Prelims Probability

James Martin
martin@stats.ox.ac.uk

Michaelmas Term 2019

(Version of 20 October 2019)

Background

Probability theory is one of the fastest growing areas of mathematics. Probabilistic arguments are used in a tremendous range of applications from number theory to genetics, from physics to finance. It is a core part of computer science and a key tool in analysis. And of course it underpins statistics. It is a subject that impinges on our daily lives: we come across it when we go to the doctor or buy a lottery ticket, but we're also using probability when we listen to the radio or use a mobile phone, or when we enhance digital images and when our immune system fights a cold. Whether you knew it or not, from the moment you were conceived, probability played an important role in your life.

We all have some idea of what probability is: maybe we think of it as an approximation to long run frequencies in a sequence of repeated trials, or perhaps as a measure of degree of belief warranted by some evidence. Each of these interpretations is valuable in certain situations. For example, the probability that I get a head if I flip a coin is sensibly interpreted as the proportion of heads I get if I flip that same coin many times. But there are some situations where it simply does not make sense to think of repeating the experiment many times. For example, the probability that 'UK interest rates will be more than 6% next March' or the probability that 'I'll be involved in a car accident in the next twelve months' cannot be determined by repeating the experiment many times and looking for a long run frequency.

The philosophical issue of interpretation is not one that we'll resolve in this course. What we *will* do is set up the abstract framework necessary to deal with complicated probabilistic questions.

These notes are intended to complement the contents of the lectures. They contain more material than the lectures and, in particular, a few more examples. To get the most out of the course, I strongly encourage you to attend all of the lectures. These notes are heavily based on a previous version by Christina Goldschmidt, who in turn drew on versions by Alison Etheridge, Neil Laws and Jonathan Marchini. I'm very glad to receive any comments or corrections at `martin@stats.ox.ac.uk`.

The synopsis and reading list from the course handbook are reproduced on the next page for your convenience. The suggested texts are an excellent source of further examples.

I hope you enjoy the course!

Overview

An understanding of random phenomena is becoming increasingly important in today's world within social and political sciences, finance, life sciences and many other fields. The aim of this introduction to probability is to develop the concept of chance in a mathematical framework. Random variables are introduced, with examples involving most of the common distributions.

Learning Outcomes

Students should have a knowledge and understanding of basic probability concepts, including conditional probability. They should know what is meant by a random variable, and have met the common distributions and their probability mass functions. They should understand the concepts of expectation and variance of a random variable. A key concept is that of independence which will be introduced for events and random variables.

Synopsis

Sample space, events, probability measure. Permutations and combinations, sampling with or without replacement. Conditional probability, partitions of the sample space, law of total probability, Bayes' Theorem. Independence.

Discrete random variables, probability mass functions, examples: Bernoulli, binomial, Poisson, geometric. Expectation, expectation of a function of a discrete random variable, variance. Joint distributions of several discrete random variables. Marginal and conditional distributions. Independence. Conditional expectation, law of total probability for expectations. Expectations of functions of more than one discrete random variable, covariance, variance of a sum of dependent discrete random variables.

Solution of first and second order linear difference equations. Random walks (finite state space only).

Probability generating functions, use in calculating expectations. Examples including random sums and branching processes.

Continuous random variables, cumulative distribution functions, probability density functions, examples: uniform, exponential, gamma, normal. Expectation, expectation of a function of a continuous random variable, variance. Distribution of a function of a single continuous random variable. Joint probability density functions of several continuous random variables (rectangular regions only). Marginal distributions. Independence. Expectations of functions of jointly continuous random variables, covariance, variance of a sum of dependent jointly continuous random variables.

Random sample, sums of independent random variables. Markov's inequality, Chebyshev's inequality, Weak Law of Large Numbers.

Textbooks

1. G. R. Grimmett and D. J. A. Welsh, *Probability: An Introduction*, 2nd edition, Oxford University Press, 2014, Chapters 1–5, 6.1–6.3, 7.1–7.3, 7.5 (Markov's inequality), 8.1-8.2, 10.4.
2. J. Pitman, *Probability*, Springer-Verlag, 1993.
3. S. Ross, *A First Course In Probability*, Prentice-Hall, 1994.
4. D. Stirzaker, *Elementary Probability*, Cambridge University Press, 1994, Chapters 1–4, 5.1–5.6, 6.1–6.3, 7.1, 7.2, 7.4, 8.1, 8.3, 8.5 (excluding the joint generating function).

A brief preview

Here are three problems that the techniques of this course will equip us to solve.

Euler's formula

One of the most important so-called "special functions" of mathematics is the Riemann zeta function. It is the function on which the famous Riemann hypothesis is formulated and it has long been known to have deep connections with the prime numbers. It is defined on the complex numbers, but here we just consider it as a function on real numbers s with $s > 1$. Then it can be written as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

An early connection between the zeta-function and the primes was established by Euler who showed that

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

There is a beautiful probabilistic proof of this theorem.

False positives

A lot of medical tests give a probabilistic outcome. Suppose that a laboratory test is 95% effective in detecting a certain disease when it is in fact present. However, the test also gives a *false positive* result for 1% of healthy people tested. (That is, if a healthy person is tested, then with probability 0.01 the test will imply that he has the disease.)

If 0.5% of the population actually has the disease, what is the probability that a randomly tested person has the disease given that the test result was positive?

Test your intuition by taking a guess now. This is the same sort of question that judges often face when presented with things like DNA evidence in court.

Gambler's ruin

A gambler enters a casino with £k. She repeatedly plays a game in which she wins £1 with probability p and loses £1 with probability $1-p$. She will leave the casino if she loses all her money or if her holding reaches the 'house limit' of £N.

What is the probability that she leaves with nothing?

What is the average number of games until she leaves?

Chapter 1

Events and probability

1.1 Introduction

We will think of performing an experiment which has a set of possible *outcomes* Ω. We call Ω the *sample space*. For example,

(a) tossing a coin: $\Omega = \{H, T\}$;

(b) throwing two dice: $\Omega = \{(i, j) : 1 \leq i, j \leq 6\}$.

A subset of Ω is called an *event*. An event $A \subseteq \Omega$ *occurs* if, when the experiment is performed, the outcome $\omega \in \Omega$ satisfies $\omega \in A$. You should think of events as things you can decide have or have not happened by looking at the outcome of your experiment. For example,

(a) coming up heads: $A = \{H\}$;

(b) getting a total of 4: $A = \{(1, 3), (2, 2), (3, 1)\}$.

The complement of A is $A^c := \Omega \setminus A$ and means "A does not occur". For events A and B,

$A \cup B$ means "at least one of A and B occurs";

$A \cap B$ means "both A and B occur";

$A \setminus B$ means "A occurs but B does not".

If $A \cap B = \emptyset$ we say that A and B are *disjoint* – they cannot both occur.

We assign a *probability* $\mathbb{P}(A) \in [0, 1]$ to each (suitable) event. For example,

(a) for a fair coin, $\mathbb{P}(A) = 1/2$;

(b) for two unweighted dice, $\mathbb{P}(A) = 1/12$.

(b) demonstrates the importance of *counting* in the situation where we have a finite number of possible outcomes to our experiment, all equally likely. For (b), Ω has 36 elements (6 ways of choosing i and 6 ways of choosing j). Since $A = \{(1,3),(2,2),(3,1)\}$ contains 3 sample points, and all sample points are equally likely, we get $\mathbb{P}(A) = 3/36 = 1/12$.

We want to be able to tackle much more complicated counting problems.

1.2 Counting

Most of you will have seen this before. If you haven't, or if you find it confusing, then you can find more details in the first chapter of *Introduction to Probability* by Ross.

Arranging distinguishable objects

Suppose that we have n distinguishable objects (e.g. the numbers $1, 2, \ldots, n$). How many ways to order them (*permutations*) are there? If we have three objects a, b, c then the answer is 6: *abc*, *acb*, *bac*, *bca*, *cab* and *cba*.

In general, there are n choices for the first object in our ordering. Then, whatever the first object was, we have $n-1$ choices for the second object. Carrying on, we have $n-m+1$ choices for the mth object and, finally, a single choice for the nth. So there are

$$n(n-1)\ldots 2.1 = n!$$

different orderings.

Since $n!$ increases extremely fast, it is sometimes useful to know Stirling's formula:

$$n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n},$$

where $f(n) \sim g(n)$ means $f(n)/g(n) \to 1$ as $n \to \infty$. This is astonishingly accurate even for quite small n. For example, the error is of the order of 1% when $n = 10$.

Arrangements when not all objects are distinguishable

What happens if not all the objects are distinguishable? For example, how many different arrangements are there of a, a, a, b, c, d?

If we had a_1, a_2, a_3, b, c, d, there would be 6! arrangements. Each arrangement (e.g. b, a_2, d, a_3, a_1, c) is one of 3! which differ only in the ordering of a_1, a_2, a_3. So the 6! arrangements fall into groups of size 3! which are indistinguishable when we put $a_1 = a_2 = a_3$. We want the number of groups which is just $6!/3!$.

We can immediately generalise this. For example, to count the arrangements of a, a, a, b, b, d, play the same game. We know how many arrangements there are if the b's are distinguishable, but then all such arrangements fall into pairs which differ only in the ordering of b_1, b_2, and we see that the number of arrangements is $6!/3!2!$.

Lemma 1.1. *The number of arrangements of the n objects*

$$\underbrace{\alpha_1, \ldots, \alpha_1}_{m_1 \text{ times}}, \underbrace{\alpha_2, \ldots, \alpha_2}_{m_2 \text{ times}}, \ldots, \underbrace{\alpha_k, \ldots, \alpha_k}_{m_k \text{ times}}$$

where α_i appears m_i times and $m_1 + \cdots + m_k = n$ is

$$\frac{n!}{m_1! m_2! \cdots m_k!}. \tag{1.1}$$

Example 1.2. *The number of arrangements of the letters of STATISTICS is $\frac{10!}{3!3!2!}$.*

If there are just two types of object then, since $m_1 + m_2 = n$, the expression (1.1) is just a binomial coefficient, $\binom{n}{m_1} = \frac{n!}{m_1!(n-m_1)!} = \binom{n}{m_2}$.

Note: we will always use the notation

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}.$$

Recall the binomial theorem,

$$(x+y)^n = \sum_{m=0}^{n} \binom{n}{m} x^m y^{n-m}.$$

You can see where the binomial coefficient comes from because writing

$$(x+y)^n = (x+y)(x+y)\cdots(x+y)$$

and multiplying out, each term involves one pick from each bracket. The coefficient of $x^m y^{n-m}$ is the number of sequences of picks that give x exactly m times and y exactly $n-m$ times and that's the number of ways of choosing the m "slots" for the x's.

The expression (1.1) is called a *multinomial* coefficient because it is the coefficient of $a_1^{m_1} \cdots a_k^{m_k}$ in the expansion of

$$(a_1 + \cdots + a_k)^n$$

where $m_1 + \cdots m_k = n$. We sometimes write

$$\binom{n}{m_1 \; m_2 \; \ldots \; m_k}$$

for the multinomial coefficient.

Instead of thinking in terms of arrangements, we can think of our binomial coefficient in terms of choices. For example, if I have to choose a committee of size k from n people, there are $\binom{n}{k}$ ways to do it. To see how this ties in, stand the n people in a line. For each arrangement of k 1's and $n-k$ 0's I can create a different committee by picking the ith person for the committee if the ith term in the arrangement is a 1.

Many counting problems can be solved by finding a bijection (that is, a one-to-one correspondence) between the objects we want to enumerate and other objects that we already know how to enumerate.

Example 1.3. *How many distinct non-negative integer-valued solutions of the equation*

$$x_1 + x_2 + \cdots + x_m = n$$

are there?

Solution. Consider a sequence of n $\star$'s and $m-1$ |'s. There is a bijection between such sequences and non-negative integer-valued solutions to the equation. For example, if $m=4$ and $n=3$,

$$\underbrace{\star\,\star}_{x_1=2}\Big|\underbrace{\Big|}_{x_2=0}\Big|\underbrace{\star}_{x_3=1}\Big|\underbrace{\Big|}_{x_4=0}$$

There are $\binom{n+m-1}{n}$ sequences of n $\star$'s and $m-1$ |'s and, hence, the same number of solutions to the equation.

It is often possible to perform quite complex counting arguments by manipulating binomial coefficients. Conversely, sometimes one wants to prove relationships between binomial coefficients and this can most easily be done by a counting argument. Here is one famous example:

Lemma 1.4 (Vandermonde's identity)**.** *For $k, m, n \geq 0$,*

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}, \tag{1.2}$$

where we use the convention $\binom{m}{j} = 0$ for $j > m$.

Proof. Suppose we choose a committee consisting of k people from a group of m men and n women. There are $\binom{m+n}{k}$ ways of doing this which is the left-hand side of (1.2).

Now the number of men in the committee is some $j \in \{0, 1, \ldots, k\}$ and then it contains $k-j$ women. The number of ways of choosing the j men is $\binom{m}{j}$ and for each such choice there are $\binom{n}{k-j}$ choices for the women who make up the rest of the committee. So there are $\binom{m}{j}\binom{n}{k-j}$ committees with exactly j men and summing over j we get that the total number of committees is given by the right-hand side of (1.2). □

"Breaking things down" is an important technique in counting - and also, as we'll see, in probability.

An aside on sizes of sets

In this course, we will often deal with finite collections of objects, as in our counting examples. We will also want to be able to deal with infinite sets, and we will want to distinguish between those that are *countable* and those that are *uncountable*. A countable set S is one which is either finite or such that all of its elements can be labelled by a natural number in such a way that we can write them in a list: $S = \{x_1, x_2, x_3, \ldots\} = \{x_i : i \in \mathbb{N}\}$. If a set is not countable, it is uncountable. The natural numbers are themselves countable (take $x_i = i$), as are the rational numbers, but the real numbers are not. (Those of you doing Analysis I will see much more about this there.)

1.3 The axiomatic approach

Definition 1.5. *A* probability space *is a triple $(\Omega, \mathcal{F}, \mathbb{P})$ where*

1. *Ω is the* sample space,

2. *$\mathcal{F}$ is a collection of subsets of Ω, called* events*, satisfying axioms $\boldsymbol{F}_1$–$\boldsymbol{F}_3$ below,*

3. *$\mathbb{P}$ is a* probability measure*, which is a function $\mathbb{P} : \mathcal{F} \to [0,1]$ satisfying axioms $\boldsymbol{P}_1$–$\boldsymbol{P}_3$ below.*[1]

Before formulating the axioms $\mathbf{F}_1$–$\mathbf{F}_3$ and $\mathbf{P}_1$–$\mathbf{P}_3$ we should do an example. Many of the more abstract books on probability start every section with "Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space" but we shouldn't allow ourselves to be intimidated. Here's an example to see why.

Example 1.6. *We set up a probability space to model each of the following experiments:*

1. *A single roll of a fair die in which the outcome we observe is the number thrown;*

2. *A single roll of two fair dice in which the outcome we observe is the sum of the two numbers thrown (so in particular we may not see what the individual numbers are).*

Single die. The set of outcomes of our experiment, that is our *sample space*, is $\Omega_1 = \{1,2,3,4,5,6\}$. The *events* are all possible subsets of this; denote the set of all subsets of Ω_1 by $\mathcal{F}_1$. For example, $E_1 = \{6\}$ is the event "the result is a 6" and $E_2 = \{2,4,6\}$ is the event "the result is even".
We're told that the die is fair so $\mathbb{P}_1(\{i\})$ is just $1/6$ and $\mathbb{P}_1(E) = \frac{1}{6}|E|$ where $|E|$ is the number of distinct elements in the subset E. Hence, $\mathbb{P}_1(E_1) = \frac{1}{6}$ and $\mathbb{P}_1(E_2) = \frac{1}{2}$.
Formally, $\mathbb{P}_1$ is a function on $\mathcal{F}_1$ which assigns a number from $[0,1]$ to each element of $\mathcal{F}_1$.

The total on two dice. The set of outcomes that we can actually observe is $\Omega_2 = \{2,3,4,\ldots,12\}$. We take $\mathcal{F}_2$ to be the set of all subsets of Ω_2. So for example $E_3 = \{2,4,6,8,10,12\}$ is the event "the outcome is even", $E_4 = \{2,3,5,7,11\}$ is the event "the outcome is prime" and so on.
Notice now however that *not all outcomes are equally likely.* However, tabulating all possible numbers shown on the two dice we get

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

and all of these outcomes *are* equally likely. So now we can just count to work out the probability of each event in $\mathcal{F}_2$. For example $\mathbb{P}_2(\{12\}) = \frac{1}{36}$, $\mathbb{P}_2(\{7\}) = \frac{1}{6}$, $\mathbb{P}_2(E_3) = \frac{1}{2}$ and $\mathbb{P}_2(E_4) = \frac{15}{36}$.
The probability measure is still a $[0,1]$-valued function on $\mathcal{F}_2$, but this time it is a more interesting one.

This second example raises a very important point. The sample space that we use in modelling a particular experiment is *not unique*. In fact, to calculate the probabilities $\mathbb{P}_2$, in effect we took a larger sample space $\Omega_2' = \{(i,j) : i,j \in \{1,2,\ldots,6\}\}$ that records the pair of numbers thrown. But the only events that we are interested in for this particular experiment are those that tell us something about the *sum* of the numbers thrown.

In order to make sure that the theory we build is internally consistent, we need to make some assumptions about $\mathcal{F}$ and $\mathbb{P}$, in the form of axioms. Informally, we would like to have that

[1] $\mathbb{P} : \mathcal{F} \to [0,1]$ means that to each element A of $\mathcal{F}$, we associate a number between 0 and 1 which we call $\mathbb{P}(A)$. $\mathbb{P}$ is a *function* or *mapping* from $\mathcal{F}$ to $[0,1]$. Compare to a situation you are more familiar with: if $f(x) = x^2$ then we say that f is a function from $\mathbb{R}$ to $\mathbb{R}$ (or $f : \mathbb{R} \to \mathbb{R}$ for short).

1. the probability of any event A is between 0 and 1;
2. the event $\emptyset$ that "no outcome occurs" has probability 0 and the event Ω consisting of *all* possible outcomes has probability 1;
3. if A and B are *disjoint events* (i.e. $A \cap B = \emptyset$), with the interpretation that A and B cannot occur simultaneously, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$.

(We already used 3. to calculate the probabilities of events in our examples.)

Formally, our axioms are as follows:

The axioms of probability

$\mathcal{F}$ is a collection of subsets of Ω, with:
$\mathbf{F}_1$: $\emptyset \in \mathcal{F}$.
$\mathbf{F}_2$: If $A \in \mathcal{F}$, then also $A^c \in \mathcal{F}$.
$\mathbf{F}_3$: If $\{A_i, i \in I\}$ is a finite or countably infinite collection of members of $\mathcal{F}$, then $\cup_{i \in I} A_i \in \mathcal{F}$.

$\mathbb{P}$ is a function from $\mathcal{F}$ to $\mathbb{R}$, with:
$\mathbf{P}_1$: For all $A \in \mathcal{F}$, $\mathbb{P}(A) \geq 0$.
$\mathbf{P}_2$: $\mathbb{P}(\Omega) = 1$.
$\mathbf{P}_3$: If $\{A_i, i \in I\}$ is a finite or countably infinite collection of members of $\mathcal{F}$, and $A_i \cap A_j = \emptyset$ for $i \neq j$, then $\mathbb{P}(\cup_{i \in I} A_i) = \sum_{i \in I} \mathbb{P}(A_i)$.

Note that in particular:

$$\mathbf{P}_3 \text{ (special case): If } A, B \in \mathcal{F} \text{ with } A \cap B = \emptyset, \text{ then } \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B).$$

(You may wonder whether this special case is enough to imply the full statement of $\mathbf{P}_3$. It's easy to show, for example by induction, that this special case of the statement for two events implies the statement for all finite collections of events. It turns out that the statement about countably infinite collections is genuinely stronger – but this is rather a subtle point involving intricacies of set theory!)

In the examples above, Ω was finite. In general Ω may be finite, countably infinite, or uncountably infinite. If Ω is finite or countable, as it usually will be for the first half of this course, then we normally take $\mathcal{F}$ to be the set of all subsets of Ω (the *power set* of Ω). (You should check that, in this case, $\mathbf{F}_1$–$\mathbf{F}_3$ are satisfied.) If Ω is uncountable, however, the set of all subsets turns out to be *too large*: it ends up containing sets to which we cannot consistently assign probabilities. This is an issue which some of you will see discussed properly in next year's Part A Integration course; for the moment, you shouldn't worry about it, just make a mental note that there is something to be resolved here.

Example 1.7. *Consider a countable set $\Omega = \{\omega_1, \omega_2, \ldots\}$ and an arbitrary collection $(p_1, p_2, \ldots)$ of non-negative numbers with sum $\sum_{i=1}^{\infty} p_i = 1$. Put*

$$\mathbb{P}(A) = \sum_{i : \omega_i \in A} p_i.$$

Then $\mathbb{P}$ satisfies $\mathbf{P}_1$–$\mathbf{P}_3$. The numbers $(p_1, p_2, \ldots)$ are called a probability distribution.

Example 1.8. *Pick a team of m players from a squad of n, all possible teams being equally likely. Set*

$$\Omega = \left\{ (i_1, i_2, \ldots, i_n) : i_k = 0 \text{ or } 1 \text{ and } \sum_{k=1}^{n} i_k = m \right\},$$

where

$$i_k = \begin{cases} 1 & \text{if player } k \text{ is picked,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $A = \{$*player 1 is in the team*$\}$*. Then*

$$\mathbb{P}(A) = \frac{\#\text{teams that include player 1}}{\#\text{possible teams}} = \frac{\binom{n-1}{m-1}}{\binom{n}{m}} = \frac{m}{n}.$$

We can derive some useful consequences of the axioms.

Theorem 1.9. *Suppose that* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a probability space and that* $A, B \in \mathcal{F}$*. Then*

1. $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$*;*
2. *If* $A \subseteq B$ *then* $\mathbb{P}(A) \leq \mathbb{P}(B)$*.*

Proof. 1. Since $A \cup A^c = \Omega$ and $A \cap A^c = \emptyset$, by $\mathbf{P}_3$, $\mathbb{P}(\Omega) = \mathbb{P}(A) + \mathbb{P}(A^c)$. By $\mathbf{P}_2$, $\mathbb{P}(\Omega) = 1$ and so $\mathbb{P}(A) + \mathbb{P}(A^c) = 1$, which entails the required result.

2. Since $A \subseteq B$, we have $B = A \cup (B \cap A^c)$. Since $B \cap A^c \subseteq A^c$, it must be disjoint from A. So by $\mathbf{P}_3$, $\mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(B \cap A^c)$. Since by $\mathbf{P}_1$, $\mathbb{P}(B \cap A^c) \geq 0$, we thus have $\mathbb{P}(B) \geq \mathbb{P}(A)$. □

Some other useful consequences are on the problem sheet.

1.4 Conditional probability

We have seen how to formalise the notion of probability. So for each event, which we thought of as an observable outcome of an experiment, we have a probability (a likelihood, if you prefer). But of course our assessment of likelihoods changes as we acquire more information and our next task is to formalise that idea. First, to get a feel for what I mean, let's look at a simple example.

