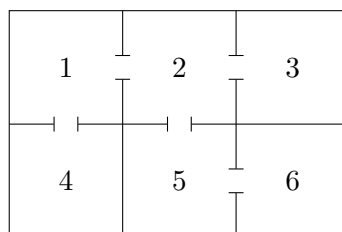


## Problem Sheet 5

Please send any comments or corrections to [martin@stats.ox.ac.uk](mailto:martin@stats.ox.ac.uk).

1. A bug jumps around the vertices of a triangle. At every jump, it moves from its current position to either of the other two vertices with probability  $1/2$  each (independently of how it arrived at its current position). The bug starts at vertex 1. Let  $p_n$  be the probability that it is at vertex 1 after  $n$  jumps. Find the value of  $p_n$  for each  $n$ . (*Hint: find an appropriate first-order linear recurrence relation.*) What happens to  $p_n$  as  $n \rightarrow \infty$ ?
2. The diagram below shows the floor plan of a house with six rooms: in room 1 is a mouse which will change rooms every minute, first moving at  $t = 1$  and choosing a door to an adjoining room at random. In room 6 is a sleeping but hungry cat which will instantly wake if the mouse should enter. How long on average can we expect the mouse to survive?



3. (**Gambler's ruin, symmetric case.**) A gambler starts a game with a bankroll of  $\mathcal{L}n$  where  $n \in \{1, 2, \dots, M-1\}$ . At each step of the game, he wins  $\mathcal{L}1$  with probability  $1/2$  and loses  $\mathcal{L}1$  with probability  $1/2$ , independently for different steps. The game ends when the gambler's bankroll reaches  $\mathcal{L}0$  or  $\mathcal{L}M$ .

In lectures we saw that the probability the gambler finishes with  $\mathcal{L}M$  is  $n/M$ .

- (a) What is the expected amount of money that the gambler has at the end of the game?
  - (b) Suppose we know that the gambler ends the game with  $\mathcal{L}M$ . What is the conditional probability that he won  $\mathcal{L}1$  on the first step?
  - (c) Let  $e_n$  be the expected length of the game. Find  $e_n$  for each  $n$ . For which  $n$  is  $e_n$  largest?
4. (a) Suppose that  $X$  has a geometric distribution with parameter  $p$ . Show that the probability generating function of  $X$  is

$$G_X(s) = \frac{ps}{1 - (1-p)s}, \quad \text{for } |s| < \frac{1}{1-p}.$$

- (b) Use this to calculate the mean and variance of  $X$ .

5. (a) A fair coin is tossed  $n$  times. Let  $r_n$  be the probability that the sequence of tosses never has a head followed by a head. Show that

$$r_n = \frac{1}{2}r_{n-1} + \frac{1}{4}r_{n-2}, \quad n \geq 2.$$

Find  $r_n$  using the conditions  $r_0 = r_1 = 1$ . Check that the value you get for  $r_2$  is correct.

- (b) Let  $X$  be the number of coin tosses needed until you first get two heads in a row. (Note that  $X \geq 2$ .) Find the probability mass function of  $X$ .
- (c) Find the probability generating function of  $X$ . Use this to calculate the mean of  $X$ . (*You may wish to check that your answer agrees with what you got for Q6 on Problem Sheet 3!*)
- (d) Let  $Y$  be the number of coin tosses needed until you first see a tail followed by a head. On any two particular coin tosses, the probability of seeing the pattern TH is  $1/4$ , the same as the probability of seeing the pattern HH. So  $\mathbb{P}(Y = 2) = \mathbb{P}(X = 2) = 1/4$ . Find  $\mathbb{P}(Y > n)$  for  $n \geq 1$  and compare it to  $\mathbb{P}(X > n)$ . Is your answer surprising?

6. (Optional. If you liked the coupon collector problem on sheet 3, you may enjoy this question too!)

Consider a symmetric random walk on a cycle with  $N$  sites, labelled  $0, 1, 2, \dots, N-1$ . A particle starts at site 0, and at each step it jumps from its current site  $i$  to one of its two neighbours  $i+1 \bmod N$  and  $i-1 \bmod N$  with equal probability (independently of how it arrived at its current position).

- (a) Find the expected number of steps until every site has been visited. [*Hint: just after a new site has been visited, what does the set of visited sites look like? The value  $e_1 = M-1$  from question 3(c) may be useful!*]
- (b) For each  $k = 1, \dots, N-1$ , what is the probability that  $k$  is the last site to be visited? [*Hint: before visiting site  $k$ , the walk must visit either site  $k-1$  or site  $k+1$ . What needs to happen from that point onwards?*]