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IFAC PapersOnLine 56-2 (2023) 9185–9190

Adaptive control of LTV systems with uncertain periodic coefficients ⋆

D. Gerasimov * A. Popov * N.D. Hien * V. Nikiforov *

* ITMO University, Kronverkskiy av. 49, Saint Petersburg, 197101, Russia (corresponding author e-mail: dngerasimov@itmo.ru).

Abstract: The paper considers the control problem for a class of uncertain linear time-varying systems. The time-varying (TV) parameters are represented by multisinusoidal functions with a priori unknown amplitudes, phases, and frequencies of harmonics. The maximum numbers of the harmonics are known. To compensate the influence of the TV parameters on the closed-loop system, based on *internal model principle*, TV parameters observers are designed, and the plant model is represented in the parameterized form with constant unknown parameters. This form is used for design of adaptive backstepping controller with *modular identifiers* with improved parameters tuning.

Keywords: Adaptive control, systems with periodic coefficients, adaptive backstepping

1. INTRODUCTION

In this paper, we develop the control technique for a class of linear time-varying (LTV) continuous plants with uncertain multisinusoidal parameters and without disturbance.

As the literature survey shows, the most of the approaches of adaptive controllers design for systems with periodic coefficients assume that the period (frequency) of parameters variation is known (see Abidi (2014); Ahn (2004); Glower (1996); Narendra and Esfandiari (2018); Xu (2004); Yakubovich and Starzhinskiy (1975); Zhu and Sun (2020)). In Yu and Huang (2015), the period is assumed unknown, however its value can be factorized as product of some unknown integer and a known constant.

The contribution of the present paper is to relax the assumption about the knowledge of the period of parameters variations and propose a solution to the control problem for LTV plants with multisinusoidal parameters with unknown amplitudes, phases, and frequencies of harmonics.

To compensate the effect of multisinusoidal variations, the *Internal Model Principle* (IMP) (see Francis and Wonham (1975) or Chapter 1 in Nikiforov and Gerasimov (2022)) is applied. According to this principle, the TV parameters are modelled as the outputs of autonomous dynamic systems, refer to as *exosystems*, with constant coefficients. If the coefficients and the states of the exosystems are known, they can be invoked in the structure of the controller to ensure zero steady state control error. Since in the problem considered the TV plant parameters are uncertain, the exosystems states are not accessible for measurement, and the exosystems coefficients are unknown. To estimate the states, special observers are applied, then the plant is represented in a special parameterized form. This form is used for design a robust adaptive backstepping controller with *modular identifiers* and improved parameters tuning (see Gerasimov et al. (2020); Nikiforov et al. (2022)). To ensure the robustness properties, the standard modification of adaptation algorithm with leakage factor is proposed.

The remainder of the paper is organized as follows. In the second section, the problem of control is formulated. In the third section, the TV parameters of the plant are represented in the forms of linear regressions. In the fourth section, a study case example is considered and discussed. In the fifth section, the plant is parameterized. In the sixth section, the adaptive controller is designed using a backstepping procedure. In the seventh section, an adaptation algorithm is derived, in the eighth section, simulation results are demonstrated and discussed.

Notations: I_i is the $i \times i$ identity matrix; $O_{i\times j}$ and O_i are the $i \times j$ zero matrix and i-th dimensional vector, respectively; $\mathcal{L}_\infty$ is the space of bounded functions; $\|f(t)\|_\infty$, $\|W(s)\|_\infty$ is the infinity-norm of a function $f(t)$ and a transfer function $W(s)$, respectively; $|\cdot|$ is the Euclidean norm; $s = d/dt$ is the time derivative operator.

2. PROBLEM STATEMENT

Consider the plant

$$\begin{cases} \dot{x} = A(t)x + bu, \ x(0), \\ y = c^\top x, \end{cases} \tag{1}$$

where $x = [x_1, x_2, \ldots, x_n]^\top \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the control signal, $y \in \mathbb{R}$ is the output variable,

$$A(t) = \begin{bmatrix} \psi_1(t) & 1 & 0 & \cdots & 0 \\ 0 & \psi_2(t) & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \psi_n(t) \end{bmatrix}, b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

For the plant, we accept the following assumptions.

Assumption 1. The plant model (1) is such that:

⋆ This work was supported by the Ministry of Science and Higher Education of Russian Federation (Project Goszadanie, 2019-0898).

10.1016/j.ifacol.2023.10.160

1.1 the parameters $\psi_i(t)$ are representable in the form

$$\psi_i(t) = \sum_{j=1}^{r_i} a_{i,j} \sin(\omega_{i,j} t + \phi_{i,j}) \quad i = 1, 2, \ldots, n \quad (2)$$

with unknown $a_{i,j}$, $\phi_{i,j}$, and $\omega_{i,j}$. The maximum numbers of harmonics r_i are known;

1.2 the plant is controllable;

1.3 the state x is accessible for measurement. ■

Assumption 1.1, is necessary for complete compensation of uncertain TV parameters by involving the IMP. Assumptions 1.2 and 1.3 are typical for problems of control.

