1. Consider the linear programming problem

$$P: \ \max \ \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} \leqslant \mathbf{b}, \ \mathbf{x} \geqslant \mathbf{0}.$$

Write down the dual problem D, and convert it into another problem of the same form as P. Verify that the dual of the dual problem D is the original problem P.

2. Write down the dual of the linear programming problem:

$$\begin{array}{ll} \text{maximise} & x_1 + x_3 + 3x_4 \\ \text{subject to} & 3x_1 - 2x_2 + 3x_3 + x_4 \leqslant 3 \\ & -x_1 + 3x_2 + 2x_3 + 2x_4 \leqslant 6 \\ & x_1, \ldots, x_4 \geqslant 0. \end{array}$$

Use the duality theorem to show that the vectors $\mathbf{x} = (0,0,0,3)^T$ and $\mathbf{y} = (\frac{5}{7}, \frac{8}{7})^T$ are optimal solutions of the original (primal) and dual problems respectively. Verify the complementary slackness conditions.

3. By considering the dual problem, or otherwise, show that the problem:

$$\begin{array}{ll} \text{maximise} & 8x_1 + 3x_2 + 4x_3 \\ \text{subject to} & 3x_1 + x_2 - 2x_3 \leqslant 5 \\ & 4x_1 - x_2 + x_3 \leqslant 4 \\ & x_1, x_2, x_3 \geqslant 0 \end{array}$$

has no bounded optimal solution.

4. Consider the problem

$$\begin{array}{ll} \text{minimise} & 2x_1 + 3x_2 + 5x_3 + 2x_4 + 3x_5 \\ \text{subject to} & x_1 + x_2 + 2x_3 + x_4 + 3x_5 \geqslant 4 \\ & 2x_1 - 2x_2 + 3x_3 + x_4 + x_5 \geqslant 3 \\ & x_1, x_2, x_3, x_4, x_5 \geqslant 0. \end{array}$$

Write down the dual problem, and solve it graphically. Use complementary slackness to deduce the optimal solution to the primal problem.

5. Consider the linear program P in the 'standard form' given in question 1. We have seen that each constraint in the primal corresponds to a variable in the dual (and vice versa). Suppose that the constraints $A\mathbf{x} \leqslant \mathbf{b}$ are replaced by $A\mathbf{x} = \mathbf{b}$. Show that the effect on the dual problem is that the variables in the dual become free variables (i.e. not constrained to be non-negative).

[*Hint: replace the equalities $A\mathbf{x} = \mathbf{b}$ by two inequalities to put the problem back into 'standard form'; these two inequalities give rise to two sets of variables in the dual problem which can be combined into a single vector of free variables.*]

(This can be extended into a general correspondence between primal (dual) constraints and dual (primal) variables: restricted variable $\leftrightarrow$ inequality constraint, free variable $\leftrightarrow$ equality constraint.)

6. We showed in lectures that Player II's problem can be represented by

$$P: \max_{\mathbf{x}} \sum_{j=1}^{n} x_j \quad \text{subject to} \quad A\mathbf{x} \leqslant \mathbf{1},\ \mathbf{x} \geqslant \mathbf{0}.$$

Write down the dual problem D of P.

Player I's problem is

$$\max_{\mathbf{p}} \left\{ \min_{j} \sum_{i=1}^{m} a_{ij} p_i \right\} \quad \text{subject to} \quad \sum_{i=1}^{m} p_i = 1,\ \mathbf{p} \geqslant \mathbf{0}.$$

Following the same kind of transformation that we used in lectures for Player II's problem, show that Player I's problem is equivalent to D.

7. Consider the following two-player zero-sum game. Fabien chooses an integer in $\{1, 2, 3\}$ and Fabienne chooses an integer in $\{2, 3, 4\}$. If the chosen numbers are the same, no money changes hands. If the numbers are different, the person who chooses the larger number wins 1 euro, *unless* the numbers differ by 1 in which case the person choosing the *smaller* number wins 1 euro.

Using the simplex method or otherwise, find the value of the game and the optimal strategies.

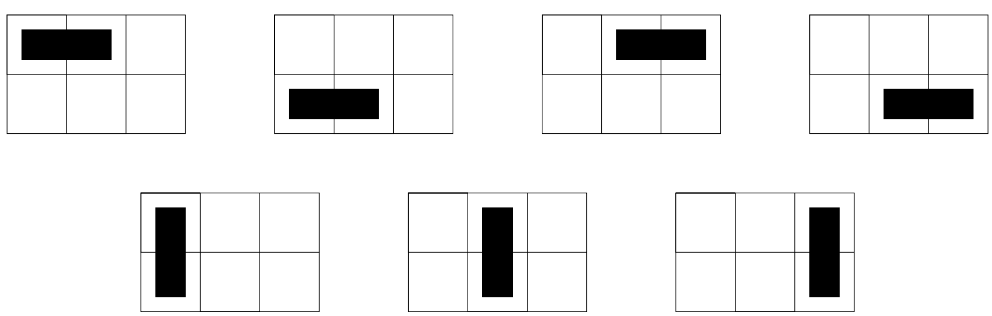
8. Find the optimal strategies of both players, and the value of the game, for the following two payoff matrices:

$$A = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 2 & 4 \\ -1 & 5 & 3 \\ 3 & 3 & 2 \end{pmatrix}.$$

9. The $n \times n$ matrix of a two-person zero-sum game is such that the row and column sums all equal s. Show that the game has value s/n.

[*Hint: guess a solution and show that it is optimal.*]

10. A domino can be placed on a 2×3 board in seven different ways:



The first player places the domino and the second player selects one of the six squares. If the selected square is covered by the domino, then the second player wins; otherwise the first player wins. What are the optimal strategies, and what chance do they give each player of winning? Can you exploit the symmetries? How do your findings generalise to larger boards?