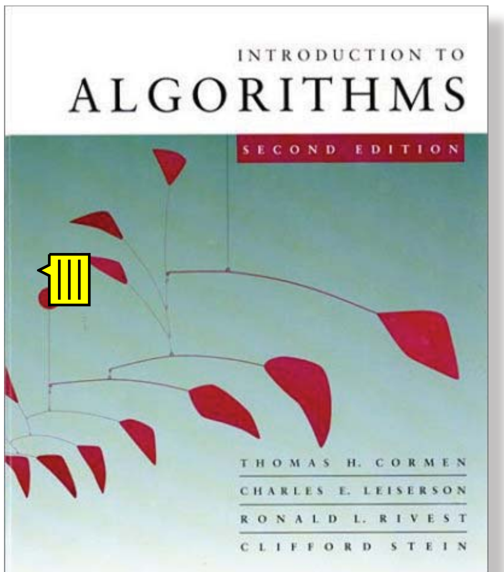
Design and Analysis of Algorithms

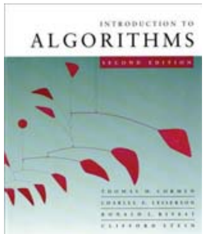
6.046J/18.401J



LECTURE 13

Network Flow

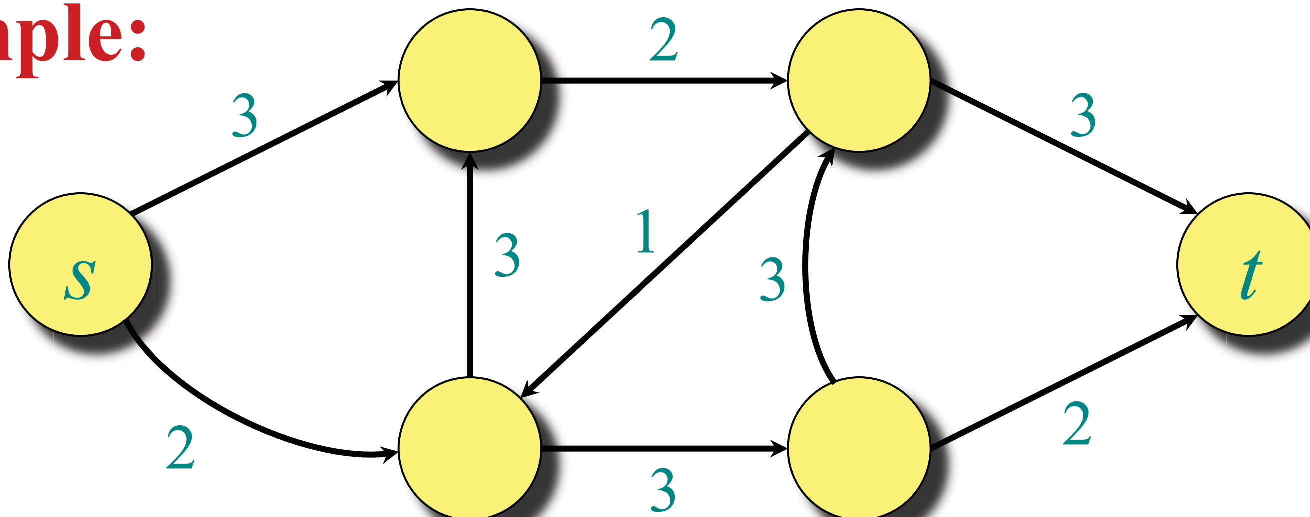
- Flow networks
- Maximum-flow problem
- Cuts
- Residual networks
- Augmenting paths
- Max-flow min-cut theorem
- Ford Fulkerson algorithm