The model (1) can describe oscillatory systems (Blekhman and Sorokin (2010); Yakubovich and Starzhinskiy (1975)), resonance systems (Fossen and Nijmeijer (2012)), electric circuits (Richard (1983)), models in vibration mechanics (Camino and Santos (2019); Hartono and Van Der Burgh (2004); Rand (2012); Tehrani and Kalkowski (2015)), of the models designed via approximations of process with some periodic excitation (e.g. Wang et al. (2006)), etc.

The objective is to design a control law that ensures the boundedness of all closed-loop signals in the system and:

- if $|x_i(t)| \leq x_{0i}$, for at least one i from 1 to n, for all $t \geq t_0 \geq 0$, where $x_{0i} > 0$ are some predefined constants, then it provides the inequality
$$|\varepsilon(t)| \leq \varepsilon_0 \quad \forall t \geq T_0, \quad (3)$$
where $\varepsilon = y_m - y$ is the tracking error, $y_m(t)$ is a continuous bounded function with bounded and measurable high-order time derivatives (HOTD) $y_m^{(i)}$ $(i = 1, 2, \ldots, n)$, $\varepsilon_0 > 0$ is the maximum value of the steady-state error, $T_0 > 0$ is the settling time. It is assumed that ε_0 and T_0 can be tightened by the control law parameters;
- otherwise it provides the limiting equality
$$\lim_{t \to \infty} |\varepsilon(t)| = 0. \quad (4)$$

Remark 1. It is seen from the structure of the plant that when at least one of the states $x_i(t)$ approaches or crosses zero, the parameters in $A(t)$ and their variations are no longer to be "observable" and the corresponding TV parameters observers are no longer to provide precise information about the parameters variations. As a result, an observer based controller does not compensate completely the effect of these variations. ■

Using Assumption 1.1 and according to the IMP (see Francis and Wonham (1975) or Chapter 2 in Nikiforov and Gerasimov (2022)), the functions $\psi_i(t)$ $(i = 1, 2, \ldots, n)$ can be represented as the outputs of autonomous systems

$$\begin{cases} \dot{z}_i = \Gamma_i z_i, \\ \psi_i = h_i^\top z_i, \end{cases} \quad (5)$$

where $z_i \in \mathbb{R}^{m_i}$ are the inaccessible for measurement states with unknown initial conditions but known dimensions m_i (due to Assumption 1.1), $\Gamma_i \in \mathbb{R}^{m_i \times m_i}$, $h_i \in \mathbb{R}^{m_i}$ are constant unknown matrices and vectors, respectively.

3. PARAMETERIZATION OF THE FUNCTIONS $\psi(t)$

To cope with uncertain TV parameters, we represent the latter in the forms of linear regressions (see Lemma 2.2 in Nikiforov and Gerasimov (2022)).

Lemma 1. Under Assumption 1.1, the functions $\psi_i(t)$ can be represented in the forms of linear regressors

$$\psi_i = \theta_i^\top \xi_i, \quad (6)$$

in which $\theta_i \in \mathbb{R}^{m_i}$ are vectors of unknown constant parameters, $\xi_i \in \mathbb{R}_i^m$ are the states of the filters

$$\dot{\xi}_i = G_i \xi_i + l_i \psi_i \quad (7)$$

with certain initial conditions $\xi_i(0)$, arbitrary Hurwitz matrices $G_i \in \mathbb{R}^{m_i \times m_i}$, and vectors $l_i \in \mathbb{R}^{m_i}$ selected so that the pairs (G_i, l_i) are controllable. ■

Replacing (6) in (7), we obtain the autonomous models

$$\dot{\xi}_i = \overline{G}_i \xi_i, \quad (8)$$

where $\overline{G}_i = G_i + l_i \theta_i^\top$.

Remark 2. Note that since ψ_i are uncertain, the states ξ_i of the filters (7) (and, hence, the model (8)) are not measurable. To overcome this problem and design a linear regression model for ψ with a measurable regressor, we will propose a special TV parameters observer. ■

Now, for the sake of simplicity of paper presentation, we will start from an example of adaptive control for a scalar plant. Then extend the idea to the general case.

4. STUDY CASE

Let the plant be given by

$$\dot{y} = \psi(t) y + u, \quad (9)$$

where y is the scalar output, $\psi(t) = \sum_{j=1}^{r} a_j \sin(\omega_j t + \phi_j)$ is the unknown parameter with uncertain a_j, ω_j, and ϕ_j.

The problem is to design a control ensuring the control objective (3), (4) for the plant (9).

By Lemma 1, $\psi(t)$ is representable in the form

$$\psi = \theta^\top \xi, \quad (10)$$

$$\dot{\xi} = G\xi + l\psi, \ \xi(0) \quad (11)$$

with unmeasurable regressor ξ (see Remark 2), the vector of unknown constant parameters $\theta \in \mathbb{R}^m$. To recover the vector ξ, we propose the following nonlinear observer.

