2024

Application Task

SOFTINWAY

DUC PHI NGO

Contents

1D fluid flow modeling

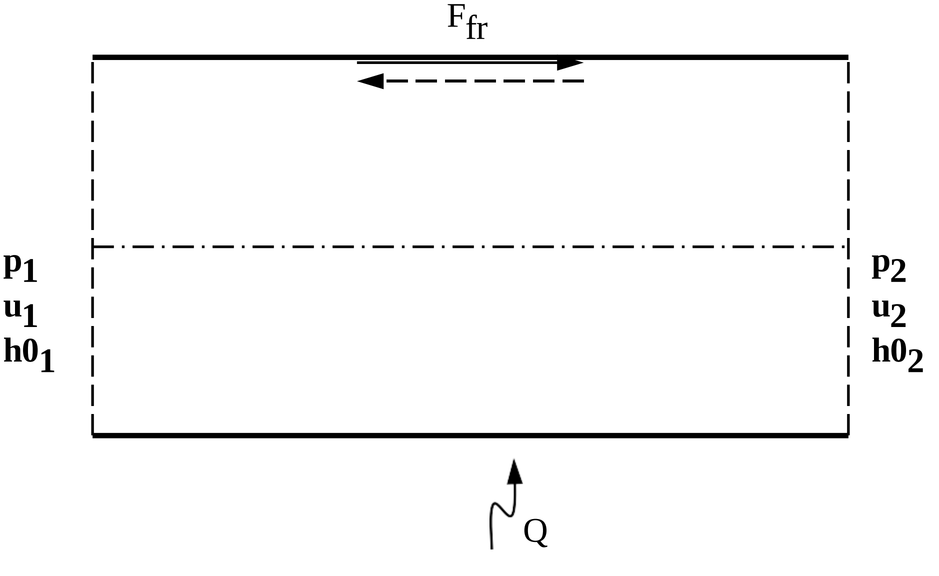
Consider a solver that solves a system of non-linear equations.

1) Write a system of 1D equations to model the transient fluid flow of an ideal gas through a channel without discretization in space. The variables for this system of equations should be:

- p_1 - static pressure at the 1st cross-section
- u_1 - velocity at the 1st cross-section
- $h0_1$ - total enthalpy at the 1st cross-section
- p_2 - static pressure at the 2nd cross-section
- u_2 - velocity at the 2nd cross-section
- $h0_2$ - total enthalpy at the 2nd cross-section

Assume that the channel has a constant cross-sectional area A. Take into account the constant frictional force F_{fr} that acts in the opposite of the fluid flow direction. Also, take into account the heat flow Q applied to the fluid flow in the channel.

2) According to the proposed system of equations, which variables must be defined and which ones could be calculated by the solver if we want to model the fluid flow just through this channel?



Notation

F_{fr}	Frictional force	$\frac{kg\,m}{s^2}$
$\dot{F}_{fr}$	Frictional force per unit volume	$\frac{kg}{s^2 m^2}$
Q	Heat flow	W
$\dot{q}$	Heat flow per unit volume	$\frac{W}{m^3}$
u_e, u_2	Velocity at the east face	$\frac{m}{s}$
u_w, u_1	Velocity at the west face	$\frac{m}{s}$
u_P	Velocity at the center control volume at point P	$\frac{m}{s}$
h_{0e}, h_{02}	Total enthalpy at the east face per unit mass	J/kg
h_{0w}, h_{01}	Total enthalpy at the west face per unit mass	J/kg
h_{0P}	Total enthalpy at the central of control volume	J/kg
p_e, p_2	Pressure at the east face	Pa
p_w, p_1	Pressure at the west face	Pa
p	Old pressure at center of cell	Pa
T	Temperature at center of control volume	K
ρ_P^0	Old density at center volume at point P at time t	$\frac{kg}{m^3}$
ρ_P	New density at time $t + \Delta t$	$\frac{kg}{m^3}$
T_P^0	Old temperature at center volume at time t	K
T_P	New temperature at center volume at time $t + \Delta t$	K
T^0	Old temperature at time t	K
e	Values at the east face	-
w	Values at the west face	-
ΔV	Volume	m^3
k	Thermal Conductivity	W/m.K

1. Governing equation

1.1 Mass conservation

The product of density, area, and the velocity component normal to the face yields the mass flow rate across the element's face.