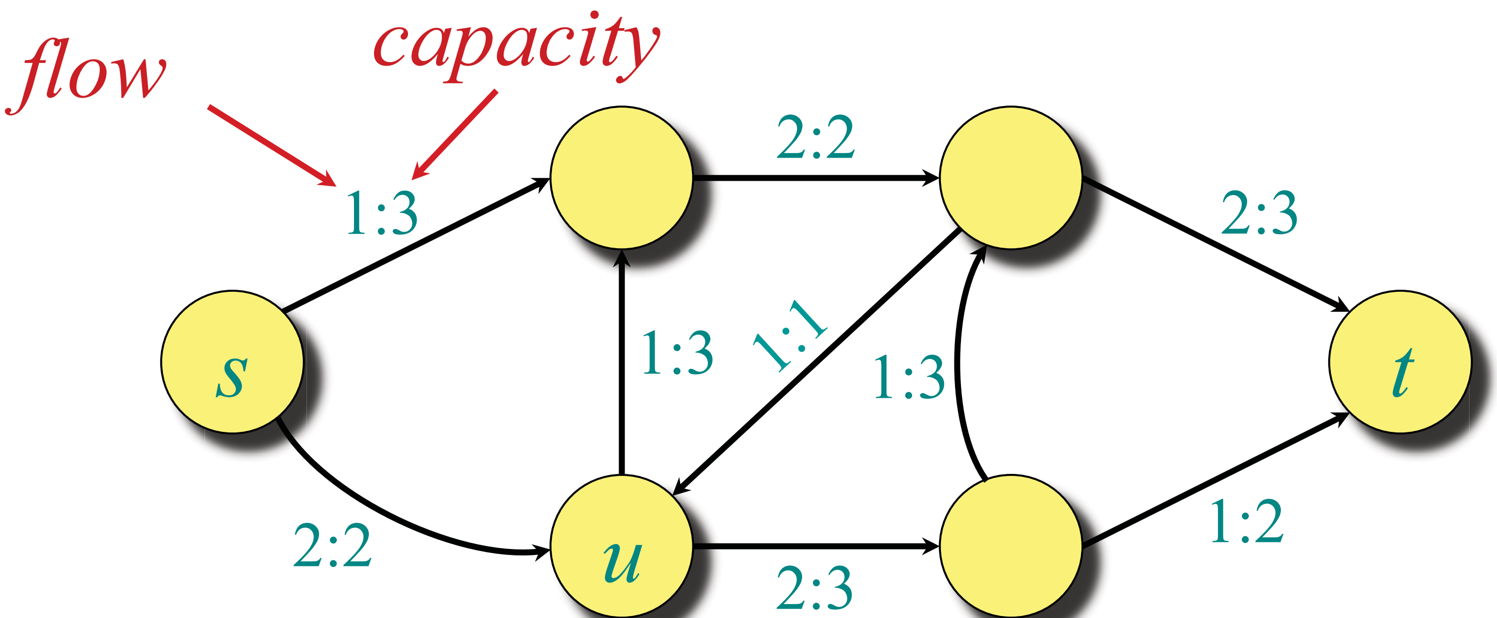
Flow networks

Definition. A ***flow network*** is a directed graph $G = (V, E)$ with two distinguished vertices: a ***source*** s and a ***sink*** t. Each edge $(u, v) \in E$ has a nonnegative ***capacity*** $c(u, v)$. If $(u, v) \notin E$, then $c(u, v) = 0$.

Example:



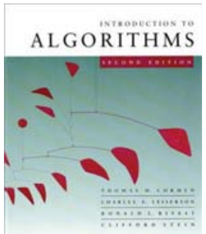
A flow on a network



Flow conservation (like Kirchoff's current law):

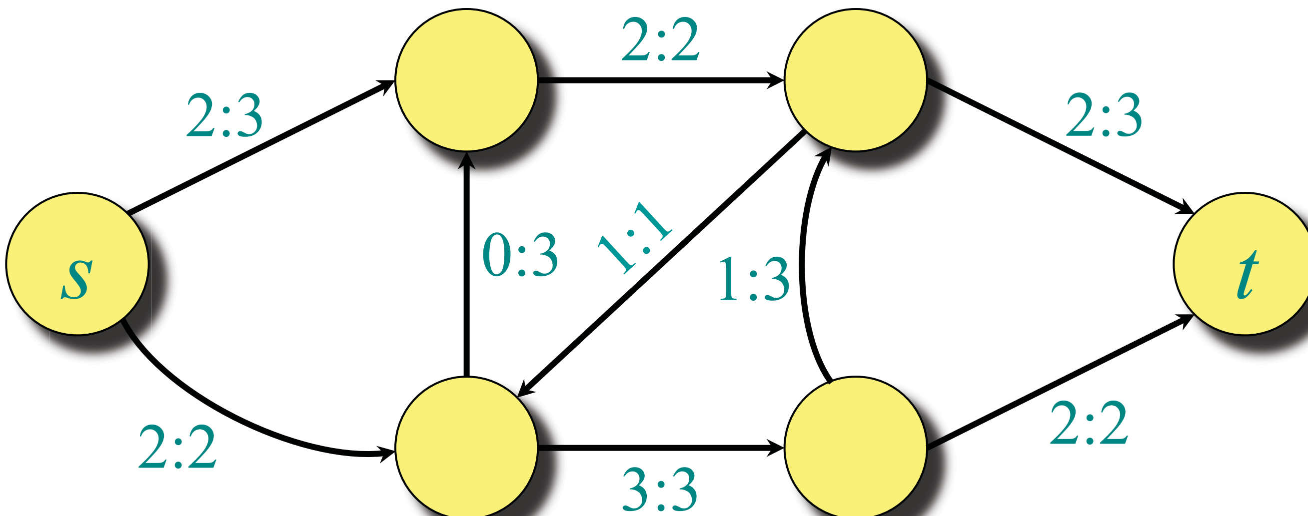
- Flow into u is $2 + 1 = 3$.
- Flow out of u is $1 + 2 = 3$.