Lemma 2. The observer

$$\hat{\xi} = \zeta + N(y, \sigma), \quad (12)$$

$$\dot{\zeta} = G\zeta + GN(y, \sigma) - \frac{\partial N(y, \sigma)}{\partial y} u, \ \zeta(0), \quad (13)$$

where

$$N(y, \sigma) = \frac{1}{2} l \ln\left(y^2 + \sigma^2\right),$$

$\sigma(t)$ is the state of the filter

$$\dot{\sigma}(t) = -r\sigma(t) + \bar{\sigma}(t), \quad \sigma(0) > 0, \quad (14)$$

$$\bar{\sigma}(t) = \begin{cases} 0 & \text{if } |y(t)| \geq x_{01}, \\ \sigma_0 & \text{otherwise} \end{cases} \quad (15)$$

with some design parameters $x_{01}, r, \sigma_0 > 0$, generates the estimate $\hat{\xi}$ of ξ satisfying the equality

$$\epsilon(t) \triangleq \xi(t) - \hat{\xi}(t) = e^{Gt}\epsilon(0) \quad (16)$$
$$+ (sI_m - G)^{-1} l \left[\frac{\sigma^2(t)}{y^2(t) + \sigma^2(t)} \left(\psi(t) + r - \frac{\bar{\sigma}(t)}{\sigma(t)}\right)\right].$$

The norm $||\epsilon(t)||_\infty$ is bounded and can be arbitrarily reduced by reduction of σ_0 and/or x_{01}. ■

The lemma can be proved by evaluation of the time derivative $\dot{\epsilon} = \dot{\xi} - \dot{\hat{\xi}}$ in view of (11)–(15) and (9).

Therefore, in view of (10) and Lemma 2, the plant model (9) can be rewritten in the following parameterized form suitable for certainty equivalent controller design:

$$\dot{y} = \theta^\top \hat{\xi} y + u + \bar{\epsilon} y, \tag{17}$$

where $\bar{\epsilon} \triangleq \psi - \theta^\top \hat{\xi} = \theta^\top \epsilon$ is the signal considered as a bounded parametric disturbance.

Now, we calculate the time derivative of the tracking error $\dot{\varepsilon} = \dot{y}_m - \dot{y}$ and in view of (17) design the error equation

$$\dot{\varepsilon} = -c_0\varepsilon + \left(\dot{y}_m + c_0\varepsilon - \hat{\theta}^\top \hat{\xi} y - u\right) - \tilde{\theta}^\top \hat{\xi} y - \bar{\epsilon} y, \tag{18}$$

where $c_0 > 0$ is a constant, $\tilde{\theta} = \theta - \hat{\theta}$ is the vector of parametric errors.

Selecting the adjustable controller in the form

$$u = \left(c_0 + \mu_0 y^2\right)\varepsilon - \hat{\theta}^\top \hat{\xi} y + \dot{y}_m, \tag{19}$$

where $\mu_0 > 0$ is the damping term coefficient, and replacing it in (18), we obtain the tracking error model

$$\dot{\varepsilon} = -\left(c_0 + \mu_0 y^2\right)\varepsilon - \tilde{\theta}^\top \hat{\xi} y - \bar{\epsilon} y. \tag{20}$$

The term $\mu_0 y^2 \varepsilon$ in (19) is included to provide the *input-to-state* (ISS) stability of the closed-loop system (20) with the input $\tilde{\theta}$ affected by the *disturbance* $\bar{\epsilon} y$ (see more general result of Lemma 5).

The model (20) can be used for design of the robust adaptation algorithm (see Ioannou and Sun (1996))

$$\dot{\hat{\theta}} = -\gamma \hat{\xi} y \varepsilon - \gamma\gamma_0 \sigma_\theta(\hat{\theta})\hat{\theta}, \tag{21}$$

where $\gamma, \gamma_0 > 0$ are the gains,

$$\sigma_\theta(\hat{\theta}) = \begin{cases} 0 & \text{if } |\hat{\theta}| < \theta^*, \\ \dfrac{|\hat{\theta}|}{\theta^*} - 1 & \text{if } \theta^* \le |\hat{\theta}| \le 2\theta^*, \\ 1 & \text{if } |\hat{\theta}| > 2\theta^*, \end{cases} \tag{22}$$

is parametric feedback gain with a preselected large enough constant $\theta^* > 0$. The algorithm (21) referred to as *σ-modification* is applied to avoid unbounded parameteric drift that may be caused by the *disturbance* $\bar{\epsilon} y$ in (20). At the same time, the switching mechanism in (22) enables us to zero the tracking error ε if this disturbance approaches zero (the case $|y(t)| > x_{01}$).

Now, we are in position to formulate the following statement.