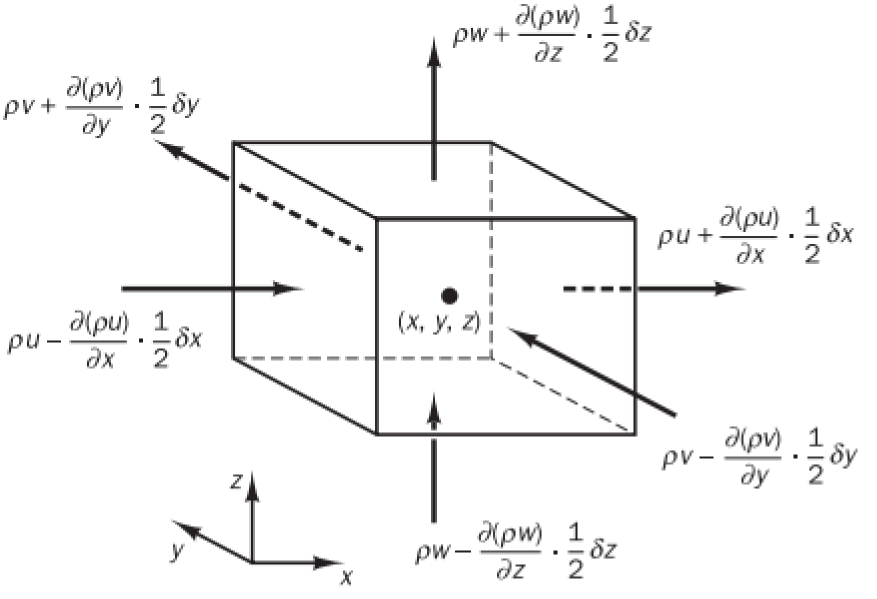


Figure 1: Entry and exit mass flow of a fluid element [1]

In three dimensions, according to H. Versteeg et al. the continuity equation is:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho u)}{\partial y} + \frac{\partial(\rho u)}{\partial z} = 0$$

Therefore, in 1 dimension, the continuity equation is:

The continuity equation
$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \qquad (1)$$

This is the unsteady, one-dimensional continuity or mass conservation equation for a compressible fluid at a specific point in time. The density's (mass per unit volume) rate of change over time is the first term on the left. The second term, known as the convective term, describes the net mass flow out of the element across its limits.

1.2 Momentum equation

Rate of increase of ϕ of fluid element + Net rate of flow ϕ out of fluid element = Rate of increase of ϕ for a fluid particle.

For 1 dimension, the rates of increase of x-momentum per unit volume of a fluid particle are given by $\rho\frac{Du}{Dt}$ [1]:

$$\rho\frac{Du}{Dt} \qquad \frac{\partial(\rho u)}{\partial t}+\frac{\partial u^2}{\partial x} \qquad (2)$$

According to Newton's second law, a fluid particle's rate of change of momentum is equal to the sum of the forces acting on it.

Rate of increase of momentum of fluid particle = Sum of forces on fluid-particle [1]**.**

We distinguish two types of forces on fluid particles including surface forces such as pressure forces, viscous force, gravity forces, and body forces like centrifugal force, coriolis force, and electromagnetic force.

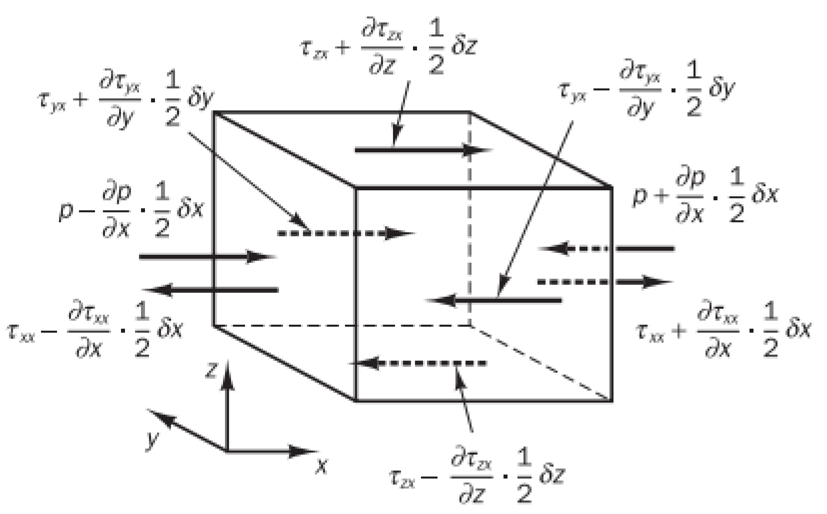


Figure 2: Component of stress in direction x [1]

The rate of change of the fluid particle's momentum is equal to the total force acting on the element in the x-direction as a result of surface stresses plus the rate at which momentum is increasing as a result of sources, and this yields the x-component of the momentum equation [1].

