FE5222 ADP Project Two

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1 Introduction

Spread options have terminal payoffs based on the difference in prices between two underlying assets together with a strike. When the strike equals 0, the spread option is equivalent to an option to exchange one asset for another, where the Margrabe's formula works as an explicit solution **[1], [2]**. The Kirk's approximation as published in 1995, is a valid formula when the strike is small but non-zero, where a special sigma is adopted into the generalized Black-Scholes option pricing **[3]–[5]**.

In this project, we would investigate the spread option pricing via Kirk's approximation, and employ the Monte Carlo (MC) simulation output as a benchmark. Results from both methods are compared and discussed in various scenarios.

2 Materials and Methods

2.1 Monte Carlo Simulation

Pricing spread call option by Monte Carlo is similar to pricing vanilla European option. The initial step is to generate terminal prices for the two correlated stocks' geometric Brownian motions:

$$S_1(T) = S_1(0)e^{(r-0.5\sigma_1^2)T+\sigma_1 W_1(T)}$$
$$S_2(T) = S_2(0)e^{(r-0.5\sigma_2^2)T+\sigma_2 W_2(T)}$$
$$W_2(T) = \rho W_1(T) + \sqrt{1-\rho^2}Z(T)$$

When generating standard Brownian motions $W_1(T)$ and $Z(T)$, the antithetic variates method could be used to reduce variance and accelerate convergence. Its core idea is to sample random variable pairs from the same distribution such that their corresponding cumulative distribution functions add up to one. In this project, we could generate paired multivariate independent Normal distributed random variables $(W_1(T), Z(T))'$ and $-(W_1(T), Z(T))'$ to calculate terminal prices. As a linear combination of Gaussian random variables, $W_2(T)$ is also following Gaussian distribution.

The simulation is repeated multiple times, where for each simulation we calculate the payoff at expiry:

$$payoff = \max\{S_1(T) - S_2(T) - K, 0\}$$

By taking expectation and using discount factors, we get the Monte Carlo estimation of the spread call value.

2.2 Kirk's Approximation

Kirk's approximation represents a closed form formula to price a European spread call option,

$$C(S_1, S_2, T) = S_1\mathcal{N}(d_1) - (S_2 + Ke^{-rT})\mathcal{N}(d_2)$$

where S_i is the stock's price, $\mathcal{N}$ is the distribution function of a standard normal distribution

$$d_{1,2} = \frac{\ln\left(S_1/(S_2 + Ke^{-rT})\right) \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

And,

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2\left(\frac{S_2}{S_2 + Ke^{-rT}}\right)^2 - 2\rho\sigma_1\sigma_2\left(\frac{S_2}{S_2 + Ke^{-rT}}\right)}$$

Proof:

Let $S_1(t)$ and $S_2(t)$ be the prices of two stocks such that

$$\frac{dS_i(t)}{S_i(t)} = rdt + \sigma_i dW_i(t), \quad i = 1,2$$

And,

$$dW_1(t)dW_2(t) = \rho dt$$

By risk neutral pricing on Measure P, we have the value of the spread option:

$$V(0) = E[e^{-\mathrm{rT}}(S_1(T) - S_2(T) - K)^+]$$

Let $Y(t) = S_2(t) + Ke^{-r(T-t)}$. Hence,

$$V(0) = E\left[e^{-\mathrm{rT}}\left(S_1(T) - Y(T)\right)^+\right]$$

By Ito's formula,

$$dY(t) = ke^{-r(T-t)}rdt + dS_2(t) = rY(t)dt + S_2(t)\sigma_2 dW_2(t)$$

Given the correlation assumption and $W_2(t)$ is a standard Brownian motion, we have:

$$dY(t) = rY(t)dt + S_2(t)\sigma_2\left(\rho dW_1(t) + \sqrt{1-\rho^2}dZ(t)\right)$$

where $Z(t)$ is a standard Brownian motion independent from $W_1(t)$.

To prove $dW_2(t) = \left(\rho dW_1(t) + \sqrt{1-\rho^2}dZ(t)\right)$, is equivalent to prove its integrated form:

$$W_2(t) = \rho W_1(t) + \sqrt{1-\rho^2}Z_t$$

In this formula, we observe that $W_2(t)$ is a Brownian motion since the sum of independent Brownian motions is Brownian motion which can be shown Levy's Lemma and its mean is zero. Hence, we only need to prove its variance is t and its correlation with $W_1(t)$ is ρ.