Proposition 1. Under Assumption 1.1, the control law (19) together with the observer (12)–(15) and the adaptation algorithm (21), (22) ensures the boundedness of all closed-loop signals and the objective (3), (4). The maximum steady-state error ε_0 can be decreased by decreasing of γ_0 and σ_0 and/or increasing of μ_0 and c_0. □□□

Proof. The boundness property and the objective (3) can be proved using the Lyapunov function $V = \frac{1}{2}\varepsilon^2 + \frac{1}{2\gamma}|\tilde{\theta}|^2$ and its time derivative calculated in view of (20), (21), and the identity $\dot{\hat{\theta}} = -\dot{\tilde{\theta}}$:

$$\begin{aligned}\dot{V} &= -c_0\varepsilon^2 - \mu_0 y^2\varepsilon^2 - \bar{\epsilon} y \varepsilon + \gamma_0\sigma_\theta\tilde{\theta}^\top\theta - \gamma_0\sigma_\theta\tilde{\theta}^\top\tilde{\theta} \\ &= -\frac{c_0}{2}\varepsilon^2 - \frac{\gamma_0\sigma_\theta}{2}|\tilde{\theta}|^2 + \frac{\gamma_0\sigma_\theta}{2}|\theta|^2 + \frac{\bar{\epsilon}^2}{4\mu_0} - \frac{\gamma_0\sigma_\theta}{2}|\hat{\theta}|^2 \\ &\quad - \left(\sqrt{\mu_0} y\varepsilon - \frac{\bar{\epsilon}}{2\sqrt{\mu_0}}\right)^2 \\ &\le -\frac{c_0}{2}\varepsilon^2 - \frac{\gamma_0\sigma_\theta}{2}|\tilde{\theta}|^2 + \frac{\gamma_0\sigma_\theta}{2}|\theta|^2 + \frac{\bar{\epsilon}^2}{4\mu_0}.\end{aligned}$$

In view of definition of $\bar{\epsilon}$, positive property of $\sigma(t)$, and the properties of norms, we have

$$\bar{\epsilon}^2(t) \le \left(\theta^\top e^{Gt}\epsilon(0) + \|W(s)\|_\infty \|\psi(t) + r\|_\infty\right)^2,$$

where $W(s) = \theta^\top (sI_m - G)^{-1} l$. Hence,

$$\dot{V} \le -\frac{c_0}{2}\varepsilon^2 - \frac{\gamma_0\sigma_\theta}{2}|\tilde{\theta}|^2 + \Delta \le -\kappa_0 V + \Delta, \tag{23}$$

where $\kappa_0 = \min\{c_0, \gamma\gamma_0\sigma_\theta\}$, $|\sigma_\theta| \le 1$,

$$\Delta(t) = \frac{\gamma_0}{2}|\theta|^2 + \frac{1}{4\mu_0}\left(\theta^\top e^{Gt}\epsilon(0) + \|W(s)\|_\infty \|\psi(t) + r\|_\infty\right)^2.$$

It is seen from the latter that $\Delta(t) \in \mathcal{L}_\infty$. Therefore, it follows from (23) that $\varepsilon, \tilde{\theta} \in \mathcal{L}_\infty$. Since $\varepsilon, \hat{\xi}, y_m, \dot{y}_m \in \mathcal{L}_\infty$, then $\dot{\hat{\theta}}, u \in \mathcal{L}_\infty$. This completes the proof of the closed-loop signals boundness. The objective (3) is followed from the analysis of the term Δ.

Now, prove the objective (4) (the case $|y(t)| > x_{01}$). For this case, $\sigma(t) \to 0$ as $t \to \infty$ due to (14), (15) and, as a result, $\epsilon(t), \bar{\epsilon}(t) = \theta\epsilon(t) \to 0$ exponentially fast according to (16). Hence, the disturbance $\bar{\epsilon} y$ approaches zero exponentially fast, and due to the properties of the robust adaptation algorithm in absence of a disturbance (Ioannou and Sun, 1996, Section 8.4.1), $\sigma_\theta(t) \equiv 0$. Therefore, $\Delta(t) \to 0$ exponentially fast, and it follows from (23) that $\varepsilon(t) \to 0$ as $t \to \infty$. ■

Now, extend this particular solution for the plant of general order (1).

5. PLANT PARAMETERIZATION

To represent the plant (1) in the form suitable for design of a certainty equivalent controller, we extend Lemma 2 and design the TV parameters observers.

Lemma 3. The observers

$$\hat{\xi}_i = \zeta_i + N_i(x_i, \sigma_i), \tag{24}$$

$$\dot{\zeta}_i = G_i\zeta_i + G_i N(x_i, \sigma_i) - \frac{\partial N_i}{\partial x_i} x_{i+1}, \ \zeta_i(0), \tag{25}$$

$$\hat{\xi}_n = \zeta_n + N_n(x_n, \sigma_n), \tag{26}$$

$$\dot{\zeta}_n = G_n\zeta_n + G_n N(x_n, \sigma_n) - \frac{\partial N_n(x_n, \sigma_n)}{\partial x_n} u, \ \zeta_n(0), \tag{27}$$

where $i = 1, 2, \ldots, n-1$,

$$N_i(x_i, \sigma_i) = \frac{1}{2} l_i \ln\left(x_i^2 + \sigma_i^2\right), \tag{28}$$

$\sigma_i(t)$ are the states of the filters

$$\dot{\sigma}_i(t) = -r_i\sigma_i(t) + \bar{\sigma}_i(t), \quad \sigma_i(0) > 0, \tag{29}$$

$$\bar{\sigma}_i(t) = \begin{cases} 0 & \text{if } |x_i(t)| \ge x_{0i}, \\ \sigma_{0i} & \text{otherwise}, \end{cases} \tag{30}$$

