GROUP ASSIGNMENT

Part 3: COUNTING, ALGORITHM, GRAPHS & TREES AND PROBABILITY

GROUP MEMBERS:

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Counting (12 marks)

1. Count the number of ways to choose 3 students from a group of 7 students:

Using the formula: $C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

We get $C(7,3) = \frac{7}{3} = \frac{7!}{3!(7-3)!} = \frac{7\times6\times5\times4!}{3\times2\times4!} = 35$

Therefore, there are 35 ways for the seven students to choose three of their group to talk with the chairperson.

2.

Red	Black
10	5

We select four students.

i. Two students wearing red shirts and two student wearing black shirts:

The total number of these two combinations:

$C(10,2) \times C(5,2) = \frac{10!}{2!(10-2)!} \times \frac{5!}{2!(5-2)!} = \frac{10\times9\times8!}{2\times8!} \times \frac{5\times4\times3!}{2\times3!} = 45 \times 10 = 450$ ways

Therefore, there are 450 ways to select four student if 2 student are wearing red shirts and 2 are wearing black shirted.

ii. No more than 3 are wearing black shirts:

No more than 3 black, it means there can be 3 black or 2 black or 1 black or no black at all

But there is an another way, we select 4 students from 15 student $C(15,4)$ and subtraction 4 students from 5 of students are wearing black $C(5,4)$.

The total number of these combinations:

$C(15,4) - C(5,4) = \frac{15!}{4!\,11!} - 5 = \frac{15\times14\times13\times12\times11!}{4\times3\times2\times11!} - 5 = 1365 - 5 = 1360$ ways

Therefore, there are 1360 ways to select four student if no more than 3 are wearing black shirts.

iii. The queue of four students can be formed by using permutation:

$P(15,4) = \frac{15!}{(15-4)!} = \frac{15\times14\times13\times12\times11!}{11!} = 32\,760$ ways

Therefore, there are 32 760 ways the queue can be formed.

3.

i. Find the number of different activity patterns that can be formed from those five activites:

For this case, we will use Subsets of permutations because we have to calculate the number of way to arrang these five activities for a day.

Using the formula: $P(n,r) = P\binom{n}{r} = \frac{n!}{(n-r)!}$

We get $P(5,5) = P\binom{5}{5} = \frac{5!}{(5-5)!} = 5! = 5\times4\times3\times2 = 120$ patterns

Therefore, there are 120 different activity patterns that can be formed from those five activites.

ii. Find the number of patterns that contain only three activites:

$C(5,3) = \frac{5!}{3!\,2!} = \frac{5\times4\times3!}{3!\,2} = 10$

Therefore, there are 10 patterns that contain only three activites.

iii. Find the number of patterns with at least four different activities that start with go shopping.

We consider two cases:

Case 1: Exactly 4 activites in a day start with go shopping

then we need to choose three activities from the remaining four activites $C(4,3) = 4$

and arrange the chosen 4 activities $= 3!$

$4 \times 3! = 4 \times 3 \times 2 = 24$ patterns

Case 2: Exactly 5 activies in a day start with go shopping

then we need to choose four activities from the remaining four activites $C(4,4) = 1$

and arrange the chosen 4 activities $= 4!$

$1 \times 4! = 1 \times 4 \times 3 \times 2 = 24$ patterns

Required number of patterns = case 1 + case 2 = $24 + 24 = 48$

Therefore, there are 48 patterns with at least four different activities start with going shopping.

4. i. The number of ways these 6 ingredients can be combined:

Using the formula $C(n,r) = \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

We consider 6 cases:

Case 1: Pick 1 out of 6

$$C(6,1) = \frac{6!}{(6-1)!1!} = 6 \text{ ways}$$

Case 2: Pick 2 out of 6

$$C(6,2) = \frac{6!}{2!(6-2)!} = 15 \text{ ways}$$

Case 3: Pick 3 out of 6

$$C(6,3) = \frac{6!}{3!\,3!} = 20 \text{ ways}$$

Case 4: Pick 4 out of 6

$$C(6,4) = \frac{6!}{4!\,(6-4)!} = 15 \text{ ways}$$

Case 5: Pick 5 out of 6

$$C(6,5) = \frac{6!}{5!(6-5)!} = 6 \text{ ways}$$

Case 6: Pick all of them

$$C(6,6) = \frac{6!}{6!(6-6)!} = 1 \text{ way}$$

Required number of ways = Case 1 + Case 2 + Case 3 + Case 4 + Case 5 + Case 6

$= 6 + 15 + 20 + 15 + 6 + 1$

$= 63$ ways

Therefore, there are 63 ways to combine all six ingredients.

ii. The number of ways that we use three ingredients only:

Using the formula $C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

We get $C(6,3) = \binom{6}{3} = \frac{6!}{3!(6-3)!} = \frac{6\times5\times4\times3!}{3\times2\times3!} = 20$

$\boxed{\text{Therefore, there are 20 ways to use three ingredients only.}}$

5.