Intuition: View flow as a *rate*, not a *quantity*.



The maximum-flow problem

Maximum-flow problem: Given a flow network G, find a flow of maximum value on G.

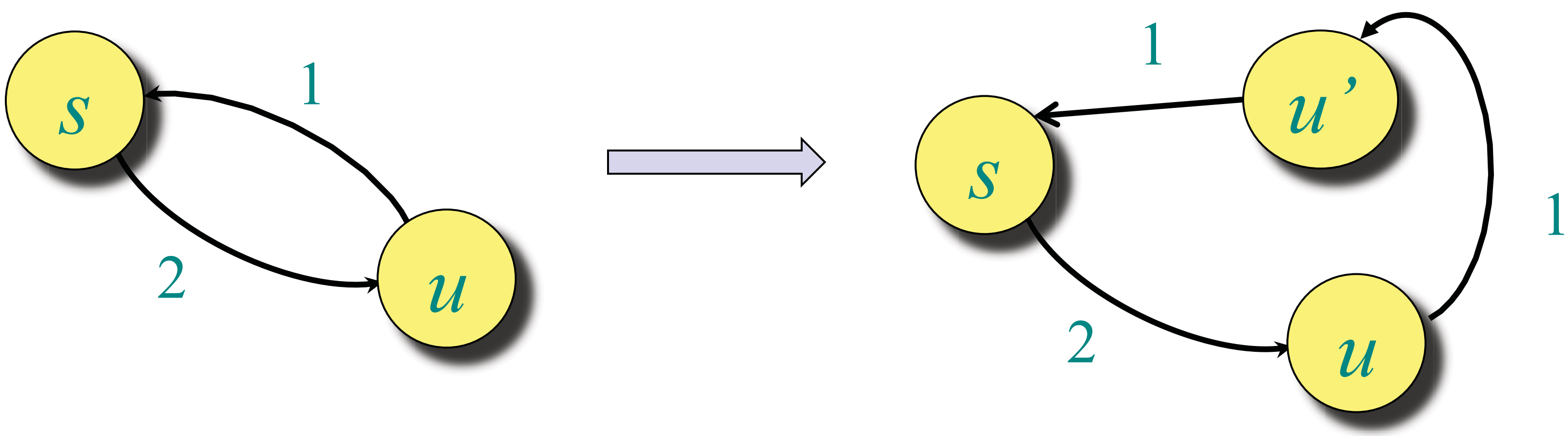


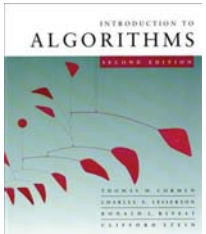
The value of the maximum flow is 4.

Flow network Assumptions

Assumption. If edge $(u, v) \in E$ exists, then $(v, u) \notin E$.

Assumption. No self-loop edges (u, u) exist





Net Flow

Definition. A ***(net) flow*** on G is a function $f : V \times V \rightarrow \mathbb{R}$ satisfying the following:

- ***Capacity constraint:*** For all $u, v \in V$,
$$f(u, v) \le c(u, v).$$
- ***Flow conservation:*** For all $u \in V - \{s, t\}$,
$$\sum_{v \in V} f(u, v) = 0.$$
- ***Skew symmetry:*** For all $u, v \in V$,
$$f(u, v) = -f(v, u).$$

Note: CLRS defines positive flows and net flows; these are equivalent for our flow networks obeying our assumptions.

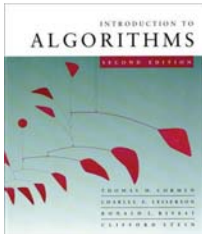
Notation

Definition. The ***value*** of a flow f, denoted by $|f|$, is given by

$$|f| = \sum_{v \in V} f(s, v)$$
$$= f(s, V).$$

Implicit summation notation: A set used in an arithmetic formula represents a sum over the elements of the set.

- **Example** — flow conservation:
  $f(u, V) = 0$ for all $u \in V - \{s, t\}$.



Simple properties of flow

Lemma.