Example 1.10. *Suppose that in a single roll of a fair die we know that the outcome is an even number. What is the probability that it is in fact a six?*

Solution. Let $B = \{$result is even$\} = \{2, 4, 6\}$ and $C = \{$result is a six$\} = \{6\}$. Then $\mathbb{P}(B) = \frac{1}{2}$ and $\mathbb{P}(C) = \frac{1}{6}$, but if I *know* that B has happened, then $\mathbb{P}(C|B)$ (read "the probability of C given B") is $\frac{1}{3}$ because given that B happened, we know the outcome was one of $\{2, 4, 6\}$ and since the die is fair, in the absence of any other information, we assume each of these is equally likely.

Now let $A = \{$result is divisible by 3$\} = \{3, 6\}$. If we know that B happened, then the only way that A can also happen is if the outcome is in $A \cap B$, in this case if the outcome is $\{6\}$ and so $\mathbb{P}(A|B) = \frac{1}{3}$ again which is $\mathbb{P}(A \cap B)/\mathbb{P}(B)$.

Definition 1.11. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space. If* $A, B \in \mathcal{F}$ *and* $\mathbb{P}(B) > 0$ *then the* conditional probability of A given B *is*

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

(If $\mathbb{P}(B) = 0$*, then* $\mathbb{P}(A|B)$ *is not defined.)*

We should check that this new notion fits with our idea of probability. The next theorem says that it does.

Theorem 1.12. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $B \in \mathcal{F}$ satisfy $\mathbb{P}(B) > 0$. Define a new function $\mathbb{Q} : \mathcal{F} \to \mathbb{R}$ by $\mathbb{Q}(A) = \mathbb{P}(A|B)$. Then $(\Omega, \mathcal{F}, \mathbb{Q})$ is also a probability space.*

Proof. Because we're using the same $\mathcal{F}$, we need only check axioms $\mathbf{P}_1$–$\mathbf{P}_3$.

$\mathbf{P}_1$. For any $A \in \mathcal{F}$,

$$\mathbb{Q}(A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \geq 0.$$

$\mathbf{P}_2$. By definition,

$$\mathbb{Q}(\Omega) = \frac{\mathbb{P}(\Omega \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1.$$

$\mathbf{P}_3$. For disjoint events $A_1, A_2, \ldots$,

$$\begin{aligned}
\mathbb{Q}(\cup_{i=1}^{\infty} A_i) &= \frac{\mathbb{P}((\cup_{i=1}^{\infty} A_i) \cap B)}{\mathbb{P}(B)} \\
&= \frac{\mathbb{P}\left(\cup_{i=1}^{\infty} (A_i \cap B)\right)}{\mathbb{P}(B)} \\
&= \frac{\sum_{i=1}^{\infty} \mathbb{P}(A_i \cap B)}{\mathbb{P}(B)} \quad \text{(because } A_i \cap B, \, i \geq 1, \text{ are disjoint)} \\
&= \sum_{i=1}^{\infty} \mathbb{Q}(A_i).
\end{aligned}$$

□

From the definition of conditional probability, we get a very useful multiplication rule:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\,\mathbb{P}(B). \tag{1.3}$$

This generalises to

$$\mathbb{P}(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n) = \mathbb{P}(A_1)\,\mathbb{P}(A_2|A_1)\,\mathbb{P}(A_3|A_1 \cap A_2) \ldots \mathbb{P}(A_n|A_1 \cap A_2 \cap \ldots \cap A_{n-1}) \tag{1.4}$$

(you can prove this by induction).

Example 1.13. *An urn contains 8 red balls and 4 white balls. We draw 3 balls at random* without replacement. *Let $R_i = \{$the ith ball is red$\}$ for $1 \leq i \leq 3$. Then*

$$\mathbb{P}(R_1 \cap R_2 \cap R_3) = \mathbb{P}(R_1)\,\mathbb{P}(R_2|R_1)\,\mathbb{P}(R_3|R_1 \cap R_2) = \frac{8}{12} \cdot \frac{7}{11} \cdot \frac{6}{10} = \frac{14}{55}.$$

Example 1.14. *A bag contains 26 tickets, one with each letter of the alphabet. If six tickets are drawn at random from the bag (without replacement), what is the chance that they can be rearranged to spell CALVIN ?*

Solution. Write A_i for the event that the ith ticket drawn is from the set $\{C, A, L, V, I, N\}$. By (1.4),

$$\mathbb{P}(A_1 \cap \ldots \cap A_6) = \frac{6}{26} \cdot \frac{5}{25} \cdot \frac{4}{24} \cdot \frac{3}{23} \cdot \frac{2}{22} \cdot \frac{1}{21}.$$

Example 1.15. *A bitstream when transmitted has*

$$\mathbb{P}(0 \text{ } sent) = \frac{4}{7}, \qquad \mathbb{P}(1 \text{ } sent) = \frac{3}{7}.$$

Owing to noise,

$$\mathbb{P}(1 \text{ } received \mid 0 \text{ } sent) = \frac{1}{8},$$
$$\mathbb{P}(0 \text{ } received \mid 1 \text{ } sent) = \frac{1}{6}.$$

What is $\mathbb{P}(0 \text{ } sent \mid 0 \text{ } received)$*?*

Solution. Using the definition of conditional probability,

$$\mathbb{P}(0 \text{ sent} \mid 0 \text{ received}) = \frac{\mathbb{P}(0 \text{ sent and } 0 \text{ received})}{\mathbb{P}(0 \text{ received})}.$$

Now

$$\mathbb{P}(0 \text{ received}) = \mathbb{P}(0 \text{ sent and } 0 \text{ received}) + \mathbb{P}(1 \text{ sent and } 0 \text{ received}).$$

Now we use (1.3) to get

$$\begin{aligned}\mathbb{P}(0 \text{ sent and } 0 \text{ received}) &= \mathbb{P}(0 \text{ received} \mid 0 \text{ sent})\mathbb{P}(0 \text{ sent}) \\ &= \big(1 - \mathbb{P}(1 \text{ received} \mid 0 \text{ sent})\big)\mathbb{P}(0 \text{ sent}) \\ &= \left(1 - \frac{1}{8}\right)\frac{4}{7} = \frac{1}{2}.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathbb{P}(1 \text{ sent and } 0 \text{ received}) &= \mathbb{P}(0 \text{ received} \mid 1 \text{ sent})\mathbb{P}(1 \text{ sent}) \\ &= \frac{1}{6} \cdot \frac{3}{7} = \frac{1}{14}.\end{aligned}$$

Putting these together gives

$$\mathbb{P}(0 \text{ received}) = \frac{1}{2} + \frac{1}{14} = \frac{8}{14}$$

and

$$\mathbb{P}(0 \text{ sent} \mid 0 \text{ received}) = \frac{\frac{1}{2}}{\frac{8}{14}} = \frac{7}{8}.$$

1.5 Independence

Of course, knowing that B has happened doesn't always influence the chances of A.

Definition 1.16. *1. Events A and B are* independent *if* $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

2. More generally, a family of events $\mathcal{A} = \{A_i : i \in I\}$ is independent *if*

$$\mathbb{P}\left(\bigcap_{i \in J} A_i\right) = \prod_{i \in J} \mathbb{P}(A_i)$$

for all finite subsets J of I.

3. A family $\mathcal{A}$ of events is pairwise independent *if $\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i)\mathbb{P}(A_j)$ whenever $i \neq j$.*

WARNING: PAIRWISE INDEPENDENT DOES NOT IMPLY INDEPENDENT.

See the problem sheet for an example of this.

Suppose that A and B are independent. Then if $\mathbb{P}(B) > 0$, we have $\mathbb{P}(A|B) = \mathbb{P}(A)$, and if $\mathbb{P}(A) > 0$, we have $\mathbb{P}(B|A) = \mathbb{P}(B)$. In other words, knowledge of the occurrence of B does not influence the probability of A, and vice versa.

Example 1.17. *Suppose we have two fair dice. Let*

$$A = \{\textit{first die shows 4}\}, \quad B = \{\textit{total score is 6}\} \quad \textit{and} \quad C = \{\textit{total score is 7}\}.$$

Then

$$\mathbb{P}(A \cap B) = \mathbb{P}(\{(4,2)\}) = \frac{1}{36}$$

but

$$\mathbb{P}(A)\,\mathbb{P}(B) = \frac{1}{6} \cdot \frac{5}{36} \neq \frac{1}{36}.$$

So A and B are not independent. However, A and C are *independent (you should check this).*

Theorem 1.18. *Suppose that A and B are independent. Then*

(a) A and B^c are independent;

(b) A^c and B^c are independent.

Proof. (a) We have $A = (A \cap B) \cup (A \cap B^c)$, where $A \cap B$ and $A \cap B^c$ are disjoint, so using the independence of A and B,

$$\mathbb{P}(A \cap B^c) = \mathbb{P}(A) - \mathbb{P}(A \cap B) = \mathbb{P}(A) - \mathbb{P}(A)\,\mathbb{P}(B) = \mathbb{P}(A)\,(1 - \mathbb{P}(B)) = \mathbb{P}(A)\,\mathbb{P}(B^c)\,.$$

(b) Apply part (a) to the events B^c and A. □

More generally, if $\{A_i, i \in I\}$ is any family of independent events, then also the family $\{A_i^c, i \in I\}$ is independent. Proof: exercise! (We need to show that the product formula in Definition 1.16 holds for all finite subsets $\{A_i^c, i \in J\}$). One approach is to use the inclusion-exclusion formula from Problem Sheet 1; various induction arguments are also possible.)

1.6 The law of total probability and Bayes' theorem

Definition 1.19. *A family of events $\{B_1, B_2, \ldots\}$ is a* partition *of Ω if*

1. *$\Omega = \bigcup_{i \geq 1} B_i$ (so that at least one B_i must happen), and*
2. *$B_i \cap B_j = \emptyset$ whenever $i \neq j$ (so that no two can happen together).*

Theorem 1.20 (The law of total probability)**.** *Suppose* $\{B_1, B_2, \ldots\}$ *is a partition of* Ω *by sets from* $\mathcal{F}$*, such that* $\mathbb{P}(B_i) > 0$ *for all* $i \geq 1$*. Then for any* $A \in \mathcal{F}$*,*

$$\mathbb{P}(A) = \sum_{i \geq 1} \mathbb{P}(A|B_i)\mathbb{P}(B_i).$$

This result is sometimes also called the *partition theorem*. We used it in our bitstream example to calculate the probability that 0 was received.

Proof. We have

$$\begin{aligned}
\mathbb{P}(A) &= \mathbb{P}(A \cap (\cup_{i \geq 1} B_i)), \text{ since } \cup_{i \geq 1} B_i = \Omega \\
&= \mathbb{P}(\cup_{i \geq 1}(A \cap B_i)) \\
&= \sum_{i \geq 1} \mathbb{P}(A \cap B_i) \text{ by axiom } \mathbf{P}_3, \text{ since } A \cap B_i,\ i \geq 1 \text{ are disjoint} \\
&= \sum_{i \geq 1} \mathbb{P}(A|B_i)\,\mathbb{P}(B_i).
\end{aligned}$$

□

Note that if $\mathbb{P}(B_i) = 0$ for some i, then the expression in Theorem 1.20 wouldn't make sense, since $\mathbb{P}(A|B_i)$ is undefined. (Although we could agree a convention by which $\mathbb{P}(A|B)\mathbb{P}(B)$ means 0 whenever $\mathbb{P}(B) = 0$; then we can make sense of the expression in Theorem 1.20 even if some of the B_i have zero probability.) In any case, we can still write $\mathbb{P}(A) = \sum_i \mathbb{P}(A \cap B_i)$.

Example 1.21. *Crossword setter I composes* m *clues; setter II composes* n *clues. Alice's probability of solving a clue is* α *if the clue was composed by setter I and* β *if the clue was composed by setter II.*

Alice chooses a clue at random. What is the probability she solves it?

Solution. Let

$$\begin{aligned}
A &= \{\text{Alice solves the clue}\} \\
B_1 &= \{\text{clue composed by setter I}\}, \\
B_2 &= \{\text{clue composed by setter II}\}.
\end{aligned}$$

Then

$$\mathbb{P}(B_1) = \frac{m}{m+n}, \quad \mathbb{P}(B_2) = \frac{n}{m+n}, \quad \mathbb{P}(A|B_1) = \alpha, \quad \mathbb{P}(A|B_2) = \beta.$$

By the law of total probability,

$$\mathbb{P}(A) = \mathbb{P}(A|B_1)\,\mathbb{P}(B_1) + \mathbb{P}(A|B_2)\,\mathbb{P}(B_2) = \frac{\alpha m}{m+n} + \frac{\beta n}{m+n} = \frac{\alpha m + \beta n}{m+n}.$$

In our solution to Example 1.15, we combined the law of total probability with the definition of conditional probability. In general, this technique has a name:

Theorem 1.22 (Bayes' Theorem)**.** *Suppose that* $\{B_1, B_2, \ldots\}$ *is a partition of* Ω *by sets from* $\mathcal{F}$ *such that* $\mathbb{P}(B_i) > 0$ *for all* $i \geq 1$*. Then for any* $A \in \mathcal{F}$ *such that* $\mathbb{P}(A) > 0$*,*

$$\mathbb{P}(B_k|A) = \frac{\mathbb{P}(A|B_k)\mathbb{P}(B_k)}{\sum_{i \geq 1} \mathbb{P}(A|B_i)\mathbb{P}(B_i)}.$$

Proof. We have

$$\begin{aligned}\mathbb{P}(B_k|A) &= \frac{\mathbb{P}(B_k \cap A)}{\mathbb{P}(A)} \\ &= \frac{\mathbb{P}(A|B_k)\mathbb{P}(B_k)}{\mathbb{P}(A)}.\end{aligned}$$

Now substitute for $\mathbb{P}(A)$ using the law of total probability. □

In Example 1.15, we calculated $\mathbb{P}(0 \text{ sent} \mid 0 \text{ received})$ by taking $\{B_1, B_2, \ldots\}$ to be $B_1 = \{0 \text{ sent}\}$ and $B_2 = \{1 \text{ sent}\}$ and A to be the event $\{0 \text{ received}\}$.

Example 1.23. *Recall Alice, from Example 1.21. Suppose that she chooses a clue at random and solves it. What is the probability that the clue was composed by setter I?*

Solution. Using Bayes' theorem,

$$\begin{aligned}\mathbb{P}(B_1|A) &= \frac{\mathbb{P}(A|B_1)\mathbb{P}(B_1)}{\mathbb{P}(A|B_1)\mathbb{P}(B_1) + \mathbb{P}(A|B_2)\mathbb{P}(B_2)} \\ &= \frac{\frac{\alpha m}{m+n}}{\frac{\alpha m}{m+n} + \frac{\beta n}{m+n}} \\ &= \frac{\alpha m}{\alpha m + \beta n}.\end{aligned}$$

Example 1.24 (Simpson's paradox)**.** *Consider the following table showing a comparison of the outcomes of two types of surgery for the removal of kidney stones (from Charig et al, 1986):*

	Number	Success rate
Treatment A (open surgery)	350	(273/350 =) 0.78
Treatment B (nephrolithotomy)	350	(289/350 =) 0.83

On the basis of this comparison, it looks like Treatment B has performed slightly better than Treatment A. A closer analysis of the data divides the patients into two groups, according to the sizes of the stones:

	Type I (stone < 2cm)		Type II (stone > 2cm)	
	Number	Success rate	Number	Success rate
Treatment A	87	(81/87 =) 0.93	263	(192/263 =) 0.73
Treatment B	270	(234/270 =) 0.87	80	(55/80 =) 0.69

Now Treatment A appears to beat Treatment B both in patients of Type I, and in patients of Type II. Our initial analysis seems to have been misleading because of a "confounding variable", the severity of the case. Looking at the second table, we can see that patients of Type II are harder to treat; Treatment A was more often given to these harder cases, and Treatment B to easier cases. This made Treatment B appear to perform better overall.

This is Simpson's paradox*; in conditional probability language, it consists of the fact that for events E, F, G, we can have*

$$\begin{aligned}\mathbb{P}(E|F \cap G) &> \mathbb{P}(E|F^c \cap G) \\ \mathbb{P}(E|F \cap G^c) &> \mathbb{P}(E|F^c \cap G^c)\end{aligned}$$

and yet

$$\mathbb{P}(E|F) < \mathbb{P}(E|F^c).$$

(Exercise: identify corresponding events E, F and G in the example above.)

1.7 Some useful rules for calculating probabilities

If you're faced with a probability calculation you don't know how to do, here are some things to try.

- **AND**: Try using the multiplication rule:

  $$\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\,\mathbb{P}(B) = \mathbb{P}(B|A)\,\mathbb{P}(A)$$

  or its generalisation:

  $$\mathbb{P}(A_1 \cap A_2 \cap \ldots \cap A_n) = \mathbb{P}(A_1)\,\mathbb{P}(A_2|A_1)\ldots\mathbb{P}(A_n|A_1 \cap A_2 \cap \ldots \cap A_{n-1})$$

  (as long as all of the conditional probabilities are defined).

- **OR**: If the events are disjoint, use

  $$\mathbb{P}(A_1 \cup A_2 \cup \ldots \cup A_n) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \cdots + \mathbb{P}(A_n).$$

  Otherwise, try taking complements:

  $$\mathbb{P}(A_1 \cup A_2 \cup \ldots \cup A_n) = 1 - \mathbb{P}((A_1 \cup A_2 \cup \ldots \cup A_n)^c) = 1 - \mathbb{P}(A_1^c \cap A_2^c \cap \ldots \cap A_n^c)$$

  ("the probability at least one of the events occurs is 1 minus the probability that none of them occur"). If that's no use, try using the inclusion-exclusion formula (see the problem sheet):

  $$\mathbb{P}(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \cdots + (-1)^{n+1}\mathbb{P}(A_1 \cap A_2 \cap \ldots \cap A_n).$$

- If you can't calculate the probability of your event directly, try splitting it up according to some partition of Ω and using the law of total probability.

> Useful check: any probability that you calculate should be in the interval $[0, 1]$! If not, something, has gone wrong....

Chapter 2

Discrete random variables

Interesting information about the outcome of an experiment can often be encoded as a number. For example, suppose that I am modelling the arrival of telephone calls at an exchange. Modelling this directly could be very complicated: my sample space should include all of the possible starting and finishing times of calls, all possible numbers of calls and so on. But if I am just interested in the number of calls that arrive in some time interval $[0, t]$, then I can take my sample space to be just $\Omega = \{0, 1, 2, \ldots\}$. We'll return to this example later.

Even if we are not counting something, we may be able to *encode* the result of an experiment as a number. As a trivial example, the result of a flip of a coin can be coded by letting 1 denote "head" and 0 denote "tail", say.

Real-valued discrete random variables are essentially real-valued measurements of this kind. Here's a formal definition.

Definition 2.1. *A* discrete random variable X *on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a function* $X : \Omega \to \mathbb{R}$ *such that*

(a) $\{\omega \in \Omega : X(\omega) = x\} \in \mathcal{F}$ *for each* $x \in \mathbb{R}$,

(b) $\mathrm{Im}X := \{X(\omega) : \omega \in \Omega\}$ *is a finite or countable subset of* $\mathbb{R}$.

We often abbreviate "random variable" to "r.v.".

This looks very abstract, so give yourself a moment to try to understand what it means.

- (a) says that $\{w \in \Omega : X(\omega) = x\}$ is an event to which we can assign a probability. We will usually abbreviate this event to $\{X = x\}$ and write $\mathbb{P}(X = x)$ to mean $\mathbb{P}(\{\omega \in \Omega : X(\omega) = x\})$. If these abbreviations confuse you at first, put in the ω's to make it clearer what is meant.
- (b) says that X can only take countably many values. Often $\mathrm{Im}X$ will be some subset of $\mathbb{N}$.
- If Ω is countable, (b) holds automatically because we can think of $\mathrm{Im}X$ as being indexed by Ω, and so, therefore, $\mathrm{Im}X$ must itself be countable. If we also take $\mathcal{F}$ to be the set of all subsets of Ω then (a) is also immediate.

- Later in the course, we will deal with continuous random variables, which take uncountably many values; we have to be a a bit more careful about what the correct analogue of (a) is; we will end up requiring that sets of the form $\{X \leq x\}$ are events to which we can assign probabilities.

Example 2.2. *Roll two dice and take* $\Omega = \{(i,j) : 1 \leq i,j \leq 6\}$. *Take*

$$X(i,j) = \max\{i,j\}, \quad \textit{the maximum of the two scores}$$
$$Y(i,j) = i + j, \quad \textit{the total score.}$$

A given probability space has lots of random variables associated with it. So, for example, in our telephone exchange we might have taken the "time in minutes until the arrival of the third call" in place of the number of calls by time t, say.

Definition 2.3. *The* probability mass function *(p.m.f.) of* X *is the function* $p_X : \mathbb{R} \to [0,1]$ *defined by*

$$p_X(x) = \mathbb{P}(X = x).$$

If $x \notin \mathrm{Im}X$ (that is, $X(\omega)$ never equals x) then $p_X(x) = \mathbb{P}(\{\omega : X(\omega) = x\}) = \mathbb{P}(\emptyset) = 0$. Also

$$\begin{aligned}\sum_{x \in \mathrm{Im}X} p_X(x) &= \sum_{x \in \mathrm{Im}X} \mathbb{P}(\{\omega : X(\omega) = x\}) \\ &= \mathbb{P}\left(\bigcup_{x \in \mathrm{Im}X} \{\omega : X(\omega) = x\}\right) \text{ since the events are disjoint} \\ &= \mathbb{P}(\Omega) \text{ since every } \omega \in \Omega \text{ gets mapped somewhere in } \mathrm{Im}X \\ &= 1.\end{aligned}$$

Example 2.4. *Fix an event* $A \in \mathcal{F}$ *and let* $X : \Omega \to \mathbb{R}$ *be the function given by*

$$X(\omega) = \begin{cases} 1 & \textit{if } \omega \in A, \\ 0 & \textit{otherwise.} \end{cases}$$

Then X *is a random variable with probability mass function*

$$p_X(0) = \mathbb{P}(X = 0) = \mathbb{P}(A^c) = 1 - \mathbb{P}(A), \qquad p_X(1) = \mathbb{P}(X = 1) = \mathbb{P}(A)$$

and $p_X(x) = 0$ *for all* $x \neq 0, 1$. *We will usually write* $X = \mathbb{1}_A$ *and call this the* indicator function *of the event* A.