Since $\mathrm{V}ar(W_1(t)) = t$,

$$Var\left(W_2(t)\right) = \rho^2 Var\left(W_1(t)\right) + (1-\rho^2)Var(Z_t) = \rho^2 t + (1-\rho^2)t = t$$

From the formula and since Z(t) is a standard Brownian motion independent from $W_1(t)$, we can see that

$$cov\big(W_1(t), W_2(t)\big) = \rho t + 0 \ = \rho t$$

Hence,

$$\mathrm{corr}\big(W_1(t), W_2(t)\big) = \frac{cov\big(W_1(t), W_2(t)\big)}{\mathrm{t}} = \rho$$

To make both $Y(t)$ and $S_1(t)$ using the same Brownian motion, we rewrite them in vector form,

$$\frac{dS_1(t)}{S_1(t)} = rdt + \Sigma \cdot d\widetilde{W}(t)$$

And,

$$\frac{dY(t)}{Y(t)} = rdt + \Sigma' \cdot d\widetilde{W}(t)$$

Where $\Sigma = \big(\sigma_{1,} 0\big)$ and $\Sigma' = \left(\rho \frac{S_2(t)}{Y(t)} \sigma_2, S_2(t)\sigma_2\sqrt{1-\rho^2}\right)$, and

$$d\widetilde{W}(t) = \big(dW_1(t), d\mathrm{Z}(t)\big)$$

By similar procedures in deriving the exchange option pricing formula from lecture notes we have the modified σ (shown in Fig.1) as:

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2\left(\frac{S_2}{S_2 + Ke^{-rT}}\right)^2 - 2\rho\sigma_1\sigma_2\left(\frac{S_2}{S_2 + Ke^{-rT}}\right)}$$

For K << S2, assuming Y (t) also follows log-normal (which is an assumption of Black-Scholes formula which in turn used in our derivation for Exchange Option’s formula), and applying the formula for exchange option we derived in class, we will obtain:

$$\begin{aligned} C(S_1, S_2, T) &= V(0) = S_1\mathcal{N}(d_1) - Y(0)\mathcal{N}(d_2) \\ &= S_1\mathcal{N}(d_1) - (S_2 + Ke^{-rT})\mathcal{N}(d_2) \end{aligned}$$

And,

$$d_{1,2} = \frac{\ln\big(S_1/(S_2 + Ke^{-rT})\big) \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

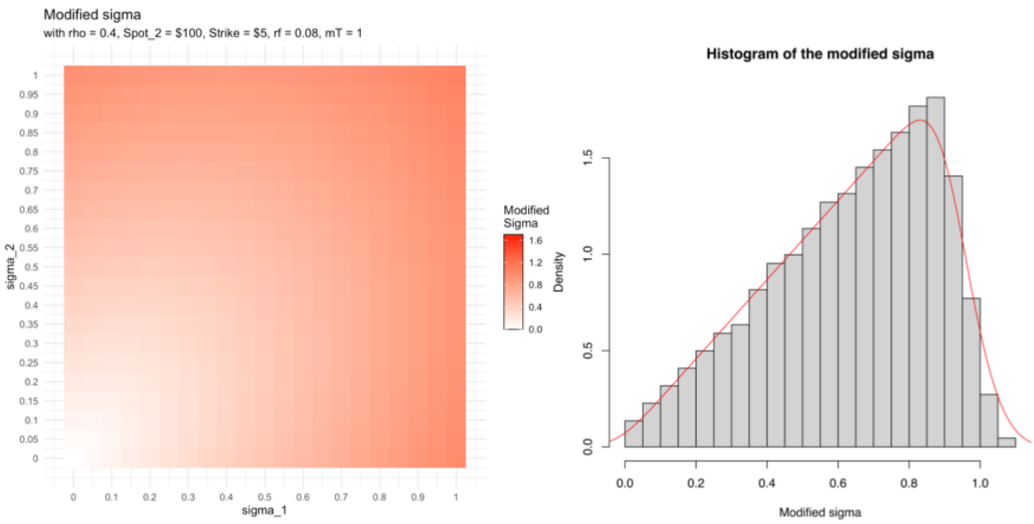


Figure 1. The modified σ, as calculated using σ_1 and σ_2, ranges from 0.00 to 1.07 with a negative skewness of -0.51 and a negative excess kurtosis of -0.60.

2.3 Comparison and Visualization

To compare the two investigated methods with respect to pricing parameters, we further performed pairwise pricing and visualized their differences (Kirk's – MC's) over two dimensional grids with red-white-blue color scales.