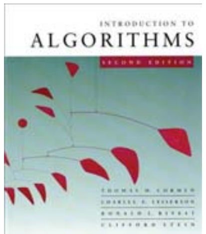
- $f(X, X) = 0$,
- $f(X, Y) = -f(Y, X)$,
- $f(X \cup Y, Z) = f(X, Z) + f(Y, Z)$ if $X \cap Y = \emptyset$. $\square$

Theorem. $|f| = f(V, t)$.

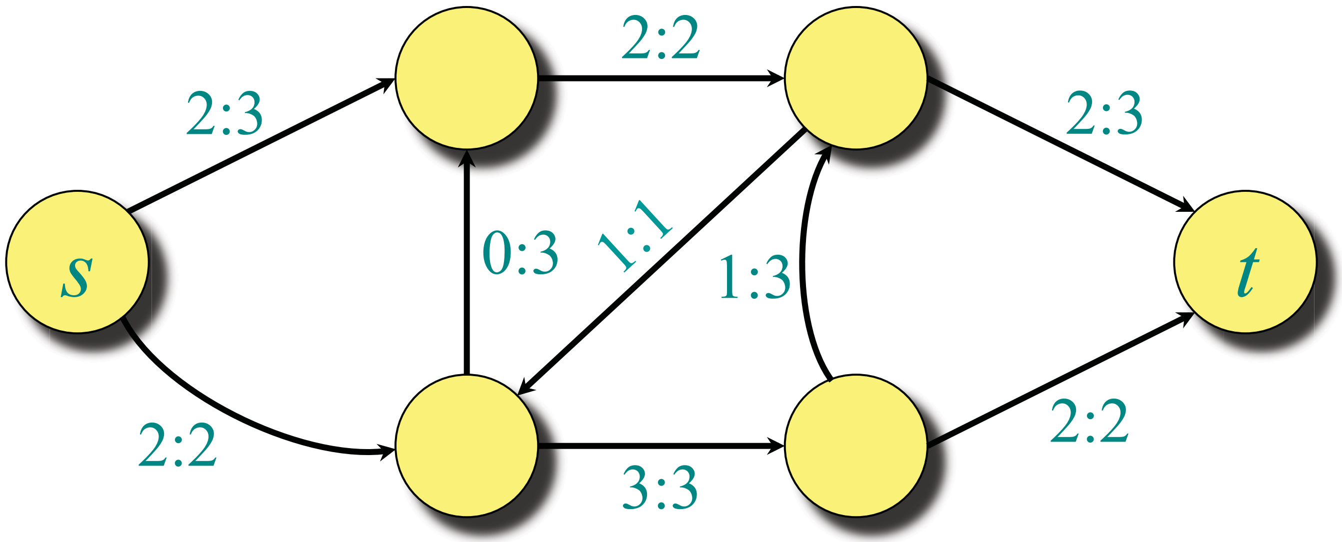
Proof.

$$
\begin{aligned}
|f| &= f(s, V) \\
&= f(V, V) - f(V-s, V) \\
&= f(V, V-s) \\
&= f(V, t) + f(V, V-s-t) \\
&= f(V, t). \quad \square
\end{aligned}
$$

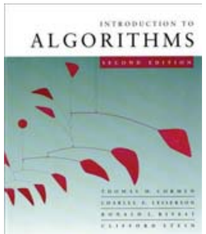
Omit braces.



