

1. If  $X_1, \dots, X_n$  is a random sample from a geometric distribution with parameter  $p$ , find the maximum likelihood estimator  $\hat{p}$  of  $p$ .

Let  $\theta = 1/p$ . Find the likelihood as a function of  $\theta$ , the maximum likelihood estimator  $\hat{\theta}$ , and verify that  $\hat{\theta} = 1/\hat{p}$ .

Show that  $\hat{\theta}$  is unbiased. In the case  $n = 1$  show that  $E(\hat{p}) > p$ . [In the  $n = 1$  case, having first shown  $E(\hat{p}) > p$ , can you find the value of  $E(\hat{p})$ ?

2. Suppose  $X_1, \dots, X_n$  is a random sample from a  $N(\mu, \sigma^2)$  distribution, where  $\mu = \sigma^2 = \theta$ . Show that the maximum likelihood estimator of  $\theta$  is

$$\hat{\theta} = \frac{1}{2} \left\{ \left( 1 + \frac{4}{n} \sum_{j=1}^n X_j^2 \right)^{1/2} - 1 \right\}.$$

3. A researcher wishes to estimate the density  $\rho$  of organisms per unit volume in a liquid. She conducts  $n$  independent experiments: in experiment  $i = 1, \dots, n$ , she takes a fixed volume  $v_i$  of liquid and measures the number of organisms  $X_i$  in this volume – she assumes  $X_i$  has a Poisson distribution with mean  $\rho v_i$ . Find the maximum likelihood estimator  $\hat{\rho}$  and find the bias of  $\hat{\rho}$ .

If the total volume taken is fixed,  $\sum_{i=1}^n v_i = a$  say, show that the variance of  $\hat{\rho}$  only depends on  $v_1, \dots, v_n$  via their sum  $a$ .

4. Suppose  $X_1, \dots, X_n$  is a random sample from a distribution with probability density function

$$f(x; \theta) = \begin{cases} e^{-(x-\theta)} & \text{if } x \geq \theta, \\ 0 & \text{otherwise.} \end{cases}$$

Find the maximum likelihood estimator of  $\theta$ .

5. The following data (from Dyer (1981)) are annual wages (in multiples of 100 US dollars) of a random sample of 30 production line workers in a large American industrial firm.

Annual wages (hundreds of US \$)									
112	154	119	108	112	156	123	103	115	107
125	119	128	132	107	151	103	104	116	140
108	105	158	104	119	111	101	157	112	115

- (a) A standard probability model used for data on wages is the Pareto distribution, which has probability density function

$$f(x; \theta) = \theta \alpha^\theta x^{-(\theta+1)} \quad \text{for } x \geq \alpha,$$

where  $\theta > 0$  and the constant  $\alpha$  represents a statutory minimum wage. Find the maximum likelihood estimator of  $\theta$  from a random sample  $X_1, X_2, \dots, X_n$ , and, assuming  $\alpha = 100$ , the maximum likelihood estimate for the above dataset (for which  $\sum \log x_i = 143.5$ ).

- (b) Now suppose there is no statutory minimum wage, so that  $\alpha$  is also an unknown parameter.

(i) Show that the MLE for  $\alpha$  is  $\hat{\alpha} = \min_i X_i$ . What is the MLE for  $\theta$ ?

(ii) Show that

$$P(\hat{\alpha} > y) = \left(\frac{\alpha}{y}\right)^{n\theta} \quad \text{for } y \geq \alpha.$$

[Use the fact that  $\min_i X_i > y \iff \{X_i > y \text{ for } 1 \leq i \leq n\}$ .]

(iii) Deduce that, for each  $\epsilon > 0$ ,

$$P(|\hat{\alpha} - \alpha| > \epsilon) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

**6.** (Optional: using R or Matlab)

R: work through Rdemo-1 on the course website. At this stage the idea is to get some experience with R and to look at simple plots of the `trees` data. In a few lectures' time we will fit linear regression models to data of this type.

Also, look at [http://www.stats.ox.ac.uk/~marchini/data\\_analysis/R\\_Intro.pdf](http://www.stats.ox.ac.uk/~marchini/data_analysis/R_Intro.pdf)

Working through this document will give you more practice and prepare you for the R exercises later in the course.

Matlab: work through the Matlab section on the final page of Rdemo-1.