**FE5222 ADP Project One**

Ho Ngok Chao, Gao Jichen, Cheng Tuoyuan

# Introduction

In this project, we would implement the least square Monte Carlo (LSMC) method for American put options pricing, and employ the binomial Black-Scholes model with Richardson extrapolation (BBSR) as a benchmark. Results from both methods are compared and discussed in various scenarios. The effects of numerical parameters such as the number of simulation paths as well as the number of discrete time steps are further investigated and visualized.

As a simulation-based model, LSMC can be extended with factor models. It allows for path-dependent and early exercise features, and permits block parallelization. Overall, it’s intuitive to understand, and transparent or flexible to implement. At the same time, the simulation per path and regression per time-step inside are time-consuming, which asks up to balance between MC simulation counts and time step counts [2,3,6].

As a tree-based model, BBS is improved from traditional binomial trees by applying BSM on option values at the step, which were difficult for traditional binomial trees to consider the time value of OTM options. BBSR further involved the Richardson extrapolation technique to cancel out higher-order error terms while keeping its simplicity and adding limited computational costs. Disadvantages from binomial trees were still inherited inside. Moreover, it’s not flexible enough to cope with incomplete market or extreme scenario tests [1].

# Materials and Methods

## Least Square Monte Carlo (LSMC)

Summarizing from the lecture notes while using more generalized timestamp, if the American Put option is alive within the time horizon [0, T], early exercise is only allowed at discrete times . For a particular exercise date , early exercise is performed if the payoff from immediate exercise exceeds the continuation value. This continuation value can be expressed as conditional expectation.

The initial step of the actual algorithm is to determine the cashflow vector: at the last timestep whose continuation values are zero. For each stock price path :

Second, we need to simulate the exercise value at timestamp :

In order to obtain continuation values, we regress the discounted future value realized from continuing ) onto polynomial of the spot price with least square fit. The regression is done by using the values from all of the stock price paths and after which we obtain the coefficients-betas. The continuation value for a stock price path with values at time is:

Once we have the continuation values and exercise values, we will perform early exercise if exercise value is larger than continuation value and obtain the value of the option discounted to timestamp :

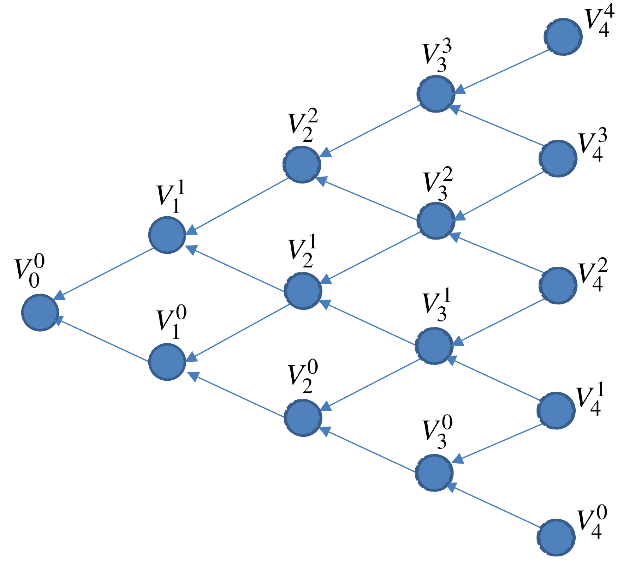
We repeat this process for backward one step at a time . At each time-step, we keep the value of the put option discounted to .

Finally, we take average of and get the value of the option at time 0.

## BBSR

### Binomial tree

Binomial tree method is frequently used in the calculation of option prices. The time between intervals is , time to final maturity is . A multi-period binomial can be shown as:



For European options, the option price at node is calculated by , where . For the American put option that we need to price in the project, its option price is calculated by .

### Black-Scholes binomial tree method

BBS method is a modification to the binomial method where the Black-Scholes formula replaces the usual “continuation value” at the time step just before option maturity.

To compute the price of an American put option, we start with the leaf nodes . These nodes correspond to and the value of option is . At time , the continuation value is equivalent to the price of a European put option, replace it with BS formula for put option, that is:

Notice that is the price of a European put option with time to maturity and spot .

### Binomial Black–Scholes Method with Richardson Extrapolation

In numerical analysis, Richardson extrapolation is a sequence acceleration method, used to improve the rate of convergence of a sequence of estimates of some value The formula is:

Using the step sizes and for some constant , the two formulas for are:

Multiplying the second equation and subtracting the first equation gives:

Then we can solve the equation to get . Combined BBS model and Richardson Extrapolation, the option price we want is in the above formula. That is the BBSR model.