To improve comparability, we kept the color scale bar centered around \$0. Investigated pricing parameters with default values includes:

Spot difference between the two stocks: $Spotdiff = S_{0,1} - S_{0,2} = \$110 - \$100 = \$10$;

Strike as in percentage of $Spotdiff$: $\frac{K}{SpotDiff} = \frac{5}{10} = 50\%$;

Volatility of the first stock: $\sigma_1 = 0.2$;

Volatility of the second stock: $\sigma_2 = 0.2$;

Instantaneous correlation: $\rho = 0.4$;

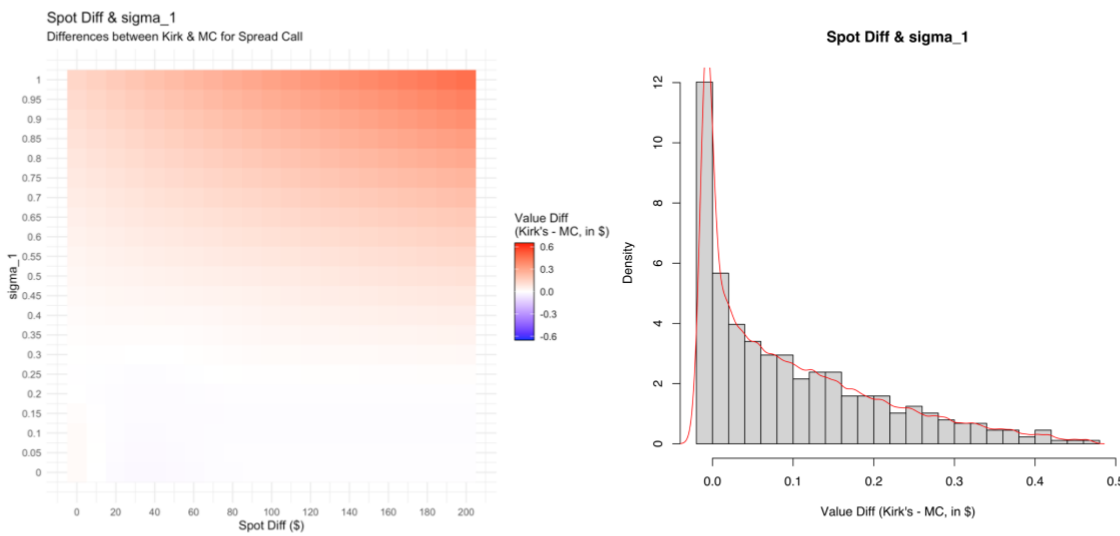
Interest rate: $r = 0.08$;

Time to maturity: $mT = 1$.

They are explored using equal spaced grids in corresponding plots but kept constant otherwise. To keep reproducibility and accelerate convergence, all Monte Carlo pricing schemes share the same initial random seed with 1,000,000 paths and antithetic variates. Visualizations are implemented in RStudio via packages 'tidyverse' and 'ggplot2' **[6], [7]**.

3 Results and Discussion

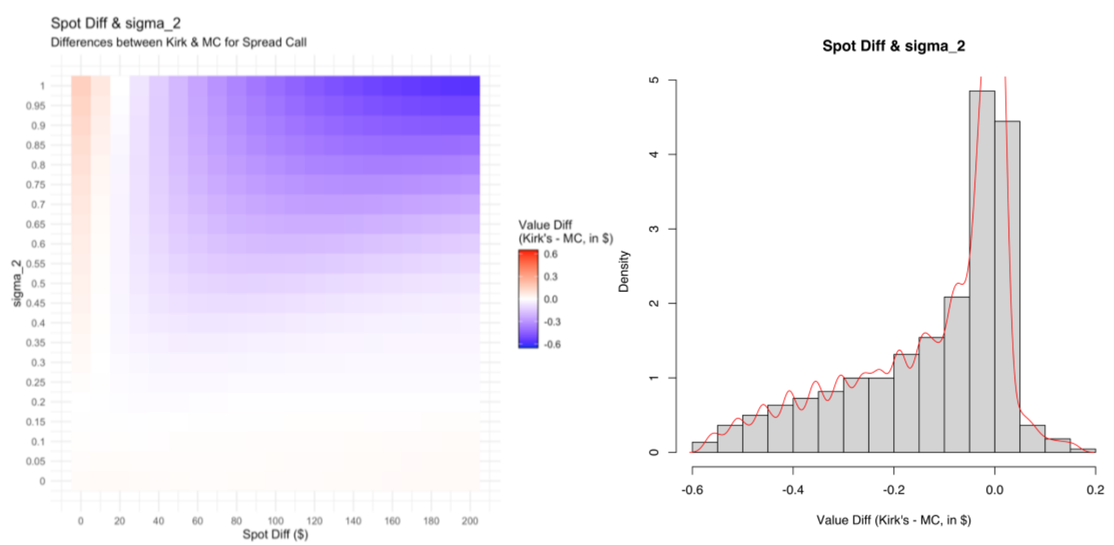
3.1 Grid comparison for pricing parameters



(a) (b)

Figure 2. The differences between Kirk's and MC among various $SpotDiff$ and σ_1, as in (a) grid plot, and in (b) histogram.