S_{M_x}: internal moment source

$$\rho\frac{Du}{Dt}=\frac{\partial(-p+\tau_{xx})}{\partial x}+S_{M_x}$$

In this case, gravitation is equal to 0, and there is only a frictional force on the wall, therefore:

$$S_{M_x} = \dot{F}_{fr}$$

Hence the equation becomes:

$$\rho\frac{Du}{Dt} = \frac{\partial(-p + \tau_{xx})}{\partial x}$$

The viscous component can be expressed as:

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} + \lambda\frac{\partial u}{\partial x}, with\ \lambda = \frac{2}{3}\mu$$

then:

$$\rho\frac{Du}{Dt} = \frac{\partial(-p + \tau_{xx})}{\partial x} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x}$$

Substituting τ_{xx} to the previous equation:

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x} + \lambda\frac{\partial u}{\partial x}\right]$$

Finally we have the momentum equation:

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial u^2}{\partial x} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} \qquad (3)$$

Or

momentum equation

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial u^2}{\partial x} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x} + \lambda\frac{\partial u}{\partial x}\right] \qquad (4)$$

1.3 Energy equation

Rate of increase of energy of fluid particle = Net rate of heat added to fluid particle + net rate of work done on fluid-particle [1].

The rate of work done on the fluid particle in the element by a surface force **is equal to the product of the force and velocity component in the direction of the force.**

$$\rho\frac{DE}{Dt}=\frac{\partial(pu)}{\partial x}+\frac{\partial(u\tau_{xx})}{\partial x}+\frac{\partial}{\partial x}(k\frac{\partial T}{\partial x})+S_E$$

The sum of energy equation of a fluid is the sum of internal (thermal) energy + kinetic energy + gravitational energy. By equating the fluid particle's rate of change in energy to the total of its net rate of work done on it, its net rate of heat addition, and its rate of energy increase owing to the source, the fluid particle's energy is conserved.

$$\rho\frac{DE}{Dt}=\frac{\partial(pu)}{\partial x}+\frac{\partial(u\tau_{xx})}{\partial x}+\frac{\partial}{\partial x}(k\frac{\partial T}{\partial x})$$

$$q=k\frac{\partial T}{\partial x}$$

The energy equation is for 1 dimension:

$$\frac{\partial\rho h_0}{\partial t}+\frac{\partial\rho h_0 u}{\partial x}=\frac{\partial u\tau_{xx}}{\partial x}+\frac{\partial p}{\partial t}+\frac{\partial}{\partial x}(k\frac{\partial T}{\partial x}) \tag{5}$$

According to H.Versteeg et al. we can obtain the internal energy after rearrangement from equation (5):

$$\frac{\partial\rho i}{\partial t}+\frac{\partial\rho iu}{\partial x}=-p\frac{\partial u}{\partial x}+\frac{\partial}{\partial x}(k\frac{\partial T}{\partial x}) \tag{6}$$

For the ideal gas $i = C_v T$, (6) can be written as below:

$$\frac{\partial\rho C_v T}{\partial t}+\frac{\partial\rho C_v Tu}{\partial x}=-p\frac{\partial u}{\partial x}+\frac{\partial}{\partial x}(k\frac{\partial T}{\partial x}) \tag{7}$$

2. Governing Equation for 1 dimension for transient flow through a channel

In general, we have the governing equations for 1 dimension as below:

Continuity equation:

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial t}=0$$

Momentum equation:

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial u^2}{\partial x}=-\frac{\partial p}{\partial x}+\frac{\partial\tau_{xx}}{\partial x}$$

Energy equation :

$$\frac{\partial\rho h_0}{\partial t}+\frac{\partial\rho h_0 u}{\partial x}=\frac{\partial u\tau_{xx}}{\partial x}+\frac{\partial p}{\partial t}+\frac{\partial}{\partial x}(k\frac{\partial T}{\partial x})$$

Or we have energy equation based on temperature.

$$\frac{\partial\rho C_v T}{\partial t}+\frac{\partial\rho C_v T u}{\partial x}=\frac{\partial u\tau_{xx}}{\partial x}-p\frac{\partial u}{\partial x}+\frac{\partial}{\partial x}(k\frac{\partial T}{\partial x})$$

Because we do not consider space discretization:

- The moment create by shear force

$$\frac{\partial\tau_{xx}}{\partial x}=\dot{F}_{fr}$$

- The energy created by shear stress and heat flow is :

$$\frac{\partial u\tau_{xx}}{\partial x}=\dot{F}_{fr}u$$

$$\frac{\partial}{\partial x}(k\frac{\partial T}{\partial x})=\dot{q}$$

Continuity equation:	$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}=0$

The dimension of continuity equation is calculated as following:

$$\frac{\partial\rho}{\partial t}=\frac{kg}{m^3}\frac{1}{s}=\frac{kg}{m^3 s}$$

$$\frac{\partial(\rho u)}{\partial x}=\frac{kg}{m^3}\frac{m}{s}\frac{1}{m}=\frac{kg}{m^3 s}$$

So, the Continuity has a consistency dimension

Momentum equation:	$\frac{\partial(\rho u)}{\partial t}+\frac{\partial u^2}{\partial x}=-\frac{\partial p}{\partial x}+\dot{F}_{fr}$

The dimension of momentum equation is calculated as following:

$$\frac{\partial(\rho u)}{\partial t}=\frac{kg}{m^3}\frac{1}{s}\frac{m}{s}=\frac{kg}{m^2 s^2}$$

$$\dot{F}_{fr}=\frac{1}{m^3}\frac{kgm}{s^2}=\frac{kg}{m^2 s^2}$$

$$\frac{\partial\rho u^2}{\partial x}=\frac{kg}{m^3}\frac{m^2}{s^2}\frac{1}{m}=\frac{kg}{m^2 s^2}$$

$$\frac{\partial p}{\partial x}=\frac{kgm}{s^2}\frac{1}{m^3}\frac{1}{m^2}=\frac{kg}{m^2 s^2}$$

Then the dimension of the momentum equation is consistency

Energy equation	$\frac{\partial\rho h_0}{\partial t}+\frac{\partial\rho h_0 u}{\partial x}=\frac{\partial p}{\partial t}+\dot{q}+\dot{F}_{fr}u$
Or	

$$\frac{\partial \rho C_v T}{\partial t} + \frac{\partial \rho C_v T u}{\partial x} = -p\frac{\partial u}{\partial x} + \dot{q} + \dot{F}_{fr} u$$

Energy equation Dimension is calculated as following:

$$\frac{\partial \rho h_0}{\partial t} = \frac{kg}{m^3}\frac{J}{kg}\frac{1}{s} = \frac{J}{m^3 s} = \frac{kg}{ms^3}$$

$$\frac{\partial \rho h_0 u}{\partial x} = \frac{kg}{m^3}\frac{J}{kg}\frac{m}{s}\frac{1}{m} = \frac{J}{m^3 s} = \frac{kg}{ms^3}$$

$$\frac{\partial p}{\partial t} = \frac{kgm}{s^2}\frac{1}{m^2}\frac{1}{s} = \frac{kg}{ms^3}$$

$$\dot{F}_{fr} u = \frac{kgm}{s^2}\frac{1}{m^3}\frac{m}{s^2} = \frac{kg}{ms^3}$$

$$\dot{q} = \frac{Q}{V} = \frac{J}{s\, m^3} = \frac{kg}{ms^3}$$

Hence the dimension of the energy equation is consistency

- **For an ideal flow:**

p = ρRT and i = CvT

The total enthalpy: $h_0 = h + \frac{1}{2}u^2, h = i + \frac{P}{\rho}$

$h_0 = \mathrm{C_v T} + \frac{P}{\rho} + \frac{1}{2}u^2 = (\mathrm{C_v} + \mathrm{R})\mathrm{T} + \frac{1}{2}u^2$

$\Leftrightarrow \mathrm{T} = \dfrac{h_0 - \frac{1}{2}u^2}{(\mathrm{C_v} + \mathrm{R})}$	(8)

From (8), we can calculate temperature from total enthalpy for the calculated domain.

3. Case Study for the fluid flows through the channel.

Initial conditions for unsteady flow:

- Everywhere in the solution region ρ, u, T must be given at time t =0, in case of incompressible we do not need to provide ρ

Boundary conditions for unsteady flow:

- On solid walls: u = u_w, k∂T/∂n = −q_w (fixed heat flux)
- On fluid boundary:
    - Inlet: ρ, u, T *mus be known*
    - Outlet: −p + μ∂un/∂n = F_n and μ∂ut /∂n = F_t (shear force)

Because the requirement does not mention anything about the type of flow, we consider the system in 2 small study cases to answer question number 2, as below:

- The flow is the incompressible flow.

- The flow is the compressible flow.