Notice that given a probability mass function p_X, we can always write down a probability space and a random variable defined on it with that probability mass function. For simplicity, suppose that $\mathrm{Im}X = \{0, 1, \ldots\}$. Then let $\Omega = \{0, 1, \ldots\}$, let $\mathcal{F}$ be the power set of Ω, set

$$\mathbb{P}(\{\omega\}) = p_X(\omega) \quad \text{for each } \omega \in \Omega$$

and then take X to be the identity function i.e. $X(\omega) = \omega$. However, this is often not the most natural probability space to take. For example, suppose that X represents the number of heads obtained in a sequence of three fair coin tosses. Then we could proceed as just outlined. But we could also take $\Omega = \{(i,j,k) : i,j,k \in \{0,1\}\}$, with a 0 representing a tail and a 1 representing a head, so that an element of Ω tells us exactly what the three coin tosses were. Then take $\mathcal{F}$ to be the power set of Ω,

$$\mathbb{P}(\{(i,j,k)\}) = 2^{-3} \quad \text{for all } i,j,k \in \{0,1\},$$

so that every sequence of coin tosses is equally likely, and finally set $X((i,j,k)) = i + j + k$. In both cases, X has the same distribution, but the probability spaces are quite different.

Although in our examples so far, the sample space has been explicitly present, we *can* and *will* talk about random variables X without mentioning Ω.

2.1 Some classical distributions

Before introducing concepts related to discrete random variables, we introduce a stock of examples to try these concepts out on. All are classical and ubiquitous in probabilistic modelling. They also have beautiful mathematical structure, some of which we'll uncover over the course of the term.

1. **The Bernoulli distribution.** X has the Bernoulli distribution with parameter p (where $0 \leq p \leq 1$) if
$$\mathbb{P}(X = 0) = 1 - p, \quad \mathbb{P}(X = 1) = p.$$
We often write $q = 1 - p$. (Of course since $(1 - p) + p = 1$, we must have $\mathbb{P}(X = x) = 0$ for all other values of x.) We write $X \sim \text{Ber}(p)$.

   We showed in Example 2.4 that the indicator function $\mathbb{1}_A$ of an event A is an example of a Bernoulli random variable with parameter $p = \mathbb{P}(A)$, constructed on an explicit probability space.

   The Bernoulli distribution is used to model, for example, the outcome of the flip of a coin with "1" representing heads and "0" representing tails. It is also a basic building block for other classical distributions.

2. **The binomial distribution.** X has a binomial distribution with parameters n and p (where n is a positive integer and $p \in [0, 1]$) if
$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}.$$
We write $X \sim \text{Bin}(n, p)$.

   X models the number of heads obtained in n independent coin flips, where p is the probability of a head. To see this, note that the probability of any particular sequence of length n of heads and tails containing exactly k heads is $p^k(1 - p)^{n-k}$ and there are exactly $\binom{n}{k}$ such sequences.

3. **The geometric distribution.** X has a geometric distribution with parameter p
$$\mathbb{P}(X = k) = p(1 - p)^{k-1}, \quad k = 1, 2, \ldots.$$
Notice that now X takes values in a countably infinite set – the whole of the positive integers. We write $X \sim \text{Geom}(p)$.

   We can use X to model the number of independent trials needed until we see the first success, where p is the probability of success on a single trial.

   WARNING: there is an alternative and also common definition for the geometric distribution as the distribution of the number of failures, Y, before the first success. This corresponds to $X - 1$ and so
$$\mathbb{P}(Y = k) = p(1 - p)^k, \quad k = 0, 1, \ldots.$$
If in doubt, state which one you are using.

4. **The Poisson distribution.** X has the Poisson distribution with parameter $\lambda \geq 0$ if
$$\mathbb{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, \ldots.$$
We write $X \sim \text{Po}(\lambda)$.

   This distribution arises in many applications. For example, the number of calls to arrive at a telephone exchange in a given time period or the number of electrons emitted by a radioactive source in a given time and so on. It can be extended, as we'll see, to something that evolves with time. The other setting in which we encounter it is as an approximation to a binomial distribution with a large number of trials but a low success probability for each one (see the problem sheet).

Exercise 2.5. *Check that each of these really does define a probability mass function. That is:*

- $p_X(x) \geq 0$ *for all* x,
- $\sum_x p_X(x) = 1$.

You may find it useful to refer to the reminders about series which you can find in the Appendix at the end of these notes.

Given any function p_X which is non-zero for only a finite or countably infinite number of values x and satisfying these two conditions we can define the corresponding discrete random variable – we have not produced an exhaustive list!

2.2 Expectation

By plotting the probability mass funtion for the different random variables, we get some idea of how each one will behave, but often such information can be difficult to parse and we'd like what a statistician would call "summary statistics" to give us a feel for how they behave.

The first summary statistic tells us the "average value" of our random variable.

Definition 2.6. *The* expectation *(or* expected value *or* mean*) of* X *is*

$$\mathbb{E}[X] = \sum_{x \in \mathrm{Im}X} x\mathbb{P}(X = x) \tag{2.1}$$

provided that $\sum_{x\in \mathrm{Im}X} |x|\mathbb{P}(X = x) < \infty$. *If* $\sum_{x\in \mathrm{Im}X} |x|\mathbb{P}(X = x)$ *is infinite, we say that the* expectation does not exist.

The reason we insist that $\sum_{x\in \mathrm{Im}X} |x|\mathbb{P}(X = x)$ is finite, that is that the sum on the right-hand side of equation (2.1) is *absolutely convergent*, is that we need the expectation to take the same value regardless of the order in which we sum the terms. See Section A.1 for a discussion of absolute convergence.

(The problems with different orders of summation giving different answer concern cases when there are both positive and negative terms in the sum. If X is positive, i.e. $\mathrm{Im}X \subseteq \mathbb{R}_+$, and if $\sum_{x\in \mathrm{Im}X} x\mathbb{P}(X = x)$ diverges, then there is no issue with the order of summation. In this case, we sometimes write $\mathbb{E}[X] = \infty$.)

The expectation of X is the 'average' value which X takes – if we were able to take many independent copies of the experiment that X describes, and take the average of the outcomes, then we should expect that average to be close to $\mathbb{E}[X]$. We will come back to this idea at the end of the course when we look at the *Law of Large Numbers*.

Example 2.7. *1. Suppose that* X *is the number obtained when we roll a fair die. Then*

$$\begin{aligned}\mathbb{E}[X] &= 1 \cdot \mathbb{P}(X = 1) + 2 \cdot \mathbb{P}(X = 2) + \ldots + 6 \cdot \mathbb{P}(X = 6)\\ &= 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \ldots + 6 \cdot \frac{1}{6} = 3.5.\end{aligned}$$

Of course, you'll never throw 3.5 on a single roll of a die, but if you throw a lot of times you expect the average number thrown to be close to 3.5.

2. *Suppose $A \in \mathcal{F}$ is an event and $\mathbb{1}_A$ is its indicator function. Then*

$$\mathbb{E}\left[\mathbb{1}_A\right] = 0 \cdot \mathbb{P}\left(A^c\right) + 1 \cdot \mathbb{P}\left(A\right) = \mathbb{P}\left(A\right).$$

3. *Suppose that $\mathbb{P}\left(X = n\right) = \frac{6}{\pi^2}\frac{1}{n^2}$, $n \geq 1$. Then*

$$\sum_{n=1}^{\infty} n\mathbb{P}\left(X = n\right) = \frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n} = \infty$$

and so the expectation does not exist (or we may say $\mathbb{E}\left[X\right] = \infty$).

4. *Let $X \sim \mathrm{Po}(\lambda)$. Then*

$$\begin{aligned}
\mathbb{E}\left[X\right] &= \sum_{k=0}^{\infty} k\frac{e^{-\lambda}\lambda^k}{k!} \\
&= e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^k}{(k-1)!} \\
&= \lambda e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!} \\
&= \lambda e^{-\lambda}e^{\lambda} \\
&= \lambda.
\end{aligned}$$

You will find some more examples on the problem sheet.

Let $h : \mathbb{R} \to \mathbb{R}$. Then if X is a discrete random variable, $Y = h(X)$ is also a discrete random variable.

Theorem 2.8. *If $h : \mathbb{R} \to \mathbb{R}$, then*

$$\mathbb{E}\left[h(X)\right] = \sum_{x \in \mathrm{Im}X} h(x)\mathbb{P}\left(X = x\right)$$

provided that $\sum_{x\in\mathrm{Im}X} |h(x)|\mathbb{P}\left(X = x\right) < \infty$.

Proof. Let $A = \{y : y = h(x) \text{ for some } x \in \mathrm{Im}X\}$. Then, starting from the right-hand side,

$$\begin{aligned}
\sum_{x\in\mathrm{Im}X} h(x)\mathbb{P}\left(X = x\right) &= \sum_{y\in A}\sum_{x\in\mathrm{Im}X:h(x)=y} h(x)\mathbb{P}\left(X = x\right) \\
&= \sum_{y\in A}\sum_{x\in\mathrm{Im}X:h(x)=y} y\mathbb{P}\left(X = x\right) \\
&= \sum_{y\in A} y\sum_{x\in\mathrm{Im}X:h(x)=y} \mathbb{P}\left(X = x\right) \\
&= \sum_{y\in A} y\mathbb{P}\left(h(X) = y\right) \\
&= \mathbb{E}\left[h(X)\right].
\end{aligned}$$

□

Example 2.9. *Take $h(x) = x^k$. Then $\mathbb{E}[X^k]$ is called the* kth moment *of X, when it exists.*

Let us now prove some properties of the expectation which will be useful to us later on.

Theorem 2.10. *Let X be a discrete random variable such that $\mathbb{E}[X]$ exists.*

(a) If X is non-negative then $\mathbb{E}[X] \geq 0$.

(b) If $a, b \in \mathbb{R}$ then $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$.

Proof. (a) We have $\mathrm{Im}X \subseteq [0, \infty)$ and so

$$\mathbb{E}[X] = \sum_{x \in \mathrm{Im}X} x\mathbb{P}(X = x)$$

is a sum whose terms are all non-negative and so must itself be non-negative.

(b) Exercise. □

The problem with using the expectation as a summary statistic is that it is too blunt an instrument in many circumstances. For example, suppose that you are investing in the stock market. If two different stocks increase at about the same rate on the average, you may still not consider them to be equally good investments. You'd like to also know something about the size of the fluctuations about that average rate.

Definition 2.11. *For a discrete random variable X, the* variance *of X is defined by*

$$\mathrm{var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

provided that this quantity exists.

(This is $\mathbb{E}[f(X)]$ where f is given by $f(x) = (x - \mathbb{E}[X])^2$ – remember that $\mathbb{E}[X]$ is just a number.)

Note that, since $(X - \mathbb{E}[X])^2$ is a non-negative random variable, by part (a) of Theorem 2.10, $\mathrm{var}(X) \geq 0$. The variance is a measure of how much the distribution of X is spread out about its mean: the more the distribution is spread out, the larger the variance. If X is, in fact, deterministic (i.e. $\mathbb{P}(X = a) = 1$ for some $a \in \mathbb{R}$) then $\mathbb{E}[X] = a$ also and so $\mathrm{var}(X) = 0$: *only randomness gives rise to variance.*

Writing $\mu = \mathbb{E}[X]$ and expanding the square we see that

$$\begin{aligned}
\mathrm{var}(X) &= \mathbb{E}\left[(X - \mu)^2\right] \\
&= \sum_{x \in \mathrm{Im}X} (x^2 - 2\mu x + \mu^2) p_X(x) \\
&= \sum_{x \in \mathrm{Im}X} x^2 p_X(x) - 2\mu \sum_{x \in \mathrm{Im}X} x p_X(x) + \mu^2 \sum_{x \in \mathrm{Im}X} p_X(x) \\
&= \mathbb{E}\left[X^2\right] - 2\mu\mathbb{E}[X] + \mu^2 \\
&= \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2.
\end{aligned}$$

This is often an easier expression to work with.

Those of you who have done statistics at school will have seen the *standard deviation*, which is $\sqrt{\mathrm{var}(X)}$. In probability, we usually work with the variance instead because it has natural mathematical properties.

Theorem 2.12. *Suppose that X is a discrete random variable whose variance exists. Then if a and b are (finite) fixed real numbers, then the variance of the discrete random variable $Y = aX + b$ is given by*

$$\mathrm{var}(Y) = \mathrm{var}(aX + b) = a^2 \mathrm{var}(X).$$

The proof is an exercise, but notice that of course b doesn't come into it because it simply shifts the whole distribution – and hence the mean – by b, whereas variance measures relative to the mean.

In view of Theorem 2.12, why do you think statisticians often prefer to use the standard deviation rather than variance as a measure of spread?

2.3 Conditional distributions

Back in Section 1.4 we talked about conditional probability $\mathbb{P}(A|B)$. In the same way, for a discrete random variable X we can define its *conditional distribution*, given the event B. This is what it sounds like: the mass function obtained by conditioning on the outcome B.

Definition 2.13. *Suppose that B is an event such that $\mathbb{P}(B) > 0$. Then the* conditional distribution of X given B *is*

$$\mathbb{P}(X = x|B) = \frac{\mathbb{P}(\{X = x\} \cap B)}{\mathbb{P}(B)},$$

for $x \in \mathbb{R}$. The conditional expectation of X given B *is*

$$\mathbb{E}[X|B] = \sum_x x\mathbb{P}(X = x|B),$$

whenever the sum converges absolutely. We write $p_{X|B}(x) = \mathbb{P}(X = x|B)$.

Theorem 2.14 (Partition theorem for expectations)**.** *If $\{B_1, B_2, \ldots\}$ is a partition of Ω such that $\mathbb{P}(B_i) > 0$ for all $i \geq 1$ then*

$$\mathbb{E}[X] = \sum_{i \geq 1} \mathbb{E}[X|B_i]\,\mathbb{P}(B_i),$$

whenever $\mathbb{E}[X]$ exists.

Proof.

$$\begin{aligned}
\mathbb{E}[X] &= \sum_x x\mathbb{P}(X = x) \\
&= \sum_x x \left(\sum_i \mathbb{P}(X = x|B_i)\mathbb{P}(B_i) \right) \text{ by the law of total probability} \\
&= \sum_x \sum_i x\mathbb{P}(X = x|B_i)\mathbb{P}(B_i) \\
&= \sum_i \mathbb{P}(B_i) \left(\sum_x x\mathbb{P}(X = x|B_i) \right) \\
&= \sum_i \mathbb{E}[X|B_i]\mathbb{P}(B_i).
\end{aligned}$$

□

Example 2.15. *Let X be the number of rolls of a fair die required to get the first 6. (So X is geometrically distributed with parameter $1/6$.) Find $\mathbb{E}[X]$ and $\mathrm{var}(X)$.*

Solution. Let B_1 be the event that the first roll of the die gives a 6, so that B_1^c is the event that it does

not. Then

$$\begin{aligned}\mathbb{E}[X] &= \mathbb{E}[X|B_1]\mathbb{P}(B_1) + \mathbb{E}[X|B_1^c]\mathbb{P}(B_1^c) \\ &= \frac{1}{6} + \frac{5}{6}\mathbb{E}[1+X] \quad \text{(successive rolls are independent)} \\ &= \frac{1}{6} + \frac{5}{6}(1+\mathbb{E}[X]).\end{aligned}$$

Rearrange to get $\mathbb{E}[X] = 6$ (as our intuition would have us guess). Similarly,

$$\begin{aligned}\mathbb{E}[X^2] &= \mathbb{E}[X^2|B_1]\mathbb{P}(B_1) + \mathbb{E}[X^2|B_1^c]\mathbb{P}(B_1^c) \\ &= \frac{1}{6} + \frac{5}{6}\mathbb{E}[(1+X)^2] \\ &= \frac{1}{6} + \frac{5}{6}(1 + 2\mathbb{E}[X] + \mathbb{E}[X^2]).\end{aligned}$$

Rearranging and using the previous result ($\mathbb{E}[X] = 6$) gives $\mathbb{E}[X^2] = 66$ and so $\text{var}\,(X) = 30$.

Compare this solution to a direct calculation using the probability mass function:

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} kpq^{k-1}, \quad \mathbb{E}[X^2] = \sum_{k=1}^{\infty} k^2 pq^{k-1},$$

with $p = \frac{1}{6}$ and $q = \frac{5}{6}$.

We'll see a powerful approach to moment calculations in §4, but first we must find a way to deal with more than one random variable at a time.

2.4 Joint distributions

Suppose that we want to consider two discrete random variables, X and Y, defined on the same probability space. In the same way as a single random variable was characterised in terms of its probability mass function, $p_X(x)$ for $x \in \mathbb{R}$, so now we must specify $p_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y)$. It's not enough to specify $\mathbb{P}(X = x)$ and $\mathbb{P}(Y = y)$ because the events $\{X = x\}$ and $\{Y = y\}$ might not be independent (think of the case $Y = X^2$, for example).

Definition 2.16. *Given two random variables X and Y their* joint distribution *(or* joint probability mass function*) is*

$$p_{X,Y}(x,y) = \mathbb{P}\left(\{X = x\} \cap \{Y = y\}\right), \qquad x, y \in \mathbb{R}.$$

We usually write the right-hand side simply as $\mathbb{P}(X = x, Y = y)$. We have $p_{X,Y}(x,y) \geq 0$ for all $x, y \in \mathbb{R}$ and $\sum_x \sum_y p_{X,Y}(x,y) = 1$. The marginal distribution *of X is*

$$p_X(x) = \sum_y p_{X,Y}(x,y)$$

and the marginal *distribution of Y is*

$$p_Y(y) = \sum_x p_{X,Y}(x,y).$$

The marginal distribution of X tells you what the distribution of X is if you have no knowledge of Y.

We can write the joint mass function as a table.

Example 2.17. *Suppose that X and Y take only the values 0 or 1 and their joint mass function is given by*

Y \ X	0	1
0	$\frac{1}{3}$	$\frac{1}{2}$
1	$\frac{1}{12}$	$\frac{1}{12}$

Observe that $\sum_{x,y} p_{X,Y}(x,y) = 1$ (always a good check when modelling).

The marginals are found by summing the rows and columns:

Y \ X	0	1	$p_Y(y)$
0	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{5}{6}$
1	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$
$p_X(x)$	$\frac{5}{12}$	$\frac{7}{12}$	

Notice that $\mathbb{P}(X=1) = \frac{7}{12}$, $\mathbb{P}(Y=1) = \frac{1}{6}$ and $\mathbb{P}(X=1, Y=1) = \frac{1}{12} \neq \frac{7}{12} \times \frac{1}{6}$ so $\{X=1\}$ and $\{Y=1\}$ are not *independent events.*

Whenever $p_X(x) > 0$ for some $x \in \mathbb{R}$, we can also write down the *conditional distribution of Y given that $X = x$*:

$$p_{Y|X=x}(y) = \mathbb{P}(Y = y | X = x) = \frac{p_{X,Y}(x,y)}{p_X(x)} \quad \text{for } y \in \mathbb{R}.$$

The *conditional expectation of Y given that $X = x$* is then

$$\mathbb{E}[Y|X=x] = \sum_y y p_{Y|X=x}(y),$$

whenever the sum converges absolutely.

Example 2.18. *For the joint distribution in Example 2.17, we have*

$$p_{Y|X=0}(0) = \frac{4}{5}, \quad p_{Y|X=0}(1) = \frac{1}{5}$$

and

$$\mathbb{E}[Y|X=0] = \frac{1}{5}.$$

Definition 2.19. *Discrete random variables X and Y are* independent *if*

$$\mathbb{P}(X=x, Y=y) = \mathbb{P}(X=x)\mathbb{P}(Y=y) \quad \textit{for all } x, y \in \mathbb{R}.$$

In other words, X and Y are independent if and only if the events $\{X = x\}$ and $\{Y = y\}$ are independent **for all** choices of x and y. We can also write this as

$$p_{X,Y}(x, y) = p_X(x)p_Y(y) \quad \text{for all } x, y \in \mathbb{R}.$$

Example 2.20 (Part of an old exam question)**.** *A coin when flipped shows heads with probability p and tails with probability $q = 1 - p$. It is flipped repeatedly. Assume that the outcome of different flips is independent. Let U be the length of the initial run and V the length of the second run. Find $\mathbb{P}(U = m, V = n)$, $\mathbb{P}(U = m)$, $\mathbb{P}(V = m)$. Are U and V independent?*

Solution. We condition on the outcome of the first flip and use the law of total probability.

$$\begin{aligned}
\mathbb{P}(U = m, V = n) &= \mathbb{P}(U = m, V = n \mid \text{1st flip H})\mathbb{P}(\text{1st flip H}) + \mathbb{P}(U = m, V = n \mid \text{1st flip T})\mathbb{P}(\text{1st flip T}) \\
&= pp^{m-1}q^n p + qq^{m-1}p^n q \\
&= p^{m+1}q^n + q^{m+1}p^n. \\
\mathbb{P}(U = m) &= \sum_{n=1}^{\infty}(p^{m+1}q^n + q^{m+1}p^n) = p^{m+1}\frac{q}{1-q} + q^{m+1}\frac{p}{1-p} \\
&= p^m q + q^m p. \\
\mathbb{P}(V = n) &= \sum_{m=1}^{\infty}(p^{m+1}q^n + q^{m+1}p^n) = q^n\frac{p^2}{1-p} + p^n\frac{q^2}{1-q} \\
&= p^2q^{n-1} + q^2p^{n-1}.
\end{aligned}$$

We have $\mathbb{P}(U = m, V = n) \neq f(m)g(n)$ unless $p = q = \frac{1}{2}$. So U, V are not independent unless $p = \frac{1}{2}$. To see why, suppose that $p < \frac{1}{2}$, then knowing that U is small, say, tells you that the first run is more likely to be a run of H's and so V is likely to be longer. Conversely, knowing that U is big will tell us that V is likely to be small. U and V are *negatively correlated*.

In the same way as we defined expectation for a single discrete random variable, so in the bivariate case we can define expectation of any function of the random variables X and Y. Let $h : \mathbb{R}^2 \to \mathbb{R}$. Then $h(X, Y)$ is itself a random variable, and

$$\begin{aligned}
\mathbb{E}[h(X, Y)] &= \sum_x \sum_y h(x, y)\mathbb{P}(X = x, Y = y) \\
&= \sum_x \sum_y h(x, y)p_{X,Y}(x, y), \qquad (2.2)
\end{aligned}$$

provided the sum converges absolutely.

Theorem 2.21. *Suppose X and Y are discrete random variables and $a, b \in \mathbb{R}$ are constants. Then*

$$\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$$

provided that both $\mathbb{E}[X]$ and $\mathbb{E}[Y]$ exist.

Proof. Setting $h(x,y) = ax + by$, we have

$$\begin{aligned}
\mathbb{E}[aX + bY] &= \mathbb{E}\left[h(X,Y)\right] \\
&= \sum_x \sum_y (ax + by) p_{X,Y}(x,y) \\
&= a \sum_x \sum_y x p_{X,Y}(x,y) + b \sum_x \sum_y y p_{X,Y}(x,y) \\
&= a \sum_x x \left(\sum_y p_{X,Y}(x,y) \right) + b \sum_y y \left(\sum_x p_{X,Y}(x,y) \right) \\
&= a \sum_x x p_X(x) + b \sum_y y p_Y(y) \\
&= a\mathbb{E}[X] + b\mathbb{E}[Y].
\end{aligned}$$

□

Theorem 2.21 tells us that **expectation is linear**. This is a very important property. We can easily extend by induction to get $\mathbb{E}[a_1X_1 + \cdots + a_nX_n] = a_1\mathbb{E}[X_1] + \cdots + a_n\mathbb{E}[X_n]$ for any finite collection of random variables $X_1, \ldots, X_n$. Note that we don't need to make any assumption about independence of the random variables.

