# The Stieltjes Transform and its Role in Eigenvalue Behavior of Large Dimensional Random Matrices

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**1. Introduction**. Let $\mathcal{M}(\mathbb{R})$ denote the collection of all subprobability distribution functions on $\mathbb{R}$. We say for $\{F_n\} \subset \mathcal{M}(\mathbb{R})$, $F_n$ converges vaguely to $F \in \mathcal{M}(\mathbb{R})$ (written $F_n \xrightarrow{v} F$) if for all $[a, b]$, $a, b$ continuity points of $F$, $\lim_{n\to\infty} F_n\{[a, b]\} = F\{[a, b]\}$. We write $F_n \xrightarrow{D} F$, when $F_n$, $F$ are probability distribution functions (equivalent to $\lim_{n\to\infty} F_n(a) = F(a)$ for all continuity points $a$ of $F$).

For $F \in \mathcal{M}(\mathbb{R})$,

$$m_F(z) \equiv \int \frac{1}{x - z} dF(x), \quad z \in \mathbb{C}^+ \equiv \{z \in \mathbb{C} : \Im z > 0\}$$

is defined as the Stieltjes transform of $F$.

Properties:

1. $m_F$ is an analytic function on $\mathbb{C}^+$.
2. $\Im m_F(z) > 0$.
3. $|m_F(z)| \leq \frac{1}{\Im z}$.
4. For continuity points $a < b$ of $F$

$$F\{[a, b]\} = \frac{1}{\pi} \lim_{\eta\to 0^+} \int_a^b \Im m_F(\xi + i\eta) d\xi,$$

since the right hand side

$$= \frac{1}{\pi} \lim_{\eta\to 0^+} \int_a^b \int \frac{\eta}{(x - \xi)^2 + \eta^2} dF(x) d\xi = \frac{1}{\pi} \lim_{\eta\to 0^+} \int \int_a^b \frac{\eta}{(x - \xi)^2 + \eta^2} d\xi dF(x)$$

$$= \frac{1}{\pi} \lim_{\eta\to 0^+} \int \left[\mathrm{Tan}^{-1}\left(\frac{b - x}{\eta}\right) - \mathrm{Tan}^{-1}\left(\frac{a - x}{\eta}\right)\right] dF(x)$$

$$= \int I_{[a,b]} dF(x) = F\{[a, b]\}.$$

5. If, for $x_0 \in \mathbb{R}$, $\Im m_F(x_0) \equiv \lim_{z\in\mathbb{C}^+\to x_0} \Im m_F(z)$ exists, then $F$ is differentiable at $x_0$ with value $(\frac{1}{\pi})\Im m_F(x_0)$ (Silverstein and Choi (1995)).

Let $S \subset \mathbb{C}^+$ be countable with a cluster point in $\mathbb{C}^+$. Using 4., the fact that $F_n \xrightarrow{v} F$ is equivalent to

$$\int f_n(x) dF_n(x) \to \int f(x) dF(x)$$

for all continuous $f$ vanishing at $\pm\infty$, and the fact that an analytic function defined on $\mathbb{C}^+$ is uniquely determined by the values it takes on $S$, we have

$$F_n \xrightarrow{v} F \quad \iff \quad m_{F_n}(z) \to m_F(z) \quad \text{for all } z \in S.$$

The fundamental connection to random matrices:

For any Hermitian $n \times n$ matrix $A$, we let $F^A$ denote the *empirical distribution function* (e.d.f.) of its eigenvalues:

$$F^A(x) = \frac{1}{n}(\text{number of eigenvalues of } A \leq x).$$

Then

$$m_{F^A}(z) = \frac{1}{n} \mathsf{tr}\, (A - zI)^{-1}.$$

So, if we have a sequence $\{A_n\}$ of Hermitian random matrices, to show, with probability one, $F^{A_n} \xrightarrow{v} F$ for some $F \in \mathcal{M}(\mathbb{R})$, it is equivalent to show for any $z \in \mathbb{C}^+$

$$\frac{1}{n} \mathsf{tr}\, (A_n - zI)^{-1} \to m_F(z) \quad a.s.$$

The main goal of the lectures is to show the importance of the Stieltjes transform to limiting behavior of certain classes of random matrices. We will begin with an attempt at providing a systematic way to show a.s. convergence of the e.d.f.'s of the eigenvalues of three classes of large dimensional random matrices via the Stieltjes transform approach. Essential properties involved will be emphasized in order to better understand where randomness comes in and where basic properties of matrices are used.

Then it will be shown, via the Stieltjes transform, how the limiting distribution can be numerically constructed, how it can explicitly (mathematically) be derived in some cases, and, in general, how important qualitative information can be inferred. Other results will be reviewed, namely the exact separation properties of eigenvalues, and distributional behavior of linear spectral statistics.

It is hoped that with this knowledge other ensembles can be explored for possible limiting behavior.

Each theorem below corresponds to a matrix ensemble. For each one the random quantities are defined on a common probability space. They all assume:

For $n = 1, 2, \ldots$ $X_n = (X_{ij}^n)$, $n \times N$, $X_{ij}^n \in \mathbb{C}$, i.d. for all $n, i, j$, independent across $i, j$ for each $n$, $\mathsf{E}|X_{1\,1}^1 - \mathsf{E}X_{1\,1}^1|^2 = 1$, and $N = N(n)$ with $n/N \to c > 0$ as $n \to \infty$.

Theorem 1.1 (Marčenko and Pastur (1967), Silverstein and Bai (1995)). *Assume:*

a) $T_n = \mathit{diag}(t_1^n, \ldots, t_n^n)$, $t_i^n \in \mathbb{R}$, *and the e.d.f. of* $\{t_1^n, \ldots, t_n^n\}$ *converges weakly, with probability one to a nonrandom probability distribution function* $H$ *as* $n \to \infty$.

b) $A_n$ *is a random* $N \times N$ *Hermitian random matrix for which* $F^{A_n} \xrightarrow{v} \mathcal{A}$ *where* $\mathcal{A}$ *is nonrandom (possibly defective).*

c) $X_n$, $T_n$, *and* $A_n$ *are independent.*

*Let* $B_n = A_n + (1/N)X_n^* T_n X_n$. *Then, with probability one* $F^{B_n} \xrightarrow{v} \hat{F}$ *as* $n \to \infty$ *where for each* $z \in \mathbb{C}^+$ $m = m_{\hat{F}}(z)$ *satisfies*

$$m = m_{\mathcal{A}}\left(z - c \int \frac{t}{1 + tm} dH(t)\right). \tag{1.1}$$

*It is the only solution to (1.1) with positive imaginary part.*

THEOREM 1.2 (Yin (1986), Silverstein (1995)). *Assume:*

*$T_n$ $n \times n$ is random Hermitian non-negative definite, independent of $X_n$ with $F^{T_n} \xrightarrow{D} H$ a.s. as $n \to \infty$, $H$ nonrandom.*

*Let $T_n^{1/2}$ denote any Hermitian square root of $T_n$, and define $B_n = (1/N)T_n^{1/2}XX^*T_n^{1/2}$. Then, with probability one $F^{B_n} \xrightarrow{D} F$ as $n \to \infty$ where for each $z \in \mathbb{C}^+$ $m = m_F(z)$ satisfies*

$$m = \int \frac{1}{t(1-c-czm)-z} dH(t). \tag{1.2}$$

*It is the only solution to (1.2) in the set $\{m \in \mathbb{C} : -(1-c)/z + cm \in \mathbb{C}^+\}$.*

THEOREM 1.3 (Dozier and Silverstein a)). *Assume:*

*$R_n$ $n \times N$ is random, independent of $X_n$, with $F^{(1/N)R_nR_n^*} \xrightarrow{D} H$ a.s. as $n \to \infty$, $H$ nonrandom.*

*Let $B_n = (1/N)(R_n + \sigma X_n)(R_n + \sigma X_n)^*$ where $\sigma > 0$, nonrandom. Then, with probability one $F^{B_n} \xrightarrow{D} F$ as $n \to \infty$ where for each $z \in \mathbb{C}^+$ $m = m_F(z)$ satisfies*

$$m = \int \frac{1}{\frac{t}{1+\sigma^2 cm} - (1+\sigma^2 cm)z + \sigma^2(1-c)} dH(t) \tag{1.3.}$$

*It is the only solution to (1.3) in the set $\{m \in \mathbb{C}^+ : \Im(mz) \geq 0\}$.*

Remark: In Theorem 1.1 if $A_n = 0$ for all $n$ large, then $m_{\mathcal{A}}(z) = -1/z$ and we find that $m_F$ has an inverse

$$z = -\frac{1}{m} + c \int \frac{t}{1+tm} dH(t). \tag{1.4}$$

Since

$$F^{(1/N)X_n^*T_nX_n} = \left(1 - \frac{n}{N}\right) I_{[0,\infty)} + \frac{n}{N} F^{(1/N)T_n^{1/2}X_nX_n^*T_n^{1/2}}$$

we have

(1.5)

$$m_{F^{(1/N)X_n^* T_n X_n}}(z) = -\frac{1 - n/N}{z} + \frac{n}{N} m_{F^{(1/N)T_n^{1/2} X_n X_n^* T_n^{1/2}}}(z) \quad z \in \mathbb{C}^+,$$

so we have

$$m_{\hat{F}}(z) = -\frac{1-c}{z} + c m_F(z). \tag{1.6}$$

Using this identity, it is easy to see that (1.2) and (1.4) are equivalent.