Flow into the sink

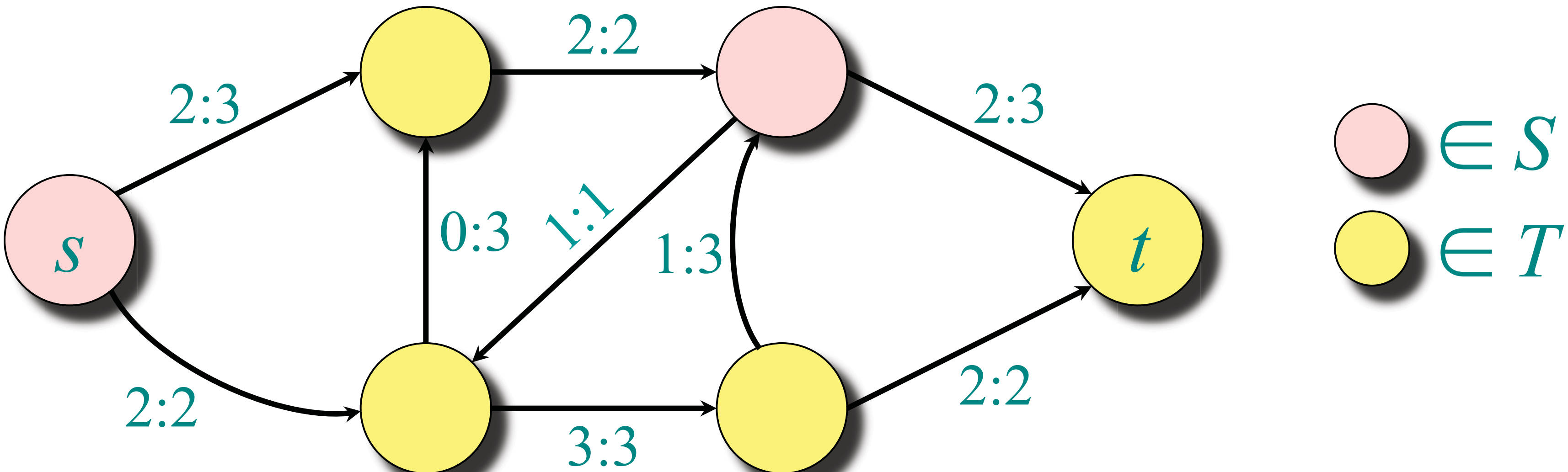


$$|f| = f(s, V) = 4 \qquad\qquad f(V, t) = 4$$

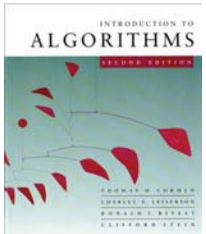


Cuts

Definition. A ***cut*** (S, T) of a flow network $G = (V, E)$ is a partition of V such that $s \in S$ and $t \in T$. If f is a flow on G, then the ***flow across the cut*** is $f(S, T)$.



$$f(S, T) = (2 + 2) + (-2 + 1 - 1 + 2) = 4$$

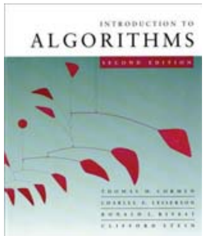


Another characterization of flow value

Lemma. For any flow f and any cut (S, T), we have $|f| = f(S, T)$.

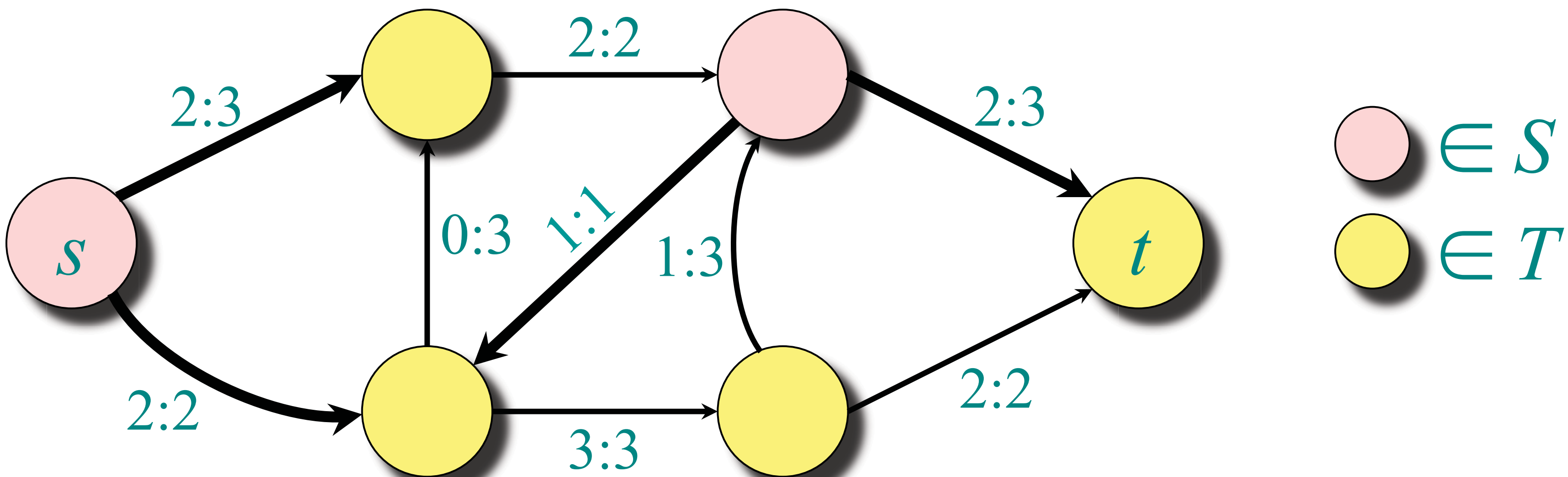
Proof.