$x_{0i}, \sigma_{0i} > 0$ are some constant parameters, generate the estimates $\hat{\xi}_i$ of ξ_i satisfying the equality

$$\epsilon_i(t) \triangleq \xi_i(t) - \hat{\xi}_i(t) = e^{G_i t}\epsilon_i(0) + \left(sI_{m_i} - G_i\right)^{-1} l_i \left[\frac{\sigma_i^2(t)}{x_i^2(t) + \sigma_i^2(t)}\left(\psi_i(t) + r_i - \frac{\bar{\sigma}_i(t)}{\sigma_i(t)}\right)\right].$$

The norms $||\epsilon_i(t)||_\infty$ are bounded and can be reduced by reduction of the corresponding gains σ_{0i} and/or x_{0i}. ■

The lemma is proved calculating the time derivatives $\dot{\epsilon}_i = \dot{\xi}_i - \dot{\hat{\xi}}_i$ in view of (7), (24)-(30), and (1).

For controller design, it will be needed to implement the HOTD of $\hat{\xi}_i$. To this end, we formulate the following corollary of Lemma 3 (see Appendix A in Nikiforov and Gerasimov (2022) or Gerasimov et al. (2020)).

Corollary 1. The estimates $\hat{\xi}_i$ $(i = 1, 2, \ldots, n)$ can be represented in the form

$$\hat{\xi}_i = T_i \hat{\xi}_{fi} + \epsilon_{Ti}, \tag{31}$$

where $\epsilon_{Ti} = -T_i \frac{\kappa_0}{\kappa(s)}[\epsilon_i]$,

$$\hat{\xi}_{fi} = \frac{\kappa_0}{\kappa(s)}[\hat{\xi}_i] = \frac{\kappa_0}{s^{n-1} + \kappa_{n-2}s^{n-1} + \ldots + \kappa_0}[\hat{\xi}_i], \tag{32}$$

κ_j $(j = 0, 1, \ldots, n-2)$ are the coefficients chosen so that the polynomial $\kappa(s)$ is Hurwitz, $T_i \in \mathbb{R}^{m_i \times m_i}$ are matrices given by

$$\kappa_0 T_i = \overline{G}_i^n + \kappa_{n-1}\overline{G}_i^{n-1} + \kappa_{n-2}\overline{G}_i^{n-2} + \ldots + \kappa_0 I_{m_i},$$

$\overline{G}_i$ are the matrices defined in (8). ■

Applying the result of Lemma 3 and Corollary 1 and introducing new notations

$$\vartheta = [\theta_1^\top T_1, \theta_2^\top T_2, \ldots, \theta_n^\top T_n]^\top,$$

$$\varpi_i = [O_{m_1}^\top, \ldots, O_{m_{i-1}}^\top, \hat{\xi}_{fi}^\top, O_{m_{i+1}}^\top, \ldots, O_{m_n}^\top]^\top \in \mathbb{R}^{\sum_{i=1}^n m_i}$$

we represent the TV plant parameters in the form

$$\psi_i = \vartheta^\top \varpi_i + \bar{\epsilon}_i$$

with $\bar{\epsilon}_i = \theta_i^\top(\epsilon_{Ti} + \epsilon_i)$.

Hence, the plant (1) can be represented in the form

$$\begin{cases} \dot{x}_i = \vartheta^\top \varpi_i x_i + \bar{\epsilon}_i x_i + x_{i+1},\ i = 1, 2, \ldots, n-1, \\ \dot{x}_n = \vartheta^\top \varpi_n x_n + \bar{\epsilon}_n x_n + u, \\ \ y = x_1, \end{cases} \tag{33}$$

It is seen from the form (33) that the uncertanty ϑ is not matched with the control. This fact motivates application of adaptive backstepping for controller design.

6. ADAPTIVE CONTROLLER

For the parameterization (33), we use a slight modification of the algorithm of adaptive backstepping compensation with *modular identifiers* recently proposed in Gerasimov et al. (2020); Nikiforov et al. (2022):

$$\alpha_1 = (c_1 + s_1)\, z_1 - x_1 \varpi_1^\top \hat{\vartheta} + \dot{y}_m, \tag{34}$$

$$\begin{aligned} \alpha_i = {} & z_{i-1} + (c_i + s_i)\, z_i + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} x_{j+1} \\ & + \left(\sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} x_j \varpi_j^\top - x_i \varpi_i^\top \right) \hat{\vartheta} + \sum_{j=0}^{i-2} \frac{\partial \alpha_{i-1}}{\partial \hat{\vartheta}^{(j)}} \hat{\vartheta}^{(j+1)} \\ & + \sum_{k=1}^{i-1} \sum_{j=0}^{i-1-k} \frac{\partial \alpha_{i-1}}{\partial \varpi_k^{(j)}} \varpi_k^{(j+1)} + \sum_{j=0}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_m^{(j)}} y_m^{(j+1)}, \end{aligned} \tag{35}$$

where $i = 2, 3, \ldots, n$; $c_i, \mu_1, \mu_i > 0$ are constants; s_i are the damping terms given by

$$s_1 = \mu_1 x_1^2,\ s_i = \mu_i \sum_{j=1}^{i-1} \left| \frac{\partial \alpha_{i-1}}{\partial x_j} x_j \right|^2 + \mu_i x_i^2; \tag{36}$$

$z_1 = y_m - y = \varepsilon$, $z_i = \alpha_{i-1} - x_i$ are the control errors.