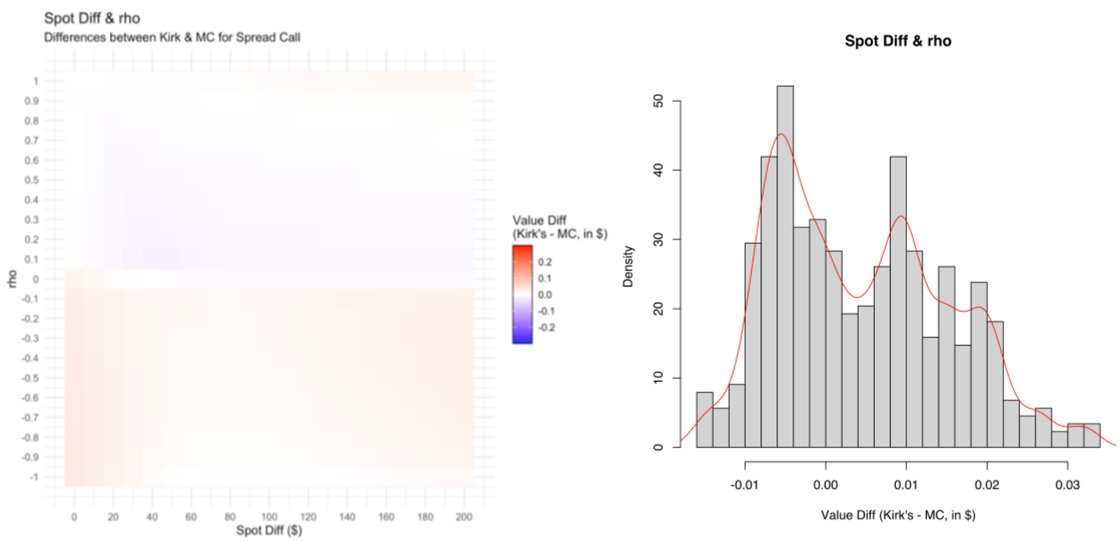
Corresponding to Scenario 1, the value differences between Kirk's and MC's among the inspected $SpotDiff$ vs. σ_1 pairs are plotted in Fig.2. The value difference ranging from \$-0.02 to \$0.46 has its mean at \$0.10 and median at \$0.06 with a positive skewness of 1.10 and a positive excess kurtosis of 0.46. Generally, larger spot differences or larger σ_1 would bring positive value differences.



(a) (b)

Figure 3. The differences between Kirk's and MC among various $SpotDiff$ and σ_2, as in (a) grid plot, and in (b) histogram.

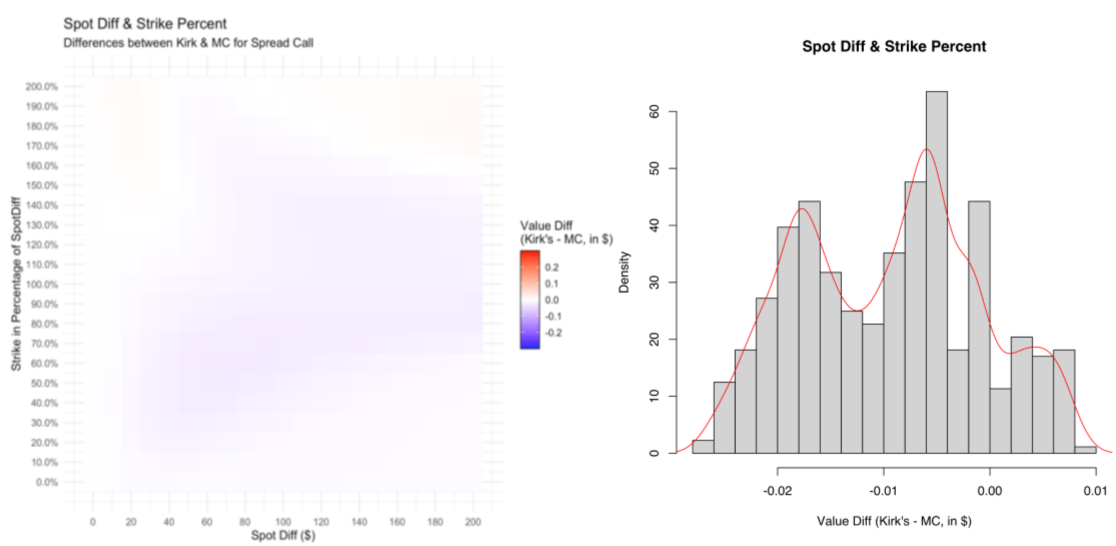
Corresponding to Scenario 1, the value differences between Kirk's and MC's among the inspected $SpotDiff$ vs. σ_2 pairs are plotted in Fig.3. The value difference ranging from \$-0.57 to \$0.16 has its mean at \$-0.12 and median at \$-0.05 with a negative skewness of -1.04 and a positive excess kurtosis of 0.10. At higher σ_2 level, larger spot differences would bring negative value differences.