**2. Why these theorems are true.** We begin with three facts which account for most of why the limiting results are true, and the appearance of the limiting equations for the Stieltjes transforms.

LEMMA 2.1 *For $n \times n$ $A$, $q \in \mathbb{C}^n$, and $t \in \mathbb{C}$ with $A$ and $A + tqq^*$ invertible, we have*

$$q^*(A + tqq^*)^{-1} = \frac{1}{1 + tq^* A^{-1} q} q^* A^{-1}$$

(since $q^* A^{-1}(A + tqq^*) = (1 + tq^* A^{-1} q) q^*$).

COROLLARY 2.1 *For $q = a + b$, $t = 1$ we have*

$$a^*(A + (a+b)(a+b)^*)^{-1} = a^* A^{-1} - \frac{a^* A^{-1}(a+b)}{1 + (a+b)^* A^{-1}(a+b)}(a+b)^* A^{-1}$$

$$= \frac{1 + b^* A^{-1}(a+b)}{1 + (a+b)^* A^{-1}(a+b)} a^* A^{-1} - \frac{a^* A^{-1}(a+b)}{1 + (a+b)^* A^{-1}(a+b)} b^* A^{-1}.$$

Proof: Using Lemma 2.1 we have

$$\begin{aligned}(A + (a+b)(a+b)^*)^{-1} - A^{-1} &= -(A + (a+b)(a+b)^*)^{-1}(a+b)(a+b)^* A^{-1} \\ &= -\frac{1}{1 + (a+b)^* A^{-1}(a+b)} A^{-1}(a+b)(a+b)^* A^{-1}\end{aligned}$$

Multiplying both sides on the left by $a^*$ gives the result.

LEMMA 2.2 *For $n \times n$ $A$ and $B$, with $B$ Hermitian, $z \in \mathbb{C}^+$, $t \in \mathbb{R}$, and $q \in \mathbb{C}^n$, we have*

$$|\mathsf{tr}\,[(B-zI)^{-1}-(B+tqq^*-zI)^{-1}]A| = \left| t\frac{q^*(B-zI)^{-1}A((B-zI)^{-1}q}{1+tq^*(B-zI)^{-1}q} \right| \le \frac{\|A\|}{\Im z}.$$

Proof. The identity follows from Lemma 2.1. We have

$$\left| t\frac{q^*(B-zI)^{-1}A((B-zI)^{-1}q}{1+tq^*(B-zI)^{-1}q} \right| \le \|A\||t|\|\frac{\|(B-zI)^{-1}q\|^2}{|1+tq^*(B-zI)^{-1}q|}.$$

Write $B = \sum_i \lambda_i e_i e_i^*$, its spectral decomposition. Then

$$\|(B-zI)^{-1}q\|^2 = \sum_i \frac{|e_i^* q|^2}{|\lambda_i - z|^2}$$

and

$$|1+tq^*(B-zI)^{-1}q| \ge |t|\Im(q^*(B-zI)^{-1}q) = |t|\Im z \sum_i \frac{|e_i^* q|^2}{|\lambda_i - z|^2}.$$

LEMMA 2.3. *For $X = (X_1, \ldots, X_n)^T$ i.i.d. standardized entries, $C$ $n \times n$, we have for any $p \ge 2$*

$$\mathsf{E}|X^*CX - \mathsf{tr}\,C|^p \le K_p\big(\big(\mathsf{E}|X_1|^4 \mathsf{tr}\,CC^*\big)^{p/2} + \mathsf{E}|X_1|^{2p}\mathsf{tr}\,(CC^*)^{p/2}\big)$$

*where the constant $K_p$ does not depend on $n$, $C$, nor on the distribution of $X_1$.* (Proof given in Bai and Silverstein (1998).)

From these properties, roughly speaking, we can make observations like the following: for $n \times n$ Hermitian $A$, $q = (1/\sqrt{n})(X_1, \ldots, X_n)^T$, with $X_i$ i.i.d. standardized and independent of $A$, and $z \in \mathbb{C}^+$, $t \in \mathbb{R}$

$$\begin{aligned} tq^*(A+tqq^*-zI)^{-1}q &= \frac{tq^*(A-zI)^{-1}q}{1+tq^*(A-zI)^{-1}q} = 1 - \frac{1}{1+tq^*(A-zI)^{-1}q} \\ &\approx 1 - \frac{1}{1+t(1/n)\mathsf{tr}\,(A-zI)^{-1}} \approx 1 - \frac{1}{1+t\,m_{A+tqq^*}(z)}. \end{aligned}$$

Making this and other observations rigorous requires technical considerations, the first being truncation and centralization of the elements of $X_n$, and truncation of the eigenvalues of $T_n$ in Theorem 1.2 (not needed in Theorem 1.1) and $(1/n)R_nR_n^*$ in Theorem 1.3, all at a rate slower than $n$ ($a \ln n$ for some positive $a$ is sufficient). The truncation and centralization steps will be outlined later. We are at this stage able to go through algebraic manipulations, keeping in mind the above three lemmas, and intuitively derive the equations appearing in each of the three theorems. At the same time we can see what technical details need to be worked out.

Before continuing, two more basic properties of matrices is included here.

LEMMA 2.4 *Let* $z_1, z_2 \in \mathbb{C}^+$ *with* $\max(\Im z_1, \Im z_2) \geq v > 0$, $A$ *and* $B$ $n \times n$ *with* $A$ *Hermitian, and* $q \in \mathbb{C}^n$. *Then*

$$|\mathsf{tr}\, B((A - z_1 I)^{-1} - (A - z_2 I)^{-1})| \leq |z_2 - z_1| N \|B\| \frac{1}{v^2}, \text{ and}$$

$$|q^* B(A - z_1 I)^{-1} q - q^* B(A - z_2 I)^{-1} q| \leq |z_2 - z_1| \, \|q\|^2 \|B\| \frac{1}{v^2}.$$

Consider first the $B_n$ in Theorem 1.1. Let $q_i$ denote $1/\sqrt{N}$ times the $i^{\text{th}}$ column of $X_n^*$. Then

$$(1/N) X_n^* T_n X_n = \sum_{i=1}^{n} t_i q_i q_i^*.$$

Let $B_{(i)} = B_n - t_i q_i q_i^*$. For any $z \in \mathbb{C}^+$ and $x \in \mathbb{C}$ we write

$$B_n - zI = A_n - (z - x)I + (1/N) X_n^* T_n X_n - xI.$$

Taking inverses we have

$$\begin{aligned}&(A_n - (z - x)I)^{-1} \\ &\qquad = (B_n - zI)^{-1} + (A_n - (z - x)I)^{-1}((1/N) X_n^* T_n X_n - xI)(B_n - zI)^{-1}.\end{aligned}$$

Dividing by $N$, taking traces and using Lemma 2.1 we find

$$m_{F^{A_n}}(z-x)-m_{F^{B_n}}(z) = (1/N)\mathsf{tr}\,(A_n-(z-x)I)^{-1}\left(\sum_{i=1}^n t_i q_i q_i^* - xI\right)(B_n-zI)^{-1}$$

$$= (1/N)\sum_{i=1}^n \frac{t_i q_i^*(B_{(i)}-zI)^{-1}(A_n-(z-x)I)^{-1}q_i}{1+t_i q_i^*(B_{(i)}-zI)^{-1}q_i} - x(1/N)\mathsf{tr}\,(B_n-zI)^{-1}(A_n-(z-x)I)^{-1}.$$

Notice when $x$ and $q_i$ are independent, Lemmas 2.2,2.3 give us

$$q_i^*(B_{(i)}-zI)^{-1}(A_n-(z-x)I)^{-1}q_i \approx (1/N)\mathsf{tr}\,(B_n-zI)^{-1}(A_n-(z-x)I)^{-1}.$$

Letting

$$x = x_n = (1/N)\sum_{i=1}^n \frac{t_i}{1+t_i m_{F^{B_n}}(z)}$$

we have

$$m_{F^{A_n}}(z-x_n) - m_{F^{B_n}}(z) = (1/N)\sum_{i=1}^n \frac{t_i}{1+t_i m_{F^{B_n}}(z)} d_i$$

where

$$d_i = \frac{1+t_i m_{F^{B_n}}(z)}{1+t_i q_i^*(B_{(i)}-zI)^{-1}q_i} q_i^*(B_{(i)}-zI)^{-1}(A_n-(z-x_n)I)^{-1}q_i - (1/N)\mathsf{tr}\,(B_n-zI)^{-1}(A_n-(z-x_n)I)^{-1}.$$

In order to use Lemma 2.3, for each $i$, $x_n$ is replaced by

$$x_{(i)} = (1/N)\sum_{j=1}^n \frac{t_j}{1+t_j m_{F^{B_{(i)}}}(z)}.$$

An outline of the remainder of the proof is given. It is easy to argue that if $\mathcal{A}$ is the zero measure on $\mathbb{R}$ (that is, almost surely, only $o(N)$ eigenvalues of $A_n$ remain bounded), then the Stieltjes transforms of $F^{A_n}$ and $F^{B_n}$ converge a.s. to zero, the limits obviously satisfying (1.1) . So we assume $\mathcal{A}$ is not the zero measure. One can then show

$$\delta = \inf_n \Im(m_{F^{B_n}}(z))$$

is positive almost surely.