$$
\begin{aligned}
f(S, T) &= f(S, V) - f(S, S) \\
&= f(S, V) \\
&= f(s, V) + f(S-s, V) \\
&= f(s, V) \\
&= |f|. \quad \square
\end{aligned}
$$

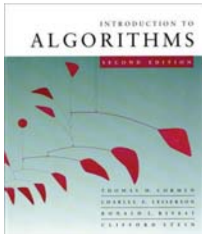


Capacity of a cut

Definition. The ***capacity of a cut*** (S, T) is $c(S, T)$.



$$c(S, T) = (3 + 2) + (1 + 3) = 9$$



Upper bound on the maximum flow value

Theorem. The value of any flow is bounded above by the capacity of any cut.

Proof.

$$
\begin{aligned}
|f| &= f(S,T) \\
&= \sum_{u \in S} \sum_{v \in T} f(u,v) \\
&\le \sum_{u \in S} \sum_{v \in T} c(u,v) \\
&= c(S,T)\,. \qquad \square
\end{aligned}
$$

Residual network

Definition. Let f be a flow on $G = (V, E)$. The ***residual network*** $G_f(V, E_f)$ is the graph with strictly positive ***residual capacities***

$$c_f(u, v) = c(u, v) - f(u, v) > 0.$$

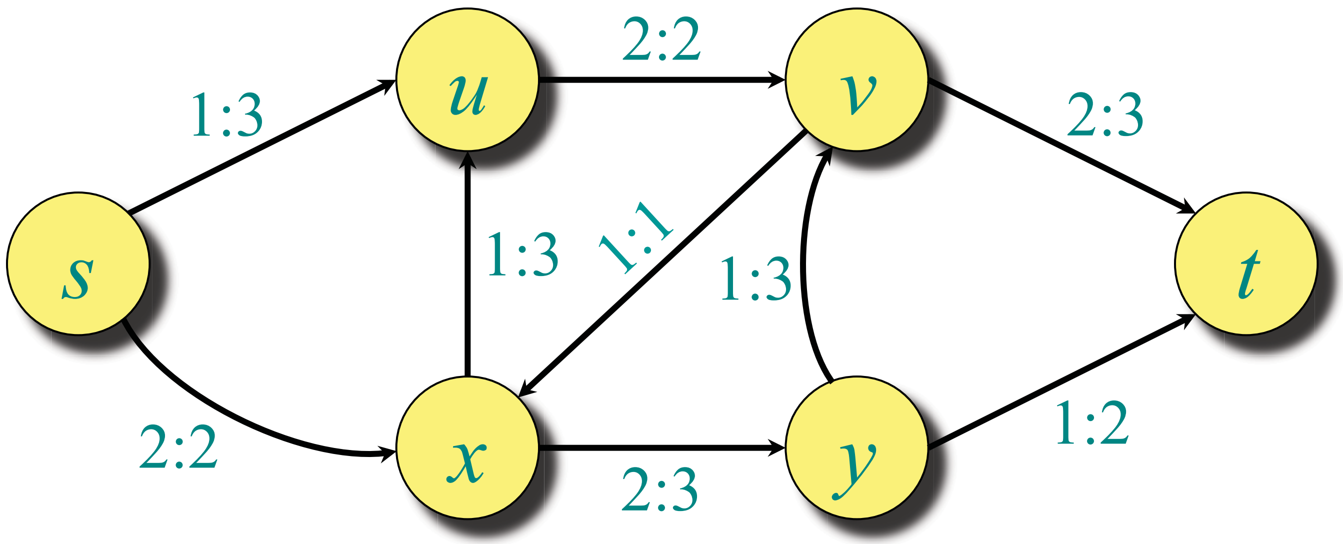
Edges in E_f admit more flow.

If $(v, u) \notin E$, $c(v, u) = 0$, but $f(v, u) = -f(u, v)$.