(a) (b)

Figure 4. The differences between Kirk's and MC among various $SpotDiff$ and ρ, as in (a) grid plot, and in (b) histogram.

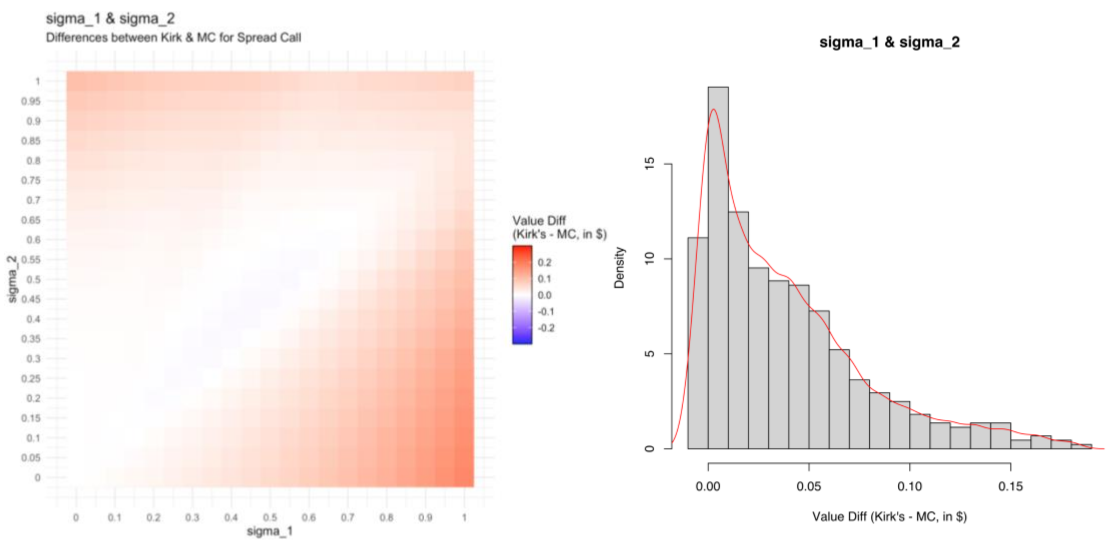
Corresponding to Scenario 1 and 3, the value differences between Kirk's and MC's among the inspected $SpotDiff$ vs. ρ pairs are plotted in Fig.4. The value difference ranging from \$-0.02 to \$0.03 has its mean at \$0.00 and median at \$0.00 with a positive skewness of 0.40 and a negative excess kurtosis of -0.70. Generally, positive correlation would bring negative value differences, and vice versa.



(a) (b)

Figure 5. The differences between Kirk's and MC among various $SpotDiff$ and Strike in percentage of $SpotDiff$, as in (a) grid plot, and in (b) histogram.

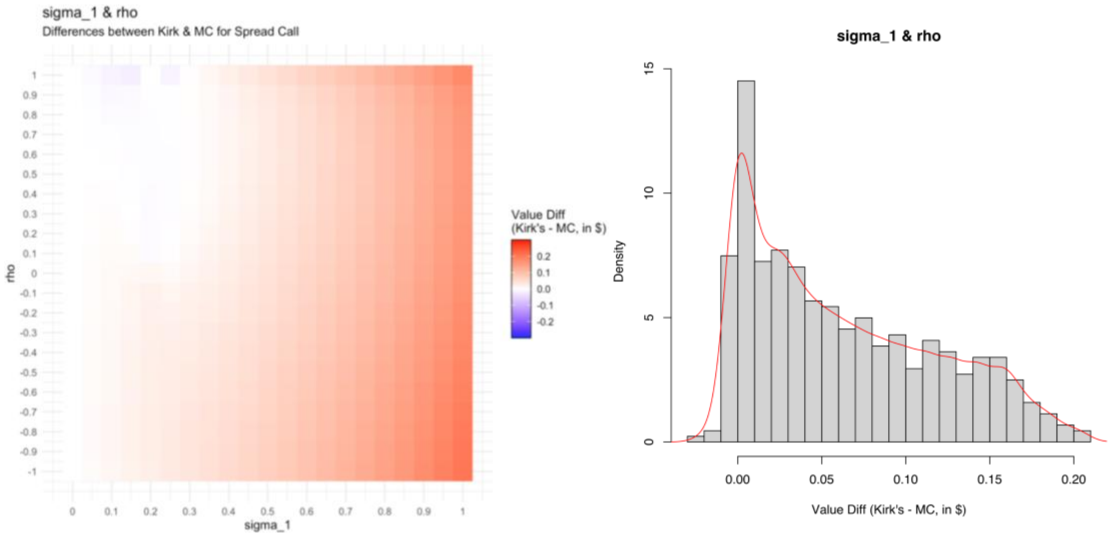
Corresponding to Scenario 1 and 5, the value differences between Kirk's and MC's among the inspected $SpotDiff$ vs. $\frac{K}{SpotDiff}$ pairs are plotted in Fig.5. The value difference ranging from \$-0.03 to \$0.01 has its mean at \$-0.01 and median at \$-0.01 with a positive skewness of 0.05 and a negative excess kurtosis of -0.92. The difference is more visible when the Strike is around the differences between two Spots.