Using Lemma 2.3 ($p = 6$ is sufficient) and the fact that all matrix inverses encountered are bounded in spectral norm by $1/\Im z$ we have from standard arguments using Boole's and Chebyshev's inequalities, almost surely

$$\text{(2.1)} \qquad \max_{i\le n} \max[|\,\|q_i\|^2 - 1|, |q_i^*(B_{(i)} - zI)^{-1}q_i - m_{F^{B_{(i)}}}(z)|,$$

$$|q_i^*(B_{(i)}-zI)^{-1}(A_n-(z-x_{(i)})I)^{-1}q_i-(1/N)\mathsf{tr}\,(B_{(i)}-zI)^{-1}(A_n-(z-x_{(i)})I)^{-1}|]$$

$$\to 0 \quad \text{as } n \to \infty.$$

Consider now a realization for which (2.1) holds, $\delta > 0$, $F^{T_n} \xrightarrow{D} H$, and $F^{A_n} \xrightarrow{v} \mathcal{A}$. From Lemma 2.2 and (2.1) we have

$$\text{(2.2)} \quad \max_{i\le n} \max[|m_{F^{B_n}}(z) - m_{F^{B_{(i)}}}(z)|, |m_{F^{B_n}}(z) - q_i^*(B_{(i)} - zI)^{-1}q_i|] \to 0,$$

and subsequently

$$\text{(2.3)} \qquad \max_{i\le n} \max\left[\left|\frac{1 + t_i m_{F^{B_n}}(z)}{1 + t_i q_i^*(B_{(i)} - zI)^{-1}q_i} - 1\right|, |x - x_{(i)}|\right] \to 0.$$

Therefore, from Lemmas 2.2,2.4, and (2.1) -(2.3), we get $\max_{i\le n} d_i \to 0$, and since

$$\left|\frac{t_i}{1 + t_i m_{F^{B_n}}(z)}\right| \le \frac{1}{\delta},$$

we conclude from (4.1) that

$$m_{A_n}(z - x_n) - m_{B_n}(z) \to 0.$$

Consider a subsequence $\{n_i\}$ on which $m_{F^{B_{n_i}}}(z)$ converges to a number $m$. It follows that

$$x_{n_i} \to c \int \frac{t}{1+tm} dH(t).$$

Therefore, $m$ satisfies (1.1). Uniqueness (to be discussed later) gives us, for this realization $m_{F^{B_n}}(z) \to m$. This event occurs with probability one.

**3. The other equations.** Let us now derive the equation for the matrix $B_n = (1/N)T_n^{1/2} X_n X_n^* T_n^{1/2}$, after the truncation steps have been taken. Let $c_n = n/N$, $q_j = (1/\sqrt{n})X_{\cdot j}$ (the $j^{\text{th}}$ column of $X_n$), $r_j = (1/\sqrt{N})T_n^{1/2} X_{\cdot j}$, and $B_{(j)} = B_n - r_j r_j^*$. Fix $z \in \mathbb{C}^+$ and let $m_n(z) = m_{F^{B_n}}(z)$, $\mathbf{m}_n(z) = m_{F^{(1/N)X_n^* T_n X_n}}(z)$. By (1.5) we have

$$\mathbf{m}_n(z) = -\frac{1-c_n}{z} + c_n m_n. \tag{3.1}$$

We first derive an identity for $\mathbf{m}_n(z)$. Write

$$B_n - zI + zI = \sum_{j=1}^{N} r_j r_j^*.$$

Taking the inverse of $B_n - zI$ on the right on both sides and using Lemma 2.1 we find

$$I + z(B_n - zI)^{-1} = \sum_{j=1}^{N} \frac{1}{1 + r_j^*(B_{(j)} - zI)^{-1} r_j} r_j r_j^* (B_{(j)} - zI)^{-1}.$$

Taking the trace on both sides and dividing by $N$ we have

$$c_n + z c_n m_n = \frac{1}{N} \sum_{j=1}^{N} \frac{r_j^*(B_{(j)} - zI)^{-1} r_j}{1 + r_j^*(B_{(j)} - zI)^{-1} r_j} = 1 - \frac{1}{N} \sum_{j=1}^{N} \frac{1}{1 + r_j^*(B_{(j)} - zI)^{-1} r_j}.$$

Therefore

$$\mathbf{m}_n(z) = -\frac{1}{N} \sum_{j=1}^{N} \frac{1}{z(1 + r_j^*(B_{(j)} - zI)^{-1} r_j)}. \tag{3.2}$$

Write $B_n - zI - \big(-z\mathbf{m}_n(z)T_n - zI\big) = \sum_{j=1}^N r_jr_j^* - (-z\mathbf{m}_n(z))T_n$. Taking inverses and using Lemma 2.1, (3.2) we have

$$(-z\mathbf{m}_n(z)T_n - zI)^{-1} - (B_n - zI)^{-1}$$
$$= (-z\mathbf{m}_n(z)T_n - zI)^{-1}\left[\sum_{j=1}^N r_jr_j^* - (-z\mathbf{m}_n(z))T_n\right](B_n - zI)^{-1}$$

$$=\sum_{j=1}^N \frac{-1}{z(1+r_j^*(B_{(j)}-zI)^{-1}r_j)}\Big[(\mathbf{m}_n(z)T_n + I)^{-1}r_jr_j^*(B_{(j)} - zI)^{-1}$$
$$- (1/N)(\mathbf{m}_n(z)T_n + I)^{-1}T_n(B_n - zI)^{-1}\Big].$$

Taking the trace and dividing by $n$ we find

$$(1/n)\mathsf{tr}\,(-z\mathbf{m}_n(z)T_n - zI)^{-1} - m_n(z) = \frac{1}{N}\sum_{j=1}^N \frac{-1}{z(1 + r_j^*(B_{(j)} - zI)^{-1}r_j)}d_j$$

where

$$d_j = q_j^*T_n^{1/2}(B_{(j)} - zI)^{-1}(\mathbf{m}_n(z)T_n + I)^{-1}T_n^{1/2}q_j$$
$$- (1/n)\mathsf{tr}\,(\mathbf{m}_n(z)T_n + I)^{-1}T_n(B_n - zI)^{-1}.$$

The derivation for Theorem 1.3 will proceed in a constructive way. Here we let $x_j$ and $r_j$ denote, respectively, the $j^{\text{th}}$ columns of $X_n$ and $R_n$ (after truncation). As before $m_n = m_{F^{B_n}}$, and let

$$\mathbf{m}_n(z) = m_{F^{(1/N)(R_n+\sigma X_n)^*(R_n+\sigma X_n)}}(z).$$

We have again the relationship (3.1) . Notice then equation (1.3) can be written

$$m = \int \frac{1}{\frac{t}{1+\sigma^2 cm} - \sigma^2 z\mathbf{m} - z}dH(t) \tag{3.3}$$

where

$$\mathbf{m} = -\frac{1-c}{z} + cm.$$

Let $B_{(j)} = B_n - (1/N)(r_j + \sigma x_j)(r_j + \sigma x_j)^*$. Then, as in (3.2) we have

$$\text{(3.4)} \quad \mathbf{m}_n(z) = -\frac{1}{N}\sum_{j=1}^{N} \frac{1}{z(1 + (1/N)(r_j + \sigma x_j)^*(B_{(j)} - zI)^{-1}(r_j + \sigma x_j))}.$$

Pick $z \in \mathbb{C}^+$. For any $n \times n$ $Y_n$ we write

$$B_n - zI - (Y_n - zI) = \frac{1}{N}\sum_{j=1}^{N}(r_j + \sigma x_j)(r_j + \sigma x_j)^* - Y_n.$$

Taking inverses, dividing by $n$ and using Lemma 2.1 we get

$$(1/n)\mathsf{tr}\,(Y_n - zI)^{-1} - m_n(z)$$

$$\frac{1}{N}\sum_{j=1}^{N} \frac{(1/n)(r_j + \sigma x_j)^*(B_{(j)} - zI)^{-1}(Y_n - zI)^{-1}(r_j + \sigma x_j)}{1 + (1/N)(r_j + \sigma x_j)^*(B_{(j)} - zI)^{-1}(r_j + \sigma x_j)}$$
$$- (1/n)\mathsf{tr}\,(Y_n - zI)^{-1}Y_n(B_n - zI)^{-1}.$$

The goal is to determine $Y_n$ so that each term goes to zero. Notice first that

$$(1/n)x_j^*(B_{(j)} - zI)^{-1}(Y_n - zI)^{-1}x_j \approx (1/n)\mathsf{tr}\,(B_n - zI)^{-1}(Y_n - zI)^{-1},$$

so from (3.4) we see that $Y_n$ should have a term

$$-\sigma^2 z\mathbf{m}_n(z)I.$$

Since for any $n \times n$ $C$ bounded in norm

$$|(1/n)x_j^*Cr_j|^2 = (1/n^2)x_j^*Cr_jr_j^*C^*x_j$$

we have from Lemma 2.3

$$|(1/n)x_j^* C r_j|^2 \approx (1/n^2)\operatorname{tr} C r_j r_j^* C^* = (1/n^2) r_j^* C^* C r_j = o(1) \tag{3.5}$$

(from truncation $(1/N)\|r_j\|^2 \leq \ln n$), so the cross terms are negligible.