Example 2.22. *Your spaghetti bowl contains n strands of spaghetti. You repeatedly choose 2 ends at random, and join them together. What is the average number of loops in the bowl, once no ends remain?*

Solution. We start with $2n$ ends, and the number decreases by 2 at each step. When we have k ends, the probability of forming a loop is $1/(k-1)$. Before the ith step, we have $2(n-i+1)$ ends, so we form a loop with probability $1/[2(n-i)+1]$.

Let X_i be the indicator function of the event that we form a loop at the ith step. Then $\mathbb{E}[X_i] = 1/[2(n-i)+1]$. Let M be the total number of loops formed. Then $M = X_1 + \cdots + X_n$, so using linearity of expectation,

$$\begin{aligned}
\mathbb{E}[M] &= \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + +\mathbb{E}[X_{n-1}] + \mathbb{E}[X_n] \\
&= \frac{1}{2n-1} + \frac{1}{2n-3} + \cdots + \frac{1}{3} + 1.
\end{aligned}$$

(If n is large, this expectation is close to $\log n$.)

Note that the probability mass function of M is not easy to obtain. So finding the expectation of M directly from the definition at (2.1) would have been very much less straightforward. □

Theorem 2.23. *If X and Y are independent discrete random variables whose expectations exist, then*

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y].$$

Proof. We have

$$\begin{aligned}
\mathbb{E}[XY] &= \sum_x \sum_y xy\mathbb{P}(X = x, Y = y) \\
&= \sum_x \sum_y xy\mathbb{P}(X = x)\mathbb{P}(Y = y) \quad \text{(by independence)} \\
&= \left(\sum_x x\mathbb{P}(X = x) \right) \left(\sum_y y\mathbb{P}(Y = y) \right) \\
&= \mathbb{E}[X]\mathbb{E}[Y].
\end{aligned}$$

□

Exercise 2.24. *Show that* $\operatorname{var}(X+Y) = \operatorname{var}(X) + \operatorname{var}(Y)$ *when* X *and* Y *are independent.*

What happens when X and Y are *not* independent? It's useful to define the *covariance*,

$$\operatorname{cov}(X,Y) = \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right].$$

Notice that $\operatorname{cov}(X,X) = \operatorname{var}(X)$.

Exercise 2.25. *Check that* $\operatorname{cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$ *and that*

$$\operatorname{var}(X+Y) = \operatorname{var}(X) + \operatorname{var}(Y) + 2\operatorname{cov}(X,Y).$$

Notice that this means that if X *and* Y *are independent, their covariance is 0. In general, the covariance can be either positive or negative valued.*

WARNING: $\operatorname{cov}(X,Y) = 0$ DOES NOT IMPLY THAT X AND Y ARE INDEPENDENT.

See the problem sheet for an example.

Definition 2.26. *We can define* multivariate distributions *analogously:*

$$p_{X_1,X_2,\ldots,X_n}(x_1,x_2,\ldots,x_n) = \mathbb{P}(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n),$$

for $x_1, x_2, \ldots, x_n \in \mathbb{R}$*, and so on.*

By analogy with the way we defined independence for a sequence of events, we can define independence for a family of random variables.

Definition 2.27. *A family* $\{X_i : i \in I\}$ *of discrete random variables are independent if for all* finite *sets* $J \subseteq I$ *and all collections* $\{A_i : i \in J\}$ *of subsets of* $\mathbb{R}$*,*

$$\mathbb{P}\left(\bigcap_{i\in J}\{X_i \in A_i\}\right) = \prod_{i\in J}\mathbb{P}(X_i \in A_i).$$

Suppose that $X_1, X_2, \ldots$ *are independent random variables which all have the same distribution. Then we say that* $X_1, X_2, \ldots$ *are* independent and identically distributed (i.i.d.).

Chapter 3

Difference equations and random walks

3.1 Difference equations

Our next topic is not probability theory, but rather a tool that you need both to answer some probability questions in the next chapter, as well as in all sorts of other areas of mathematics. Here is a famous probability problem by way of motivation.

Example 3.1 (Gambler's ruin)**.** *A gambler repeatedly plays a game in which he wins £1 with probability p and loses £1 with probability $q = 1 - p$ (independently at each play). He will leave the casino if he loses all his money or if his fortune reaches £M.*

What is the probability that he leaves with nothing if his initial fortune is £n?

Call the probability u_n and condition on the outcome of the first play to see that

$$u_n = \mathbb{P}(\text{bankruptcy} \mid \text{win 1st game})\mathbb{P}(\text{win 1st game}) + \mathbb{P}(\text{bankruptcy} \mid \text{lose 1st game})\mathbb{P}(\text{lose 1st game}).$$

If the gambler wins the first game, by *independence* of different plays it's just like starting over from an initial fortune of £$(n+1)$; similarly, if he loses the first games, it's just like starting over from an initial fortune of £$(n-1)$. This implies that

$$u_n = pu_{n+1} + qu_{n-1}, \tag{3.1}$$

which is valid for $1 \leq n \leq M-1$. We have the *boundary conditions* $u_0 = 1$, $u_M = 0$.

This is an example of a second-order recurrence relation; it is equations of this sort that we will now learn how to solve.

Definition 3.2. *A* kth order linear recurrence relation *(or* difference equation*) has the form*

$$\sum_{j=0}^{k} a_j u_{n+j} = f(n) \tag{3.2}$$

with $a_0 \neq 0$ and $a_k \neq 0$, where $a_0, \ldots, a_k$ are constants independent of n. A solution *to such a difference equation is a sequence $(u_n)_{n\geq 0}$ satisfying (3.2) for all $n \geq 0$.*

You should keep in mind what you know about solving linear ordinary differential equations like

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

for the function y, since what we do here will be completely analogous.

The next theorem says that we can split the problem of finding a solution to our difference equations into two parts.

Theorem 3.3. *The general solution $(u_n)_{n\geq 0}$ (i.e. if the boundary conditions are not specified) of*

$$\sum_{j=0}^{k} a_j u_{n+j} = f(n)$$

can be written as $u_n = v_n + w_n$ where $(v_n)_{n\geq 0}$ is a particular solution to the equation and $(w_n)_{n\geq 0}$ solves the homogeneous equation

$$\sum_{j=0}^{k} a_j w_{n+j} = 0.$$

Proof. Suppose (u_n) has the suggested form and $(\tilde{u}_n)$ is another solution which may not necessarily be expressed in this form. Then

$$\sum_{j=0}^{k} a_j(u_{n+j} - \tilde{u}_{n+j}) = 0.$$

So (u_n) and $(\tilde{u}_n)$ differ by a solution (x_n) to the homogeneous equation. In particular,

$$\tilde{u}_n = v_n + (w_n + x_n),$$

which *is* of the suggested form since $(w_n + x_n)$ is clearly a solution to the homogeneous equation. □

3.2 First order linear difference equations

We will develop the necessary methods via a series of worked examples.

Example 3.4. *Solve*

$$u_{n+1} = au_n + b$$

where $u_0 = 3$ and the constants $a \neq 0$ and b are given

Solution. The homogeneous equation is $w_{n+1} = aw_n$. "Putting it into itself", we get

$$w_n = aw_{n-1} = \ldots = a^n w_0 = Aa^n$$

for some constant A.

How about a particular solution? As in differential equations, guess a constant solution might work, so try $v_n = C$. This gives $C = aC + b$ so provided that $a \neq 1$, $C = \frac{b}{1-a}$ and we have general solution

$$u_n = Aa^n + \frac{b}{1-a}.$$

Setting $n = 0$ allows us to determine A:

$$3 = A + \frac{b}{1-a} \text{ and so } A = 3 - \frac{b}{1-a}.$$

Hence,

$$u_n = \left(3 - \frac{b}{1-a}\right) a^n + \frac{b}{1-a} = 3a^n + \frac{b(1-a^n)}{1-a}.$$

What happens if $a = 1$? An applied-maths-type approach would set $a = 1 + \epsilon$ and try to see what happens as $\epsilon \to 0$:

$$\begin{aligned} u_n &= u_0(1+\epsilon)^n + \frac{b(1-(1+\epsilon)^n)}{1-(1+\epsilon)} \\ &= u_0 + b\frac{(1-(1+n\epsilon))}{-\epsilon} + \mathcal{O}(\epsilon) \\ &= u_0 + nb + \mathcal{O}(\epsilon) \to u_0 + nb \quad \text{as } \epsilon \to 0. \end{aligned}$$

An alternative approach is to mimic what you did for differential equations and "try the next most complex thing". We have $u_{n+1} = u_n + b$ and the homogeneous equation has solution $w_n = A$ (a constant). For a particular solution try $v_n = Cn$ (note that there is no point in adding a constant term because the constant solves the homogeneous equation and so it makes no contribution to the right-hand side when we substitute).

Then $C(n+1) = Cn + b$ gives $C = b$ and we obtain once again the general solution

$$u_n = A + bn.$$

Setting $n = 0$ yields $A = 3$ and so $u_n = 3 + bn$.

Example 3.5.

$$u_{n+1} = au_n + bn.$$

Solution. As above, the homogeneous equation has solution $w_n = Aa^n$. For a particular solution, try $v_n = Cn + D$. Substituting

$$C(n+1) + D = a(Cn + D) + bn.$$

Equating coefficients of n and the constant terms gives

$$C = aC + b, \quad C + D = aD,$$

so again provided $a \neq 1$ we can solve to obtain $C = \frac{b}{1-a}$ and $D = \frac{-c}{1-a}$. Thus for $a \neq 1$

$$u_n = Aa^n + \frac{bn}{1-a} - \frac{b}{(1-a)^2}.$$

To find A, we need a boundary condition (e.g. the value of u_0).

Exercise 3.6. *Solve the equation for $a = 1$. Hint: try $v_n = Cn + Dn^2$.*

3.3 Second order linear difference equations

Consider

$$u_{n+1} + au_n + bu_{n-1} = f(n).$$

The general solution will depend on *two* constants. For the first order case, the homogeneous equation had a solution of the form $w_n = A\lambda^n$, so we try the same here. Substituting $w_n = A\lambda^n$ in

$$w_{n+1} + aw_n + bw_{n-1} = 0$$

gives

$$A\lambda^{n+1} + aA\lambda^n + bA\lambda^{n-1} = 0.$$

For a non-trivial solution we can divide by $A\lambda^{n-1}$ and see that λ must solve the quadratic equation

$$\lambda^2 + a\lambda + b = 0.$$

This is called the *auxiliary equation.* (So just as when you solve 2nd order ordinary differential equations you obtain a quadratic equation by considering solutions of the form $e^{\lambda t}$, so here we obtain a quadratic in λ by considering solutions of the form λ^n.)

If the auxiliary equation has distinct roots, λ_1 and λ_2 then the general solution to the homogeneous equation is

$$w_n = A_1\lambda_1^n + A_2\lambda_2^n.$$

If $\lambda_1 = \lambda_2 = \lambda$ try the next most complicated thing (or mimic what you do for ordinary differential equations) to get

$$w_n = (A + Bn)\lambda^n.$$

Exercise 3.7. *Check that this solution works.*

How about particular solutions? The same tricks as for the one-dimensional case apply. We can start by trying something of the same form as f, and if that fails then try the next most complicated thing. You can save yourself work by not including components that you already know solve the homogeneous equation.

Example 3.8. *Solve*

$$u_{n+1} + 2u_n - 3u_{n-1} = 1.$$

Solution. The auxiliary equation is just

$$\lambda^2 + 2\lambda - 3 = 0$$

which has roots $\lambda_1 = -3$, $\lambda_2 = 1$, so

$$w_n = A(-3)^n + B.$$

For a particular solution, we'd like to try a constant, but that won't work because we know that it solves the homogeneous equation (it's a special case of w_n). So try the next most complicated thing, which is $v_n = Cn$. Substituting, we obtain

$$C(n+1) + 2Cn - 3C(n-1) = 1,$$

which gives $C = \frac{1}{4}$. The general solution is then

$$u_n = A(-3)^n + B + \frac{1}{4}n.$$

If the boundary conditions had been specified, you could now find A and B by substitution. (Note that it takes one boundary condition to specify the solution to a first order difference equation and two to specify the solution to a 2nd order difference equation. Usually these will be the values of u_0 and u_1 but notice that in the gambler's ruin problem we are given u_0 and u_N.)

Example 3.9. *Solve*

$$u_{n+1} - 2u_n + u_{n-1} = 1.$$

Solution. The auxiliary equation $\lambda^2 - 2\lambda + 1 = 0$ has repeated root $\lambda = 1$, so the homogeneous equation has general solution

$$w_n = An + B.$$

For a particular solution, try the next most complicated thing, so $v_n = Cn^2$. (Once again there is no point in adding a $Dn + E$ term to this as that solves the homogeneous equation, so substituting it on the left cannot contribute anything to the 1 that we are trying to obtain on the right of the equation.) Substituting, we obtain

$$C(n+1)^2 - 2Cn^2 + C(n-1)^2 = 1,$$

which gives $C = \frac{1}{2}$. So the general solution is

$$u_n = An + B + \frac{1}{2}n^2.$$

Example 3.10 (The Fibonacci numbers)**.** *The Fibonacci numbers* $1, 1, 2, 3, 5, 8, 13, \ldots$ *are defined by the second-order linear difference equation*

$$f_{n+2} = f_{n+1} + f_n, \quad n \geq 0, \tag{3.3}$$

with initial conditions $f_0 = f_1 = 1$.

This is homogeneous, with auxiliary equation $\lambda^2 - \lambda - 1 = 0$. *The roots are* $\lambda = \frac{1 \pm \sqrt{5}}{2}$, *and so the general solution of (3.3) is given by*

$$f_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Putting in the initial conditions yields the simultaneous equations

$$1 = A + B, \quad 1 = A\frac{1+\sqrt{5}}{2} + B\frac{1-\sqrt{5}}{2}$$

which have solution $A = \frac{\sqrt{5}+1}{2\sqrt{5}}$, $B = \frac{\sqrt{5}-1}{2\sqrt{5}}$. *This yields the remarkable result that for* $n \geq 0$,

$$\begin{aligned} f_n &= \frac{\sqrt{5}+1}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{\sqrt{5}-1}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \\ &= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}. \end{aligned}$$

Notice that, despite the fact that $\sqrt{5}$ *is irrational, this gives an integer for every* $n \geq 0$*!*

Example 3.11. *Consider the second-order linear difference equation*

$$u_{n+2} - 2u_{n+1} + 4u_n = 0, \quad n \geq 0, \tag{3.4}$$

with initial conditions $u_0 = u_1 = 1$. The auxiliary equation is $\lambda^2 - 2\lambda + 4 = 0$, which has roots $\lambda = 1 \pm i\sqrt{3}$. So the general solution to (3.4) is

$$u_n = A(1 + i\sqrt{3})^n + B(1 - i\sqrt{3})^n.$$

Using the initial conditions, we get $A = B = \frac{1}{2}$, and so

$$u_n = \frac{1}{2}(1 + i\sqrt{3})^n + \frac{1}{2}(1 - i\sqrt{3})^n, \quad n \geq 0.$$

This is, in fact, real for every $n \geq 0$. In order to see this, recall that $1 + i\sqrt{3} = 2e^{i\pi/3}$ and $1 - i\sqrt{3} = 2e^{-i\pi/3}$. So

$$u_n = \frac{1}{2}\left(2e^{i\pi/3}\right)^n + \frac{1}{2}\left(2e^{-i\pi/3}\right)^n = 2^n \frac{e^{in\pi/3} + e^{-in\pi/3}}{2} = 2^n \cos\left(\frac{n\pi}{3}\right).$$

3.4 Random walks

We return to the gambler's ruin problem of Example 3.1. The gambler's fluctuating wealth is an example of a more general class of random processes called *random walks* (sometimes the more evocative phrase *drunkard's walk* is used). Imagine a particle moving around a network. At each step, it can move to one of the other nodes of the network: there are rules determining where the particle can move to at the next time step from that position and with what probability it moves to each of the possible new positions. The important point is that these rules *only* depend on the current position, not on the earlier positions that the particle has visited. Random walks can be used to model various real-world situations. For example, the path traced by a molecule as it moves in a liquid or a gas; the path of an animal searching for food; or the price of a particular stock every Monday morning. There are various examples on the problem sheets and later in the course.

Let's return to the setting of Example 3.1 and solve the recurrence relation we obtained there. Recall that $u_n = \mathbb{P}\,(\text{bankruptcy})$ if the gambler's initial fortune is $£n$, and that (rearranging (3.1)),

$$pu_{n+1} - u_n + qu_{n-1} = 0, \qquad 1 \leq n \leq M - 1, \tag{3.5}$$

(where $q = 1 - p$), with $u_0 = 1$, $u_M = 0$. This is a homogeneous second-order difference equation. The auxiliary equation is

$$p\lambda^2 - \lambda + q = 0$$

which factorises as

$$(p\lambda - q)(\lambda - 1) = 0.$$

So $\lambda = \frac{q}{p}$ or 1. If $p \neq \frac{1}{2}$ then

$$u_n = A + B\left(\frac{q}{p}\right)^n$$

for some constants A and B which we can find using the boundary conditions:

$$u_0 = 1 = A + B \quad \text{and} \quad u_M = 0 = A + B\left(\frac{q}{p}\right)^M.$$

These give

$$A = -\frac{\left(\frac{1-p}{p}\right)^M}{1 - \left(\frac{1-p}{p}\right)^M}, \quad B = \frac{1}{1 - \left(\frac{1-p}{p}\right)^M}$$

and so

$$u_n = \frac{\left(\frac{1-p}{p}\right)^n - \left(\frac{1-p}{p}\right)^M}{1 - \left(\frac{1-p}{p}\right)^M}.$$

Exercise 3.12. *Check that in the case* $p = \frac{1}{2}$ *we get*

$$u_n = 1 - \frac{n}{M}, \quad 0 \le n \le M.$$

Figure 3.1 shows a simulation of paths in the gambler's ruin model.

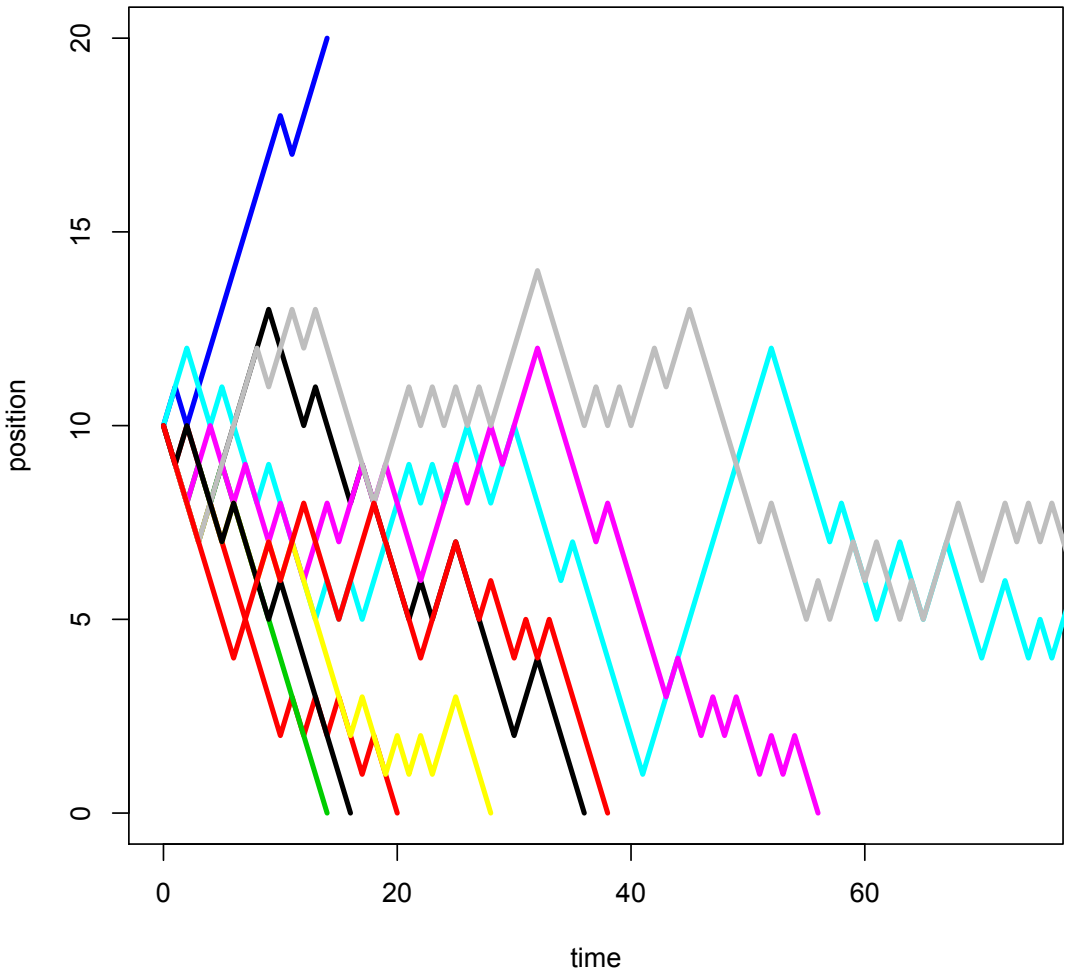


Figure 3.1: 10 simulated paths in the gambler's ruin model, with $M = 20$, $n = 10$ and $p = 0.4$. We see some get absorbed at 0, one at 20, and two which have not yet reached either boundary at time 80.

Example 3.13. *What is the expected number of plays in the gambler's ruin model before the gambler's fortune hits either* 0 *or* M*?*

Solution. Just as we used the partition theorem to get a recurrence for the probability of bankruptcy, we can use the partition theorem for expectations to get a recurrence for the expected length of the process.

Let X be the number of steps until the walk reaches one of the barriers at 0 or M. Write e_n for the expectation of X when the process is started from n. Then

$$e_n = p\mathbb{E}\left[X|\text{first step is to } n+1\right] + q\mathbb{E}\left[X|\text{first step is to } n-1\right].$$

Let's think carefully about the conditional expectations on the right-hand side. If the first step is to $n+1$, then we have already spent one step to get there, and thereafter the number of steps to reach the boundary is just the time to reach the boundary in a walk starting from $n+1$. Hence we get

$$\mathbb{E}\left[X|\text{first step is to } n+1\right] = 1 + e_{n+1},$$

and similarly

$$\mathbb{E}\left[X|\text{first step is to } n-1\right] = 1 + e_{n-1}.$$

So we obtain the recurrence

$$e_n = p(1+e_{n+1}) + q(1+e_{n-1})$$

which rearranges to give

$$pe_{n+1} - e_n + qe_{n-1} = -1. \tag{3.6}$$

Our boundary conditions are $e_0 = e_M = 0$. Note that (3.6) has exactly the same form as (3.5), except that the equation is no longer homogeneous: we have the constant -1 on the right-hand side instead of 0.