Red	Blue	Green	Pink	Orange	Yellow	Black	White	Purple	Brown	Cyan	Lime	Maroon	Rose	Banana	Gray	Tan	Coral

i. The number of different ways to split the pencils evenly amoungst you and your two friends:

Using the formula

We need to form three group of six pencil each.

- We can first choose 6 pencils for yourself from 18 pencils $C(18,6)$, then choose 6 pencils for one friend from remaining 12 pencil $C(12,6)$ and the remaining 6 pencils will go to other friend $C(6,6)$. But we need to divide with arrange the chosen 3! since the order does not matter.

We get $[C(18,6)\times C(12,6)\times C(6,6)]\div 3! = \left[\frac{18!}{6!12!}\times\frac{12!}{6!6!}\times 1\right]\div 3! = 17\,153\,136 \div 6 = 2\,858\,856$

$\boxed{\text{Therefore, there are 2 858 856 different ways to organize and split the pencils evenly among you and your two friends.}}$

ii. Show a second approach

Red	Blue	Green	Pink	Orange	Yellow	Black	White	Purple	Brown	Cyan	Lime	Maroon	Rose	Banana	Gray	Tan	Coral
6!						6!						6!					

- We can split them into three equal sets
- The number of ways to arrange the 18 pencils is 18!
- We need to divide by the number of arrangments of each sets $(6!)^3$ since the pencils of the same color are indistinguishable.
- We also need to divide by 3! for split the pencils into three sets in different ways.

We get $\frac{18!}{(6!)^3\times 3!} = 2\,858\,856$

$\boxed{\text{Therefore, there are 2 858 856 different ways to organize and split the pencils evenly among the three people.}}$

iii. Since there are 10 broken pencils, we are left with 8 pencils

Remaining 8 pencils = $\begin{cases} \text{- 2 pencils for you} = C(8,2) \\ \text{- 3 pencils for your first friend} = C(6,3) \\ \text{- 3 pencils for your second friend} = C(3,3) \end{cases}$

But we need to divide by 2! to avoid the two people who get 3 pencils each are counting twice.

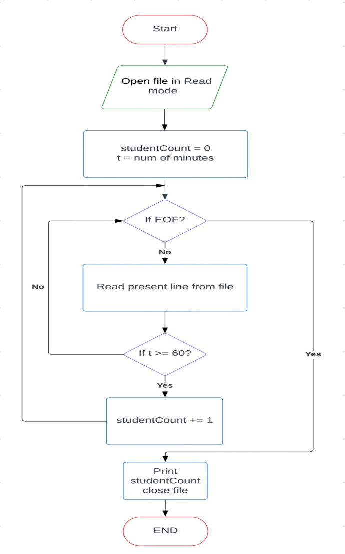
Using the formula: $C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

We get $[C(8,2)\times C(6,3)\times C(3\times3)] = \left[\frac{8!}{2!6!}\times\frac{6!}{3!3!}\times 1\right] = 560$

$\boxed{\text{Therefore, there are 560 ways to split the remaining pencils amongst the 3 of you.}}$

Algorithm (12 marks)

6. i Draw a Flowchart :



ii. The primary parameter of the equation is 'N', representing the number of data items (lines) processed in the file. The algorithm consists of a single loop that iterates 'n' times, processing each line of the file.
Within the algorithm, there are three constant operations performed before the loop, setting the variables 'studentCount' and 't', and reading the file.
Inside the loop, there are three more constant operations: reading 't', incrementing 'studentCount', and moving to the next line.
After the loop, there is one final operation: printing the value of 'studentCount'.
Thus, the time complexity of algorithm can be ÷
expressed as: $T(n) = 3 + 3n + 1$

$T(n) = 3n + 4$

$O(n) = n$

In big-O notation, I focus on dominant term, which is 'n' in this case.
Therefore, the likely complexity of the program using big-O notation is $O(n)$, indicating a linear time complexity.

iii. Pseudocode:

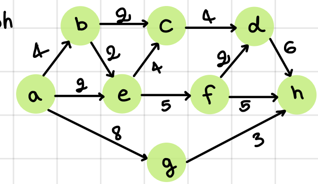
```
Start
    Open file in Read mode
    Initialize variables : studentCount=0 , t=0, totalTime=0
    While (file line != EOF)
        Read present line from file;
        If (t > 60)
            totalTime += t ;
            studentCount += 1 ;
        Read the next line from the file;
    end While
    averageTime = totalTime / studentCount ;
    print averageTime , close file
END
```

7. i. I would recommend algorithm X with a runtime complexity of $O(n)$ over algorithm Y with a runtime complexity of $O(n^2)$ as X has a faster worst-case runtime. As n increases, the runtime of algorithm X will increase linearly compared to agorithm Y which will increase quadratically.
For example, if n is 10, the runtime of algorithm X with be 10 however the runtime of algorithm Y will be 100 which is significantly higher.

ii. Yes, I would recommend algorithm Y as on average, Y would have a shorter runtime than algorithm X.
For example, if n was 100, algorithm X would have an average runtime of 200 whereas algorithm Y would have a shorter average runtime of 100.

Graphs and Trees (12 marks)

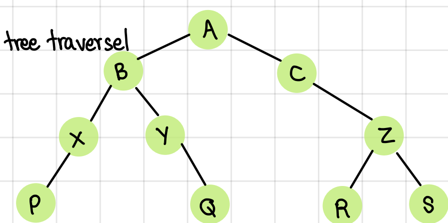
8. Given graph



Visited Node	b	c	d	e	f	g	h
a	4_a	∞	∞	2_a	∞	8_a	∞
a, e	4_a	6_e	∞	2_a	7_e	8_a	∞
a, e, b	4_a	6_b	∞	2_a	7_e	8_a	∞
a, e, b, c	4_a	6_b	10_c	2_a	7_e	8_a	∞
a, e, b, c, f	4_a	6_b	9_f	2_a	7_e	8_a	12_f
a, e, b, c, f, g	4_a	6_b	9_f	2_a	7_e	8_a	11_g
a, e, b, c, f, g, d	4_a	6_b	9_f	2_a	7_e	8_a	11_g

As we can see, the shortest path from a to h is a, g, h which has a distance of 11.

9. Given tree traversel



Preorder : A, B, X, P, Y, Q, C, Z, R, S

Inorder : P, X, B, Y, Q, A, C, R, Z, S

Postorder : P, X, Q Y, B, R, S, Z, C, A

10. Given :

$V = \{1,2,3,4,5,6,7\}$

$E = \{\{1,3\},\{1,5\},\{1,7\},\{2,3\},\{2,4\},\{2,6\},\{3,4\},\{3,5\},\{3,7\},\{5,7\},\{6,7\}\}$

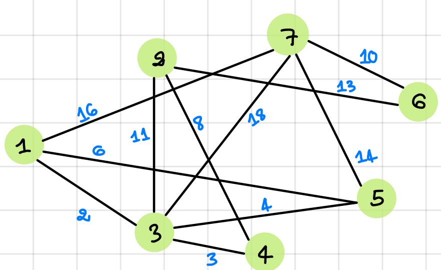
$W(1,3)=2$, $W(1,5)=6$, $W(1,7)=16$

$W(2,3)=11$, $W(2,4)=8$, $W(2,6)=13$

$W(3,4)=3$, $W(3,5)=4$, $W(3,7)=18$

$W(5,7)=14$, $W(6,7)=10$

a. Draw the graph including weights.



This is the grapth G(V, E, W)

b. To determine the order of edges as they are removed from set S and whether they are kept or discarded:

A minimum spanning tree (MST) is a subset of the edges of a connected, un directed graph that connects all the vertices with the most negligible possible total weight of the edges. MST has precisely n-1 edges, where n is the number of vertices in the graph.

Therefore graph F shoud has $7-1 = 6$ edges.

We will find MST(F) using Kruskal Algorithm following these steps:

Step 1: Create a new graph F with nodes V and no edges:

$F(V', E')$:

$V' = \{1, 2, 3, 4, 5, 6, 7\}$

$E' = \{\}$

Step 2: Add all the edges E to a set S and order them by weight starting with the minimum weight:

$S = \{\{1,3\}, \{3,4\}, \{3,5\}, \{1,5\}, \{2,4\}, \{6,7\}, \{2,3\}, \{2,6\}, \{5,7\}, \{1,7\}, \{3,7\}\}$

Step 3: While S is not empty and all the nodes V in F are not connected:

Iteration 1:

- Remove the edge $\{1,3\}$ from set S with the lowest weight.
- Check if adding $\{1,3\}$ to F create a cycle.
  - Since it does not form a cycle, keep it.

