Discrete Mathematics MT20: Problem Sheet 2

Chapters 2 (Functions) and 3 (Counting)

2.1 For each of the following functions find $\mathrm{Im}(f)$, determine whether f is 1-1 and/or onto (give brief justification), and find the inverse function whenever it exists.

(i) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 3$.

(ii) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - 2x^2 + x$.

(iii) $f : \mathbb{R} \to [-1/2, 1/2]$, $f(x) = (\sin x)(\cos x)$.

(iv) $f : \mathbb{R} \to (-1, 1)$, $f(x) = x/(1 + |x|)$.

2.2 Let a be a fixed real number and define $f : (-a, a) \to \mathbb{R}$ by $f(x) = \tan x$.

(i) What is the largest value of a which makes f a well-defined function?

(ii) For that value of a, what is $\mathrm{Im}(f)$? Is f is a bijection?

(iii) For that value of a, what is the domain and codomain of $g = f^{-1}$?

(iv) By modifying g, give a bijection from $\mathbb{R}$ to $(-1, 1)$.

2.3 Consider the function $f : \mathbb{N}^2 \to \mathbb{Z}$ given by $f(x, y) = x^2 - 4y^2$. Show that f is neither 1-1 nor onto.

Hint: it might be helpful to factorise $x^2 - 4y^2$.

2.4

(i) Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2\}$. How many different functions are there from A to B? How many of them are 1-1? How many are onto?

(ii) If $|A| = m$ and $|B| = n$, how many functions are there from A to B? How many are 1-1? (Computing how many are onto is much more difficult, although it is possible to derive a recursive formula.)

2.5

(i) How many positive integers less than a thousand have their digits sum to 8?

(ii) How many 3-digit positive integers have only their first digit above 4, and all digits sum to 8?

(iii) How many positive integers less than a thousand have exactly one digit above 4, and all digits sum to 8?

(iv) How many positive integers less than a thousand have their digits sum to 14?

Hint for (iv): apply the technique used in part (i). Initially, this will allow "digits" greater than 9. Count how many of these inadmissible answers must be removed using the technique you used for (iii), and subtract.

2.6 How many positive integers with exactly 3 digits are divisible by one or more of 5, 6, or 8?

2.7 By counting something, prove that $(mn)!$ must be divisible by $(m!)^n$.