Take the case $p \neq q$. As above, we have the general solution to the homogeneous equation

$$w_n = A + B\left(\frac{q}{p}\right)^n.$$

For a particular solution to (3.6), try $v_n = Cn$ (note that there's no point trying a constant since we already know that any constant solves the homogeneous equation). This yields

$$pC(n+1) - Cn + qC(n-1) = -1$$

and so $C = -1/(p-q)$. Putting everything together, we get

$$e_n = A + B\left(\frac{q}{p}\right)^n - \frac{n}{p-q}.$$

Using the boundary conditions, we get

$$e_0 = 0 = A + B, \quad e_M = 0 = A + B\left(\frac{q}{p}\right)^M - \frac{M}{p-q}.$$

Solving for A and B, we finally obtain

$$e_n = \frac{M}{(p-q)}\frac{1-(q/p)^n}{1-(q/p)^M} - \frac{n}{p-q}$$

for $0 \leq n \leq M$.

Exercise 3.14. *Find e_n for $p = q = 1/2$ (the expression is rather simpler in that case!).*

Finally, consider what happens if we remove the upper barrier at M, and instead have a random walk on the infinite set $\{0, 1, 2, \dots\}$, starting from some site $n > 0$. Does the walk ever reach the site 0, or does it stay strictly positive for ever? Let's look at the probability of the event that it hits 0. A natural idea is to let $M \to \infty$ in the finite problem. Write $u_n^{(M)}$ for the probability of hitting 0 before M, which we calculated above. Then we have

$$\lim_{M\to\infty} u_n^{(M)} = \begin{cases} \lim_{M\to\infty} \frac{\left(\frac{q}{p}\right)^n - \left(\frac{q}{p}\right)^M}{1-\left(\frac{q}{p}\right)^M} & \text{if } p \neq q \\ \lim_{M\to\infty} 1 - \frac{n}{M} & \text{if } p = q = 1/2 \end{cases} = \begin{cases} \left(\frac{q}{p}\right)^n & \text{if } p > q \\ 1 & \text{if } p \leq q. \end{cases}$$

It turns out that this limit as $M \to \infty$ really does give the appropriate probability that the random walk on $\{0, 1, 2, \dots\}$ hits 0. In particular, the walk has positive probability to stay away from 0 for ever if and only if $p > q$. There are various ways to prove this; the idea below is not complicated, but is nonetheless somewhat subtle.

Theorem 3.15. *(Non-examinable) Consider a random walk on the integers $\mathbb{Z}$, started from some $n > 0$, which at each step increases by 1 with probability p, and decreases by 1 with probability $q = 1 - p$. Then the probability u_n that the walk ever hits 0 is given by*

$$u_n = \begin{cases} \left(\frac{q}{p}\right)^n & \text{if } p > q, \\ 1 & \text{if } p \leq q. \end{cases}$$

Proof. In Proposition A.8 in the Appendix, we prove a useful result about *increasing sequences of events*. A sequence of events $A_k, k \geq 1$ is called *increasing* if $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$. Then Proposition A.8 says that for such a sequence of events,

$$\mathbb{P}\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} \mathbb{P}(A_k).$$

(This can be regarded as a sort of continuity result for the probability function $\mathbb{P}$.)

To apply this result to the random walk started from n, consider the event H that the random walk reaches 0, and for each M, consider the event A_M that the random walk reaches 0 before M. If the walk ever reaches 0, then there must be some M such that the walk reaches 0 before M, so that A_M occurs. Conversely, if any event A_M occurs, then clearly the event H also occurs. Hence we have $H = \bigcup_M A_M$.

Then indeed we have

$$\begin{aligned} u_n &= \mathbb{P}(H) \\ &= \mathbb{P}\left(\bigcup_{M=1}^{\infty} A_M\right) \\ &= \lim_{M\to\infty} \mathbb{P}(A_M) \\ &= \lim_{M\to\infty} u_n^{(M)}, \end{aligned}$$

as desired. □

Chapter 4

Probability generating functions

We're now going to turn to an extremely powerful tool, not just in calculations but also in proving more abstract results about discrete random variables.

From now on we consider *non-negative integer-valued* random variables i.e. X takes values in $\{0, 1, 2, \ldots\}$.

Definition 4.1. *Let X be a non-negative integer-valued random variable. Let*

$$\mathcal{S} := \left\{ s \in \mathbb{R} : \sum_{k=0}^{\infty} |s|^k \mathbb{P}(X = k) < \infty \right\}.$$

Then the probability generating function *(p.g.f.) of X is $G_X : \mathcal{S} \to \mathbb{R}$ defined by*

$$G_X(s) = \mathbb{E}[s^X] = \sum_{k=0}^{\infty} s^k \mathbb{P}\left(X = k\right).$$

Let us agree to save space by setting

$$p_k = p_X(k) = \mathbb{P}(X = k).$$

Notice that because $\sum_{k=0}^{\infty} p_k = 1$, $G_X(s)$ is certainly defined for $|s| \leq 1$ (i.e. $[-1, 1] \subseteq \mathcal{S}$) and $G_X(1) = 1$. Notice also that $G_X(s)$ is just a real-valued *function*. The parameter s is the *argument* of the function and has nothing to do with X. It plays the same role as x if I write $\sin x$, for example.[1]

Why are generating functions so useful? Because they encode all of the information about the distribution of X in a single function. It will turn out that we can get at this information by using the tools of calculus.

Theorem 4.2. *The distribution of X is uniquely determined by its probability generating function, G_X.*

[1] The probability generating function is an example of a *power series*, that is a function of the form $f(x) = \sum_{n=0}^{\infty} c_n x^n$. It may be that this sum diverges for some values of x; the *radius of convergence* is the value r such that the sum converges if $|x| < r$ and diverges if $|x| > r$. For a probability generating function, we can see that the radius of convergence must be at least 1. For the purposes of this course, you are safe to assume that the derivative of f is well-defined for $|x| < r$ and is given by

$$f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

i.e. what you would get differentiating term-by-term. Those of you who are doing Analysis I & II will learn more about power series there.

Proof. First note that $G_X(0) = p_0$. Now, for $|s| < 1$, we can differentiate $G_X(s)$ term-by-term to get

$$G'_X(s) = p_1 + 2p_2 s + 3p_3 s^2 + \cdots .$$

Setting $s = 0$, we see that $G'_X(0) = p_1$. Similarly, by differentiating repeatedly, we see that

$$\frac{d^k}{ds^k} G_X(s)\big|_{s=0} = k!\, p_k.$$

So we can recover $p_0, p_1, \ldots$ from G_X. □

Probability generating functions for common distributions.

1. **Bernoulli distribution.** $X \sim \mathrm{Ber}(p)$. Then

$$G_X(s) = \sum_k p_k s^k = qs^0 + ps^1 = q + ps$$

for all $s \in \mathbb{R}$.

2. **Binomial distribution.** $X \sim \mathrm{Bin}(n, p)$. Then

$$G_X(s) = \sum_{k=0}^{n} s^k \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (ps)^k (1-p)^{n-k} = (1 - p + ps)^n,$$

by the binomial theorem. This is valid for all $s \in \mathbb{R}$.

3. **Poisson distribution.** $X \sim \mathrm{Po}(\lambda)$. Then

$$G_X(s) = \sum_{k=0}^{\infty} s^k \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(s\lambda)^k}{k!} = e^{\lambda(s-1)}$$

for all $s \in \mathbb{R}$.

4. **Geometric distribution with parameter p.** Exercise on the problem sheet: check that

$$G_X(s) = \frac{ps}{1 - (1-p)s},$$

provided that $|s| < \frac{1}{1-p}$.

Theorem 4.3. *If X and Y are independent, then*

$$G_{X+Y}(s) = G_X(s) G_Y(s).$$

Proof. We have

$$G_{X+Y}(s) = \mathbb{E}\left[s^{X+Y}\right] = \mathbb{E}\left[s^X s^Y\right].$$

Since X and Y are independent, s^X and s^Y are independent (see a question on the problem sheet). So then by Theorem 2.23, this is equal to

$$\mathbb{E}\left[s^X\right] \mathbb{E}\left[s^Y\right] = G_X(s) G_Y(s).$$ □

This can be very useful for proving distributional relationships.

Theorem 4.4. *Suppose that $X_1, X_2, \ldots, X_n$ are independent* $\mathrm{Ber}(p)$ *random variables and let $Y = X_1 + \cdots + X_n$. Then $Y \sim \mathrm{Bin}(n, p)$.*

Proof. We have

$$G_Y(s) = \mathbb{E}[s^Y] = \mathbb{E}[s^{X_1+\cdots+X_n}] = \mathbb{E}[s^{X_1} \cdots s^{X_n}] = \mathbb{E}[s^{X_1}] \cdots \mathbb{E}[s^{X_n}] = (1 - p + ps)^n.$$

As Y has the same p.g.f. as a $\mathrm{Bin}(n, p)$ random variable, we deduce that $Y \sim \mathrm{Bin}(n, p)$. □

The interpretation of this is that X_i tells us whether the ith of a sequence of independent coin flips is heads or tails (where heads has probability p). Then Y counts the number of heads in n independent coin flips and so must be distributed as $\mathrm{Bin}(n, p)$.

Theorem 4.5. *Suppose that $X_1, X_2, \ldots, X_n$ are independent random variables such that $X_i \sim \mathrm{Po}(\lambda_i)$. Then*

$$\sum_{i=1}^{n} X_i \sim \mathrm{Po}\left(\sum_{i=1}^{n} \lambda_i\right).$$

In particular, if $\lambda_i = \lambda$ for all $1 \leq i \leq n$ then $\sum_{i=1}^{n} X_i \sim \mathrm{Po}(n\lambda)$.

Proof. Recall that $\mathbb{E}\left[s^{X_i}\right] = e^{\lambda_i(s-1)}$. By independence,

$$\mathbb{E}\left[s^{X_1+X_2+\ldots+X_n}\right] = \prod_{i=1}^{n} \mathbb{E}\left[s^{X_i}\right] = \prod_{i=1}^{n} e^{\lambda_i(s-1)} = \exp\left((s-1)\sum_{i=1}^{n} \lambda_i\right).$$

Since this is the p.g.f. of the $\mathrm{Po}(\sum_{i=1}^{n} \lambda_i)$ distribution and probability generating functions uniquely determine distributions, the result follows. □

4.1 Calculating expectations using probability generating functions

We've already seen that differentiating $G_X(s)$ and setting $s = 0$ gives us a way to get at the probability mass function of X. Derivatives at other points can also be useful. We have

$$G'_X(s) = \frac{d}{ds}\mathbb{E}[s^X] = \frac{d}{ds}\sum_{k=0}^{\infty} s^k \mathbb{P}(X = k) = \sum_{k=0}^{\infty} \frac{d}{ds} s^k \mathbb{P}(X = k) = \sum_{k=0}^{\infty} k s^{k-1} \mathbb{P}(X = k) = \mathbb{E}[X s^{X-1}].$$

So

$$G'_X(1) = \mathbb{E}[X]$$

(as long as $\mathbb{E}[X]$ exists). Differentiating again, we get

$$G''_X(1) = \mathbb{E}[X(X-1)] = \mathbb{E}[X^2] - \mathbb{E}[X],$$

and so, in particular,

$$\mathrm{var}(X) = G''_X(1) + G'_X(1) - (G'_X(1))^2.$$

In general,

$$\frac{d^k}{ds^k} G_X(s)\bigg|_{s=1} = \mathbb{E}[X(X-1)\cdots(X-k+1)].$$

Example 4.6. *Let $Y = X_1 + X_2 + X_3$, where X_1, X_2 and X_3 are independent random variables each having probability generating function*

$$G(s) = \frac{1}{6} + \frac{s}{3} + \frac{s^2}{2}.$$

1. *Find the mean and variance of X_1.*
2. *What is the p.g.f. of Y? What is $\mathbb{P}(Y = 3)$?*
3. *What is the p.g.f. of $3X_1$? Why is it not the same as the p.g.f. of Y? What is $\mathbb{P}(3X_1 = 3)$?*

Solution. 1. Differentiating the probability generating function,

$$G'(s) = \frac{1}{3} + s, \quad G''(s) = 1,$$

and so $\mathbb{E}[X_1] = G'(1) = \frac{4}{3}$ and

$$\text{var}\,(X_1) = G''(1) + G'(1) - (G'(1))^2 = 1 + \frac{4}{3} - \frac{16}{9} = \frac{5}{9}.$$

2. Just as in our derivation of the probability generating function for the binomial distribution,

$$G_Y(s) = \mathbb{E}[s^{X_1+X_2+X_3}] = \mathbb{E}[s^{X_1}]\mathbb{E}[s^{X_2}]\mathbb{E}[s^{X_3}]$$

and so

$$G_Y(s) = \left(\frac{1}{6} + \frac{s}{3} + \frac{s^2}{2}\right)^3 = \frac{1}{216}\left(1 + 6s + 21s^2 + 44s^3 + 63s^4 + 54s^5 + 27s^6\right).$$

$\mathbb{P}(Y = 3)$ is the coefficient of s^3 in $G_Y(s)$, that is $\frac{11}{54}$. (As an exercise, calculate $\mathbb{P}(Y = 3)$ directly.)

3. We have

$$G_{3X_1}(s) = \mathbb{E}[s^{(3X_1)}] = \mathbb{E}[(s^3)^{X_1}] = G_{X_1}(s^3) = \frac{1}{6} + \frac{s^3}{3} + \frac{s^6}{2}.$$

This is different from $G_Y(s)$ because $3X_1$ and S_3 have different distributions - knowing X_1 does not tell you Y, but it does tell you $3X_1$. Finally, $\mathbb{P}(3X_1 = 3) = \mathbb{P}(X_1 = 1) = \frac{1}{3}$.

Of course, for each fixed $s \in \mathbb{R}$, s^X is itself a discrete random variable. So we can use the law of total probability when calculating its expectation.

Example 4.7. *Suppose that there are n red balls, n white balls and 1 blue ball in an urn. A ball is selected at random and then replaced. Let X be the number of red balls selected before a blue ball is chosen. Find*

(a) the probability generating function of X,

(b) $\mathbb{E}[X]$,

(c) $\text{var}\,(X)$.

Solution. (a) We will use the law of total probability for expectations. Let R be the event that the first ball is red, W be the event that the first ball is white and B be the event that the first ball is blue. Then

$$G_X(s) = \mathbb{E}\left[s^X\right] = \mathbb{E}\left[s^X|R\right]\mathbb{P}(R) + \mathbb{E}\left[s^X|W\right]\mathbb{P}(W) + \mathbb{E}\left[s^X|B\right]\mathbb{P}(B).$$

Of course, the value of X is affected by the first ball which is picked. If the first ball is blue then we know that $X = 0$. If the first ball is white, we learn nothing about the value of X. If the first ball is red then effectively we start over again counting numbers of red balls, but we add 1 for the red ball we have already seen. This yields

$$\begin{aligned} G_X(s) &= \mathbb{E}\left[s^{1+X}\right]\mathbb{P}(R) + \mathbb{E}\left[s^X\right]\mathbb{P}(W) + \mathbb{E}\left[s^0\right]\mathbb{P}(B) \\ &= sG_X(s)\frac{n}{2n+1} + G_X(s)\frac{n}{2n+1} + \frac{1}{2n+1} \end{aligned}$$

and so

$$G_X(s) = \frac{1}{n+1-ns} = \frac{1/(n+1)}{1-(1-1/(n+1))s}.$$

(b) Differentiating, we get

$$G'_X(s) = \frac{n}{(n+1-ns)^2}$$

and so

$$\mathbb{E}[X] = G'_X(1) = n.$$

(c) Recall that

$$\mathrm{var}(X) = G''_X(1) + G'_X(1) - (G'_X(1))^2.$$

Differentiating the p.g.f. again we get

$$G''_X(s) = \frac{2n^2}{(n+1-ns)^3}$$

and so $G''_X(1) = 2n^2$. Hence,

$$\mathrm{var}(X) = 2n^2 + n - n^2 = n(n+1).$$

If we were just asked for $\mathbb{E}[X]$ it would be easier to calculate

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E}[X|R]\mathbb{P}(R) + \mathbb{E}[X|W]\mathbb{P}(W) + \mathbb{E}[X|B]\mathbb{P}(B) \\ &= (1+\mathbb{E}[X])\frac{N}{2n+1} + \mathbb{E}[X]\frac{n}{2n+1} + 0\cdot\frac{1}{2n+1} = n. \end{aligned}$$

In order to calculate $\mathrm{var}(X)$, however, we need both $\mathbb{E}[X]$ and $\mathbb{E}\left[X^2\right]$ and so it's easier just to find $G_X(s)$ and differentiate it.

Theorem 4.8. *Let $X_1, X_2, \ldots$ be i.i.d. non-negative integer-valued random variables with p.g.f. $G_X(s)$. Let N be another non-negative integer-valued random variable, independent of $X_1, X_2, \ldots$ and with p.g.f. $G_N(s)$. Then the p.g.f. of $\sum_{i=1}^N X_i$ is $G_N(G_X(s))$.*

Notice that the sum $\sum_{i=1}^N X_i$ has a *random* number of terms. We interpret it as 0 if $N = 0$.

Proof. We partition according to the value of N: we have

$$\begin{aligned}
\mathbb{E}\left[s^{X_1+\cdots+X_N}\right] &= \sum_{n=0}^{\infty} \mathbb{E}\left[s^{X_1+\cdots+X_N}|N=n\right]\mathbb{P}(N=n) \quad \text{by the law of total probability} \\
&= \sum_{n=0}^{\infty} \mathbb{E}\left[s^{X_1+\cdots+X_n}|N=n\right]\mathbb{P}(N=n) \\
&= \sum_{n=0}^{\infty} \mathbb{E}\left[s^{X_1+\cdots+X_n}\right]\mathbb{P}(N=n) \quad \text{by the independence of } N \text{ and } \{X_1, X_2, \ldots\} \\
&= \sum_{n=0}^{\infty} \mathbb{E}\left[s^{X_1}\right]\cdots\mathbb{E}\left[s^{X_n}\right]\mathbb{P}(N=n) \quad \text{since } X_1, X_2, \ldots \text{ are independent} \\
&= \sum_{n=0}^{\infty} (G_X(s))^n \mathbb{P}(N=n) \\
&= G_N(G_X(s)).
\end{aligned}$$

□

Corollary 4.9. *Suppose that $X_1, X_2, \ldots$ are independent and identically distributed* $\mathrm{Ber}(p)$ *random variables and that $N \sim \mathrm{Po}(\lambda)$, independently of $X_1, X_2, \ldots$. Then $\sum_{i=1}^{N} X_i \sim \mathrm{Po}(\lambda p)$.*

(Notice that we saw this result in disguise via a totally different method in a problem sheet question.)

Proof. We have $G_X(s) = 1 - p + ps$ and $G_N(s) = \exp(\lambda(s-1))$ and so by Theorem 4.8,

$$\mathbb{E}\left[s^{\sum_{i=1}^{N} X_i}\right] = G_N(G_X(s)) = \exp(\lambda(1-p+ps-1)) = \exp(\lambda p(s-1)).$$

Since this is the p.g.f. of $\mathrm{Po}(\lambda p)$ and p.g.f.'s uniquely determine distributions, the result follows. □

Example 4.10. *In a short fixed time period, a photomultiplier detects 0, 1 or 2 photons with probabilities $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{6}$ respectively. The photons detected by the photomultiplier cause it to give off a charge of 2, 3, 4 or 5 electrons (with equal probability) independently for every one photon originally detected. What is the probability generating function of the number of electrons given off in the time period? What is the probability that exactly five electrons are given off in that period?*

Solution. Let N be the number of photons detected. Then the probability generating function of N is

$$G_N(s) = \frac{1}{2} + \frac{1}{3}s + \frac{1}{6}s^2.$$

Let X_i be the number of electrons given off by the ith photon detected. Then $Y = X_1 + \cdots + X_N$ is the total number given off in the period (remember that N here is *random*). Now $G_X(s) = \frac{1}{4}(s^2+s^3+s^4+s^5)$ and so, by Theorem 4.8,

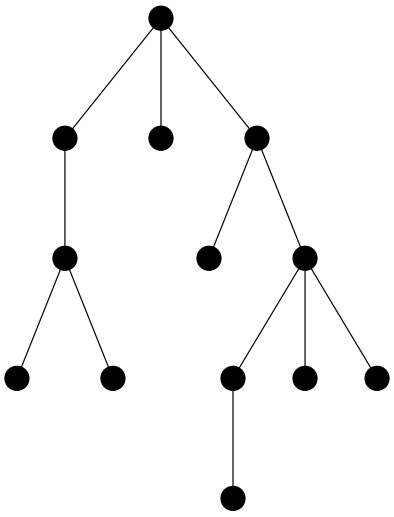
$$\begin{aligned}
G_Y(s) &= G_N(G_X(s)) \\
&= \frac{1}{2} + \frac{1}{3}G_X(s) + \frac{1}{6}(G_X(s))^2 \\
&= \frac{1}{2} + \frac{1}{12}s^2 + \frac{1}{12}s^3 + \frac{1}{12}s^4 + \frac{1}{12}s^5 + \frac{1}{96}(s^4 + 2s^5 + 3s^6 + 4s^7 + 3s^8 + 2s^9 + s^{10}).
\end{aligned}$$

The probability that five electrons are given off is the coefficient of s^5, that is $\frac{5}{48}$.

4.2 Branching processes

A really nice illustration of the power of probability generating functions is in the study of *branching processes*.

Suppose we have a population (say of bacteria). Each individual in the population lives a unit time and, just before dying, gives birth to a random number of children in the next generation. This number of children has probability mass function $p(i), i \geq 0$, called the *offspring distribution*. Different individuals reproduce independently in the same manner. Here is a possible family tree of such a population:



We start at the top of the tree, with a single individual in generation 0. Then there are 3 individuals in generations 1 and 2, 5 individuals in generation 3, a single individual in generation 4 and no individuals in subsequent generations.

Let X_n be the size of the population in generation n, so that $X_0 = 1$. Let $C_i^{(n)}$ be the number of children of the ith individual in generation $n \geq 0$, so that we may write

$$X_{n+1} = C_1^{(n)} + C_2^{(n)} + \cdots + C_{X_n}^{(n)}.$$

(We interpret this sum as 0 if $X_n = 0$.) Note that $C_1^{(n)}, C_2^{(n)}, \ldots$ are independent and identically distributed. Let $G(s) = \sum_{i=0}^{\infty} p(i)s^i$ and let $G_n(s) = \mathbb{E}\left[s^{X_n}\right]$.