$F(V', E')$:

$V' = \{1, 2, 3, 4, 5, 6, 7\}$

$E' = \{\{1,3\}\}$

$S = \{\{3,4\}, \{3,5\}, \{1,5\}, \{2,4\}, \{6,7\}, \{2,3\}, \{2,6\}, \{5,7\}, \{1,7\}, \{3,7\}\}$

Iteration 2:

- Remove the edge $\{3,4\}$ from set S with the lowest weight.
- Check if adding $\{3,4\}$ to F create a cycle.
  - Since it does not form a cycle, keep it.

$F(V', E')$:

$V' = \{1, 2, 3, 4, 5, 6, 7\}$

$E' = \{\{1,3\}, \{3,4\}\}$

$S = \{\{3,5\}, \{1,5\}, \{2,4\}, \{6,7\}, \{2,3\}, \{2,6\}, \{5,7\}, \{1,7\}, \{3,7\}\}$

Iteration 3:

- Remove the edge $\{3,5\}$ from set S with the lowest weight.
- Check if adding $\{3,5\}$ to F create a cycle.
  - Since it does not form a cycle, keep it.

$F(V', E')$:

$V' = \{1, 2, 3, 4, 5, 6, 7\}$

$E' = \{\{1,3\}, \{3,4\}, \{3,5\}\}$

$S = \{\{1,5\}, \{2,4\}, \{6,7\}, \{2,3\}, \{2,6\}, \{5,7\}, \{1,7\}, \{3,7\}\}$

Iteration 4:

- Remove the edge $\{1,5\}$ from set S with the lowest weight.
- Check if adding $\{1,5\}$ to F create a cycle.
  - Since it form a cycle, discard it.

$F(V', E')$:

$V' = \{1, 2, 3, 4, 5, 6, 7\}$

$E' = \{\{1,3\}, \{3,4\}, \{3,5\}\}$

$S = \{\{2,4\}, \{6,7\}, \{2,3\}, \{2,6\}, \{5,7\}, \{1,7\}, \{3,7\}\}$

Iteration 5 :

- Remove the edge $\{2,4\}$ from set S with the lowest weight.
- Check if adding $\{2,4\}$ to F create a cycle.
  - Since it do not form a cycle, keep it.

$F(V', E')$:

$V' = \{1, 2, 3, 4, 5, 6, 7\}$

$E' = \{\{1,3\}, \{3,4\}, \{3,5\}, \{2,4\}\}$

$S = \{\{6,7\}, \{2,3\}, \{2,6\}, \{5,7\}, \{1,7\}, \{3,7\}\}$

Iteration 6 :

- Remove the edge $\{6,7\}$ from set S with the lowest weight.
- Check if adding $\{6,7\}$ to F create a cycle.
  - Since it do not form a cycle, keep it.

$F(V', E')$:

$V' = \{1, 2, 3, 4, 5, 6, 7\}$

$E' = \{\{1,3\}, \{3,4\}, \{3,5\}, \{2,4\}, \{6,7\}\}$

$S = \{\{2,3\}, \{2,6\}, \{5,7\}, \{1,7\}, \{3,7\}\}$

Iteration 7 :

- Remove the edge $\{2,3\}$ from set S with the lowest weight.
- Check if adding $\{2,3\}$ to F create a cycle.
  - Since it form a cycle, discard it.

$F(V', E')$:

$V' = \{1, 2, 3, 4, 5, 6, 7\}$

$E' = \{\{1,3\}, \{3,4\}, \{3,5\}, \{2,4\}, \{6,7\}\}$

$S = \{\{2,6\}, \{5,7\}, \{1,7\}, \{3,7\}\}$

Iteration 8 :

- Remove the edge $\{2,6\}$ from set S with the lowest weight.
- Check if adding $\{2,6\}$ to F create a cycle.
  - Since it do not form a cycle, keep it.