This leaves us $(1/n)r_j^*(B_{(j)} - zI)^{-1}(Y_n - zI)^{-1}r_j$. Recall Corollary 2.1

$$a^*(A + (a+b)(a+b)^*)^{-1}$$

$$= \frac{1 + b^* A^{-1}(a+b)}{1 + (a+b)^* A^{-1}(a+b)} a^* A^{-1} - \frac{a^* A^{-1}(a+b)}{1 + (a+b)^* A^{-1}(a+b)} b^* A^{-1}.$$

Identify $a$ with $(1/\sqrt{N})r_j$, $b$ with $(1/\sqrt{N})\sigma x_j$, and $A$ with $B_{(j)}$. Using Lemmas 2.2, 2.3 and (3.5), we have

$$\begin{aligned}
&(1/n)r_j^*(B_n - zI)^{-1}(Y_n - zI)^{-1}r_j \\
&\approx \frac{1 + \sigma^2 c_n m_n(z)}{1 + \frac{1}{N}(r_j + \sigma x_j)^*(B_{(j)} - zI)^{-1}(r_j + \sigma x_j)} (1/n) r_j^*(B_{(j)} - zI)^{-1}(Y_n - zI)^{-1} r_j.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{1}{N}\sum_{j=1}^{N} \frac{(1/n)r_j^*(B_{(j)} - zI)^{-1}(Y_n - zI)^{-1}r_j}{1 + \frac{1}{N}(r_j + \sigma x_j)^*(B_{(j)} - zI)^{-1}(r_j + \sigma x_j)} \\
&\qquad \approx \frac{1}{N}\sum_{j=1}^{N} \frac{(1/n)r_j^*(B_n - zI)^{-1}(Y_n - zI)^{-1}r_j}{1 + \sigma^2 c_n m_n(z)} \\
&= (1/n)\frac{1}{1 + \sigma^2 c_n m_n(z)} \operatorname{tr}(1/N) R_n R_n^* (B_n - zI)^{-1}(Y_n - zI)^{-1}.
\end{aligned}$$

So we should take

$$Y_n = \frac{1}{1 + \sigma^2 c_n m_n(z)}(1/N) R_n R_n^* - \sigma^2 z \mathbf{m}_n(z) I.$$

Then $(1/n)\operatorname{tr}(Y_n - zI)^{-1}$ will approach the right hand side of (3.3).

**4. Proof of uniqueness of (1.1).** For $m \in \mathbb{C}^+$ satisfying (1.1) with $z \in \mathbb{C}^+$ we have

$$m = \int \frac{1}{\tau - \left(z - c\int \frac{t}{1+tm} dH(t)\right)} d\mathcal{A}(\tau)$$

$$= \int \frac{1}{\tau - \Re\left(z - c\int \frac{t}{1+tm} dH(t)\right) - i\left(\Im z + c\int \frac{t^2 \Im m}{|1+tm|^2} dH(t)\right)} d\mathcal{A}(\tau)$$

Therefore

(4.1)

$$\Im m = \left(\Im z + c\int \frac{t^2 \Im m}{|1+tm|^2} dH(t)\right) \int \frac{1}{\left|\tau - z + c\int \frac{t}{1+tm} dH(t)\right|^2} d\mathcal{A}(\tau)$$

Suppose $\mathbf{m} \in \mathbb{C}^+$ also satisfies (1.1). Then

(4.2)

$$m - \mathbf{m} = c\int \frac{\left[\int \frac{t}{1+t\mathbf{m}} - \frac{t}{1+tm}\right] dH(t)}{\left(\tau - z + c\int \frac{t}{1+tm} dH(t)\right)\left(\tau - z + c\int \frac{t}{1+t\mathbf{m}} dH(t)\right)} d\mathcal{A}(\tau)$$

$$(m - \mathbf{m})c\int \frac{t^2}{(1+tm)(1+t\mathbf{m})} dH(t)$$
$$\times \int \frac{1}{\left(\tau - z + c\int \frac{t}{1+tm} dH(t)\right)\left(\tau - z + c\int \frac{t}{1+t\mathbf{m}} dH(t)\right)} d\mathcal{A}(\tau).$$

Using Cauchy-Schwarz and (4.1) we have

$$\left| c\int \frac{t^2}{(1+tm)(1+t\mathbf{m})} dH(t) \right.$$
$$\left. \times \int \frac{1}{\left(\tau - z + c\int \frac{t}{1+tm} dH(t)\right)\left(\tau - z + c\int \frac{t}{1+t\mathbf{m}} dH(t)\right)} d\mathcal{A}(\tau) \right|$$

$$\leq \left( c \int \frac{t^2}{|1+tm|^2} dH(t) \int \frac{1}{\left|\tau - z + c\int \frac{t}{1+tm} dH(t)\right|^2} d\mathcal{A}(\tau) \right)^{1/2}$$

$$\times \left( c \int \frac{t^2}{|1+t\mathbf{m}|^2} dH(t) \int \frac{1}{\left|\tau - z + c\int \frac{t}{1+t\mathbf{m}} dH(t)\right|^2} d\mathcal{A}(\tau) \right)^{1/2}$$

$$= \left( c \int \frac{t^2}{|1+tm|^2} dH(t) \frac{\Im m}{\left(\Im z + c\int \frac{t^2 \Im m}{|1+tm|^2} dH(t)\right)} \right)^{1/2}$$

$$\times \left( c \int \frac{t^2}{|1+t\mathbf{m}|^2} dH(t) \frac{\Im \mathbf{m}}{\left(\Im z + c\int \frac{t^2 \Im \mathbf{m}}{|1+t\mathbf{m}|^2} dH(t)\right)} \right)^{1/2} < 1.$$

Therefore, from (4.2) we must have $m = \mathbf{m}$.

**5. Truncation and Centralization**. We outline here the steps taken to enable us to assume in the proof of Theorem 1.1, for each $n$, the $X_{ij}$'s are bounded by a multiple of $\ln n$. The following lemmas are needed.

LEMMA 5.1. *Let $X_1, \dots, X_n$ be i.i.d. Bernoulli with $p = \mathsf{P}(X_1 = 1) < 1/2$. Then for any $\epsilon > 0$ such that $p + \epsilon \le 1/2$ we have*

$$\mathsf{P}\left(\sum_{i=1}^{n} X_i - p \ge \epsilon\right) \le e^{-\frac{n\epsilon^2}{2(p+\epsilon)}}.$$

LEMMA 5.2. *Let $A$ be $N \times N$ Hermitian, $Q$, $\overline{Q}$ both $n \times N$, and $T$, $\overline{T}$ both $n \times n$ Hermitian. Then*

*a)* $$\|F^{A+Q^*TQ} - F^{A+\overline{Q}^*T\overline{Q}}\| \le \frac{2}{N}\mathrm{rank}(Q - \overline{Q})$$

*and*

*b)* $$\|F^{A+Q^*TQ} - F^{A+Q^*\overline{T}Q}\| \le \frac{1}{N}\mathrm{rank}(T - \overline{T}).$$

LEMMA 5.3. *For rectangular $A$, rank$(A) \le$ the number of nonzero entries of $A$.*

LEMMA 5.4 *For Hermitian $N \times N$ matrices $A$, $B$*

$$\sum_{i=1}^{N} (\lambda_i^A - \lambda_i^B)^2 \le \mathsf{tr}\,(A - B)^2.$$

LEMMA 5.5 *Let $\{f_i\}$ be an enumeration of all continuous functions that take a constant $\frac{1}{m}$ value ($m$ a positive integer) on $[a, b]$, where $a$, $b$ are rational, 0 on $(-\infty, a - \frac{1}{m}] \cup [b + \frac{1}{m}, \infty)$, and linear on each of $[a - \frac{1}{m}, a]$, $[b, b + \frac{1}{m}]$. Then*

a) *for $F_1, F_2 \in \mathcal{M}(\mathbb{R})$*

$$D(F_1, F_2) \equiv \sum_{i=1}^{\infty} \left| \int f_i dF_1 - \int f_i dF_2 \right| 2^{-i}$$

*is a metric on $\mathcal{M}(\mathbb{R})$ inducing the topology of vague convergence.*

b) *For $F_N, G_N \in \mathcal{M}(\mathbb{R})$*

$$\lim_{N\to\infty} \|F_N - G_N\| = 0 \implies \lim_{N\to\infty} D(F_N, G_N) = 0.$$

c) *For empirical distribution functions $F, G$ on the (respective) sets $\{x_1, \dots, x_N\}, \{y_1, \dots, y_N\}$*

$$D^2(F,G) \le \left(\frac{1}{N}\sum_{j=1}^N |x_j - y_j|\right)^2 \le \frac{1}{N}\sum_{j=1}^N (x_j - y_j)^2.$$