(a) (b)

Figure 6. The differences between Kirk's and MC among various σ_1 and σ_2, as in (a) grid plot, and in (b) histogram.

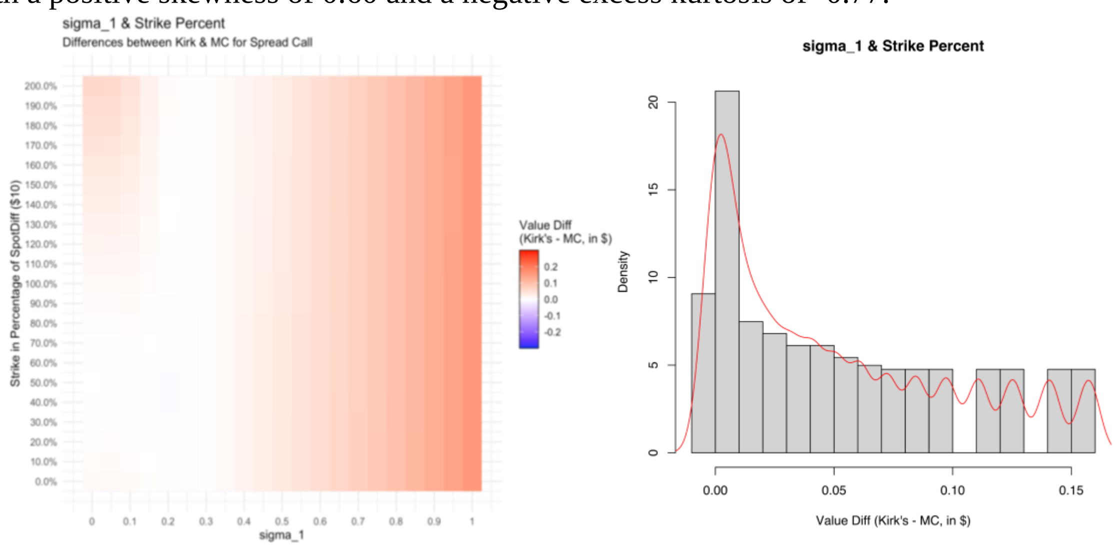
Corresponding to Scenario 2, the value differences between Kirk's and MC's among the inspected σ_1 vs. σ_2 pairs are plotted in Fig.6. The value difference ranging from \$-0.01 to \$0.18 has its mean at \$0.04 and median at \$0.03 with a positive skewness of 1.24 and a positive excess kurtosis of 1.19. Both larger σ_1 and larger σ_2 would bring positive value differences. The difference is minimized when $\sigma_1 = \sigma_2$.



(a) (b)

Figure 7. The differences between Kirk's and MC among various
σ_1 and ρ, as in (a) grid plot, and in (b) histogram.