$F(V', E')$:

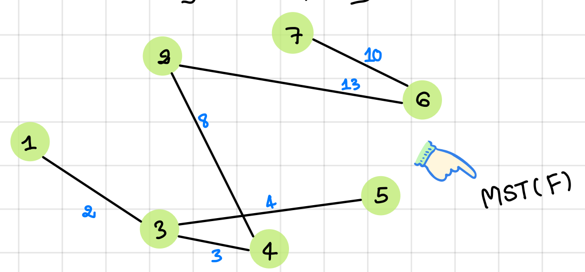
$V' = \{1, 2, 3, 4, 5, 6, 7\}$

$E' = \{\{1,3\}, \{3,4\}, \{3,5\}, \{2,4\}, \{6,7\}, \{2,6\}\}$

$S = \{\{5,7\}, \{1,7\}, \{3,7\}\}$

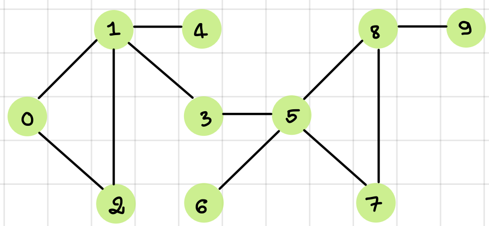
By including this node, it will include 6 edges in the MST, so you don't have to traverse any further in the sorted list.

ii. Draw the resulting minimum spanning tree F

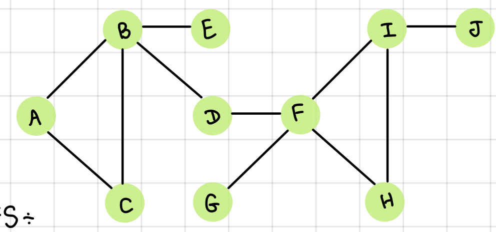


The summation of all the edges weights in MST $F(V', E')$ is equal to 40, which is the least possible edges weight for any possible spanning tree structure for this graph.

11. Given graph



As the node is strating from A but the given graph is numerically valued
So, alphabetic graph is



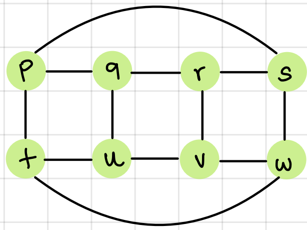
BFS ÷

1. Strating from A, visited array is [A].
2. Visit adjacent unvisited nodes of a which is B, C
   Hence, array becomes, [A, B, C].
3. Now, unvisited ajacent nodes of B is E and D
   So, array becomes [A,B,C,E,D].
4. Unvisited adjacent node of C is none.
5. For E it is none, for D unvisited node is F.
   So, array becomes [A,B,C,E,D,F].
6. Unvisited adjacent nodes for F is G, H, I.
   So, array becomes [A,B,C,E,D,F,G,H,I].
7. Now, unvisited adjacent node for I is J.
   Hence, materix becomes [A,B,C,E,D,F,G,H,I,J].
8. Now, there are no unvisited node. Hence, all node are covered.
9. The visited order in BFS traversel is [A, B, C, E, D, F, G, H, I, J].

DFS ÷

1. Visit node A, so array is [A]
2. Visit unvisited node according to alphabetical order
   so array becomes [A, B]
3. Visit unvisited node of B in alphabetical order.
   So array becomes [A, B, D].
4. Again using the same process. So array becomes [A, B, D, F].
5. Now, visited unvisited node from F in alphabetical order.
   So array becomes [A, B, D, F, G].
6. Now, G doesn't has any node. So, back track and come to F.
   Unvisited node from F is H, so array becomes [A,B,D,F,G,H].
7. Now unvisited node for F is I.
   So array becomes [A,B,D,F,G,H,I].
8. Also same for I it is J, so array becomes [A,B,D,F,G,H,I,J].
9. Keep on back tracking till reached B.
   Now, unvisited node for B is E, so array becomes [A,B,D,F,G,H,I,J,E].
10. Again back tracking and come to A, unvisited node from A is C.
    Hence, materix becomes [A, B, D, F, G, H, I, J, E, C].
    Now, there is no more adjacent nodes to visit.
    The visited order in DFS traversel is [A, B, D, F, G, H, I, J, E, C].
    We can represent as numerical values is [0, 1, 3, 5, 6, 7, 8, 9, 4, 2].

12. Find whether a given graph is Eulerian cycle or not ÷



Eulerian Cycle : An undirected graph has Eulerian cycle if following two conditions are true.

1. All vertices with non zero degree are connected. We don't care about vertices with zero degree beacause they don't belong to Eulerian Cycle or Path (we only consider all edges).
2. All vertices have even degree.