Let $p_n = \mathsf{P}(|X_{11}| \ge \sqrt{n})$. Since the second moment of $X_{11}$ is finite we have

$$np_n = o(1). \tag{5.1}$$

Let $\widehat{X}_{ij} = X_{ij} I_{(|X_{ij}|<\sqrt{n})}$ and $\widehat{B}_n = A_n + (1/N)\widehat{X}_n^* T_n \widehat{X}_n$, where $\widehat{X} = (X_{ij})$. Then from Lemmas 5.2 a) 5.3, for any positive $\epsilon$

$$\begin{aligned}\mathsf{P}(\|F^{B_n} - F^{\widehat{B}_n}\| \ge \epsilon) &\le \mathsf{P}\left(\frac{2}{N}\sum_{ij} I_{(|X_{ij}|\ge\sqrt{n})} \ge \epsilon\right)\\ &= \mathsf{P}\left(\frac{1}{Nn}\sum_{ij} I_{(|X_{ij}|\ge\sqrt{n})} - p_n \ge \frac{\epsilon}{2n} - p_n\right).\end{aligned}$$

Then by Lemma 5.1, for all $n$ large

$$\mathsf{P}(\|F^{B_n} - F^{\widehat{B}_n}\| \ge \epsilon) \le e^{-\frac{N\epsilon}{16}},$$

which is summable. Therefore

$$\|F^{B_n} - F^{\widehat{B}_n}\| \xrightarrow{a.s.} 0.$$

Let $\widetilde{B}_n = A_n + (1/N)\widetilde{X}_n T_n \widetilde{X}_n^*$ where $\widetilde{X}_n = \widehat{X}_n - \mathsf{E}\widehat{X}_n$. Since $\operatorname{rank}(\mathsf{E}\widehat{X}) \leq 1$, we have from Lemma 5.2 a)

$$\|F^{\widehat{B}_n} - F^{\widetilde{B}_n}\| \longrightarrow 0.$$

For $\alpha > 0$ define $T_\alpha = \operatorname{diag}(t_1^n I_{(|t_1^n| \leq \alpha)}, \ldots, t_n^n I_{(|t_n^n| \leq \alpha)})$, and let $Q$ be any $\times N$ matrix. If $\alpha$ and $-\alpha$ are continuity points of $H$, we have by Lemma 5.2 b)

$$\|F^{A_n + Q^* T_n Q} - F^{A_n + Q^* T_\alpha Q}\| \\ \leq \frac{1}{N}\operatorname{rank}(T_n - T_\alpha) = \frac{1}{N}\sum_{i=1}^{n} I_{(|t_i^n| > \alpha)} \xrightarrow{a.s.} cH\{[-\alpha, \alpha]^c\}.$$

It follows that if $\alpha = \alpha_n \to \infty$ then

$$\|F^{A_n + Q^* T_n Q} - F^{A_n + Q^* T_\alpha Q}\| \xrightarrow{a.s.} 0.$$

Let $\overline{X}_{ij} = \widetilde{X}_{ij} I_{(|X_{ij}| < \ln n)} - \mathsf{E}\widetilde{X}_{ij} I_{(|X_{ij}| < \ln n)}$, $\overline{X}_n = ((1/\sqrt{N})\overline{X}_{ij})$, $\overline{\overline{X}}_{ij} = \widetilde{X}_{ij} - \overline{X}_{ij}$, and $\overline{\overline{X}}_n = ((1/\sqrt{N})\overline{\overline{X}}_{ij})$. Then, from Lemmas 5.5 c) and 5.4 and simple applications of Cauchy-Schwarz we have

$$D^2(F^{A_n + \widetilde{X}_n T_\alpha \widetilde{X}_n^*}, F^{A + \overline{X}_n T_\alpha \overline{X}_n{}^*}) \leq \frac{1}{N}\mathsf{tr}\,(\widetilde{X}_n T_\alpha \widetilde{X}_n^* - \overline{X}_n T_\alpha \overline{X}_n{}^*)^2$$

$$\leq \frac{1}{N}[\mathsf{tr}\,(\overline{\overline{X}}_n T_\alpha \overline{\overline{X}}_n^*)^2 + 4\mathsf{tr}\,(\overline{\overline{X}}_n T_\alpha \overline{X}_n^* \overline{X}_n T_\alpha \overline{\overline{X}}_n^*) \\ + 4(\mathsf{tr}\,(\overline{\overline{X}}_n T_\alpha \overline{X}_n^* \overline{X}_n T_\alpha \overline{\overline{X}}_n^*)\mathsf{tr}\,(\overline{\overline{X}}_n T_\alpha \overline{\overline{X}}_n^*)^2)^{1/2}].$$

We have

$$\mathsf{tr}\,(\overline{\overline{X}}_n T_\alpha \overline{\overline{X}}_n^*)^2 \leq \alpha^2 \mathsf{tr}\,(\overline{\overline{X}}\,\overline{\overline{X}}^*)^2$$

and

$$\mathsf{tr}\,(\overline{\overline{X}}_n T_\alpha \overline{X}_n^* \overline{X} T_\alpha \overline{\overline{X}}^*) \leq (\alpha^4 \mathsf{tr}\,(\overline{\overline{X}}\,\overline{\overline{X}}^*)^2 \mathsf{tr}\,(\overline{X}\,\overline{X}^*)^2)^{1/2}.$$

Therefore, to verify

$$D(F^{A+\widetilde{X}T_\alpha\widetilde{X}^*}, F^{A+\overline{X}T_\alpha\overline{X}^*}) \xrightarrow{a.s.} 0$$

it is sufficient to find a sequence $\{\alpha_n\}$ increasing to $\infty$ so that

$$\alpha_n^4 \frac{1}{N}\mathsf{tr}\,(\overline{\overline{X}}\,\overline{\overline{X}}^*)^2 \xrightarrow{a.s.} 0 \text{ and } \frac{1}{N}\mathsf{tr}\,(\overline{X}\,\overline{X}^*)^2 = O(1) \text{ a.s..}$$

The details are omitted.

Notice the matrix $\text{diag}(\mathsf{E}|\overline{X}_{1\,1}|^2 t_1^n, \ldots, \mathsf{E}|\overline{X}_{1\,1}|^2 t_n^n)$ also satisfies assumption a) of Theorem 1.1. Just substitute this matrix for $T_n$, and replace $\overline{X}_n$ by $(1/\sqrt{\mathsf{E}|\overline{X}_{1\,1}|^2})\overline{X}_n$. Therefore we may assume

1) $X_{ij}$ are i.i.d. for fixed $n$,
2) $|X_{11}| \le a \ln n$ for some positive $a$,
3) $\mathsf{E}X_{11} = 0$, $\mathsf{E}|X_{11}|^2 = 1$.

**6. The limiting distributions**. The Stieltjes transform provides a great deal of information to the nature of the limiting distribution $\hat{F}$ when $A_n = 0$ in Theorem 1.1, and $F$ in Theorems 1.2, 1.3. For the first two

$$z = -\frac{1}{\mathbf{m}} + c\int \frac{t}{1+t\mathbf{m}}dH(t)$$

is the inverse of $\mathbf{m} = m_{\hat{F}}(z)$, the limiting Stieltjes transform of $F^{(1/N)X_n^* T_n X_n}$. Recall, when $T_n$ is nonnegative definite, the relationships between $F$, the limit of $F^{(1/N)T_n^{1/2}X_n X_n^* T_n^{1/2}}$ and $\hat{F}$

$$\hat{F}(x) = 1 - cI_{[0,\infty)}(x) + cF(x),$$

and $m_F$ and $m_{\hat{F}}$

$$m_{\hat{F}}(z) = -\frac{1-c}{z} + cm_F(z).$$

Based solely on the inverse of $m_{\hat{F}}$ the following is shown in Silverstein and Choi (1995):

1. For all $x \in \mathbb{R}$, $x \neq 0$

$$\lim_{z \in \mathbb{C}^+} m_{\hat{F}}(z) \equiv m_0(x)$$

exists. The function $m_0$ is continuous on $\mathbb{R}-\{0\}$. Consequently, by property 5. of Stieltjes transforms, $\hat{F}$ has a continuous derivative $f$ on $\mathbb{R}-\{0\}$ given by $\hat{f}(x) = \frac{1}{\pi}\Im m_0(x)$ ($F$ subsequently has derivative $f = \frac{1}{c}\hat{f}$). The density $\hat{f}$ is analytic (possess a power series expansion) for every $x \neq 0$ for which $f(x) > 0$. Moreover, for these $x$, $\pi f(x)$ is the imaginary part of the unique $\mathbf{m} \in \mathbb{C}^+$ satisfying