Theorem 4.11. *For $n \geq 0$,*

$$G_{n+1}(s) = G_n(G(s)) = \underbrace{G(G(\ldots G}_{n+1 \text{ times}}(s)\ldots)) = G(G_n(s)).$$

Proof. Since $X_0 = 1$, we have $G_0(s) = s$. Also, we get $X_1 = C_1^{(0)}$ which has p.m.f. $p(i), i \geq 0$. So $G_1(s) = \mathbb{E}\left[s^{X_1}\right] = G(s)$. Since

$$X_{n+1} = \sum_{i=1}^{X_n} C_i^{(n)},$$

by Theorem 4.8 we get

$$G_{n+1}(s) = \mathbb{E}\left[s^{X_{n+1}}\right] = \mathbb{E}\left[s^{\sum_{i=1}^{X_n} C_i^{(n)}}\right] = G_n(G(s)).$$

Hence, by induction, for $n \geq 1$,

$$G_{n+1}(s) = \underbrace{G(G(\ldots G}_{n \text{ times}}(s)\ldots)) = G(G_n(s)).$$

□

Corollary 4.12. *Suppose that the mean number of children of a single individual is μ i.e. $\sum_{i=1}^{\infty} ip(i) = \mu$. Then*

$$\mathbb{E}[X_n] = \mu^n.$$

Proof. We have $\mathbb{E}[X_n] = G_n'(1)$. By the chain rule,

$$G_n'(s) = \frac{d}{ds} G(G_{n-1}(s)) = G_{n-1}'(s) G'(G_{n-1}(s)).$$

Plugging in $s = 1$, we get

$$\mathbb{E}[X_n] = \mathbb{E}[X_{n-1}]\, G'(1) = \mathbb{E}[X_{n-1}]\, \mu = \cdots = \mu^n.$$ □

In particular, notice that we get exponential growth on average if $\mu > 1$ and exponential decrease if $\mu < 1$. This raises an interesting question: can the population die out? If $p(0) = 0$ then every individual has at least one child and so the population clearly grows forever. If $p(0) > 0$, on the other hand, then the population dies out with positive probability because

$$\mathbb{P}(\text{population dies out}) = \mathbb{P}(\cup_{n=1}^{\infty}\{X_n = 0\}) \geq \mathbb{P}(X_1 = 0) = p(0) > 0.$$

(Notice that this holds even in the cases where $\mathbb{E}[X_n]$ grows as $n \to \infty$!)

Example 4.13. *Suppose that $p(i) = (1/2)^{i+1}$, $i \geq 0$, so that each individual has a geometric number of offspring. What is the distribution of X_n?*

Solution. First calculate

$$G(s) = \sum_{k=0}^{\infty} s^k \left(\frac{1}{2}\right)^{k+1} = \frac{1}{2-s}.$$

By plugging this into itself a couple of times, we get

$$G_2(s) = \frac{2-s}{3-2s}, \quad G_3(s) = \frac{3-2s}{4-3s}.$$

A natural guess is that $G_n(s) = \frac{n-(n-1)s}{(n+1)-ns}$ which is, in fact, the case, as can be proved by induction. If we want the probability mass function of X_n, we need to expand this quantity out in powers of s. We have

$$\frac{1}{(n+1)-ns} = \frac{1}{n+1}\frac{1}{1-ns/(n+1)} = \sum_{k=0}^{\infty} \frac{n^k s^k}{(n+1)^{k+1}}.$$

Multiplying by $n-(n-1)s$, we get

$$G_n(s) = \sum_{k=0}^{\infty} \frac{n^{k+1}s^k}{(n+1)^{k+1}} - \sum_{k=1}^{\infty} \frac{n^{k-1}(n-1)s^k}{(n+1)^k} = \frac{n}{n+1} + \sum_{k=1}^{\infty} \frac{n^{k-1}s^k}{(n+1)^{k+1}}.$$

We can read off the coefficients now to see that

$$\mathbb{P}(X_n = k) = \begin{cases} \frac{n}{n+1} & \text{if } k = 0 \\ \frac{n^{k-1}}{(n+1)^{k+1}} & \text{if } k \geq 1. \end{cases}$$

Notice that $\mathbb{P}(X_n = 0) \to 1$ as $n \to \infty$, which indicates that the population dies out eventually in this case.

4.2.1 Extinction probability (*non-examinable*)

Let's return to the general case for the moment and let $q = \mathbb{P}(\text{population dies out})$. We can call q the *extinction probability* of the branching process. We can find an equation satisfied by q by conditioning on the number of children of the first individual.

$$q = \sum_{k=0}^{\infty} \mathbb{P}(\text{population dies out} | X_1 = k)\, \mathbb{P}(X_1 = k) = \sum_{k=0}^{\infty} \mathbb{P}(\text{population dies out} | X_1 = k)\, p(k).$$

Now remember that each of the k individuals in the first generation behaves exactly like the parent. In particular, we can think of each of them starting its own family, which is an independent copy of the original family. Moreover, the whole population dies out if and only if all of these sub-populations die out. If we had k families, this occurs with probability q^k. So

$$q = \sum_{k=0}^{\infty} q^k p(k) = G(q). \tag{4.1}$$

The equation $q = G(q)$ doesn't quite enable us to determine q: notice that 1 is always a solution, but it's not necessarily the only solution in $[0, 1]$.

Using Proposition A.8 about increasing sequences of events (see Appendix), we have

$$\begin{aligned} q &= \mathbb{P}\left(\bigcup_n \{X_n = 0\}\right) \\ &= \lim_{n\to\infty} \mathbb{P}(X_n = 0) \\ &= \lim_{n\to\infty} G_n(0). \end{aligned} \tag{4.2}$$

Theorem 4.14. *The extinction probability q is the smallest non-negative solution of*

$$x = G(x) \tag{4.3}$$

Proof. From (4.1) we know that q solves (4.3). Suppose some $r \geq 0$ also solves (4.3). We claim that in that case, $G_n(0) \leq r$ for all $n \geq 0$. In that case we are done, since then also $q = \lim_{n\to\infty} G_n(0) \leq r$, and so indeed q is smaller than any other solution of (4.3).

We use induction to prove the claim that $G_n(0) \leq r$ for all n. For the base case $n = 0$, we have $G_0(0) = 0 \leq r$ as required.

For the induction step, suppose that $G_{n-1}(0) \leq r$. Now notice that the generating function $G(s) = \sum_{k=0}^{\infty} p(k) s^k$ is a non-decreasing function for $s \geq 0$. Hence

$$G_n(0) = G\big(G_{n-1}(0)\big) \leq G(r) = r,$$

as required. This completes the proof. □

It turns out that the question of whether the branching process inevitably dies out is determined by the mean number of children of a single individual. To avoid a trivial case, we assume in the next result that $p(1) \neq 1$. (If $p(1) = 1$ then $X_n = 1$ with probability 1, for all n.) Then we find that there is a positive probability of survival of the process for ever if and only if $\mu > 1$.

Theorem 4.15. *Assume $p(1) \neq 1$. Then $q = 1$ if $\mu \leq 1$, and $q < 1$ if $\mu > 1$.*

Proof. Note first that there's a quick argument for the case where μ is strictly less than 1. Note that as X_n takes non-negative integer values,

$$\mathbb{P}(X_n > 0) \leq \mathbb{E}[X_n]$$

(since $\mathbb{P}(X_n > 0) = \sum_{k\geq 1} \mathbb{P}(X_n = k) \leq \sum_{k\geq 1} k\mathbb{P}(X_n = k) = \mathbb{E}[X_n]$).

But from Corollary 4.12, we have $\mathbb{E}[X_n] = \mu^n$. Hence $\mathbb{P}(X_n > 0) \to 0$ as $n \to \infty$, and so from (4.2), we get $q = 1$.

Now we give a more general argument that also covers the cases $\mu = 1$ and $\mu > 1$. First, observe that the gradient $G'(s) = \sum_{k=1} kp(k)s^{k-1}$ is non-decreasing for $s \geq 0$ (and, indeed, strictly increasing unless $p_0 + p_1 = 1$). That is, G is *convex*.

Consider the graph of $y = G(x)$, on the interval $x \in [0, 1]$. It passes through the points $(0, p_0)$ and $(1, 1)$, and at $(1, 1)$ its slope is $\mu = G'(1)$.

We have the following two cases:

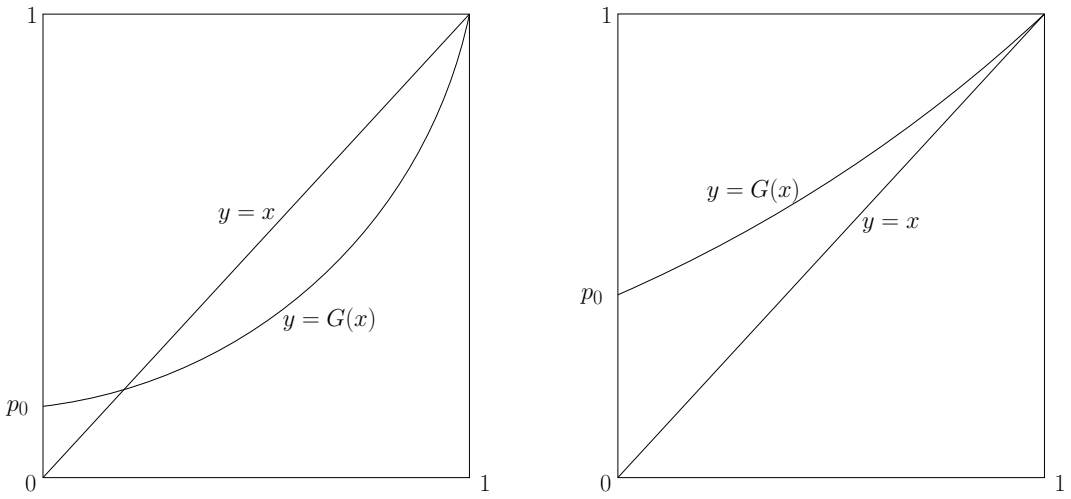


Figure 4.1: On the left, the case $\mu > 1$; on the right, the case $\mu \leq 1$.

(1) Suppose $\mu > 1$. Since the gradient of the curve $y = G(x)$ is more than 1 at $x = 1$, and the curve starts on the non-negative y-axis at $x = 0$, it must cross the line $y = x$ at some $x \in [0, 1)$. See the left side of Figure 4.1. Hence indeed the smallest non-negative fixed point q of G is less than 1.

(2) Suppose $\mu \leq 1$. The gradient at 1 is at most 1, and in fact the gradient is strictly less than 1 for all $x \in [0, 1)$. (We excluded the case $p_1 = 1$ for which the gradient is 1 everywhere.) Now the function $y = G(x)$ must stay above the line $y = x$ throughout $[0, 1)$. See the right side of Figure 4.1. So the smallest non-negative fixed point q of G is 1.

□

Chapter 5

Continuous random variables

5.1 Random variables and cumulative distribution functions

Recall that we defined a discrete random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to be a function $X : \Omega \to \mathbb{R}$ such that X can only take countably many values (and such that we can assign a probability to the event $\{X = x\}$, i.e. such that $\{X = x\} \in \mathcal{F}$). There is, however, a more general notion. The essential idea is that a random variable can be *any* (sufficiently nice) function $X : \Omega \to \mathbb{R}$, which represents some sort of observable quantity in our random experiment.

Why do we need more general random variables?

- Some outcomes are essentially continuous. In particular, many physical quantities are most naturally modelled as taking uncountably many possible values, for example, lengths, weights and speeds.
- Even for discrete quantities, it is often useful to think instead in terms of continuous approximations. For example, suppose you wish to consider the number of working adults who regularly contribute to charity. You might model this number as $X \in \{0, 1, \ldots, n\}$, where n is the total number of working adults in the UK. We could, in theory, model this as a $\text{Bin}(n, p)$ random variable where $p = \mathbb{P}(\text{adult contributes})$. But n is measured in millions. So instead model $Y \approx \frac{X}{n}$ as a continuous random variable taking values in $[0, 1]$ and giving the proportion of adults who contribute.

To give a concrete example of a random variable which is not discrete, imagine you have a board game spinner. You spin the arrow and it lands pointing at an angle somewhere between 0 and 2π in such a way that every angle is equally likely; we want to model this angle as a random variable X. How can we describe its distribution? We can't assign a positive probability to each angle – our probabilities wouldn't sum to 1. To get around this, we don't define the probability of individual sample points, but only of certain natural events. For example, by symmetry we expect that $\mathbb{P}(X \leq \pi) = 1/2$. More generally, we expect the probability that X lies in an interval $[a, b] \subseteq [0, 2\pi)$ to be proportional to the length of that interval: $\mathbb{P}(X \in [a, b]) = \frac{b-a}{2\pi}$, $0 \leq a < b < 2\pi$.

Definition 5.1. *A* random variable X *defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a function* $X : \Omega \to \mathbb{R}$ *such that* $\{\omega : X(\omega) \leq x\} \in \mathcal{F}$ *for each* $x \in \mathbb{R}$.

Let's just check that this includes our earlier definition. If X is a discrete random variable then

$$\{\omega : X(\omega) \leq x\} = \bigcup_{y \leq x : y \in \mathrm{Im}X} \{\omega : X(\omega) = y\}.$$

Since $\mathrm{Im}X$ is countable, this is a countable union of events in $\mathcal{F}$ and, therefore, itself belongs to $\mathcal{F}$.

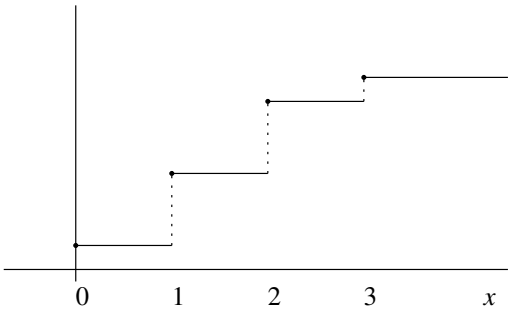
Of course, $\{\omega : X(\omega) \leq x\} \in \mathcal{F}$ means precisely that we can assign a probability to this event. The collection of these probabilities as x varies in $\mathbb{R}$ will play a central part in what follows.

Definition 5.2. *The* cumulative distribution function (c.d.f.) *of a random variable X is the function $F_X : \mathbb{R} \to [0,1]$ defined by*

$$F_X(x) = \mathbb{P}(X \leq x).$$

Example 5.3. *Let X be the number of heads obtained in three tosses of a fair coin. Then $\mathbb{P}(X = 0) = \frac{1}{8}$, $\mathbb{P}(X = 1) = \mathbb{P}(X = 2) = \frac{3}{8}$ and $\mathbb{P}(X = 3) = \frac{1}{8}$. So*

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{8} & \text{if } 0 \leq x < 1 \\ \frac{1}{8} + \frac{3}{8} = \frac{1}{2} & \text{if } 1 \leq x < 2 \\ \frac{1}{8} + \frac{3}{8} + \frac{3}{8} = \frac{7}{8} & \text{if } 2 \leq x < 3 \\ 1 & \text{if } x \geq 3. \end{cases}$$



Example 5.4. *Let X be the angle of the board game spinner. Then*

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{x}{2\pi} & \text{if } 0 \leq x < 2\pi, \\ 1 & \text{if } x \geq 2\pi. \end{cases}$$

We can immediately write down some properties of the c.d.f. F_X corresponding to a general random variable X.

Theorem 5.5. *1. F_X is non-decreasing.*

2. $\mathbb{P}(a < X \leq b) = F_X(b) - F_X(a)$ for $a < b$.

3. As $x \to -\infty$, $F_X(x) \to 0$.

4. As $x \to \infty$, $F_X(x) \to 1$.

Proof. 1. If $a < b$ then $\{\omega : X(\omega) \leq a\} \subseteq \{\omega : X(\omega) \leq b\}$ and so

$$F_X(a) = \mathbb{P}(X \leq a) \leq \mathbb{P}(X \leq b) = F_X(b).$$

2. Since $\{X \leq a\}$ is a subset of $\{X \leq b\}$,

$$\mathbb{P}(a < X \leq b) = \mathbb{P}(\{X \leq b\} \setminus \{X \leq a\}) = \mathbb{P}(X \leq b) - \mathbb{P}(X \leq a) = F_X(b) - F_X(a).$$

3 & 4. (sketch) Intuitively, we want to put "$F_X(-\infty) = \mathbb{P}(X \leq -\infty)$" and then, since X can't possibly be $-\infty$ (or less!), the only sensible interpretation we could give the right-hand side would be 0. Likewise,

we would like to put "$F_X(\infty) = \mathbb{P}(X \leq \infty)$" and, since X cannot be larger than ∞, the only sensible interpretation we could give the right-hand side would be 1. The problem is that ∞ and $-\infty$ aren't real numbers, but F_X is a function on $\mathbb{R}$. The only sensible way to deal with this problem is by taking limits and to do this carefully involves using the countable additivity axiom $\mathbf{P}_3$ in a somewhat intricate way. □

Conversely, any function F satisfying conditions 1, 3 and 4 of Theorem 5.5 plus *right-continuity* is the cumulative distribution function of *some* random variable defined on *some* probability space, although we will not prove this fact.

As you can see from the coin-tossing example, F_X need not be a smooth function. Indeed, for a discrete random variable, F_X is always a step function. However, in the rest of the course, we're going to concentrate on the case where F_X is very smooth in that it has a derivative (except possibly at a collection of isolated points).

Definition 5.6. *A* continuous random variable *X is a random variable whose c.d.f. satisfies*

$$F_X(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f_X(u)du,$$

where $f_X : \mathbb{R} \to \mathbb{R}$ is a function such that

(a) $f_X(u) \geq 0$ for all $u \in \mathbb{R}$

(b) $\int_{-\infty}^{\infty} f_X(u)du = 1$.

f_X is called the probability density function (p.d.f.) *of X or, sometimes, just its* density.

Remark 5.7. *The definition of a continuous random variable leaves implicit* which *functions f_X might possibly serve as a probability density function. Part of this is a more fundamental question concerning which functions we are allowed to integrate (and for some of you, that will be resolved in the Analysis III course in Trinity Term and in Part A Integration). For the purposes of this course, you may assume that f_X is a function which has at most countably many jumps and is smooth everywhere else. Indeed, in almost all of the examples we will consider, f_X will have 0, 1 or 2 jumps.*

Remark 5.8. *The Fundamental Theorem of Calculus (which some of you will see proved in Analysis II), tells us that F_X of the form given in the definition is differentiable with*

$$\frac{dF_X(x)}{dx} = f_X(x),$$

at any point x such that $f_X(x)$ is continuous.

Example 5.9. *Suppose that X has c.d.f.*

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-x} & \text{if } x \geq 0. \end{cases}$$

Consider

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ e^{-x} & \text{for } x \geq 0. \end{cases}$$

Then

$$\int_{-\infty}^{x} f(u)du = \begin{cases} 0 & \text{if } x < 0 \\ \int_0^x e^{-u}du = 1 - e^{-x} & \text{if } x \geq 0, \end{cases}$$

and so X is a continuous random variable with density $f_X(x) = f(x)$. Notice that $f_X(0) = 1$ and so f_X has a jump at $x = 0$. On the other hand, F_X is smooth at 0, but it isn't differentiable there. To see this, if we approach 0 from the right, F_X has gradient tending to 1; if we approach 0 from the left, F_X has gradient 0 and, since these don't agree, there isn't a well-defined derivative. On the other hand, everywhere apart from 0 we do have $F'_X(x) = f_X(x)$.

Example 5.10. *Suppose that a continuous random variable X has p.d.f.*

$$f_X(x) = \begin{cases} cx^2(1-x) & \text{for } x \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

Find the constant c and an expression for the c.d.f.

Solution. To find the constant, c, note that we must have

$$1 = \int_{-\infty}^{\infty} f_X(x)dx = \int_0^1 cx^2(1-x)dx = c\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{c}{12}.$$

It follows that $c = 12$. To find the c.d.f., we simply integrate:

$$F_X(x) = \int_{-\infty}^{x} f_X(u)du = \begin{cases} 0 & \text{for } x < 0 \\ \int_0^x 12u^2(1-u)du & \text{for } 0 \leqslant x < 1 \\ 1 & \text{for } x \geqslant 1. \end{cases}$$

Since

$$\int_0^x 12u^2(1-u)du = 12\left(\frac{x^3}{3} - \frac{x^4}{4}\right),$$

we get

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ 4x^3 - 3x^4 & \text{for } 0 \leqslant x < 1 \\ 1 & \text{for } x \geqslant 1. \end{cases}$$

Example 5.11. *The duration in minutes of mobile phone calls made by students is modelled by a random variable, X, with p.d.f.*

$$f_X(x) = \begin{cases} \frac{1}{6}e^{-x/6} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

What is the probability that a call lasts

(i) between 3 and 6 minutes?

(ii) more than 6 minutes?

Solution. (i)

$$\mathbb{P}(3 < X \leq 6) = \int_3^6 f_X(x)\mathrm{d}x = \int_3^6 \frac{1}{6}e^{-x/6}dx = e^{-\frac{1}{2}} - e^{-1}.$$

(ii)

$$\mathbb{P}(X > 6) = \int_6^{\infty} f_X(x)\mathrm{d}x = \int_6^{\infty} \frac{1}{6}e^{-x/6}dx = e^{-1}.$$

We often use the p.d.f. of a continuous random variable analogously to the way we used the p.m.f. of a discrete random variable. There are several similarities between the two:

Probability density function (continuous)	Probability mass function (discrete)
$f_X(x) \geqslant 0 \quad \forall x \in \mathbb{R}$	$p_X(x) \geqslant 0 \quad \forall x \in \mathbb{R}$
$\int_{-\infty}^{\infty} f_X(x) = 1$	$\sum_{x \in \mathrm{Im} X} p_X(x) = 1$
$F_X(x) = \int_{-\infty}^{x} f_X(u)du$	$F_X(x) = \sum_{u \leqslant x:\ u \in \mathrm{Im} X} p_X(u)$

However, the analogy can be misleading. For example, there's nothing to prevent $f_X(x)$ exceeding 1.

WARNING: $f_X(x)$ IS NOT A PROBABILITY.

Suppose that $\epsilon > 0$ is small. Then, by Taylor's theorem,

$$\mathbb{P}(x < X \leq x + \epsilon) = F_X(x + \epsilon) - F_X(x) \approx f_X(x)\epsilon.$$

So $f_X(x)\epsilon$ is approximately the probability that X falls between x and $x + \epsilon$ (or, indeed, between $x - \epsilon$ and x). What happens as $\epsilon \to 0$?

Theorem 5.12. *If X is a continuous random variable with p.d.f. f_X then*

$$\mathbb{P}(X = x) = 0 \quad \textit{for all } x \in \mathbb{R}$$

and

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x)dx.$$

Proof. (Non-examinable.) We argue by contradiction. Suppose that for some $x \in \mathbb{R}$ we have $\mathbb{P}(X = x) > 0$. Let $p = \mathbb{P}(X = x)$. Then for all $n \geq 1$, $\mathbb{P}(x - 1/n < X \leq x) \geq p$. We have $\mathbb{P}(x - 1/n < X \leq x) = F_X(x) - F_X(x - 1/n)$ and so $F_X(x) - F_X(x - 1/n) \geq p$ for all $n \geq 1$. But F_X is continuous at x and so

$$\lim_{n \to \infty} (F_X(x) - F_X(x - 1/n)) = 0.$$

This gives a contradiction. So we must have $\mathbb{P}(X = x) = 0$.