3.1 The flow is the incompressible flow

Assumption	$\frac{\partial \rho}{\partial t} = 0, \frac{\partial \rho h_0}{\partial t} = 0$	
Continuity equation	$\frac{\partial(\rho u)}{\partial x} = 0$	
Momentum equation	$\frac{\partial \rho u^2}{\partial x} = -\frac{\partial p}{\partial x} + \dot{F}_{fr}$	
Energy equation	$\frac{\partial \rho h_0 u}{\partial x} = \frac{\partial p}{\partial t} + \dot{q} + \dot{F}_{fr} u$ Or $\frac{\partial \rho C_v T u}{\partial x} = -p\frac{\partial u}{\partial x} + \dot{q} + \dot{F}_{fr} u$	

3.2 The flow is the compressible flow

Continuity equation	$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial t} = 0$	
Momentum equation	$\frac{\partial(\rho u)}{\partial t} + \frac{\partial \rho u^2}{\partial x} = -\frac{\partial p}{\partial x} + \dot{F}_{fr}$	
Energy equation	$\frac{\partial \rho h_0}{\partial t} + \frac{\partial \rho h_0 u}{\partial x} = \frac{\partial p}{\partial t} + \dot{q} + \dot{F}_{fr} u$ Or $\frac{\partial \rho C_v T}{\partial t} + \frac{\partial \rho C_v T u}{\partial x} = -p\frac{\partial u}{\partial x} + \dot{q} + \dot{F}_{fr} u$	

Note: In the following sections, based on the requirement of the application task, the shear stress term can be ignored due to skipping space discretization. This means we consider only one control volume and focus solely on discretization in time.

4. Solution for case studies

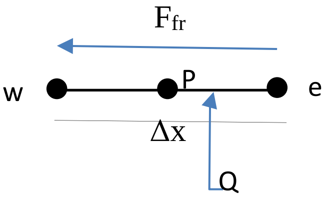
To solve the question number 2, we consider the two following cases:

4.1 Case A: The Flow is the incompressible flow.

The physical meaning of **n.a** is the projection of vector a along the direction of vector n, which is perpendicular to the surface element $\boldsymbol{dA}$. Therefore, the volume integral of the divergence of a vector is equivalent to the sum (integration) of the components of $\boldsymbol{a}$ normal to the surface that encloses the volume, over the entire surface $\boldsymbol{A}$. According to Gauss's divergence theorem, this relationship can be expressed as follows.

$$\int_{cv} div(a)\, dV = \int_{A} n.a\, dA$$

- We consider our calculated domain as following:
  - Only 1 unit volume is considered.
  - No discretization of space



$$\frac{\partial \rho u}{\partial x} = 0 \Leftrightarrow \int_{cv} \frac{\partial \rho u}{\partial x} dV = 0 \Leftrightarrow \int_{A} \rho u\, \boldsymbol{n}\, dA \ = \ 0 \Leftrightarrow (\rho u \mathrm{A})_e - (\rho u \mathrm{A})_w = 0$$

Hence, due to the assumption that our flow is incompressible, $\rho\ is\ constant$

$$u_e = u_w \tag{4.1.1}$$

For a unit control volume, the momentum equation is:

$$\frac{\partial \rho u^2}{\partial x} = -\frac{\partial p}{\partial x} + \dot{F}_{fr}$$

Applying the finite volume method:

$$\int_{cv} \frac{\partial \rho u^2}{\partial x} \mathrm{dV} = -\int_{cv} \frac{\partial p}{\partial x} \mathrm{dV} + \int_{cv} \dot{F}_{fr}\, \mathrm{dV}$$

$$\int_{A} \rho u^2\, \boldsymbol{n} dS = -\int_{A} p\boldsymbol{n}\, dS + \int_{cv} \dot{F}_{fr}\, dV$$

$$\Leftrightarrow (\rho u^2)_e - (\rho u^2)_w = -[(PA)_e - (PA)_w] + F_{fr}$$
$$\Leftrightarrow [(PA)_e - (PA)_w] = F_{fr}$$
$$\Leftrightarrow P_e = P_w + \frac{F_{fr}}{A}$$

$$P_e = P_w + \frac{F_{fr}}{A} \tag{4.1.2}$$

From Energy equation as expressed below:

$$\frac{\partial \rho C_v T u}{\partial x} = -p\frac{\partial u}{\partial x} + \dot{F}_{fr} u \ + \ \dot{q}$$

Using Finite Volume method for a unit control volume:

$$\int_A \frac{\partial \rho C_v T u}{\partial x}\mathbf{n}\mathrm{dA} = -\int_{cv} p\frac{\partial \mathrm{u}}{\partial \mathrm{x}}\mathrm{dV} + \int_{cv} \dot{\mathrm{F}}_{\mathrm{fr}}\mathrm{u}\,\mathrm{dV} + \int_{cv} \dot{\mathrm{q}}\,\mathrm{dV}$$

