Lecture 4: Divide and Conquer: van Emde Boas Trees

- Series of Improved Data Structures
- Insert, Successor
- Delete
- Space

This lecture is based on personal communication with Michael Bender, 2001.

Goal

We want to maintain n elements in the range $\{0, 1, 2, \ldots, u-1\}$ and perform Insert, Delete and Successor operations in $\mathcal{O}(\log \log u)$ time.

- If $n = n^{\mathcal{O}(1)}$ or $n^{(\log n)^{\mathcal{O}(1)}}$, then we have $\mathcal{O}(\log \log n)$ time operations
  - Exponentially faster than Balanced Binary Search Trees
  - Cooler queries than hashing
- Application: Network Routing Tables
  - u = Range of IP Addresses $\rightarrow$ port to send ($u = 2^{32}$ in IPv4)

Where might the $\mathcal{O}(\log \log u)$ bound arise ?

- Binary search over $\mathcal{O}(\log u)$ elements
- Recurrences
  - $T(\log u) = T\left(\frac{\log u}{2}\right) + \mathcal{O}(1)$
  - $T(u) = T(\sqrt{u}) + \mathcal{O}(1)$

Improvements

We will develop the van Emde Boas data structure by a series of improvements on a very simple data structure.

Bit Vector

We maintain a vector V of size u such that $V[x] = 1$ if and only if x is in the set. Now, inserts and deletes can be performed by just flipping the corresponding bit in the vector. However, successor/predecessor requires us to traverse through the vector to find the next 1-bit.

- Insert/Delete: $\mathcal{O}(1)$
- Successor/Predecessor: $\mathcal{O}(u)$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	0	0	0	0	0	0	0	1	1	0	0	0	0	1

Figure 1: Bit vector for $u = 16$. THe current set is $\{1, 9, 10, 15\}$.

Split Universe into Clusters

We can improve performance by splitting up the range $\{0, 1, 2, \ldots, u-1\}$ into $\sqrt{u}$ clusters of size $\sqrt{u}$. If $x = i\sqrt{u} + j$, then $V[x] = V.Cluster[i][j]$.

$$low(x) = x \bmod \sqrt{u} = j$$
$$high(x) = \left\lfloor \frac{x}{\sqrt{u}} \right\rfloor = i$$
$$index(i, j) = i\sqrt{u} + j$$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	0	0	0	0	0	0	0	1	1	0	0	0	0	1
$V.Cluster[0]$				$V.Cluster[1]$				$V.Cluster[2]$				$V.Cluster[3]$			

Figure 2: Bit vector ($u = 16$) split into $\sqrt{16} = 4$ clusters of size 4.

- Insert:
  - Set $V.cluster[high(x)][low(x)] = 1$ $\mathcal{O}(1)$

  - Mark cluster $high(x)$ as non-empty $\mathcal{O}(1)$

- Successor:

  - Look within cluster $high(x)$ $\mathcal{O}(\sqrt{u})$
  - Else, find next non-empty cluster i $\mathcal{O}(\sqrt{u})$
  - Find minimum entry j in that cluster $\mathcal{O}(\sqrt{u})$
  - Return $index(i, j)$ Total $= \mathcal{O}(\sqrt{u})$

Recurse

The three operations in Successor are also Successor calls to vectors of size $\sqrt{u}$. We can use recursion to speed things up.

- $V.cluster[i]$ is a size-$\sqrt{u}$ van Emde Boas structure ($\forall\ 0 \leq i < \sqrt{u}$)
- $V.summary$ is a size-$\sqrt{u}$ van Emde Boas structure
- $V.summary[i]$ indicates whether $V.cluster[i]$ is nonempty

INSERT(V, x)

$Insert(V.cluster[high(x)], low[x])$
$Insert(V.summary, high[x])$

So, we get the recurrence:

$$T(u) = 2T(\sqrt{u}) + \mathcal{O}(1)$$
$$T'(\log u) = 2T'\left(\frac{\log u}{2}\right) + \mathcal{O}(1)$$
$$\implies T(u) = T'(\log u) = \mathcal{O}(\log u)$$

SUCCESSOR(V, x)

$i = high(x)$
$j = Successor(V.cluster[i], j)$
if $j == \infty$
    $i = Successor(V.summary, i)$
    $j = Successor(V.cluster[i], -\infty)$
return $index(i, j)$

$$
\begin{aligned}
T(u) &= 3T(\sqrt{u}) + \mathcal{O}(1) \\
T'(\log u) &= 3T'\left(\frac{\log u}{2}\right) + \mathcal{O}(1) \\
\implies T(u) &= T'(\log u) = \mathcal{O}((\log u)^{\log 3}) \approx \mathcal{O}((\log u)^{1.585})
\end{aligned}
$$

To obtain the $\mathcal{O}(\log \log u)$ running time, we need to reduce the number of recursions to one.