Finally, $\mathbb{P}(a \leq X \leq b) = \mathbb{P}(X = a) + \mathbb{P}(a < X \leq b)$ and so, since $\mathbb{P}(X = a) = 0$, we get

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x)dx. \qquad \square$$

So for a continuous r.v. X, the probability of getting any fixed value x is 0! Why doesn't this break our theory of probability? We have

$$\{\omega : X(\omega) \leq x\} = \bigcup_{y \leq x} \{\omega : X(\omega) = y\}$$

and the right-hand side is an *uncountable* union of disjoint events of probability 0. If the union were countable, this would entail that the left-hand side had probability 0 also, which wouldn't make much sense. But because the union is uncountable, we cannot expect to "sum up" these zeros in order to get the probability of the left-hand side. The right way to resolve this problem is using a probability density function.

Remark 5.13. *There do exist random variables which are neither discrete nor continuous. To give a slightly artificial example, suppose that we flip a fair coin. If it comes up heads, sample U uniformly from $[0,1]$ and set X to be the value obtained; if it comes up tails, let $X = 1/2$. Then X can take uncountably many values but does not have a density. Indeed, as you can check,*

$$\mathbb{P}(X \leq x) = \begin{cases} \frac{x}{2} & \text{if } 0 \leq x < 1/2 \\ \frac{x+1}{2} & \text{if } 1/2 \leq x \leq 1, \end{cases}$$

and there does not exist a function f_X which integrates to give this.

The theory is particularly nice in the discrete and continuous cases because we can work with probability mass functions and probability density functions respectively. But the cumulative distribution function is a more general concept which makes sense for all *random variables.*

5.2 Some classical distributions

As we did for discrete distributions, we introduce a stock of examples of continuous distributions which will come up time and again in this course.

1. **The uniform distribution.** X has the uniform distribution on an interval $[a,b]$ if it has p.d.f.

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

   We write $X \sim \mathrm{U}[a,b]$.

2. **The exponential distribution.** X has the exponential distribution with parameter $\lambda \geq 0$ if it has p.d.f.

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0.$$

   We write $X \sim \mathrm{Exp}(\lambda)$. The exponential distribution is often used to model lifetimes or the time elapsing between unpredictable events (such as telephone calls, arrivals of buses, earthquakes, emissions of radioactive particles, etc).

3. **The gamma distribution.** X has the gamma distribution with parameters $\alpha > 0$ and $\lambda \geq 0$ if it has p.d.f.

$$f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad x \geq 0.$$

   Here, $\Gamma(\alpha)$ is the so-called *gamma function*, which is defined by

$$\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u} du$$

   for $\alpha > 0$. For most values of α this integral does not have a closed form. However, for a strictly positive integer n, we have $\Gamma(n) = (n-1)!$. (See the Wikipedia "Gamma function" page for lots more information about this fascinating function!)

If X has the above p.d.f. we write $X \sim \text{Gamma}(\alpha, \lambda)$. The gamma distribution is a generalisation of the exponential distribution and possesses many nice properties. The *Chi-squared distribution with d degrees of freedom*, χ^2_d, which you may have seen at 'A' Level, is the same as $\text{Gamma}(d/2, 1/2)$ for $d \in \mathbb{N}$.

4. **The normal (or Gaussian) distribution.** X has the normal distribution with parameters $\mu \in \mathbb{R}$ and $\sigma^2 > 0$ if it has p.d.f.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad x \in \mathbb{R}.$$

We write $X \sim \text{N}(\mu, \sigma^2)$. The *standard normal distribution* is $\text{N}(0, 1)$. The normal distribution is used to model all sorts of characteristics of large populations and samples. Its fundamental importance across Probability and Statistics is a consequence of the Central Limit Theorem, which you will use in Prelims Statistics and see proved in Part A Probability.

Exercise 5.14. *For the uniform and exponential distributions:*

- *Check that for each of these f_X really is a p.d.f. (i.e. that it is non-negative and integrates to 1).*
- *Calculate the corresponding c.d.f.'s.*

Example 5.15. *Show that*

$$I := \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx = 1.$$

Solution. We first change variables in the integral. Set $z = (x-\mu)/\sigma$. Then

$$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz.$$

It follows that

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} \exp\left(-\frac{(x^2+y^2)}{2}\right) dxdy. \end{aligned}$$

Now convert to polar co-ordinates: let r and θ be such that $x = r\cos\theta$ and $y = r\sin\theta$. Then the Jacobian is $|J| = r$ and so we get

$$\int_0^{2\pi}\int_0^{\infty} \frac{1}{2\pi} r \exp\left(-\frac{r^2}{2}\right) drd\theta = \left[-e^{-r^2/2}\right]_0^{\infty} = 1.$$

Since I is clearly non-negative (it's the integral of a non-negative function), we must have $I = 1$.

The c.d.f. of the standard normal distribution,

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du,$$

cannot be written in a closed form, but can be found by numerical integration to an arbitrary degree of accuracy. This very important function is usually called Φ and if you did some Statistics at 'A' Level you will certainly have come across tables of its values.

5.3 Expectation

Recall that for a discrete r.v. we defined

$$\mathbb{E}[X] = \sum_{x \in \mathrm{Im} X} x p_X(x) \tag{5.1}$$

whenever the sum is absolutely convergent and, more generally, for any function $h : \mathbb{R} \to \mathbb{R}$, we had

$$\mathbb{E}[h(X)] = \sum_{x \in \mathrm{Im} X} h(x) p_X(x) \tag{5.2}$$

whenever this sum is absolutely convergent. We want to make an analogous definition for continuous random variables. Suppose X has a smooth p.d.f. f_X. Then for any x and small $\delta > 0$,

$$\mathbb{P}(x \leq X \leq x + \delta) \approx f_X(x)\delta$$

and, in particular,

$$\mathbb{P}(n\delta \leq X \leq (n+1)\delta) \approx f_X(n\delta)\delta.$$

So for the expectation, we want something like

$$\sum_{n=-\infty}^{\infty} (n\delta) f_X(n\delta)\delta.$$

We now want to take $\delta \to 0$; intuitively, we should obtain an integral.

Definition 5.16. *Let X be a continuous random variable with probability density function f_X. The* expectation *or* mean *of X is defined to be*

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx \tag{5.3}$$

whenever $\int_{-\infty}^{\infty} |x| f_X(x) dx < \infty$. Otherwise, we say that the mean is undefined (or as in the discrete case, if only the positive tail diverges, we might say that $\mathbb{E}[X] = \infty$.)

Theorem 5.17. *Let X be a continuous random variable with probability density function f_X, and let h be a function from $\mathbb{R}$ to $\mathbb{R}$. Then*

$$\mathbb{E}[h(X)] = \int_{-\infty}^{\infty} h(x) f_X(x) dx. \tag{5.4}$$

(whenever $\int_{-\infty}^{\infty} |h(x)| f_X(x) dx < \infty$).

Notice that (5.4) is analogous to (5.2) in the same way that (5.3) is analogous to (5.1). Proving Theorem 5.17 in full generality, for any function h, is rather technical. Here we just give an idea of one approach to the proof for a particular class of functions.

Proof of Theorem 5.17 (outline of idea, non-examinable). First we claim that if X is a non-negative continuous random variable, then $\mathbb{E}[X] = \int_0^x \mathbb{P}(X > x)\, dx$. To show this, we can write the

expectation as a double integral and change the order of integration:

$$\begin{aligned}\mathbb{E}[X] &= \int_{x=0}^{\infty} x f_X(x) dx \\ &= \int_{x=0}^{\infty} \int_{y=0}^{x} f_X(x) dy\, dx \\ &= \int_{y=0}^{\infty} \int_{x=y}^{\infty} f_X(x) dx\, dy \\ &= \int_{y=0}^{\infty} \mathbb{P}(X > y)\, dy,\end{aligned}$$

giving the claim as required.

So now suppose h is such that $h(X)$ is a non-negative continuous random variable. Then

$$\begin{aligned}\mathbb{E}[h(X)] &= \int_{y=0}^{\infty} \mathbb{P}(h(X) > y)\, dy \\ &= \int_{y=0}^{\infty} \int_{x:h(x)>y} f_X(x) dx\, dy \\ &= \int_{x=0}^{\infty} f_X(x) \int_{y:y<h(x)} dy\, dx \\ &= \int_{x=0}^{\infty} f_X(x) h(x) dx,\end{aligned}$$

giving the desired formula in this case. □

As in the case of discrete random variables, we define the *variance* of X to be

$$\mathrm{var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$$

whenever the right-hand side is defined. For simplicity of notation, write $\mu = \mathbb{E}[X]$. Then we have

$$\begin{aligned}\mathrm{var}(X) &= \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x) dx \\ &= \int_{-\infty}^{\infty} (x^2 - 2x\mu + \mu^2) f_X(x) dx \\ &= \int_{-\infty}^{\infty} x^2 f_X(x) dx - 2\mu \int_{-\infty}^{\infty} x f_X(x) dx + \mu^2 \int_{-\infty}^{\infty} f_X(x) dx \\ &= \mathbb{E}\left[X^2\right] - \mu^2,\end{aligned}$$

since $\int_{-\infty}^{\infty} x f_X(x) dx = \mu$ and $\int_{-\infty}^{\infty} f_X(x) dx = 1$. So we recover the expression

$$\mathrm{var}(X) = \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2.$$

Just as in the discrete case, expectation has a *linearity property.*

Theorem 5.18. *Suppose X is a continuous random variable with p.d.f. f_X. Then if $a, b \in \mathbb{R}$ then $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$ and $\mathrm{var}(aX + b) = a^2 \mathrm{var}(X)$.*

Proof. We have

$$\mathbb{E}\left[aX+b\right] = \int_{-\infty}^{\infty}(ax+b)f_X(x)dx = a\int_{-\infty}^{\infty}xf_X(x)dx + b\int_{-\infty}^{\infty}f_X(x)dx = a\mathbb{E}\left[X\right]+b,$$

as required, since the density integrates to 1. Moreover,

$$\mathrm{var}\left(aX+b\right) = \mathbb{E}\left[(aX+b-a\mathbb{E}\left[X\right]-b)^2\right] = \mathbb{E}\left[a^2(X-\mathbb{E}\left[X\right])^2\right] = a^2\mathbb{E}\left[(X-\mathbb{E}\left[X\right])^2\right] = a^2\mathrm{var}\left(X\right).$$

□

Example 5.19. *Suppose $X \sim \mathrm{N}(\mu,\sigma^2)$. Then*

- *X has the same distribution as $\mu+\sigma Z$, where $Z \sim \mathrm{N}(0,1)$.*
- *X has c.d.f. $F_X(x) = \Phi((x-\mu)/\sigma)$, where Φ is the standard normal c.d.f.*
- *$\mathbb{E}\left[X\right] = \mu$.*
- *$\mathrm{var}\left(X\right) = \sigma^2$.*

Solution. First suppose that $\mu = 0$ and $\sigma^2 = 1$. Then the first two assertions are trivial and

$$\mathbb{E}\left[X\right] = \int_{-\infty}^{\infty}\frac{x}{\sqrt{2\pi}}e^{-x^2/2}dx$$

which must equal 0 since the integrand is an odd function. Since the mean is 0,

$$\mathrm{var}\left(X\right) = \mathbb{E}\left[X^2\right] = \int_{-\infty}^{\infty}\frac{x^2}{\sqrt{2\pi}}e^{-x^2/2}dx = \int_{-\infty}^{\infty}x\cdot\frac{xe^{-x^2/2}}{\sqrt{2\pi}}dx.$$

Integrating by parts, we get that this equals

$$\left[-x\cdot\frac{e^{-x^2/2}}{\sqrt{2\pi}}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx = 1.$$

So $\mathrm{var}\left(X\right) = 1$.

Suppose now that $Z \sim \mathrm{N}(0,1)$. Then

$$\mathbb{P}\left(\mu+\sigma Z \le x\right) = \mathbb{P}\left(Z \le (x-\mu)/\sigma\right) = \Phi((x-\mu)/\sigma).$$

Let $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, the standard normal density. Differentiating $\mathbb{P}\left(\mu+\sigma Z \le x\right)$ in x, we get

$$\frac{1}{\sigma}\phi((x-\mu)/\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

So $\mu+\sigma Z \sim \mathrm{N}(\mu,\sigma^2)$. Finally,

$$\mathbb{E}\left[X\right] = \mathbb{E}\left[\mu+\sigma Z\right] = \mu+\sigma\mathbb{E}\left[Z\right] = \mu$$

and

$$\mathrm{var}\left(X\right) = \mathrm{var}\left(\mu+\sigma Z\right) = \sigma^2\mathrm{var}\left(Z\right) = \sigma^2.$$

Exercise 5.20. *Show that if* $X \sim \mathrm{U}[a,b]$ *and* $Y \sim \mathrm{Exp}(\lambda)$ *then*

$$\mathbb{E}[X] = \frac{a+b}{2}, \quad \mathrm{var}(X) = \frac{(b-a)^2}{12}, \quad \mathbb{E}[Y] = \frac{1}{\lambda}, \quad \mathrm{var}(Y) = \frac{1}{\lambda^2}.$$

Notice, in particular, that the parameter of the Exponential distribution is the reciprocal *of its mean.*

Example 5.21. *Suppose that* $X \sim Gamma(2,2)$*, so that it has p.d.f.*

$$f_X(x) = \begin{cases} 4xe^{-2x} & \text{for } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Find $\mathbb{E}[X]$ *and* $\mathbb{E}\left[\frac{1}{X}\right]$.

Solution. We have

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \cdot 4xe^{-2x}dx = \int_{-\infty}^{\infty} \frac{2^3}{2!}x^{3-1}e^{-2x}dx$$

and, since $\Gamma(3) = 2!$ we recognise the integrand as the density of a Gamma$(3,2)$ random variable. So it must integrate to 1 and we get $\mathbb{E}[X] = 1$.

On the other hand,

$$\mathbb{E}\left[\frac{1}{X}\right] = \int_{-\infty}^{\infty} \frac{1}{x} \cdot 4xe^{-2x}dx = 2\int_{-\infty}^{\infty} 2e^{-2x}dx$$

and again we recognise the integrand as the density of an Exp(2) random variable which must integrate to 1. So we get $\mathbb{E}\left[\frac{1}{X}\right] = 2$.

WARNING: IN GENERAL, $\mathbb{E}\left[\frac{1}{X}\right] \neq \frac{1}{\mathbb{E}[X]}$.

5.4 Examples of functions of continuous random variables

Example 5.22. *Imagine a forest. Suppose that* R *is the distance from a tree to the nearest neighbouring tree. Suppose that* R *has p.d.f.*

$$f_R(r) = \begin{cases} re^{-r^2/2} & \text{for } r \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Find the distribution of the area of the tree-free circle around the original tree.

Solution. Let A be the area of the tree-free circle; then $A = \pi R^2$. We begin by finding the c.d.f. of R and then use it to find the c.d.f. of A. $F_R(r)$ is clearly 0 for $r < 0$. For $r \geq 0$,

$$F_R(r) = \mathbb{P}(R \leq r) = \int_0^r se^{-s^2/2}ds = \left[-e^{-s^2/2}\right]_0^r = 1 - e^{-r^2/2}.$$

Hence, using the fact that R can't take negative values,

$$F_A(a) = \mathbb{P}(A \leq a) = \mathbb{P}\left(\pi R^2 \leq a\right) = \mathbb{P}\left(R \leq \sqrt{\frac{a}{\pi}}\right) = F_R\left(\sqrt{\frac{a}{\pi}}\right) = 1 - e^{-a/(2\pi)}$$

for $a \geq 0$. Of course, $F_A(a) = 0$ for $a < 0$. Differentiating for $a \geq 0$, we get

$$f_A(a) = \frac{1}{2\pi} e^{-a/(2\pi)}.$$

So, recognising the p.d.f., we see that A is distributed exponentially with parameter $1/(2\pi)$.

Remark 5.23. *The distribution of R in Example 5.22 is called the* Rayleigh distribution. *One way in which this distribution occurs is as follows. Pick a point in $\mathbb{R}^2$ such that the x and y co-ordinates are independent $N(0,1)$ random variables. Then the Euclidean distance of that point from the origin $(0,0)$ has the Rayleigh distribution (see Part A Probability for a proof of this fact; there is a connection to Example 5.15).*

We can generalise the idea in Example 5.22 to prove the following theorem.

Theorem 5.24. *Suppose that X is a continuous random variable with density f_X and that $h : \mathbb{R} \to \mathbb{R}$ is a differentiable function which is strictly increasing (i.e. $\frac{dh(x)}{dx} > 0$ for all x). Then $Y = h(X)$ is a continuous random variable with p.d.f.*

$$f_Y(y) = f_X(h^{-1}(y)) \frac{d}{dy} h^{-1}(y),$$

where h^{-1} is the inverse function of h.

Proof. Since h is strictly increasing, $h(X) \leq y$ if and only if $X \leq h^{-1}(y)$. So the c.d.f. of Y is

$$F_Y(y) = \mathbb{P}\left(h(X) \leq y\right) = \mathbb{P}\left(X \leq h^{-1}(y)\right) = F_X(h^{-1}(y)).$$

Differentiating with respect to y using the chain rule, we get

$$f_Y(y) = f_X(h^{-1}(y)) \frac{d}{dy} h^{-1}(y). \qquad \square$$

There is a similar result in the case where h is strictly decreasing. In any case, you may find it easier to remember the proof than the statement of the theorem!

What if the function h is not one-to-one? It's best to treat these on a case-by-case basis and think them through carefully. Here's an example.

Example 5.25. *Suppose that a point is chosen uniformly from the perimeter of the unit circle. What is the distribution of its x-co-ordinate?*

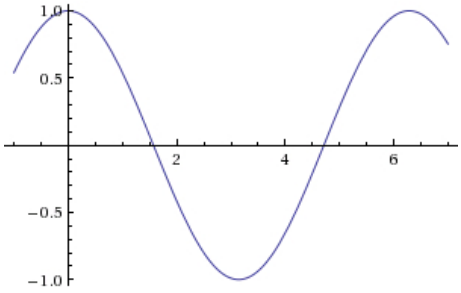
Solution. Represent the chosen point by its angle, Θ. So then Θ has a uniform distribution on $[0, 2\pi)$, with p.d.f.

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & \text{for } 0 \leq \theta < 2\pi \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the x-co-ordinate is $X = \cos\Theta$, which takes values in $[-1, 1]$. We again work via c.d.f.'s:

$$F_\Theta(\theta) = \begin{cases} 0 & \text{for } \theta < 0 \\ \frac{\theta}{2\pi} & \text{for } 0 \leq \theta < 2\pi \\ 1 & \text{for } \theta \geq 2\pi. \end{cases}$$

Notice that there are two angles in $[0, 2\pi)$ corresponding to each x-co-ordinate in $(-1, 1)$:



Then $F_X(x) = 0$ for $x \leq -1$, $F_X(x) = 1$ for $x \geq 1$ and, for $x \in (-1, 1)$,

$$\begin{aligned} F_X(x) &= \mathbb{P}(\cos\Theta \leq x) \\ &= \mathbb{P}(\arccos x \leq \Theta \leq 2\pi - \arccos x) \\ &= F_\Theta(2\pi - \arccos x) - F_\Theta(\arccos x) \\ &= 1 - \frac{\arccos x}{2\pi} - \frac{\arccos x}{2\pi} \\ &= 1 - \frac{1}{\pi}\arccos x. \end{aligned}$$

This completely determines the distribution of X, but we might also be interested in the p.d.f. Differentiating F_X, we get

$$\frac{dF_X(x)}{dx} = \begin{cases} \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}} & \text{for } -1 < x < 1 \\ 0 & \text{for } x < -1 \text{ or } x > 1 \\ \text{undefined} & \text{for } x = -1 \text{ or } x = 1. \end{cases}$$

So we can take

$$f_X(x) = \begin{cases} \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}} & \text{for } -1 < x < 1 \\ 0 & \text{for } x \leq -1 \text{ or } x \geq 1 \end{cases}$$

and get $F_X(x) = \int_{-\infty}^{x} f_X(u)du$.

Notice that $f_X(x) \to \infty$ as $x \to 1$ or $x \to -1$ even though $\int_{-\infty}^{\infty} f_X(x)dx = 1$.

5.5 Joint distributions

We will often want to think of different random variables defined on the same probability space. In the discrete case, we studied pairs of random variables via their joint probability mass function. For a pair of arbitrary random variables, we use instead the *joint cumulative distribution function*, $F_{X,Y} : \mathbb{R}^2 \to [0, 1]$, given by

$$F_{X,Y}(x, y) = \mathbb{P}(X \leq x, Y \leq y).$$

It's again possible to show that this function is non-decreasing in each of its arguments, and that

$$\lim_{x\to-\infty}\lim_{y\to-\infty} F_{X,Y}(x, y) = 0$$

and

$$\lim_{x\to\infty}\lim_{y\to\infty} F_{X,Y}(x, y) = 1.$$

Definition 5.26. *Let X and Y be random variables such that*

$$F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(u,v)dudv$$

for some function $f_{X,Y} : \mathbb{R}^2 \to \mathbb{R}$ such that

(a) $f_{X,Y}(u,v) \geq 0$ *for all* $u, v \in \mathbb{R}$

(b) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(u,v)dudv = 1$.

Then X and Y are jointly continuous *and $f_{X,Y}$ is their* joint density function.

If $f_{X,Y}$ is sufficiently smooth at (x,y), we get

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y).$$

For a single continuous random variable X, it turns out that the probability that it lies in some nice set $A \in \mathbb{R}$ (see Part A Integration to see what we mean by "nice", but note that any set you can think of or write down will be!) can be obtained by integrating its density over A:

$$\mathbb{P}(X \in A) = \int_A f_X(x)dx.$$

Likewise, for nice sets $B \subseteq \mathbb{R}^2$ we obtain the probability that the pair (X,Y) lies in B by integrating the joint density over the set B:

$$\mathbb{P}((X,Y) \in B) = \int\int_{(x,y)\in B} f_{X,Y}(x,y)dxdy.$$

We will show here that this works for rectangular regions B.

Theorem 5.27. *For a pair of jointly continuous random variables X and Y, we have*

$$\mathbb{P}(a < X \leq b, c < Y \leq d) = \int_c^d \int_a^b f_{X,Y}(x,y)dxdy,$$

for $a < b$ and $c < d$.

Proof. We have

$$\begin{aligned}
&\mathbb{P}(a < X \leq b, c < Y \leq d)\\
&= \mathbb{P}(X \leq b, Y \leq d) - \mathbb{P}(X \leq a, Y \leq d) + \mathbb{P}(X \leq a, Y \leq c) - \mathbb{P}(X \leq b, Y \leq c)\\
&= F_{X,Y}(b,d) - F_{X,Y}(a,d) + F_{X,Y}(a,c) - F_{X,Y}(b,c)\\
&= \int_c^d \int_a^b f_{X,Y}(x,y)dxdy.
\end{aligned}$$

□

Theorem 5.28. *Suppose X and Y are jointly continuous with joint density $f_{X,Y}$. Then X is a continuous random variable with density*

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy,$$

and similarly Y is a continuous random variable with density

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx.$$

In this context the one-dimensional densities f_X and f_Y are called the *marginal distributions* of the joint distribution with density $f_{X,Y}$, just as in the discrete case at Definition 2.16.