The procedure is finished by design of the actual control

$$u = \alpha_n. \tag{37}$$

Lemma 4. The error model for the closed-loop system consisting of the plant (1), admitting the form (33), and the control law (34)–(37) can be represented as

$$\dot{z} = A_z z + B_z \tilde{\vartheta} + D_z \epsilon_z, \tag{38}$$

where $\tilde{\vartheta} = \vartheta - \hat{\vartheta}$ is the $\sum_{i=1}^n m_i$-dimensional vector of parametric errors, $\epsilon_z = [\bar{\epsilon}_1, \bar{\epsilon}_2, \ldots, \bar{\epsilon}_n]^\top$,

$$A_z = \begin{bmatrix} -c_1 - s_1 & 1 & \cdots & 0 & 0 \\ -1 & -c_1 - s_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & -1 & -c_n - s_n \end{bmatrix},$$

$$D_z = \begin{bmatrix} -x_1 & 0 & \cdots & 0 & 0 \\ \frac{\partial \alpha_1}{\partial x_1} x_1 & -x_2 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \frac{\partial \alpha_{n-1}}{\partial x_1} x_1 & \frac{\partial \alpha_{n-1}}{\partial x_2} x_2 & \cdots & \frac{\partial \alpha_{n-1}}{\partial x_{n-1}} x_{n-1} & -x_n \end{bmatrix},$$

$$B_z = D_z W_z, \quad W_z = [\varpi_1 \vdots \varpi_2 \vdots \cdots \vdots \varpi_n]^\top.$$ ■

The lemma can be proved by evaluation of the time derivatives $\dot{z}_1 = \dot{y}_m - \dot{y}$, $\dot{z}_i = \dot{\alpha}_{i-1} - \dot{x}_i$ in view of (33)–(37).

As shown in Lemma 5.8 in Krstić et al. (1995), the damping terms (36) are involved in the control law to ensure the ISS stability of the model (38). Using this lemma, we formulate the following statement.

Lemma 5. If $\hat{\vartheta}(t) \in \mathcal{L}_\infty$, then $z(t) \in \mathcal{L}_\infty$ for any $t \geq 0$. ■

Proof. Selecting the Lyapunov function $V = \frac{1}{2} z^\top z$, we calculate its time derivative in view of (38):

$$\begin{aligned} \dot{V} &= \frac{1}{2} z^\top \left(A_z^\top + A_z\right) z + z^\top B_z \tilde{\vartheta} + z^\top D_z \epsilon_z \\ &= -\sum_{i=1}^n (c_i + s_i)\, z_i^2 + z^\top D_z \left(W_z \tilde{\vartheta} + \epsilon_z\right) \leq -c_0 |z|^2 \\ &\quad - \sum_{i=1}^n \left(\sqrt{\mu_i |D_{zi}^\top|} |z| - \frac{1}{2\sqrt{\mu_i}} \left| W_z \tilde{\vartheta} + \epsilon_z \right| \right)^2 \\ &\quad + \frac{\mu_0}{4} \left(W_z \tilde{\vartheta} + \epsilon_z\right)^2 \\ &\leq -\frac{c_0}{2} V + \frac{\mu_0}{4} ||W_z||_\infty^2 ||\tilde{\vartheta}||_\infty^2 + \frac{\mu_0}{4} ||\epsilon_z||_\infty^2, \end{aligned}$$

where $c_0 = \min_i \{c_i\}$, $D_{zi}^\top$ is the i-th row of D_z, $\mu_0 = \sum_{i=1}^n \frac{1}{\mu_i}$.

It follows from the latter inequality that

$$|z(t)| \leq \frac{\mu_0}{2c_0} \left(||W_z||_\infty^2 ||\tilde{\vartheta}||_\infty^2 + ||\epsilon_z||_\infty^2 \right). \tag{39}$$

Since $\varpi_i, \epsilon_i \in \mathcal{L}_\infty$ (due to the result of Lemma 3), then $||W_z||_\infty, ||\epsilon_z||_\infty \in \mathcal{L}_\infty$ are bounded. Therefore, taking into account (39), we complete the proof. ■

7. ADAPTATION ALGORITHM

In order to derive an adaptation algorithm that can guarantee the conditions of Lemma 5 and the objective (3), (4), we need to represent the model (38) in a static form. To this end, we formulate the swapping lemma.