$\Leftrightarrow (\rho C_v T u A)_e - (\rho C_v T u A)_w = -[p(uA)_e - p(uA)_w] + F_{fr}u + Q$
From (4.1.1) $p(\mathrm{uA})_\mathrm{e} - p(\mathrm{uA})_\mathrm{w} = 0$
$\Leftrightarrow (\rho C_v T\mathrm{uA})_\mathrm{e} - (\rho C_v T\mathrm{uA})_\mathrm{w} = \mathrm{F}_{\mathrm{fr}}\mathrm{u} + \mathrm{Q}$
$\Leftrightarrow (\rho C_v T u A)_e - (\rho C_v T u A)_w = F_{fr}u + Q$
$\Leftrightarrow T_e = \frac{1}{\rho C_v u_e A}\left[F_{fr}u + Q + (\rho C_v T u A)_w\right]$

$$T_e = \frac{1}{\rho C_v u_e A}\left[F_{fr}u + Q + (\rho C_v T u A)_w\right] \tag{4.1.3}$$

In which T_w can be calculated from h_{01} for the boundary condition through this formulation with ideal gas:

$$\mathrm{T} = \frac{h_0 - \frac{1}{2}u^2}{(\mathrm{C_v} + \mathrm{R})}$$

Hence:
From these equations (4.1.1), (4.1.2), (4.1.3), (4.1.4) we have the answer for question number 2 for this incompressible case:
We need to know the variables at the west face of the control volume of the calculated domain, and values at the east face could be calculated from calculation.

Known variables at the west face:	Calculated variables at the east face:
$u_w = u_1$	$u_e = u_2 = u_1$
$p_w = p_1$	$p_e = p_2$
$h_{0w} = h_{01}$	$h_{0e} = h_{02}$
F_{fr}	
Q	

4.2 Case B: The Flow is the compressible flow

Then 3 governing equations as below:

Continuity:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0$$

Momentum equation:

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial \rho u^2}{\partial x} = -\frac{\partial p}{\partial x} + F_{fr}$$

Energy equation in this case $\frac{\partial}{\partial x}(k\frac{\partial T}{\partial x})$ *is calculated from* $Q = k\frac{\partial T}{\partial x}$ to find temperature at the center of a control volume

$$\frac{\partial \rho C_v T}{\partial t} + \frac{\partial \rho C_v T u}{\partial x} = -p\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}(k\frac{\partial T}{\partial x}) + \dot{F}_{fr}u$$

- **Finite Volume Method:**

Consider 1 dimension of 1 control volume, FVM:

Continuity equation

$$\int_t^{t+\Delta t}\int_{cv}\frac{\partial \rho}{\partial t}dt\,dV + \int_t^{t+\Delta t}\int_{cv}\frac{\partial \rho u}{\partial x}dt\,dV = 0$$

Momentum Equation

$$\int_t^{t+\Delta t}\int_{cv}\frac{\partial \rho u}{\partial t}dt\,dV + \int_t^{t+\Delta t}\int_{cv}\frac{\partial \rho u^2}{\partial x}dt\,dV = -\int_t^{t+\Delta t}\int_{cv}\frac{\partial P}{\partial x}dt\,dV + \int_t^{t+\Delta t}\int_{cv}F_{fr}dt\,dV$$

Energy Equation

$$\int_t^{t+\Delta t}\int_{cv}\frac{\partial \rho C_v T}{\partial t}dt\,dV + \int_t^{t+\Delta t}\int_{cv}\frac{\partial \rho C_v T u}{\partial x}dt\,dV = -\int_t^{t+\Delta t}\int_{cv}p\frac{\partial u}{\partial x}dt\,dV + \int_t^{t+\Delta t}\int_{cv}\frac{\partial}{\partial x}(k\frac{\partial T}{\partial x})dt\,dV + \int_t^{t+\Delta t}\int_{cv}\dot{F}_{fr}u\,dt\,dV$$

- Continuity equation:

$$\int_w^e\left[\int_t^{t+\Delta t}\frac{\partial \rho}{\partial t}dt\right]dV + \int_t^{t+\Delta t}[(\rho u A)_e - (\rho u A)_w]dt = 0$$

Using backward differencing scheme (first order), for the first term of LHS of the equation

$$\int_w^e\left[\int_t^{t+\Delta t}\frac{\partial \rho}{\partial t}dt\right]dV = (\rho_P - \rho_P^0)\,\Delta V$$

ρ_P^0: refer to old density at time t
ρ_P: refer to new density at time $t + \Delta t$
The second term of LHS of the equation:

$$\int_t^{t+\Delta t}[(\rho u A)_e - (\rho u A)_w]dt = [(\rho u A)_e - (\rho u A)_w]\Delta t$$