Proof. If f_X is defined by $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$, then we have

$$\begin{aligned}\int_{-\infty}^{x} f_X(u)du &= \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{X,Y}(u,y)dy\,du \\ &= \mathbb{P}\left(X \leq x\right),\end{aligned}$$

so indeed X has density f_X (and the case of f_Y is identical). □

The definitions and results above generalise straightforwardly to the case of n random variables, $X_1, X_2, \ldots, X_n$.

Example 5.29. *Let*

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2}(x+y) & \text{for } 0 \leq x \leq 1,\ 1 \leq y \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Check that $f_{X,Y}(x,y)$ is a joint density. What is $\mathbb{P}\left(X \leq \frac{1}{2}, Y \geq \frac{3}{2}\right)$? What are the marginal densities? What is $\mathbb{P}\left(X \geq \frac{1}{2}\right)$?

Solution. Clearly, $f_{X,Y}(x,y) \geq 0$ for all $x, y \in \mathbb{R}$. We have

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy &= \int_1^2\int_0^1 \frac{1}{2}(x+y)dxdy \\ &= \int_1^2 \left[\frac{1}{4}x^2 + \frac{1}{2}xy\right]_0^1 dy \\ &= \int_1^2 \left(\frac{1}{4} + \frac{1}{2}y\right) dy \\ &= \left[\frac{1}{4}y + \frac{1}{4}y^2\right]_1^2 \\ &= 1.\end{aligned}$$

We have

$$\begin{aligned}\mathbb{P}\left(X \leq \frac{1}{2}, Y \geq \frac{3}{2}\right) &= \int_{3/2}^2\int_0^{1/2} \frac{1}{2}(x+y)dxdy \\ &= \int_{3/2}^2 \left[\frac{1}{4}x^2 + \frac{1}{2}xy\right]_0^{1/2} dy \\ &= \int_{3/2}^2 \left(\frac{1}{16} + \frac{1}{4}y\right) dy \\ &= \left[\frac{1}{16}y + \frac{1}{8}y^2\right]_{3/2}^2 \\ &= \frac{1}{4}.\end{aligned}$$

Integrating out y we get

$$f_X(x) = \int_1^2 \frac{1}{2}(x+y)dy = \frac{1}{2}x + \frac{3}{4}$$

for $x \in [0,1]$, and integrating out x we get

$$f_Y(y) = \int_0^1 \frac{1}{2}(x+y)dx = \frac{1}{4} + \frac{1}{2}y$$

for $y \in [1,2]$. Using the marginal density of X,

$$\mathbb{P}\left(X \geq \frac{1}{2}\right) = \int_{\frac{1}{2}}^1 \left(\frac{1}{2}x + \frac{3}{4}\right) dx = \frac{9}{16}.$$

Definition 5.30. *Jointly continuous random variables X and Y with joint density $f_{X,Y}$ are* independent *if*

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

for all $x, y \in \mathbb{R}$. Likewise, jointly continuous random variables $X_1, X_2, \ldots, X_n$ with joint density $f_{X_1,X_2,\ldots,X_n}$ are independent *if*

$$f_{X_1,X_2,\ldots,X_n}(x_1,x_2,\ldots,x_n) = f_{X_1}(x_1)f_{X_2}(x_2)\ldots f_{X_n}(x_n)$$

for all $x_1, x_2, \ldots, x_n \in \mathbb{R}$.

Note that if X and Y are independent then it follows easily that

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)$$

for all $x, y \in \mathbb{R}$.

Example 5.31. *Consider the set-up of Example 5.29. Since there exist x and y such that*

$$\frac{1}{2}(x+y) \neq \left(\frac{1}{2}x + \frac{3}{4}\right)\left(\frac{1}{4} + \frac{1}{2}y\right),$$

X and Y are not independent.

5.5.1 Expectation

We can write the function h of a pair of jointly continuous random variables in a natural way.

Theorem 5.32.

$$\mathbb{E}\left[h(X,Y)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)f_{X,Y}(x,y)dxdy.$$

As in the case of Theorem 5.17, the proof of this result is rather techincal, and we don't cover it here. However, note again that there is a very direct analogy with the discrete case which we saw in equation (2.2).

In particular, the *covariance* of X and Y is

$$\operatorname{cov}(X,Y) = \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right] = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$$

(exercise: check the second equality).

Exercise 5.33. *Check that*

$$\mathbb{E}\left[aX + bY\right] = a\mathbb{E}\left[X\right] + b\mathbb{E}\left[Y\right]$$

and

$$\mathrm{var}\left(X + Y\right) = \mathrm{var}\left(X\right) + \mathrm{var}\left(Y\right) + 2\mathrm{cov}\left(X, Y\right).$$

Remark 5.34. *We have now shown that the rules for calculating expectations (and derived quantities such as variances and covariances) of continuous random variables are* exactly the same *as for discrete random variables. This isn't a coincidence! We can make a more general definition of expectation which covers both cases (and more besides) but in order to do so we need a more general theory of integration, which some of you will see in the Part A Integration course.*

Example 5.35. *Let $-1 < \rho < 1$. The standard bivariate normal distribution has joint density*

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right)$$

for $x, y \in \mathbb{R}$. What are the marginal distributions of X and Y? Find the covariance of X and Y.

Proof. We have

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right) dy \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}[(y-\rho x)^2 + x^2(1-\rho^2)]\right) dy \\ &= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{(y-\rho x)^2}{2(1-\rho^2)}\right) dy. \end{aligned}$$

But the integrand is now the density of a normal random variable with mean ρx and variance $1 - \rho^2$. So it integrates to 1 and we are left with

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

So $X \sim \mathrm{N}(0, 1)$ and, by symmetry, the same is true for Y. Notice that X and Y are only independent if $\rho = 0$.

Since X and Y both have mean 0, we only need to calculate $\mathbb{E}\left[XY\right]$. We can use a similar trick:

$$\begin{aligned} \mathbb{E}\left[XY\right] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{xy}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right) dydx \\ &= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} e^{-x^2/2} \int_{-\infty}^{\infty} \frac{y}{\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{(y-\rho x)^2}{2(1-\rho^2)}\right) dydx. \end{aligned}$$

The inner integral now gives us the mean of a $\mathrm{N}(\rho x, 1 - \rho^2)$ random variable, which is ρx. So we get

$$\mathrm{cov}\left(X, Y\right) = \int_{-\infty}^{\infty} \frac{\rho x^2}{\sqrt{2\pi}} e^{-x^2/2} dx = \rho\mathbb{E}\left[X^2\right] = \rho,$$

since $\mathbb{E}\left[X^2\right] = 1$. □

This yields the interesting conclusion that standard bivariate normal random variables X and Y are independent if and only if their covariance is 0. This is a nice property of normal random variables which is *not true* for general random variables, as we have already observed in the discrete case.

Chapter 6

Random samples and the weak law of large numbers

One of the reasons that we are interested in sequences of i.i.d. random variables is that we can view them as repeated samples from some underlying distribution.

Definition 6.1. *Let $X_1, X_2, \ldots, X_n$ denote i.i.d. random variables. Then these random variables are said to constitute a* random sample *of size n from the distribution.*

Statistics often involves random samples where the underlying distribution (the *"parent distribution"*) is unknown. A realisation of such a random sample is used to make inferences about the parent distribution. Suppose, for example, we want to know about the mean of the parent distribution. An important estimator is the sample mean.

Definition 6.2. *The* sample mean *is defined to be* $\bar{X}_n = \dfrac{1}{n}\sum_{i=1}^{n} X_i$.

This is a key random variable which itself has an expectation and a variance. Recall that for random variables X and Y (discrete or continuous),

$$\text{var}\,(X+Y) = \text{var}\,(X) + \text{var}\,(Y) + 2\text{cov}\,(X,Y)\,.$$

We can extend this (by induction) to n random variables as follows:

$$\begin{aligned}\text{var}\left(\sum_{i=1}^{n} X_i\right) &= \sum_{i=1}^{n}\text{var}\,(X_i) + \sum_{i\neq j}\text{cov}\,(X_i,X_j)\\ &= \sum_{i=1}^{n}\text{var}\,(X_i) + 2\sum_{i<j}\text{cov}\,(X_i,X_j)\,.\end{aligned}$$

Theorem 6.3. *Suppose that $X_1, X_2, \ldots, X_n$ form a random sample from a distribution with mean μ and variance σ^2. Then the expectation and variance of the sample mean are*

$$\mathbb{E}\left[\bar{X}_n\right] = \mu \quad \textit{and} \quad \text{var}\left(\bar{X}_n\right) = \frac{1}{n}\sigma^2.$$

Proof. We have $\mathbb{E}[X_i] = \mu$ and $\text{var}(X_i) = \sigma^2$ for $1 \leq i \leq n$. So

$$\mathbb{E}\left[\bar{X}_n\right] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i] = \mu,$$

$$\text{var}\left(\bar{X}_n\right) = \text{var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\text{var}\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n}\text{var}(X_i) = \frac{1}{n}\sigma^2,$$

since independence implies that $\text{cov}(X_i, X_j) = 0$ for all $i \neq j$. □

Example 6.4. *Let $X_1, \ldots, X_n$ be a random sample from a Bernoulli distribution with parameter p. Then $\mathbb{E}[X_i] = p$, $\text{var}(X_i) = p(1-p)$ for all $1 \leq i \leq n$. Hence, $\mathbb{E}\left[\bar{X}_n\right] = p$ and $\text{var}\left(\bar{X}_n\right) = p(1-p)/n$.*

In order for $\bar{X}_n$ to be a good estimator of the mean, we would like to know that for large sample sizes n, $\bar{X}_n$ is not too far away from μ i.e. that $|\bar{X}_n - \mu|$ is small. The result which tells us that this is true is called the *law of large numbers* and is of fundamental importance in probability. Before we state it, let's step away from the sample mean and consider a more basic situation.

Suppose that A is an event with probability $\mathbb{P}(A)$ and write $p = \mathbb{P}(A)$. Let X be the indicator function of the event A i.e. the random variable defined by

$$X(\omega) = \mathbb{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A. \end{cases}$$

Then $X \sim \text{Ber}(p)$ and $\mathbb{E}[X] = p$. Suppose now that we perform our experiment repeatedly and let X_i be the indicator of the event that A occurs on the ith trial. Our intuitive notion of probability leads us to believe that if the number n of trials is large then the *proportion* of the time that A occurs should be close to p i.e.

$$\left|\frac{1}{n}\sum_{i=1}^{n} X_i - p\right|$$

should be small. So proving that the sample mean is close to the true mean in this situation will also provide some justification for the way we have set up our mathematical theory of probability.

Theorem 6.5 (Weak law of large numbers)**.** *Suppose that $X_1, X_2, \ldots$ are independent and identically distributed random variables with mean μ. Then for any fixed $\epsilon > 0$,*

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| > \epsilon\right) \to 0$$

as $n \to \infty$.

(Equivalently, we could have put

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \leq \epsilon\right) \to 1$$

as $n \to \infty$.)

In other words, the probability that the sample mean deviates from the true mean by more than some small quantity ϵ tends to 0 as $n \to \infty$. Notice that the result only depends on the underlying distribution through its mean.

We will give a proof of the weak law under an additional assumption that the variance of the distribution is finite. To do that, we'll first prove a couple of very useful inequalities.

Theorem 6.6 (Markov's inequality)**.** *Suppose that Y is a non-negative random variable whose expectation exists. Then*

$$\mathbb{P}(Y \geq t) \leq \frac{\mathbb{E}[Y]}{t}$$

for all $t > 0$.

Proof. Let $A = \{Y \geq t\}$. We may assume that $\mathbb{P}(A) \in (0,1)$, since otherwise the result is trivially true. Then by the law of total probability for expectations,

$$\mathbb{E}[Y] = \mathbb{E}[Y|A]\,\mathbb{P}(A) + \mathbb{E}[Y|A^c]\,\mathbb{P}(A^c) \geq \mathbb{E}[Y|A]\,\mathbb{P}(A),$$

since $\mathbb{P}(A^c) > 0$ and $\mathbb{E}[Y|A^c] \geq 0$. Now, we certainly have $\mathbb{E}[Y|A] = \mathbb{E}[Y|Y \geq t] > t$. So, rearranging, we get

$$\mathbb{P}(Y \geq t) \leq \frac{\mathbb{E}[Y]}{t}$$

as we wanted. □

Theorem 6.7 (Chebyshev's inequality)**.** *Suppose that Z is a random variable with a finite variance. Then for any $t > 0$,*

$$\mathbb{P}(|Z - \mathbb{E}[Z]| \geq t) \leq \frac{\operatorname{var}(Z)}{t^2}.$$

Proof. Note that $\mathbb{P}(|Z - \mathbb{E}[Z]| \geq t) = \mathbb{P}\left((Z - \mathbb{E}[Z])^2 \geq t^2\right)$ and then apply Markov's inequality to the non-negative random variable $Y = (Z - \mathbb{E}[Z])^2$. □

Proof of Theorem 6.5 (under the assumption of finite variance). Suppose the common distribution of the random variables X_i has mean μ and variance σ^2. Set

$$Z = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Then

$$\mathbb{E}[Z] = \mu \quad \text{and} \quad \operatorname{var}(Z) = \operatorname{var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{\sigma^2}{n}.$$

So by Chebyshev's inequality,

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| > \epsilon\right) \leq \frac{\sigma^2}{n\epsilon^2}.$$

Since $\epsilon > 0$ is fixed, the right-hand side tends to 0 as $n \to \infty$. □

Appendix

A.1 Useful ideas from Analysis

Here are brief details of some ideas about sets, sequences, and series, that it will be useful to make reference to. Those doing the Analysis I course in Maths this term will see all of this in much greater detail!

Countability

A set S is *countable* if either it's finite, or its elements can be written as a list: $S = \{x_1, x_2, x_3, \dots\}$. Put another way, S is countable if there is a bijection from a subset of $\mathbb{N}$ to S. The set $\mathbb{N}$ itself is countable; so is the set of rational numbers $\mathbb{Q}$, for example. The set of real numbers $\mathbb{R}$ is not countable.

Limits

Even if you haven't seen a definition, you probably have an idea of what it means for a sequence to converge to a limit. Formally, we say that a sequence of real numbers $(a_1, a_2, a_3, \dots)$ converges to a limit $L \in \mathbb{R}$ if the following holds: for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n - L| \leq \epsilon$ whenever $n \geq N$.

Then we may write "$L = \lim_{n\to\infty} a_n$", or "$a_n \to L$ as $n \to \infty$".

Infinite sums

Finite sums are easy. If we have a sequence $(a_1, a_2, a_3, \dots)$, then for any $n \in \mathbb{N}$ we can define

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

What do we mean by the infinite sum $\sum_{k=1}^{\infty} a_k$? An infinite sum is really a sort of limit. If the limit $L = \lim_{n\to\infty} s_n$ exists, then we say that *the series* $\sum_{k=1}^{\infty} a_k$ *converges*, and that its sum is L. If the sequence $(s_n, n \in \mathbb{N})$ does not have a limit, then we say that the series $\sum_{k=1}^{\infty} a_k$ *diverges*.

An important idea for our purposes will be *absolute convergence* of a series. We say that the series $\sum_{k=1}^{\infty} a_k$ *converges absolutely* if the series $\sum_{k=1}^{\infty} |a_k|$ converges. If a series converges absolutely, then it

also converges.

One reason why absolute convergence is important is that it guarantees that the value of a sum doesn't depend on the order of the terms. In the definition of expectation of a discrete random variable, for example, we may have an infinite sum and no reason to take the terms in any particular order. Formally, suppose f is a bijection from $\mathbb{N}$ to $\mathbb{N}$, and define $b_k = a_{f(k)}$. If the series $\sum_{k=1}^{\infty} a_k$ converges absolutely, then so does the series $\sum_{k=1}^{\infty} b_k$, and the sums $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are equal.

An example of a series that converges but does not converge absolutely is the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$, whose sum is $\ln 2$.

If we reordered the terms as $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \ldots$, then the sum instead becomes $\frac{3}{2} \ln 2$.

Power series

A (real) power series is a function of the form

$$f(x) = \sum_{k=0}^{\infty} c_k x^k$$

where the coefficients $c_k, k \geq 0$ are real constants. For any such series, there exists a *radius of convergence* $R \in [0, \infty) \cup \infty$, such that $\sum_{k=0}^{\infty} c_k x^k$ converges absolutely for $|x| < R$, and not for $|x| > R$.

In this course we will meet a particular class of power series called *probability generating functions*, with the property that the coefficients c_k are non-negative and sum to 1. In that case, R is at least 1.

Power series behave well when differentiated! A power series $f(x) = \sum_{k=0}^{\infty} c_k x^k$ with radius of convergence R is differentiable on the interval $(-R, R)$, and its derivative is also a power series with radius of convergence R, given by

$$f'(x) = \sum_{k=0}^{\infty} (k+1) c_{k+1} x^k.$$

Series identities

Here is a reminder of some useful identities:

Geometric series: if $a \in \mathbb{R}$ and $0 \leq r < 1$ then

$$\sum_{k=0}^{n-1} ar^k = \frac{a(1-r^n)}{1-r}$$

and

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}.$$

Exponential function: for $\lambda \in \mathbb{R}$,

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} = e^{\lambda}.$$

Binomial theorem: for $x, y \in \mathbb{R}$ and $n \geq 0$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Differentiation and integration give us variants of these. For example, for $0 < r < 1$,

$$\sum_{k=1}^{\infty} k r^{k-1} = \frac{d}{dr}\left(\sum_{k=0}^{\infty} r^k\right)$$

and

$$\sum_{k=1}^{\infty} \frac{r^k}{k} = \int_0^r \left(\sum_{k=0}^{\infty} t^k\right) dt.$$

A.2 Increasing sequences of events

We mentioned the following result in the later part of the course. A sequence of events $A_n, n \geq 1$ is called *increasing* if $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$.

Proposition A.8. *If $A_n, n \geq 1$ is an increasing sequence of events, then*

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mathbb{P}(A_n).$$

Proof. The proof uses countable additivity. Using the fact that the sequence is increasing, we can write $\bigcup_{n=1}^{\infty} A_n$ as a disjoint union:

$$\bigcup_{n=1}^{\infty} A_n = A_1 \cup (A_2 \setminus A_1) \cup (A_3 \setminus A_2) \cup \ldots.$$

and similarly each individual A_n as a disjoint union:

$$A_n = A_1 \cup (A_2 \setminus A_1) \cup (A_3 \setminus A_2) \cup \cdots \cup (A_n \setminus A_{n-1}).$$

Then applying the countable additivity axiom twice, we have

$$\begin{aligned}
\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) &= \mathbb{P}(A_1) + \mathbb{P}(A_2 \setminus A_1) + \mathbb{P}(A_3 \setminus A_2) + \ldots \\
&= \lim_{n\to\infty} \left[\mathbb{P}(A_1) + \mathbb{P}(A_2 \setminus A_1) + \mathbb{P}(A_3 \setminus A_2) + \cdots + \mathbb{P}(A_n \setminus A_{n-1})\right]
\end{aligned}$$

(since by definition of an infinite sum, $\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$.)

$$= \lim_{n\to\infty} \mathbb{P}(A_n).$$

□

Common discrete distributions

Distribution	Probability mass function	Mean	Variance	Generating function
Uniform $\mathrm{U}\{1,2,\ldots,n\}$, $n \in \mathbb{N}$	$\mathbb{P}(X=k) = \frac{1}{n}, 1 \le k \le n$	$\frac{n+1}{2}$	$\frac{n^2-1}{12}$	$G_X(s) = \frac{s-s^{n+1}}{n(1-s)}$
Bernoulli $\mathrm{Ber}(p)$, $p \in [0,1]$	$\mathbb{P}(X=1) = p$, $\mathbb{P}(X=0) = 1-p$	p	$p(1-p)$	$G_X(s) = 1-p+ps$
Binomial $\mathrm{Bin}(n,p)$, $n \in \mathbb{N}, p \in [0,1]$	$\mathbb{P}(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0,1,\ldots,n$	np	$np(1-p)$	$G_X(s) = (1-p+ps)^n$
Poisson $\mathrm{Po}(\lambda)$, $\lambda \ge 0$	$\mathbb{P}(X=k) = \frac{\lambda^k}{k!}\mathrm{e}^{-\lambda}$, $k = 0,1,2,\ldots$	λ	λ	$G_X(s) = \mathrm{e}^{\lambda(s-1)}$
Geometric $\mathrm{Geom}(p)$, $p \in [0,1]$	$\mathbb{P}(X=k) = (1-p)^{k-1}p$, $k = 1,2,\ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$G_X(s) = \frac{ps}{1-(1-p)s}$
Alternative geometric, $p \in [0,1]$	$\mathbb{P}(X=k) = (1-p)^k p$, $k = 0,1,\ldots$	$\frac{1-p}{p}$	$\frac{1-p}{p^2}$	$G_X(s) = \frac{p}{1-(1-p)s}$
Negative binomial $\mathrm{NegBin}(k,p)$, $k \in \mathbb{N}, p \in [0,1]$	$\mathbb{P}(X=n) = \binom{n-1}{k-1}(1-p)^{n-k}p^k$, $n = k, k+1, \ldots$	$\frac{k}{p}$	$\frac{k(1-p)}{p^2}$	$G_X(s) = \left(\frac{ps}{1-(1-p)s}\right)^k$

Common continuous distributions

Distribution	Probability density function	Cumulative distribution function	Mean	Variance
Uniform $\mathrm{U}[a,b]$, $a<b$	$f_X(x)=\dfrac{1}{b-a},\ a\le x\le b$	$F_X(x)=\dfrac{x-a}{b-a},\ a\le x\le b$	$\dfrac{a+b}{2}$	$\dfrac{(b-a)^2}{12}$
Exponential $\mathrm{Exp}(\lambda)$, $\lambda\ge 0$	$f_X(x)=\lambda \mathrm{e}^{-\lambda x},\ x\ge 0$	$F_X(x)=1-\mathrm{e}^{-\lambda x},\ x>0$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$
Gamma $\mathrm{Gamma}(\alpha,\lambda)$, $\alpha>0,\lambda\ge 0$	$f_X(x)=\dfrac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\mathrm{e}^{-\lambda x},\ x\ge 0$		$\dfrac{\alpha}{\lambda}$	$\dfrac{\alpha}{\lambda^2}$
Normal $\mathrm{N}(\mu,\sigma^2)$, $\mu\in\mathbb{R}$, $\sigma^2>0$	$f_X(x)=\dfrac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}},\ x\in\mathbb{R}$	$F_X(x)=\Phi\left(\dfrac{x-\mu}{\sigma}\right)$	μ	σ^2
Standard Normal $\mathrm{N}(0,1)$	$f_X(x)=\dfrac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}},\ x\in\mathbb{R}$	$F_X(x)=\Phi(x)=\displaystyle\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}\,du$	0	1
Beta $\mathrm{Beta}(\alpha,\beta)$	$f_X(x)=\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1},\ x\in[0,1]$		$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$