CONVEX FUNCTION

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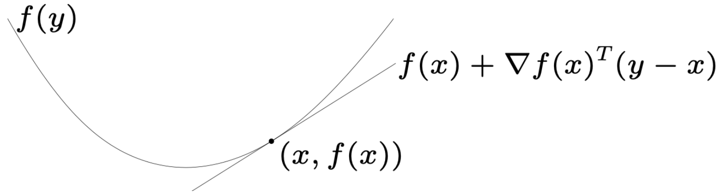
1. First-order condition

Let $f : \mathbb{R}^n \to \mathbb{R}$. If f is differentiable on $\mathrm{dom} f$ ($\mathrm{dom} f$ is open) then ∇f exists and

$$\nabla f = \Big(\frac{\partial f(x)}{\partial x_1}, \ldots, \frac{\partial f(x)}{\partial x_n} \Big).$$



Theorem 1. *(first-order condition) A differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ with convex domain is convex if and only if*

$$f(y) \geq f(x) + \nabla f(x)^T (y - x), \text{ for every } x, y \in \mathrm{dom} f.$$

Proof. We only consider the case $n = 1$. We show that a differentiable function $f : \mathbb{R} \to \mathbb{R}$ is convex if and only if

$$f(y) \geq f(x) + f(x)'(y - x), \text{ for every } x, y \in \mathrm{dom} f. \tag{1}$$

($\Rightarrow$) Assume that f is convex. Let any $x, y \in \mathrm{dom} f$ and every $0 < t \leq 1$. Then by convexity of f one gets that

$$f(x + t(y - x)) = f((1 - t)x + ty) \leq (1 - t)f(x) + tf(y).$$

This implies that

$$\begin{aligned} f(y) &\geq f(x) + \frac{f(x + t(y - x)) - f(x)}{t} \\ &= f(x) + (y - x)\frac{f(x + t(y - x)) - f(x)}{t(y - x)} \quad (x \neq y). \end{aligned}$$

Hence, by taking $t \to 0$ we get the result.

($\Leftarrow$) Suppose that we have (1). Let any $x \neq y \in \mathrm{dom} f$ and for all $t \in [0, 1]$ we set $z = tx + (1 - t)y$. By using (1) we get that

$$f(x) \geq f(z) + f'(z)(x - z) \text{ and } f(y) \geq f(z) + f'(z)(y - z).$$

Therefore

$$tf(x) + (1-t)f(y) \geq f(z).$$

This means that f is convex. $\square$

Problem 1. By using the first-order condition, check the following function is convex, concave or not.

a) $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$.
b) $f : \mathbb{R} \to \mathbb{R}, f(x) = \log x$.
c) $f : \mathbb{R} \times (0; +\infty) \to \mathbb{R}, f(x, y) = \dfrac{x^2}{y}$.

Solution.

a) $\text{dom} f = \mathbb{R}$ is convex, for every $x, y \in \mathbb{R}$, we have

$$\begin{aligned} f(y) - [f(x) + f'(x)(y-x)] &= y^2 - [x^2 + 2x(y-x)] \\ &= y^2 - [x^2 + 2xy - 2x^2] \\ &= x^2 - 2xy + y^2 \\ &= (x-y)^2 \\ &\geq 0. \end{aligned}$$

This yields, f is convex.

b) $\text{dom} f = (0; +\infty)$. For all $x, y \in \text{dom} f$ one gets that

$$\begin{aligned} f(y) - [f(x) + f'(x)(y-x)] &= \log y - [\log x + \frac{1}{x}(y-x)] \\ &= \log y - \log x - \frac{y}{x} + 1. \end{aligned}$$

It is not easy to check that $\log y - \log x - \frac{y}{x} + 1 \geq 0$ or ≤ 0 or not.

Problem 2. Consider an optimization problem

$$\min_{x \in \mathbb{R}^n} f(x),$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is convex and differentiable. Prove that any point x^* that satisfies $x^* \in \text{dom} f$ and $\nabla f(x^*) = 0$ is a global minimum.

Proof. For every $x, y \in \text{dom} f$, by the first-order condition one gets that

$$f(y) \geq f(x) + \nabla f(x)^T (y - x).$$

By replacing x by x^* we obtain that

$$f(y) \geq f(x^*) + \nabla f(x^*)^T (y - x^*), \text{ for every } y \in \text{dom} f.$$

As $\nabla f(x^*) = 0$, one has

$$f(y) \geq f(x^*), \text{ for every } y \in \text{dom} f.$$

This means that x^* is a global minimum. $\square$

Problem 3. Consider an optimization problem

$$\min_{x \in \mathbb{R}^n} f(x),$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is convex and differentiable. Prove that a point x is an optimal if and only if $x \in \text{dom} f$ and

$$\nabla f(x)^T(y - x) \geq 0, \text{ for every } y \in \text{dom} f$$

2. Second-order condition

A function $f : \mathbb{R}^n \to \mathbb{R}$ is twice differentiable if $\text{dom} f$ is open and the Hessian $\nabla^2 f$ exists.

$$\nabla^2 f(x)_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}, i, j = 1, \ldots, n.$$

Theorem 2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice differentiable with convex domain. Then*
a) f is convex if and only if

$$\nabla^2 f(x) \geq 0, \text{ for every } x \in \text{dom} f.$$

b) If $\nabla^2 f(x) > 0$, for every $x \in \text{dom} f$ then f is strictly convex.

Remark. If f is twice differentiable and $n = 1$, i.e. $f : \mathbb{R} \to \mathbb{R}$ then f is convex iff $\text{dom} f$ is convex and $f'' \geq 0$.

Problem 4. By using the second-order condition, check the following function is convex, concave or not.
a) $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$.
b) $f : \mathbb{R} \to \mathbb{R}, f(x) = \log x$.
c) $f : \mathbb{R} \times (0; +\infty) \to \mathbb{R}, f(x, y) = \dfrac{x^2}{y}$.
d) $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = \frac{1}{2} x^T P x + q^T x + r$ with $P \in M_n(\mathbb{R})$, $P \geq 0$, $q \in \mathbb{R}^n$ and $r \in \mathbb{R}$.
e) $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = \|Ax - b\|_2^2$ with $A \in M_{m \times n}(\mathbb{R})$ and $b \in \mathbb{R}^m$.

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