Therefore, the Continuity equation can be rewritten as below:

$$\frac{\rho_P - \rho_P^0}{\Delta t}\Delta V + (\rho u A)_e - (\rho u A)_w = 0$$

$$\Leftrightarrow\rho_P = \frac{\Delta t}{\Delta V}\ [(\rho u A)_w - (\rho u A)_e] + \rho_P^0$$

	$\rho_P = \dfrac{\Delta t}{\Delta V}\ [(\rho u A)_w - (\rho u A)_e] + \rho_P^0$	(4.2.1)

- Momentum equation:

$$\int_t^{t+\Delta t}\int_{cv}\frac{\partial \rho u}{\partial t}dt\,dV + \int_t^{t+\Delta t}\int_{cv}\frac{\partial \rho u^2}{\partial x}dt\,dV = -\int_t^{t+\Delta t}\int_{cv}\frac{\partial P}{\partial x}dt\,dV + \int_t^{t+\Delta t}\int_{cv}\dot{F}_{fr}dt\,dV$$

We have

$$\int_t^{t+\Delta t}\int_{cv}\frac{\partial \rho u}{\partial t}dt\,dV = (\rho_P u_P - \rho_P^0 u_P^0)\Delta V$$

$$\int_t^{t+\Delta t}\int_{cv}\frac{\partial \rho u^2}{\partial x}dt\,dV = [(\rho u^2)_e - (\rho u^2)_w]A\Delta t$$

$$\int_t^{t+\Delta t}\int_{cv}\frac{\partial p}{\partial x}dt\,dV = A(p_e - p_w)\,\Delta t$$

$$\int_t^{t+\Delta t}\int_{cv}\dot{F}_{fr}dt\,dV = \dot{F}_{fr}\Delta t\Delta V = F_{fr}\Delta t$$

$$\frac{\rho_P u_P - \rho_P^0 u_P^0}{\Delta t}\Delta V + [(\rho u^2)_e - (\rho u^2)_w]A = -A(p_e - p_w) + F_{fr}$$

Hence, we have the final momentum equation:

$\rho_P u_P = \dfrac{\Delta t\big[-A(p_e - p_w) + F_{fr} - [(\rho u^2)_e - (\rho u^2)_w]A\big]}{\Delta V} + \rho_P^0 u_P^0$	(4.2.2)

- The energy equation

$$\frac{\partial \rho C_v T}{\partial t} + \frac{\partial \rho C_v T u}{\partial x} = -p\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}(k\frac{\partial T}{\partial x}) + \dot{F}_{fr}u$$

The energy equation with FOV is:

$$\int_{t}^{t+\Delta t}\int_{cv}\frac{\partial \rho C_v T}{\partial t}dt\,dV+\int_{t}^{t+\Delta t}\int_{cv}\frac{\partial \rho C_v T u}{\partial x}dt\,dV=-\int_{t}^{t+\Delta t}\int_{cv}p\frac{\partial u}{\partial x}dt\,dV+\int_{t}^{t+\Delta t}\int_{cv}\frac{\partial}{\partial x}(k\frac{\partial T}{\partial x})dt\,dV+\int_{t}^{t+\Delta t}\int_{cv}\dot{F}_{fr}u\,dt\,dV$$

To evaluate the right-hand side of this equation we need to make an assumption about the variation of T_P, T_e and T_w with time. We could use temperatures at time t or at time t + Δt to calculate the time integral or, alternatively, a combination of temperatures at time t and t + Δt. We may generalize the approach by means of a weighting parameter θ between 0 and 1 and write the integral I_T of temperature T_P with respect to time as:

$$I_T=\int_{t}^{t+\Delta t}T_P dt=[\theta T_P+(1-\theta)T_P^0]\Delta t$$

In this case, we use $\theta=0$, hence

$$I_T=T_P^0\Delta t$$

$$\Leftrightarrow \frac{\rho_P C_v T_P-\rho_P^0 C_v T_P^0}{\Delta t}\Delta V+\left[\left(\rho C_v T^0 u\right)_e-\left(\rho C_v T^0 u\right)_w\right]A=-p[(uA)_e-(uA)_w]+\left(kA\frac{T_e^0-T_P^0}{\Delta x/2}\right)_e-\left(kA\frac{T_P^0-T_w^0}{\Delta x/2}\right)_w+F_{fr}u_{ct}$$

$$\Leftrightarrow \frac{\rho_P C_v T_P}{\Delta t}\Delta V=\frac{\rho_P^0 C_v T_P^0}{\Delta t}\Delta V-\left[\left(\rho C_v T^0 u\right)_e-\left(\rho C_v T^0 u\right)_w\right]A-p[(uA)_e-(uA)_w]+\left(kA\frac{T_e^0-T_P^0}{\Delta x/2}\right)_e-\left(kA\frac{T_P^0-T_w^0}{\Delta x/2}\right)_w+F_{fr}u_{ct}$$