Maintain Min and Max

We store the minimum and maximum entry in each structure. This gives an $\mathcal{O}(1)$ time overhead for each *Insert* operation.

SUCCESSOR(V, x)

```
i = high(x)
if low(x) < V.cluster[i].max
    j = Successor(V.cluster[i], low(x))
else i = Successor(V.summary, high(x))
    j = V.cluster[i].min
return index(i, j)
```

$$
\begin{aligned}
T(u) &= T(\sqrt{u}) + \mathcal{O}(1) \\
\implies T(u) &= \mathcal{O}(\log \log u)
\end{aligned}
$$

Don't store Min recursively

The *Successor* call now needs to check for the min separately.

$$\textbf{if } x < V.min : \textbf{return } V.min \tag{1}$$

INSERT(V, x)

```
if V.min == None
    V.min = V.max = x        ▷ O(1) time
    return
if x < V.min
    swap(x ↔ V.min)
if x > V.max
    V.max = x)
if V.cluster[high(x) == None
    Insert(V.summary, high(x))        ▷ First Call
Insert(V.cluster[high(x)], low(x))        ▷ Second Call
```

If the **first call** is executed, the **second call** only takes $\mathcal{O}(1)$ time. So

$$T(u) = T(\sqrt{u}) + \mathcal{O}(1)$$
$$\implies T(u) = \mathcal{O}(\log \log u)$$

DELETE(V, x)

```
if x == V.min        ▷ Find new min
    i = V.summary.min
    if i = None
        V.min = V.max = None        ▷ O(1) time
        return
    V.min = index(i, V.cluster[i].min)        ▷ Unstore new min
Delete(V.cluster[high(x)], low(x))        ▷ First Call
if V.cluster[high(x)].min == None
    Delete(V.summary, high(x))        ▷ Second Call
▷ Now we update V.max
if x == V.max
if V.summary.max = None
else
    i = V.summary.max
    V.max = index(i, V.cluster[i].max)
```

If the **second call** is executed, the **first call** only takes $\mathcal{O}(1)$ time. So

$$T(u) = T(\sqrt{u}) + \mathcal{O}(1)$$
$$\implies T(u) = \mathcal{O}(\log \log u)$$

Lower Bound [Patrascu & Thorup 2007]

Even for static queries (no Insert/Delete)

- $\Omega(\log \log u)$ time per query for $u = n^{(\log n)^{\mathcal{O}(1)}}$
- $\mathcal{O}(n \cdot poly(\log n))$ space

Space Improvements

We can improve from $\Theta(u)$ to $\mathcal{O}(n \log \log u)$.

- Only create nonempty clusters
  - If $V.min$ becomes $None$, deallocate V
- Store $V.cluster$ as a hashtable of nonempty clusters
- Each insert may create a new structure $\Theta(\log \log u)$ times (each empty insert)
  - Can actually happen [Vladimir Čunát]
- Charge pointer to structure (and associated hash table entry) to the structure

This gives us $\mathcal{O}(n \log \log u)$ space (but randomized).

Indirection

We can further reduce to $\mathcal{O}(n)$ space.

- Store vEB structure with $n = \mathcal{O}(\log \log u)$ using BST or even an array

  $\Longrightarrow \mathcal{O}(\log \log n)$ time once in base case
- We use $\mathcal{O}(n / \log \log u)$ such structures (disjoint)

  $\Longrightarrow \mathcal{O}(\frac{n}{\log \log u} \cdot \log \log u) = \mathcal{O}(n)$ space for small
- Larger structures "store" pointers to them

  $\Longrightarrow \mathcal{O}(\frac{n}{\log \log u} \cdot \log \log u) = \mathcal{O}(n)$ space for large
- Details: Split/Merge small structures