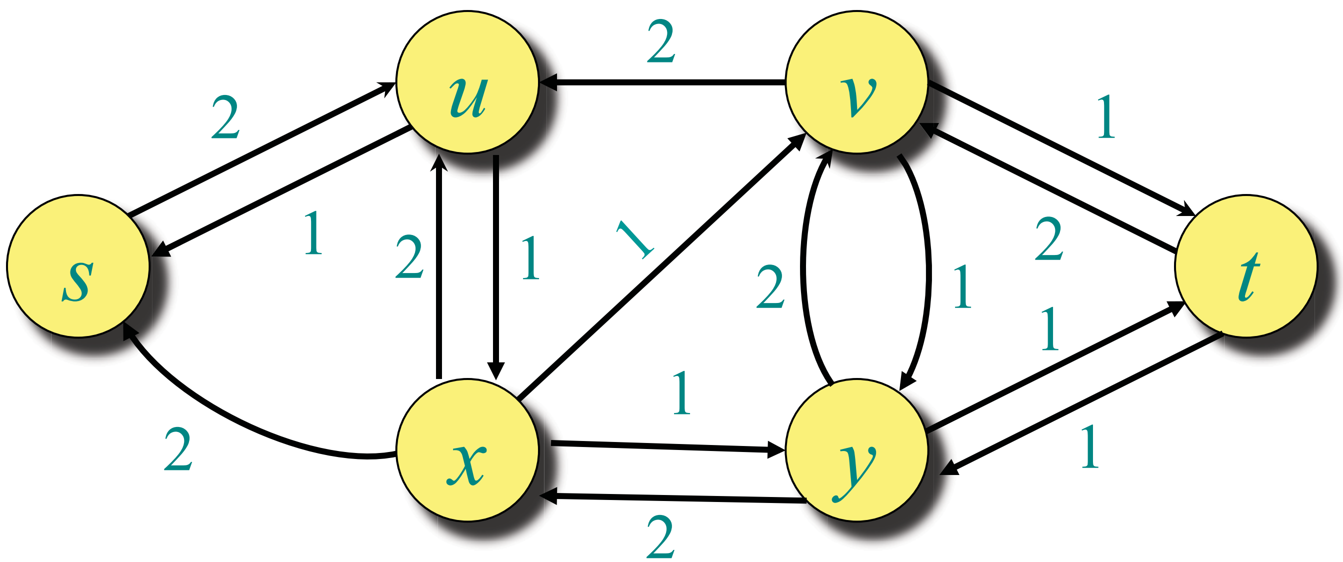
$|E_f| \le 2\,|E|$.