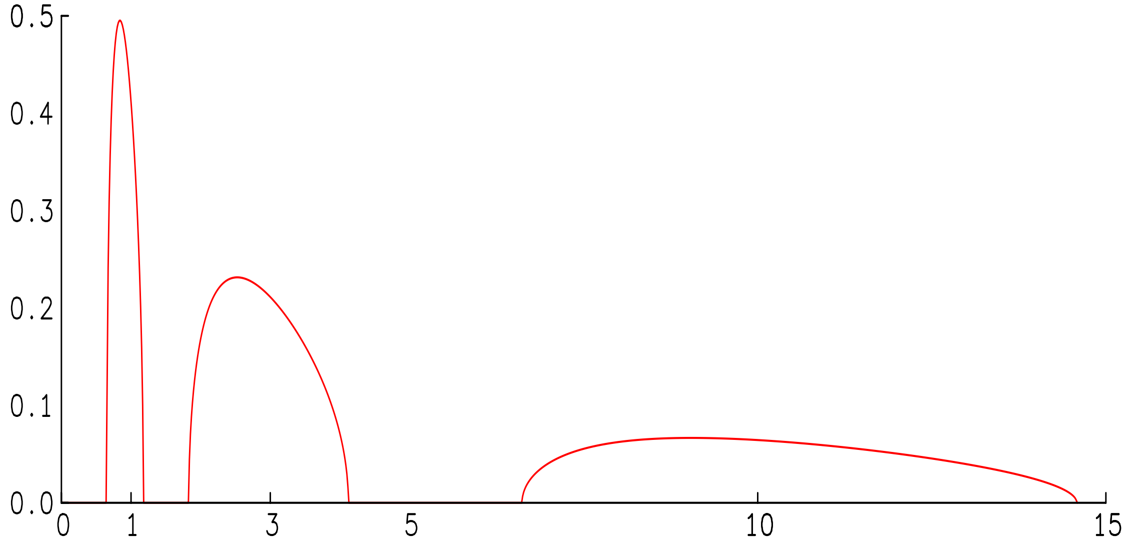
$$x = -\frac{1}{\mathbf{m}} + c\int \frac{t}{1+t\mathbf{m}}dH(t).$$

2. Let $x_{\hat{F}}$ denote the above function of $\mathbf{m}$. It is defined and analytic on $B \equiv \{\mathbf{m} \in \mathbb{R} : \mathbf{m} \neq 0, -\mathbf{m}^{-1} \in S_H^c\}$ ($S_G^c$ denoting the complement of

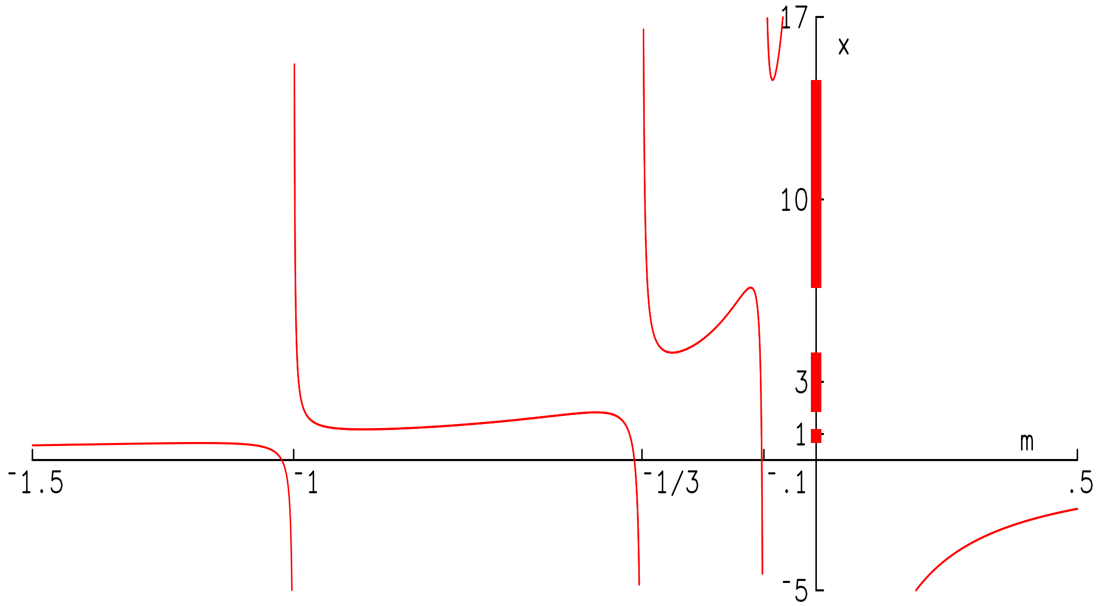
the support of distribution $G$). Then if $x \in S^c_{\hat{F}}$ we have $\mathbf{m} = m_0(x) \in B$ and $x'_{\hat{F}}(\mathbf{m}) > 0$. Conversely, if $\mathbf{m} \in B$ and $x = x'_{\hat{F}}(\mathbf{m}) > 0$, then $x \in S^c_{\hat{F}}$.

We see then a systematic way of determining the support of $\hat{F}$: Plot $x_{\hat{F}}(\mathbf{m})$ for $\mathbf{m} \in B$. Remove all intervals on the vertical axis corresponding to places where $x_{\hat{F}}$ is increasing. What remains is $S_{\hat{F}}$, the support of $\hat{F}$.

Let us look at an example where $H$ places mass at 1, 3, and 10, with respective probabilities .2, .4, and .4, and $c = .1$.




(a)




(b)

Figure (b) is the graph of

$$x_{\hat{F}}(\mathbf{m}) = -\frac{1}{\mathbf{m}} + .1\left(.2\frac{1}{1+\mathbf{m}} + .4\frac{3}{1+3\mathbf{m}} + .4\frac{10}{1+10\mathbf{m}}\right).$$

We see the support boundaries occur at relative extreme values. These values were estimated and for values of $x \in S_{\hat{F}}$, $f(x) = \frac{1}{c\pi}\Im m_0(x)$ was computed using Newton's method on $x = x_{\hat{F}}(\mathbf{m})$, resulting in figure (a).

It is possible for a support boundary to occur at a boundary of the support of $B$, which would only happen for a nondiscrete $H$. However, we have

3. Suppose support boundary $a$ is such that $m_{\hat{F}}(a) \in B$, and is a left-endpoint in the support of $\hat{F}$. Then for $x > a$ and near $a$

$$f(x) = \left(\int_a^x g(t)dt\right)^{1/2}.$$

where $g(a) > 0$ (analogous statement holds for $a$ a right-endpoint in the support of $\hat{F}$). Thus, near support boundaries, $f$ and the square root function share common features, as can be seen in figure (a).

It is remarked here that similar results have been obtained for the matrices in Theorem 1.3. See Dozier and Silverstein b).

Explicit solutions can be derived in a few cases. Consider the Mařcenko-Pastur distribution, where $T_n = I$. Then $\mathbf{m} = m_0(x)$ solves

$$x = -\frac{1}{\mathbf{m}} + c\frac{1}{1+\mathbf{m}},$$

resulting in the quadratic equation

$$x\mathbf{m}^2 + \mathbf{m}(x+1-c) + 1 = 0$$

with solution

$$m = \frac{-(x+1-c) \pm \sqrt{(x+1-c)^2 - 4x}}{2x}$$

$$= \frac{-(x+1-c) \pm \sqrt{x^2 - 2x(1+c) + (1-c)^2}}{2x}$$

$$= \frac{-(x+1-c) \pm \sqrt{(x-(1-\sqrt{c})^2)(x-(1+\sqrt{c})^2)}}{2x}$$

We see the imaginary part of $m$ is zero when $x$ lies outside the interval $[(1-\sqrt{c})^2, (1+\sqrt{c})^2]$, and we conclude that

$$f(x) = \begin{cases} \frac{\sqrt{(x-(1-\sqrt{c})^2)((1+\sqrt{c})^2-x)}}{2\pi c x} & x \in ((1-\sqrt{c})^2, (1+\sqrt{c})^2) \\ 0 & \text{otherwise} \end{cases}.$$

The Stieltjes transform in the multivariate $F$ matrix case, that is, when $T_n = ((1/N')\underline{X}_n\underline{X}_n^*)^{-1}$, $\underline{X}_n$ $n \times N'$ containing i.i.d. standardized entries, $n/N' \to c' \in (0,1)$, also satisfies a quadratic equation. Indeed, $H$ now is the distribution of the reciprocal of a Marčenko-Pasur distributed random variable which we'll denote by $X_{c'}$, the Stieltjes transform of its distribution denoted by $m_{X_{c'}}$. We have

$$x = -\frac{1}{\mathbf{m}} + c\mathsf{E}\left(\frac{\frac{1}{X_{c'}}}{1+\frac{1}{X_{c'}}\mathbf{m}}\right) = -\frac{1}{\mathbf{m}} + c\mathsf{E}\left(\frac{1}{X_{c'}+\mathbf{m}}\right)$$

$$= -\frac{1}{\mathbf{m}} + cm_{X_{c'}}(-\mathbf{m}).$$

From above we have

$$m_{X_{c'}}(z) = \frac{1-c'}{c'z} + \frac{-(z+1-c)+\sqrt{(z+1-c)^2-4z}}{2zc'}$$

$$= \frac{-z+1-c'+\sqrt{(z+1-c')^2-4z}}{2zc'}$$

(the square root defined so that the expression is a Stieltjes transform) so that $\mathbf{m} = m_0(x)$ satisfies

$$x = -\frac{1}{\mathbf{m}} + c\left(\frac{\mathbf{m}+1-c+\sqrt{(-\mathbf{m}+1-c)^2+4\mathbf{m}}}{-2\mathbf{m}c'}\right).$$

