Discrete Mathematics MT20: Problem Sheet 1

Chapter 1 (Sets)

1.1 Write the following sets explicitly, by listing their elements. In the case of infinite sets, describe all the elements concisely:

(i) $\{n \mid n \in \mathbb{Z} \text{ and } n^2 < 0\}$,

(ii) $\{n \mid n \in \mathbb{N} \text{ and } n^4 - 3n^2 + 2n = 0\}$,

(iii) $\{n^2 - n \mid n \in \mathbb{Z}_5\}$,

(iv) $\overline{\{1\} \cup \left(\bigcup_{i=2}^{\infty} A_i\right)}$, where $A_i = \{2i, 3i, 4i, \ldots\}$ and the universe is $\mathcal{U} = \mathbb{N}_+$.
Hint: compute the first few values of $\bigcup_{i=2}^{\infty} A_i$ *and see which numbers are missing.*

1.2 Suppose that $|A| = m$ and $|B| = n$. What are the maximum and minimum possible values for

(i) $|A \cup B|$, (ii) $|A \cap B|$, (iii) $|A \setminus B|$,
(iv) $|A \oplus B|$, (v) $|A \times B|$, (vi) $|\mathcal{P}(A)|$?

1.3 Which of these statements, which might be laws for symmetric difference, are true? There is no need to give proofs.

(i) Idempotence: $A \oplus A = A$,

(ii) Commutativity: $A \oplus B = B \oplus A$,

(iii) Associativity: $(A \oplus B) \oplus C = A \oplus (B \oplus C)$,

(iv) Distributivity of $\cup$ over $\oplus$: $A \cup (B \oplus C) = (A \cup B) \oplus (A \cup C)$,

(v) Distributivity of $\cap$ over $\oplus$: $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$.

1.4 Prove that $A \subseteq B$ and $A \subseteq C$ if and only if $A \subseteq B \cap C$.

Is it true that $A \subset B$ and $A \subset C$ if and only if $A \subset B \cap C$? Give a proof or counterexample.

1.5 One of the following is true and the other is false. Which is which? For the true statement give a proof, and for the false one find a counterexample.

(i) $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$;

(ii) $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$.

1.6 Using the algebraic laws for sets, in Claims 1.1 and 1.2 of the lecture notes, prove that

$$A \setminus \big((C \cap A) \cup B\big) = A \setminus (B \cup C).$$

1.7 The following is a "proof" of the **false** statement $A \setminus (B \cap C) \subseteq (A \setminus B) \cap (A \setminus C)$:

$$\begin{aligned} x \in A \setminus (B \cap C) \quad &\Rightarrow \quad x \in A \text{ and } x \notin B \cap C \\ &\Rightarrow \quad x \in A \text{ and } x \notin B \text{ and } x \notin C \\ &\Rightarrow \quad x \in A \setminus B \text{ and } x \in A \setminus C \\ &\Rightarrow \quad x \in (A \setminus B) \cap (A \setminus C) \end{aligned}$$

Find the mistake in the "proof" and give a counterexample to demonstrate that the statement is false.