Now, let's examine the degree of each vertex in the graph :

- Vertex p : degree 3
- Vertex q : degree 3
- Vertex r : degree 3
- Vertex s : degree 3
- Vertex t : degree 3
- Vertex u : degree 3
- Vertex v : degree 3
- Vertex w : degree 3

So, we can see that condition 1 satisfies as there no 0 degree vertex , but condition 2 didn't satisfy as all vertices have an odd degree(3).
Therefore , based on the degrees of the vertices, this graph does not contain any Eulerian cycles.

Hence , there is no Eulerian cycle in the given graph.

Probability (10 marks)

13.

White	Red	Blue
4	5	6

Three marbles are drawn at random from the bag

Number of ways that the sample space S can occur

$n(S) = C(15,3)$

$= \frac{15!}{3!(15-3)!} = \frac{15 \times 14 \times 13 \times 12!}{3 \times 2 \times 12!} = 455$ ways

Find the probability that all of them are red

Number of ways that event A can occur

$n(A) = C(5,3)$

$= \frac{5!}{3!(5-3)!} = \frac{5 \times 4 \times 3!}{3! \times 2} = 10$ ways

We get $P(A) = \frac{n(A)}{n(S)} = \frac{10}{455} = \frac{2}{91}$

$\boxed{P(A) = \frac{2}{91}}$

14. Let H = { Student who have been swimming at Hawthorn pool } and C = { Student who have been swimming at the City Baths },
which means

$P(H) = \frac{15}{30} = \frac{1}{2}$

$P(C) = \frac{10}{30} = \frac{1}{3}$

$P(H \cap C) = \frac{5}{30} = \frac{1}{6}$

i. The probability that they have been swimming at the City Baths given that they have been swimming at Hawthorn pool is:

Using Bayes's theorem: $P(C|H) = \frac{P(H \cap C)}{P(H)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{6} \times \frac{2}{1} = \frac{1}{3}$

$\boxed{P(C|H) = \frac{1}{3}}$

ii. The probability that they have been swimming at Hawthorn pool given that they have been swimming at the City Baths is:

Using Bayes's theorem: $P(H|C) = \frac{P(H \cap C)}{P(C)} = \frac{\frac{1}{6}}{\frac{1}{3}} = \frac{1}{6} \times \frac{3}{1} = \frac{1}{2}$

$\boxed{P(H|C) = \frac{1}{2}}$

iii. The probability that they have been swimming at either pool is:

$P(H \cup C) = P(H) + P(C) - P(H \cap C) = \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{3+2-1}{6} = \frac{4}{6} = \frac{2}{3}$

$\boxed{P(H \cup C) = \frac{2}{3}}$

15.

i. Your expectation (E) is:

$$E = \frac{\$48000}{3000} + \frac{\$28000}{3000} + \frac{\$13000}{3000} + \frac{-\$2000 \times 2997}{3000} = -\$1968.33$$

Therefore, the expected value of E is $-\$1968.33$, which means that on average, you can expect to lose money if you buy a ticket.

ii. To be fair, E must be zero, so solving for x we find:

$$\frac{\$50000 - x}{3000} + \frac{\$30000 - x}{3000} + \frac{\$15000 - x}{3000} + \frac{\$0 - 2997x}{3000} = \$0$$

$$\$50000 - x + \$30000 - x + \$15000 - x - 2997x = \$0$$

$$3000x = \$95000$$

$$x = \frac{\$95000}{3000} = \$31.67$$

Therefore, the price of the ticket should be $\$31.67$ to make the lottery fair.

16. Use the binomial distribution function with $n=6$, $p = 25\% = 0.25$ and $q = 100\% - p = 1 - 0.25 = 0.75$.

i. The probability that 2 are faulty

$$P(X=k) = C(n,k) \times p^k \times q^{(n-k)}$$

$$\Rightarrow P(X=2) = C(6,2) \times 0.25^2 \times 0.75^4 = 0.2966$$

$$\boxed{P(X=2) = 0.2966}$$

ii. The probability that at least 2 are faulty $P(X \geq 2)$:

$$P(X \geq 2) = 1 - P(X<2)$$

And $P(X<2) = P(X=0) + P(X=1)$

$$P(X=k) = C(n,k) \times p^k \times q^{(n-k)}$$

We get $P(X=0) = C(6,0) \times 0.25^0 \times 0.75^{6-0} = \left(\frac{3}{4}\right)^6$

$$P(X=1) = C(6,1) \times 0.25^1 \times 0.75^{6-1} = 0.356$$

So, $P(X \geq 2) = 1 - \left(\left(\frac{3}{4}\right)^6 + 0.356\right) = 0.466$

$$\boxed{P(X \geq 2) = 0.466}$$