It follows that $\mathbf{m}$ satisfies

$$\mathbf{m}^2(c'x^2+cx) + \mathbf{m}(2c'x - c^2 + c + cx(1-c')) + c' + c(1-c') = 0.$$

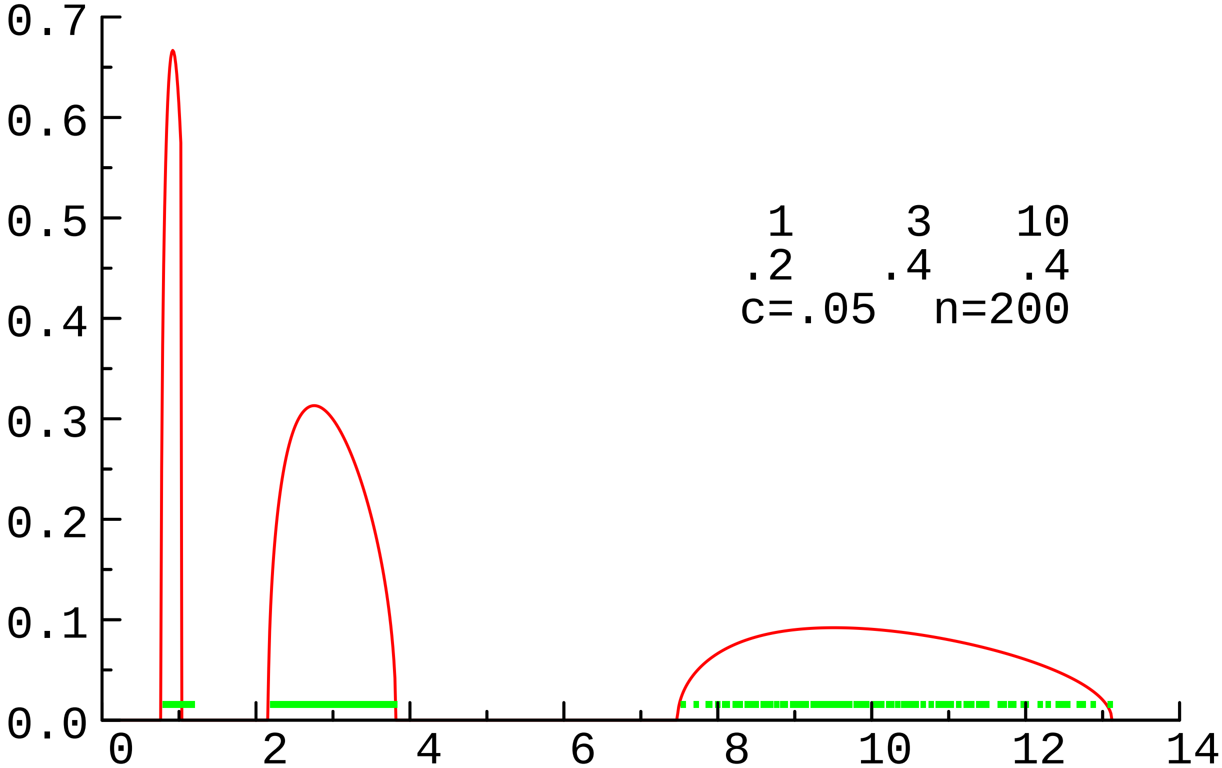
Solving for $\mathbf{m}$ we conclude that, with

$$b_1 = \left(\frac{1-\sqrt{1-(1-c)(1-c')}}{1-c'}\right)^2 \quad b_2 = \left(\frac{1-\sqrt{1-(1-c)(1-c')}}{1-c'}\right)^2$$

$$f(x) = \begin{cases} \frac{(1-c')\sqrt{(x-b_1)(b_2-x)}}{2\pi x(xc'+c)} & b_1 < x < b_2 \\ 0 & \text{otherwise.} \end{cases}$$

**7. Other uses of the Stieltjes transform**. We conclude these lectures with two results requiring Stieltjes transforms.

The first concerns the eigenvalues of matrices in Theorem 1.2 outside the support of the limiting distribution. The results mentioned so far clearly say nothing about the possibility of some eigenvalues lingering in this region. Consider this example with $T_n$ given earlier, but now $c = .05$. Below is a scatterplot of the eigenvalues from a simulation with $n = 200$ ($N = 4000$), superimposed on the limiting density.




Here the entries of $X_n$ are $N(0,1)$. All the eigenvalues appear to stay close to the limiting support. Such simulations were the prime motivation to prove

THEOREM 7.1 (Bai and Silverstein (1998)). *Let, for any $d > 0$ and d.f. $G$, $\hat{F}^{d,G}$ denote the limiting e.d.f. of $(1/N)X_n^* T_n X_n$ corresponding to limiting ratio $d$ and limiting $F^{T_n}$ $G$.*

*Assume in addition to the previous assumptions*:

a) $\mathsf{E}X_{11} = 0$, $\mathsf{E}|X_{11}|^2 = 1$, *and* $\mathsf{E}|X_{11}|^4 < \infty$.

b) $T_n$ *is nonrandom and* $\|T_n\|$ *is bounded in* $n$.

c) *The interval* $[a, b]$ *with* $a > 0$ *lies in an open interval outside the support of* $\hat{F}^{c_n,H_n}$ *for all large* $n$, *where* $H_n = F^{T_n}$.

Then

$$\mathsf{P}(\text{ no eigenvalue of } B_n \text{ appears in } [a, b] \text{ for all large } n \text{ }) = 1.$$

Steps in proof:

1. Let $\underline{B}_n = (1/N)X_n^* T_n X_n$ $\underline{m}_n = m_{F^{\underline{B}_n}}$ and $\underline{m}_n^0 = m_{\hat{F}^{c_n, H_n}}$. Then for $z = x + iv_n$

$$\sup_{x\in[a,b]} |\underline{m}_n(z) - \underline{m}_n^0(z)| = o(1/Nv_n) \quad a.s.$$

when $v_n = N^{-1/68}$.

2. The proof of 1. allows 1. to hold for $Im(z) = \sqrt{2}v_n, \sqrt{3}v_n, \ldots, \sqrt{34}v_n$. Then almost surely

$$\max_{k\in\{1,\ldots,34\}} \sup_{x\in[a,b]} |\underline{m}_n(x + i\sqrt{k}v_n) - \underline{m}_n^0(x + i\sqrt{k}v_n)| = o(v_n^{67}).$$

We take the imaginary part of these Stieltjes transforms and get

$$\max_{k\in\{1,2\ldots,34\}} \sup_{x\in[a,b]} \left| \int \frac{d(F^{\underline{B}_n}(\lambda) - \hat{F}^{c_n,H_n}(\lambda))}{(x-\lambda)^2 + kv_n^2} \right| = o(v_n^{66}) \quad a.s.$$

Upon taking differences we find with probability one

$$\max_{k_1\neq k_2} \sup_{x\in[a,b]} \left| \int \frac{v_n^2\, d(F^{\underline{B}_n}(\lambda) - \hat{F}^{c_n,H_n}(\lambda))}{((x-\lambda)^2 + k_1 v_n^2)((x-\lambda)^2 + k_2 v_n^2)} \right| = o(v_n^{66})$$

$$\max_{\substack{k_1,k_2,k_3\\ \text{distinct}}} \sup_{x\in[a,b]} \left| \int \frac{(v_n^2)^2\, d(F^{\underline{B}_n}(\lambda) - \hat{F}^{c_n,H_n}(\lambda))}{((x-\lambda)^2 + k_1 v_n^2)((x-\lambda)^2 + k_2 v_n^2)((x-\lambda)^2 + k_3 v_n^2)} \right| = o(v_n^{66})$$

$$\vdots$$

$$\sup_{x\in[a,b]} \left| \int \frac{(v_n^2)^{33}\, d(F^{\underline{B}_n}(\lambda) - \hat{F}^{c_n,H_n}(\lambda))}{((x-\lambda)^2 + v_n^2)((x-\lambda)^2 + 2v_n^2)\cdots((x-\lambda)^2 + 34v_n^2)} \right| = o(v_n^{66}).$$

Thus with probability one

$$\sup_{x\in[a,b]} \left| \int \frac{d(F^{\underline{B}_n}(\lambda) - \hat{F}^{c_n,H_n}(\lambda))}{((x-\lambda)^2 + v_n^2)((x-\lambda)^2 + 2v_n^2)\cdots((x-\lambda)^2 + 34v_n^2)} \right| = o(1)$$

Let $0 < a' < a$, $b' > b$ be such that $[a', b']$ is also in the open interval outside the support of $\hat{F}^{c_n, H_n}$ for all large $n$. We split up the integral and get with probability one

$$\sup_{x \in [a,b]} \left| \int \frac{I_{[a',b']^c}(\lambda)\, d(F^{\underline{B}_n}(\lambda) - \hat{F}^{c_n,H_n}(\lambda))}{((x-\lambda)^2 + v_n^2)((x-\lambda)^2 + 2v_n^2) \cdots ((x-\lambda)^2 + 34v_n^2)} \right.$$

$$\left. + \sum_{\lambda_j \in [a',b']} \frac{v_n^{68}}{((x-\lambda_j)^2 + v_n^2)((x-\lambda_j)^2 + 2v_n^2) \cdots ((x-\lambda_j)^2 + 34v_n^2)} \right| = o(1).$$

Now if, for each term in a subsequence satisfying the above, there is at least one eigenvalue contained in $[a, b]$, then the sum, with $x$ evaluated at these eigenvalues, will be uniformly bounded away from 0. Thus, at these same $x$ values, the integral must also stay uniformly bounded away from 0. But the integral MUST converge to zero $a.s.$ since the integrand is bounded and with probability one, both $F^{\underline{B}_n}$ and $\hat{F}^{c_n,H_n}$ converge weakly to the same limit having no mass on $\{a', b'\}$. Contradiction!