Lemma 6. Let us introduce the filters

$$\dot{\zeta}_z = A_z\zeta_z + B_z\dot{\vartheta}, \tag{40}$$

$$\dot{Z} = A_z Z + B_z \tag{41}$$

and the *augmented error*

$$\bar{z} = z + \zeta_z. \tag{42}$$

Then the vector $\bar{z}$ can be represented as the output of the static error model

$$\bar{z} = Z\vartheta + \bar{\epsilon}_z, \tag{43}$$

where $\bar{\epsilon}_z$ is the vector function satisfying the equation

$$\dot{\bar{\epsilon}}_z = A_z\bar{\epsilon}_z + D_z\epsilon_z, \quad \bar{\epsilon}_z(0) = z(0) + \zeta_z(0) - Z(0)\vartheta. \quad ■$$

The lemma is proved by evaluation of the time derivative $\dot{\bar{\epsilon}}_z = \dot{z} + \dot{\zeta}_z - \dot{Z}\vartheta$ in view of (38), (40), and (41).

Remark 3. Since the vectors ϖ_i are bounded by definition, due to the structures of the matrices A_z (with the damping terms s_i) and B_z the matrix Z is bounded irrespectively of the boundedness of B_z. Therefore, the normalization of an adaptation algorithm based on (43) is unnecessary. ■

Corollary 2. If $|x_i(t)| > x_{0i}$ for some $t \geq t_0 \geq 0$, then, starting from t_0, $\bar{\epsilon}_z(t) \to 0$ exponentially fast as $t \to \infty$. If $|x_i(t)| \leq x_{0i}$, then $\bar{\epsilon}_z(t)$ is bounded for any $t \geq 0$. ■

The corollary is followed from the result of Lemmas 6, 4, definition of $\bar{\epsilon}_i$ (see (33)), Corollary 1, and Lemma 3.

Now, using the representation (43), we design the robust adaptation algorithm with improved parametric convergence generating $\hat{\vartheta}, \dot{\hat{\vartheta}}, \ldots, \hat{\vartheta}^{(n-1)}$. To this end, we introduce a *positive* transfer function operator

$$L(s) = \frac{1}{d(s)} = \prod_{j=1}^{q} \frac{1}{s + d_j}$$

with $q \geq n-2$ and constant coefficients $d_j > 0$ and get the extended regression [1]

$$\bar{z}_L = Z_L\vartheta + \epsilon_L, \tag{44}$$

where $\bar{z}_L = L(s)[Z^\top \bar{z}]$ is the memory extended output, $Z_L = L(s)[Z^\top Z]$ is the memory extended regressor positive semidefinite by definition, $\epsilon_L = L(s)[Z^\top \bar{\epsilon}_z]$ is the bounded disturbance. Using the extended regression (44), we design the robust adaptation algorithm (see Section 7.3.3 in Nikiforov and Gerasimov (2022))

$$\dot{\hat{\vartheta}} = \gamma\left(\bar{z}_L - Z_L\hat{\vartheta}\right) - \gamma\gamma_0\bar{\sigma}(t)\hat{\vartheta}, \tag{45}$$

where $\gamma, \gamma_0 > 0$ are constants,

$$\bar{\sigma}(t) = L(s)[\sigma_\theta(\hat{\vartheta}(t))], \tag{46}$$

$$\sigma_\theta(\hat{\vartheta}) = \begin{cases} 0 & \text{if } |\hat{\vartheta}| < \vartheta^*, \\ \dfrac{|\hat{\vartheta}|}{\vartheta^*} - 1 & \text{if } \vartheta^* \leq |\hat{\vartheta}| \leq 2\vartheta^*, \\ 1 & \text{if } |\hat{\vartheta}| > 2\theta^*, \end{cases} \tag{47}$$

$\vartheta^* > 0$ is a large enough constant.

The HOTD $\dot{\hat{\vartheta}}, \ldots, \hat{\vartheta}^{(n-1)}$ for (35) are calculated as follows:

$$\hat{\vartheta}^{(j)} = \gamma\left(\bar{z}_L^{(j-1)} - \left(Z_L\hat{\vartheta} + \gamma_0\bar{\sigma}(t)\hat{\vartheta}\right)^{(j-1)}\right), \tag{48}$$

where $j = 1, 2, \ldots, n$, and the derivatives are calculated as

$$\bar{z}_L^{(k)} = \frac{s^k}{d(s)}[Z^\top \bar{z}], Z_L^{(k)} = \frac{s^k}{d(s)}[Z^\top Z], \bar{\sigma}^{(k)} = \frac{s^k}{d(s)}[\sigma_\theta(\hat{\vartheta})].$$

Now, formulate the main result of the paper.