$u_{ct}=\frac{u_e+u_w}{2}$: Velocity at the center of the control volume

Finally, we have the energy equation after discretization:

$$\frac{\rho_P C_v T_P}{\Delta t}\Delta V=\frac{\rho_P^0 C_v T_P^0}{\Delta t}\Delta V-\left[\left(\rho C_v T^0 u\right)_e-\left(\rho C_v T^0 u\right)_w\right]A-p[(uA)_e-(uA)_w]+\left(kA\frac{T_e^0-T_P^0}{\Delta x/2}\right)_e-\left(kA\frac{T_P^0-T_w^0}{\Delta x/2}\right)_w+F_{fr}u_{ct}$$	(4.2.3)

The equation 4.2.3 said that all the known values at old time (at time t) are on the right-hand side, and the new value T_P on the left-hand side at time $t+\Delta t$.
In this case, T_P^0 is calculated from Q as an initial condition at $t=0$
To solve these governing equations, we suppose to use the boundary conditions as
Inlet: as u1, p1, h01, F_{Fr}, Q

Outlet: zero gradient for pressure, enthalpy, velocity, density
From equations (4.2.1), (4.2.2), (4.2.3) we can answer the question 2 as follows:

Known variables at the west face	Calculated variables at the east face
$u_w = u_1$	$u_e = u_2$
$p_w = p_1$	$p_e = p_2$
$h_{0w} = h_{01}\ or\ T_1$	$h_{0e} = h_{02}$
F_{fr}	
Q	

Basically, we solve 3 energy equations for an unsteady flow, using the loop as the image below:

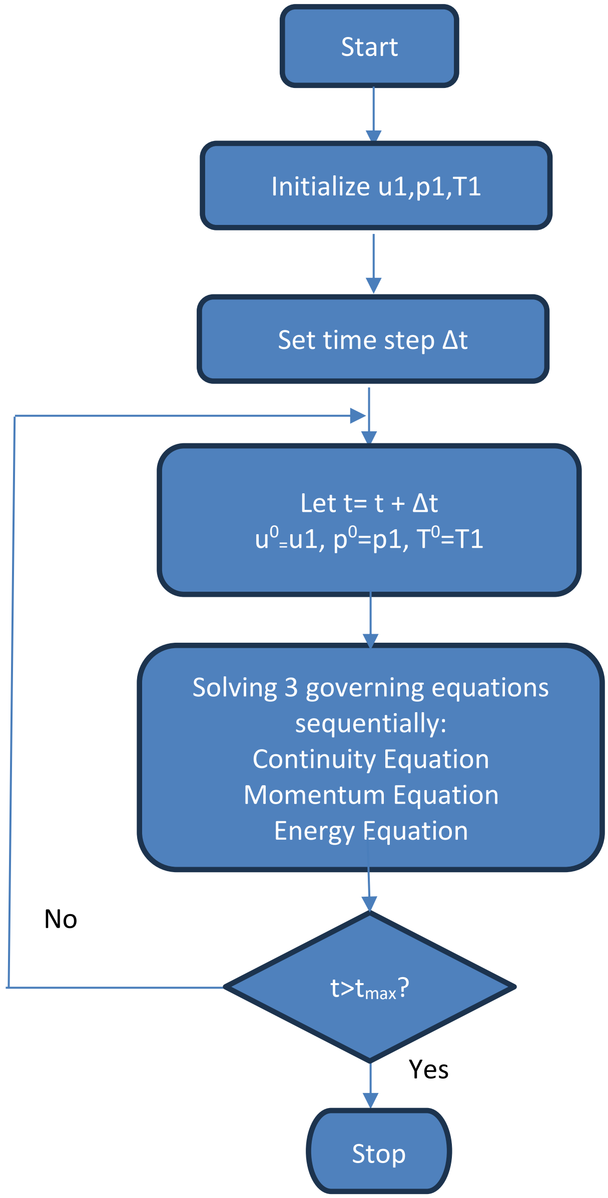


Figure 3: Transient flow and its variants.

References

[1] H. K. V. a. W. Malalasekera, "An Introduction to Computational Fluid Dynamics," *THE FINITE VOLUME METHOD,* 2007.