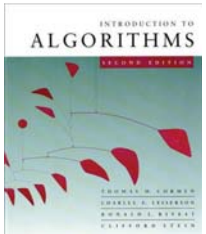
Flow and Residual Network

G



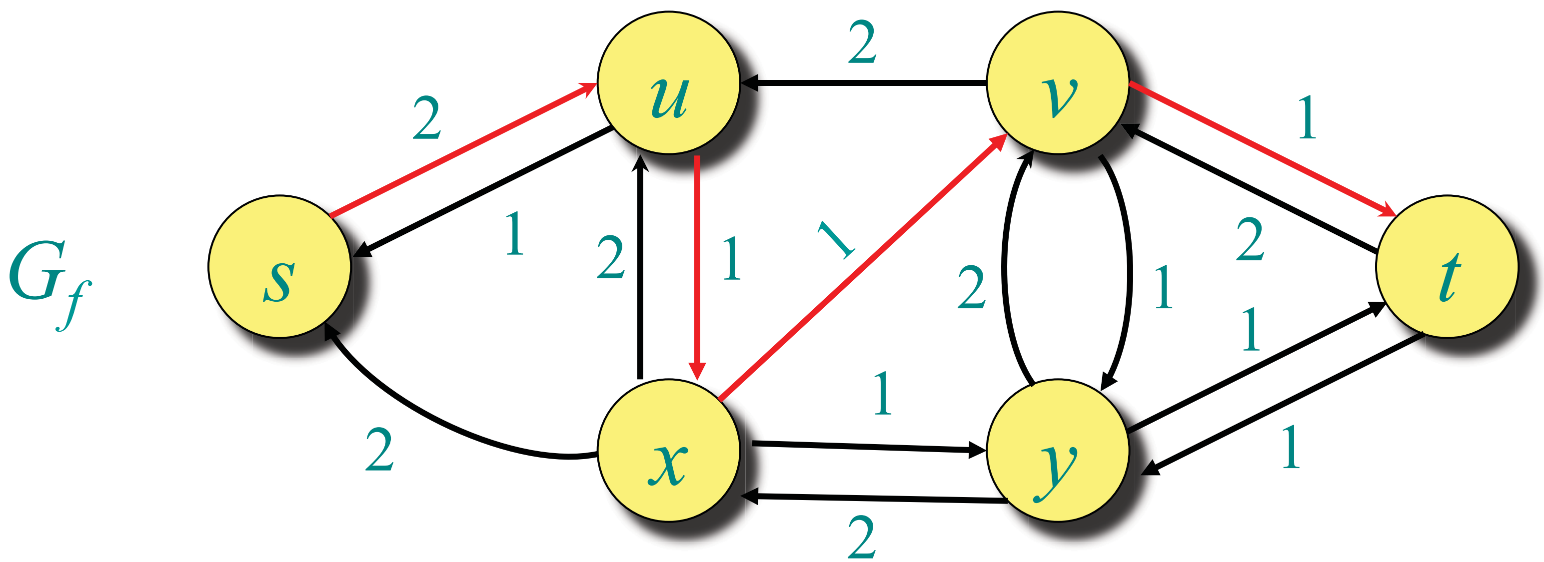
G_f



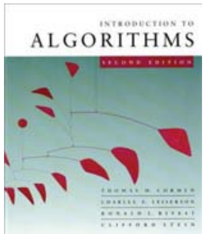


Augmenting paths

Definition. Any path from s to t in G_f is an ***augmenting path*** in G with respect to f. The flow value can be increased along an augmenting path p by $c_f(p) = \min_{(u,v)\in p} \{c_f(u,v)\}$.

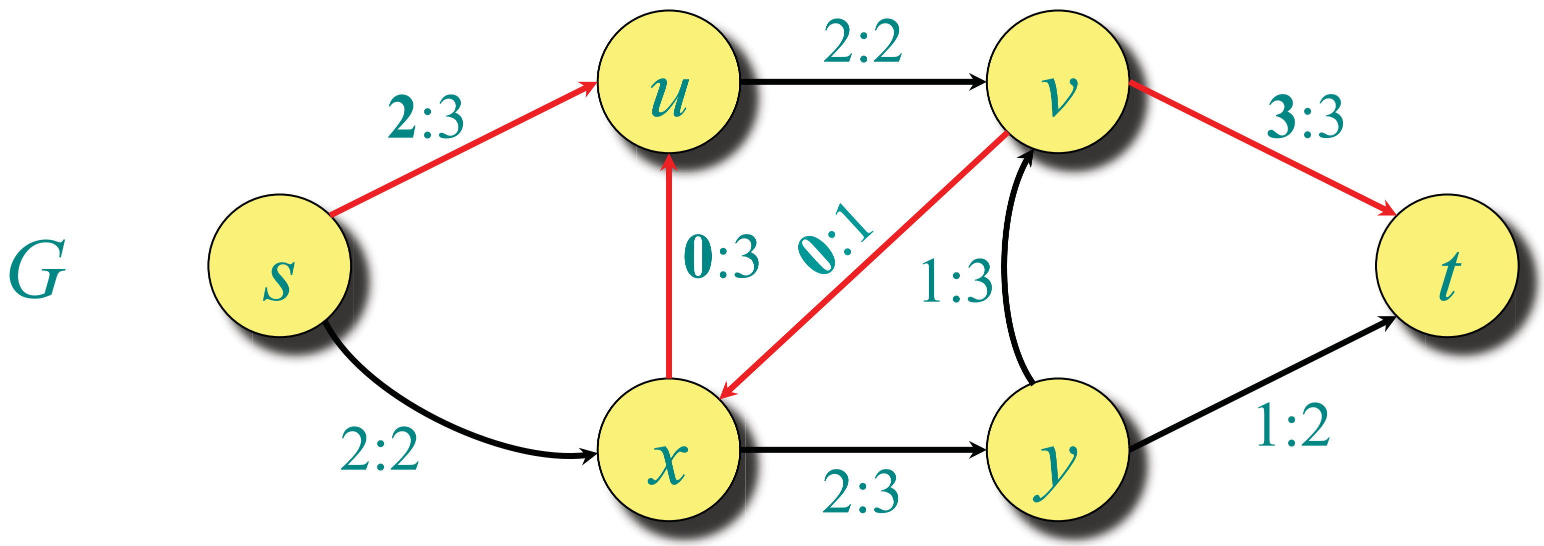


$p = \{s, u, x, v, t\}$, $c_f(p) = 1$



Augmented Flow Network

$$p = \{s, u, x, v, t\}, \;\; c_f(p) = 1$$



The value of the maximum flow is 4.

Note: Some flows on edges *decreased*.

Max-flow, min-cut theorem

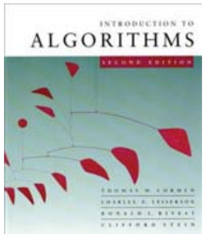
Theorem. The following are equivalent:

1. $|f| = c(S, T)$ for some cut (S, T).

2. f is a maximum flow.

3. f admits no augmenting paths.

Proof. Next time!



Ford-Fulkerson max-flow algorithm

Algorithm:

$f[u, v] \leftarrow 0$ for all $u, v \in V$

while an augmenting path p in G wrt f exists

do augment f by $c_f(p)$