Proposition 2. Under Assumption 1, the adjustable controller (34)–(37) together with the TV parameters observers (24)–(30), filters (32), swapping filters (40), (41), augmented error (42), and the adaptation algorithm (45)–(48) when applied to the plant (1) the objective (3), (4) with all the closed-loop signals bounded. The maximum steady-state error ε_0 can be decreased by decreasing of γ_0 and σ_0 and/or increasing of μ_i, and c_i. □□□

Sketch of proof. Choose the Lyapunov function $V = \frac{1}{2\gamma}\tilde{\vartheta}^\top\tilde{\vartheta}$ and evaluate its time derivative in view of (45) (for detail, see the proof of Proposition 3.4 in Nikiforov and Gerasimov (2022) extended to the adaptation algorithm with leakage factor $\bar{\sigma}$), we get

$$\dot{V} \leq -\tilde{\vartheta}^\top Z_L\tilde{\vartheta} - \frac{\gamma_0\bar{\sigma}}{2}|\tilde{\vartheta}|^2 + |\bar{\epsilon}_z|^2 + \frac{\gamma_0\bar{\sigma}}{2}|\vartheta|^2. \tag{49}$$

It follows from the latter that since $\bar{\epsilon}_z \in \mathcal{L}_\infty$ (see Lemma 3), then $\tilde{\vartheta}(t) \in \mathcal{L}_\infty$. Therefore, following the results of Lemma 5, we prove that $z(t) \in \mathcal{L}_\infty$.

If σ_0 is decreased, the maximum steady-state value of $\bar{\epsilon}_z$ is decreased. By decreasing γ_0 the term $\frac{\gamma_0\bar{\sigma}}{2}|\vartheta|^2$ is decreased. Hence, analyzing (49) and (39) it can be shown that the norm $|z|$ can be decreased by decreasing γ_0 and σ_0 and/or increasing of μ_i and c_i. Following the result of Corollary 2 and the properties of the robust adaptation algorithm (45) it can be shown that if $|x_i(t)| > x_{0i}$ for all $t \geq t_0 \geq 0$, the limiting equality (4) holds. ■

Let us make the concluding remarks regarding the tuning of controller design parameters.

- D1 if $|x_i(t)| \leq x_{0i}$ for all $t \geq t_0 \geq 0$ and for at least one of i $(i = 1, 2, \ldots, n)$, only a nonzero steady-state control error can be guaranteed. This error can be reduced by reduction of γ_0 and σ_0. or by increase of μ_i and c_i. If $|x_i(t)| > x_{0i}$ for all $t \geq t_0 \geq 0$ and i, the solution proposed ensures the asymptotic zeroing of the control error;
- D2 to maximize the efficiency of adaptation process it is recommended to increase the threshold ϑ^*;
- D3 as it shown in Chapter 3 of Nikiforov and Gerasimov (2022), if the minimum eigenvalue of $Z_L(t)$ is not integrally bounded, the rate of parametric convergence of the adaptation algorithm (45) can be increased by increasing the gain γ.

[1] In Ortega et al. (2020), this procedure is referred as to *memory regressor extension* due to the memory effect of the filter $L(s)$ enabling us to collect past weighted values of the regressor Z and design an adaptation algorithm with accelerated parameters tuning.

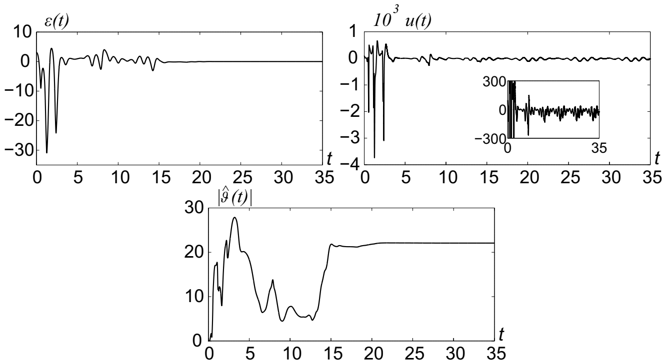

Fig. 1. Transients in the system closed by the robust adaptive backstepping controller

8. SIMULATION

Consider the second order unstable plant of the form (1) with zero initial conditions and TV parameters $\psi_1(t) = -2\cos(6t+1)+4$, $\psi_2(t) = \sin 3t - 2$ with a priori unknown amplitudes, phases, frequencies and the biases.

Problem is to design a control that will drive the plant output to the reference $y_m(t) = 2\sin t + 3$. To design the observer, according to the type of TV parameters we select the orders $m_1 = m_2 = 3$, matrices

$$G_1 = G_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad l_1 = l_2 = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix},$$

thresholds $x_{01} = x_{02} = 0.05$, the parameters $r_1 = r_2 = 20$, the initial conditions of the filters (29) $\sigma_1(0) = \sigma_2(0) = 1$, and the switching parameters $\sigma_{01} = \sigma_{02} = 0.05$.

For the backstepping procedure implemented in two steps, we choose $c_1 = c_2 = 5$, $\mu_1 = \mu_2 = 0.001$. The parameter of the filters (32) is given by $\kappa_0 = 2$.

For design of adaptation algorithm, we use the first order filter $L(s) = \frac{1}{s+d_0}$, zero initial conditions and the following set of parameters: $d_0 = 1$, $\gamma = 50$, $\gamma_0 = 1$, $\vartheta^* = 1000$.

Initial conditions for the observer equations (24)–(27), filters (32), (40), and (41) are set to zero.

It is shown by the simulation results presented in Fig.1 that the tracking error approaches zero, signal u is bounded, while the parameters $\hat{\vartheta}$ are tuned to constants. Due to the lack of space, we do not show the plots of $x_1(t)$, $x_2(t)$, in which we observe several zero-crossings that do not influence the closed-loop system stability.

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