The last result is on the rate of convergence of linear statistics of the eigenvalues of $B_n$, that is, quantities of the form

$$\int f(x)dF^{B_n}(x) = \frac{1}{n}\sum_{i=1}^{n} f(\lambda_i)$$

where $f$ is a function defined on $[0,\infty)$, and the $\lambda_i$'s are the eigenvalues of $B_n$. The result establishes the rate to be $1/n$ for analytic $f$. It considers integrals of functions with respect to

$$G_n(x) = n[F^{B_n}(x) - F^{c_n,H_n}(x)]$$

where for any $d > 0$ and d.f. $G$, $F^{d,G}$ is the limiting e.d.f. of $B_n = (1/N)T_n^{1/2}X_nX_n^*T_n^{1/2}$ corresponding to limiting ratio $d$ and limiting $F^{T_n}$ $G$.

THEOREM 7.2. *Under the assumptions in Theorem 7.1, Let $f_1, \ldots, f_r$ be $C^1$ functions on $\mathbb{R}$ with bounded derivatives, and analytic on an open interval containing*

$$[\liminf_n \lambda_{\min}^{T_n} I_{(0,1)}(c)(1-\sqrt{c})^2, \limsup_n \lambda_{\max}^{T_n}(1+\sqrt{c})^2].$$

*Let $\underline{m} = m_{\hat{F}}$. Then*

(1) *the random vector*

$$\left(\int f_1(x)\, dG_n(x), \ldots, \int f_r(x)\, dG_n(x)\right) \tag{7.1}$$

*forms a tight sequence in $n$.*

(2) *If $X_{11}$ and $T_n$ are real and $\mathsf{E}(X_{11}^4) = 3$, then (7.1) converges weakly to a Gaussian vector $(X_{f_1}, \ldots, X_{f_r})$, with means*

$$\mathsf{E}X_f = -\frac{1}{2\pi i}\int f(z)\frac{c\int \frac{\underline{m}(z)^3 t^2 dH(t)}{(1+t\underline{m}(z))^3}}{\left(1 - c\int \frac{\underline{m}(z)^2 t^2 dH(t)}{(1+t\underline{m}(z))^2}\right)^2}dz \tag{7.2}$$

and covariance function

(7.3)

$$\mathrm{Cov}(X_f, X_g) = -\frac{1}{2\pi^2} \iint \frac{f(z_1)g(z_2)}{(\underline{m}(z_1) - \underline{m}(z_2))^2} \frac{d}{dz_1}\underline{m}(z_1) \frac{d}{dz_2}\underline{m}(z_2) dz_1 dz_2$$

($f, g \in \{f_1, \ldots, f_r\}$). *The contours in (7.2) and (7.3)* (*two in (7.3), which we may assume to be non-overlapping*) *are closed and are taken in the positive direction in the complex plane, each enclosing the support of* $F^{c,H}$.

(3) *If* $X_{11}$ *is complex with* $\mathsf{E}(X_{11}^2) = 0$ *and* $\mathsf{E}(|X_{11}|^4) = 2$, *then* (2) *also holds, except the means are zero and the covariance function is 1/2 the function given in (7.3)* .

(4) *If the assumptions in (2) or (3) were to hold, then* $G_n$, *considered as a random element in* $D[0, \infty)$ *(the space of functions on* $[0, \infty)$ *right-continuous with left-hand limits, together with the Skorohod metric) cannot form a tight sequence in* $D[0, \infty)$.

The proof relies on the identity

$$\int f(x)dG(x) = -\frac{1}{2\pi i}\int f(z)m_G(z)dz$$

(f analytic on the support of $G$, contour positively oriented around the support), and establishes the following results on

$$M_n(z) = n[m_{F^{B_n}}(z) - m_{F^{c_n,H_n}}(z)].$$

a) $\{M_n(z)\}$ forms a tight sequence for $z$ in a sufficiently large contour about the origin.

b) If $X_{11}$ is complex with $\mathsf{E}(X_{11}^2) = 0$ and $\mathsf{E}(X_{11}^4) = 2$, then for $z_1, \dots, z_r$ with nonzero imaginary parts,

$$(ReM_n(z_1), ImM_n(z_1), \dots, ReM_n(z_r), ImM_n(z_r))$$

converges weakly to a mean zero Gaussian vector. It follows that $M_n$, viewed as a random element in the metric space of continuous $\mathbb{R}^2$-valued functions with domain restricted to a contour in the complex plane, converges weakly to a (2 dimensional) Gaussian process $M$. The limiting covariance function can be derived from the formula

$$\mathsf{E}(M(z_1)M(z_2)) = \frac{\underline{m}'(z_1)\underline{m}'(z_2)}{(\underline{m}(z_1) - \underline{m}(z_2))^2} - \frac{1}{(z_1 - z_2)^2}.$$

c) If $X_{11}$ is real and $\mathsf{E}(X_{11}^4) = 3$ then b) still holds, except the limiting mean can be derived from

$$\mathsf{E}M(z) = \frac{c\int \frac{\underline{m}^3 t^2 dH(t)}{(1+t\underline{m})^3}}{\left(1 - c\int \frac{\underline{m}^2 t^2 dH(t)}{(1+t\underline{m})^2}\right)^2}$$

and "covariance function" is twice that of the above function.

The difference between (2) and (3), and the difficulty in extending beyond these two cases, arise from

$$\begin{aligned}&\mathsf{E}(X_{\cdot 1}^* A X_{\cdot 1} - \operatorname{tr} A)(X_{\cdot 1}^* B X_{\cdot 1} - \operatorname{tr} B)\\ &\qquad = (\mathsf{E}(|X_{11}|^4) - |\mathsf{E}(X_{11}^2)|^2 - 2)\sum_i a_{ii}b_{ii} + |\mathsf{E}(X_{11}^2)|^2 \operatorname{tr} AB^T + \operatorname{tr} AB,\end{aligned}$$

valid for square matrices $A$ and $B$.

Can show

$$(7.2) = \frac{1}{2\pi}\int f'(x)\arg\left(1 - c\int \frac{t^2\underline{m}^2(x)}{(1+t\underline{m}(x))^2}dH(t)\right)dx$$

and

$$(7.3) = \frac{1}{\pi^2}\iint f'(x)g'(y)\ln\left|\frac{\underline{m}(x) - \overline{\underline{m}}(y)}{\underline{m}(x) - \underline{m}(y)}\right| dxdy$$

$$= \frac{1}{2\pi^2}\iint f'(x)g'(y)\ln\left(1 + 4\frac{\underline{m}_i(x)\underline{m}_i(y)}{|\underline{m}(x) - \underline{m}(y)|^2}\right)dxdy$$

where $\underline{m}_i = \Im \underline{m}$.

For case (2) with $H = I_{[1,\infty)}$ we have for $f(x) = \ln x$ and $c \in (0,1)$

$$\mathsf{E}X_{\ln} = \frac{1}{2}\ln(1-c) \qquad \text{and} \qquad \operatorname{Var} X_{\ln} = -2\ln(1-c).$$

Also, for $c > 0$

$$\mathsf{E}X_{x^r} = \frac{1}{4}((1-\sqrt{c})^{2r} + (1+\sqrt{c})^{2r}) - \frac{1}{2}\sum_{j=0}^{r}\binom{r}{j}^2 c^j$$

and

$$\operatorname{Cov}(X_{x^{r_1}}, X_{x^{r_2}}) = 2c^{r_1+r_2}\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2}\binom{r_1}{k_1}\binom{r_2}{k_2}\left(\frac{1-c}{c}\right)^{k_1+k_2} \times \sum_{\ell=1}^{r_1-k_1}\ell\binom{2r_1-1-(k_1+\ell)}{r_1-1}\binom{2r_2-1-k_2+\ell}{r_2-1}$$

(see Jonsson (1982)).