}, Z_{t-j})}_{0}$$
$$= (n+1)\sigma^2$$

$$Var[X_t - \phi^{n+1} X_{t-n-1}] = Var[X_t] + \underbrace{\phi^{2n+2}}_{1} Var[X_{t-n-1}] - 2\phi^{n+1} Cov(X_t, X_{t-n-1})$$

$$= 2\gamma_x(0) - 2\phi^{n+1}\gamma_x(n+1) \leqslant 4\gamma_x(0)$$

Từ đó suy ra $(n+1)\sigma^2 \leqslant 4\gamma_x(0)$, $\forall n$. Cho $n \to +\infty$, khi đó $\gamma_x(0) \geqslant +\infty$ (?)
Như vậy giả sử ban đầu sai.