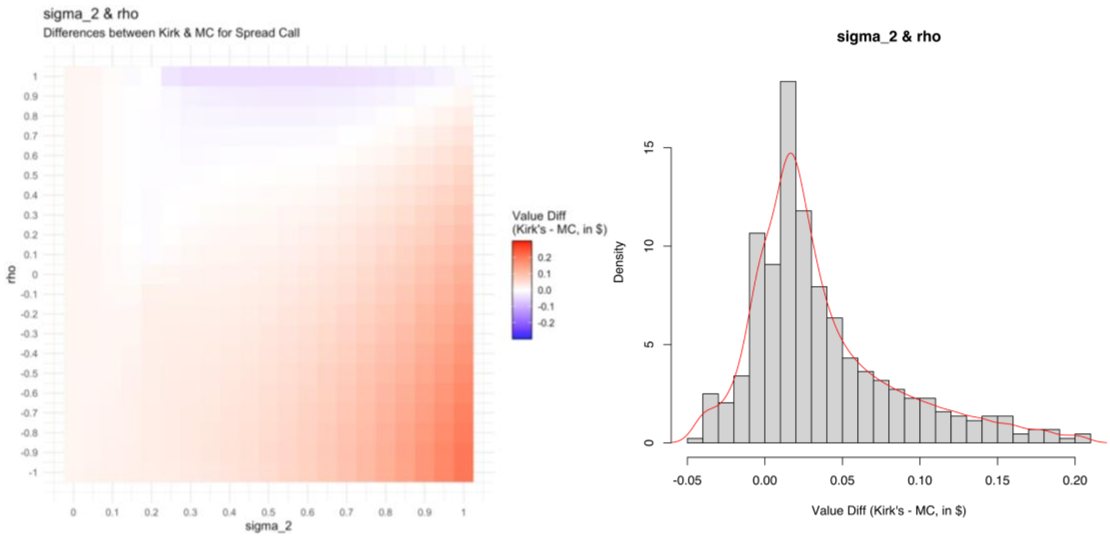
Corresponding to Scenario 2 and 3, the value differences between Kirk's and MC's among the inspected σ_1 vs. ρ pairs are plotted in Fig.7. The value difference ranging from \$-0.02 to \$0.21 has its mean at \$0.06 and median at \$0.05 with a positive skewness of 0.60 and a negative excess kurtosis of -0.77.



(a) (b)

Figure 8. The differences between Kirk's and MC among various
σ_1 and Strike in percentage of $SpotDiff$, as in (a) grid plot, and in (b) histogram.

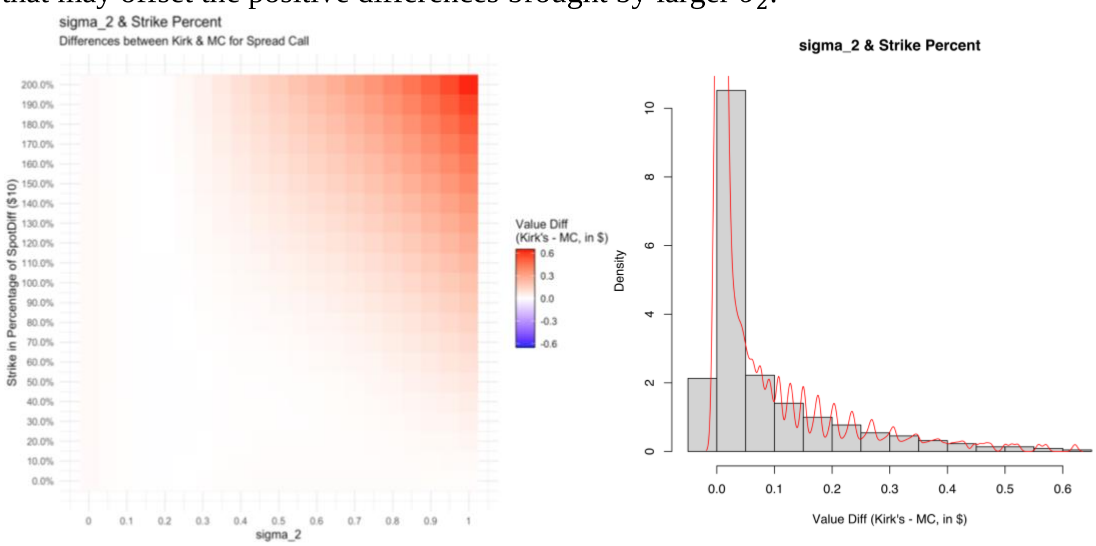
Corresponding to Scenario 2 and 4, the value differences between Kirk's and MC's among the inspected σ_1 vs. $\frac{K}{SpotDiff}$ pairs are plotted in Fig.8. The value difference ranging from \$0.00 to \$0.16 has its mean at \$0.05 and median at \$0.04 with a positive skewness of 0.71 and a negative excess kurtosis of -0.74.



(a) (b)

Figure 9. The differences between Kirk's and MC among various
σ_2 and ρ, as in (a) grid plot, and in (b) histogram.

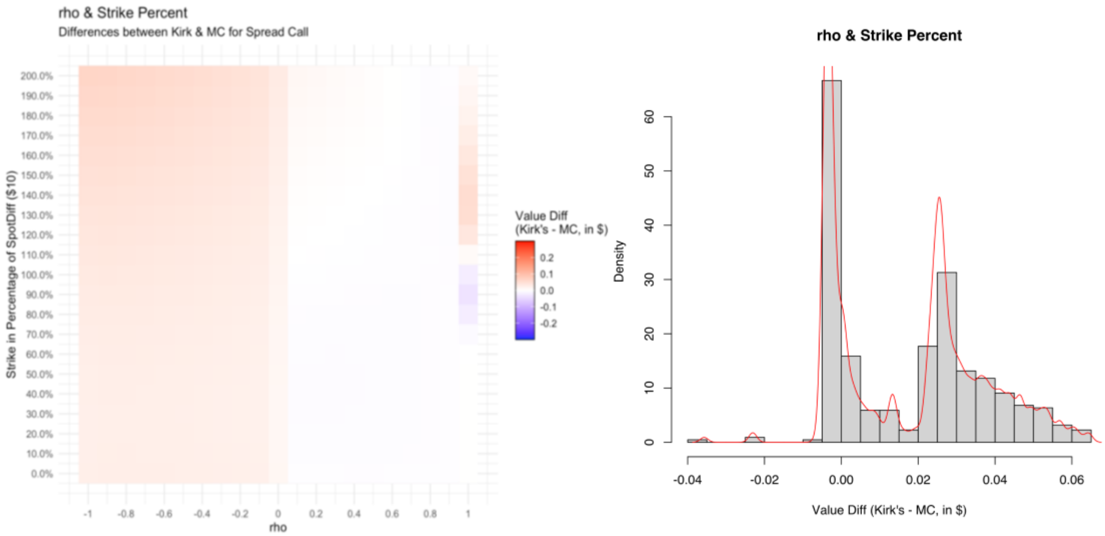
Corresponding to Scenario 2 and 3, the value differences between Kirk's and MC's among the inspected σ_2 vs. ρ pairs are plotted in Fig.9. The value difference ranging from \$-0.04 to \$0.20 has its mean at \$0.04 and median at \$0.02 with a positive skewness of 1.23 and a positive excess kurtosis of 1.34. Positive ρ would bring negative differences that may offset the positive differences brought by larger σ_2.



(a) (b)

Figure 10. The differences between Kirk's and MC among various
σ_2 and Strike in percentage of $SpotDiff$, as in (a) grid plot, and in (b) histogram.

Corresponding to Scenario 2 and 4, the value differences between Kirk's and MC's among the inspected σ_2 vs. $\frac{K}{SpotDiff}$ pairs are plotted in Fig.10. The value difference ranging from \$0.00 to \$0.62 has its mean at \$0.08 and median at $0.02 with a positive skewness of 2.08 and a positive excess kurtosis of 4.17. Both larger σ_2 and larger Strike would bring positive value differences.



(a) (b)

Figure 11. The differences between Kirk's and MC among various ρ and Strike in percentage of $SpotDiff$, as in (a) grid plot, and in (b) histogram.

The value differences between Kirk's and MC's among the inspected ρ vs. $\frac{K}{SpotDiff}$ pairs are plotted in Fig.11. The value difference ranging from $-0.04 to $0.06 has its mean at $0.02 and median at $0.02 with a positive skewness of 0.31 and a negative excess kurtosis of -0.99.

4 Conclusion

In this project we investigated and implemented Monte Carlo together with Kirk's approximation on spread call options pricing, then further compared their performances in various scenarios. Generally, larger spot differences or larger σ_1 would bring positive value differences. At higher σ_2 level, larger spot differences would bring negative value differences. Generally, positive correlation would bring negative value differences, and vice versa. The difference is more visible when the Strike is around the differences between two Spots. Both larger σ_1 and larger σ_2 would bring positive value differences. The difference is minimized when $\sigma_1 = \sigma_2$. Positive ρ would bring negative differences that may offset the positive differences brought by larger σ_2.

For future research, we would investigate the outcome of Kirk's approximation in spread put option pricing, and compare its numerical performances with other approximation methods **[4], [8]**.

5 References

[1] Paul Wilmott, *Paul Wilmott on Quantitative Finance*. 2006.

[2] W. Margrabe, "The Value of an Option to Exchange One Asset for Another," *J. Finance*, vol. 33, no. 1, p. 177, 1978.

[3] J. Choi, "Sum of all Black–Scholes–Merton models: An efficient pricing method for spread, basket, and Asian options," *J. Futur. Mark.*, vol. 38, no. 6, pp. 627–644, 2018.

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[5] E. Kirk and J. Aron, “Correlation in the energy markets,” *Manag. energy price risk*, vol. 1, pp. 71--78, 1995.

[6] H. Wickham *et al.*, “Welcome to the Tidyverse,” *J. Open Source Softw.*, vol. 4, no. 43, p. 1686, 2019.

[7] H. Wickham, *ggplot2: Elegant Graphics for Data Analysis*. 2016.

[8] N. D. Pearson, “An Efficient Approach for Pricing Spread Options,” *J. Deriv.*, vol. 3, no. 1, pp. 76–91, 1995.

6 Appendix

A. Spread Call Monte Carlo Pricing and Kirk’s Approximation in Python Codes, by Gao Jichen

B. Comparison & Visualization in R Codes, by Cheng Tuoyuan