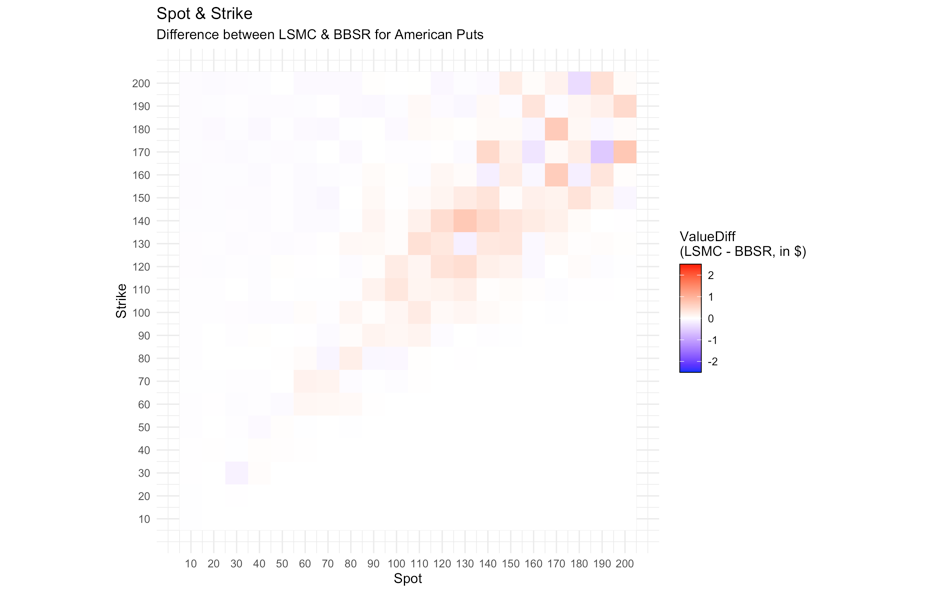
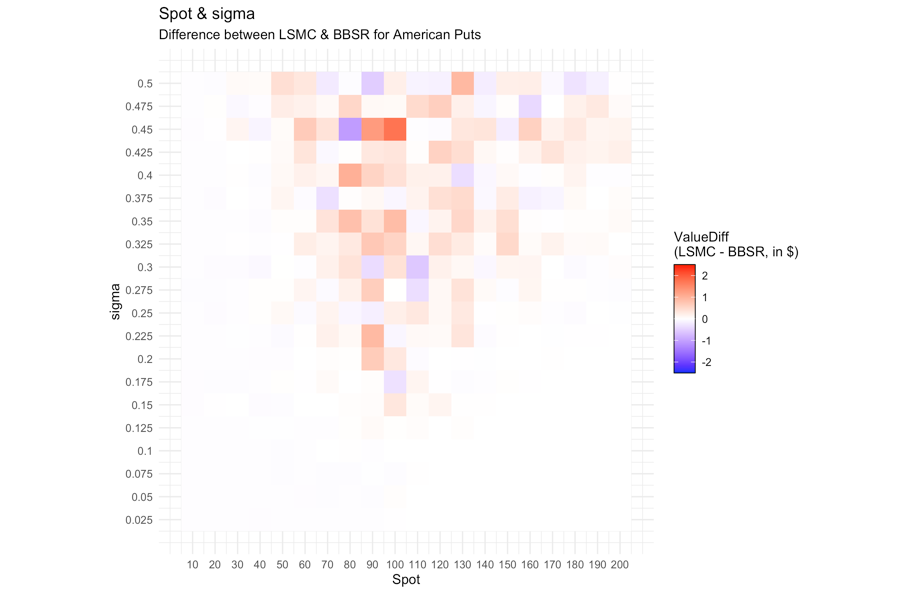
## Comparison and visualization

To compare the two investigated models with respect to the **pricing parameters** and **numerical parameters**, we further performed pairwise pricing and visualized their differences (LSMC - BBSR) over two dimensional grids with red-white-blue color scales. To improve comparability, we kept the color bar from $-2.5 to $2.5 and centered around $0. Investigated pricing parameters with default values includes: Spot = $100, Strike = $100, volatility () = 0.2, interest rate () = 0.06, and time to maturity () = 1. They are explored using equal spaced grids in corresponding plots but kept constant otherwise.

Investigated numerical parameters with default values includes: number of Monte Carlo paths () = 1000, number of time steps () = 252. We plotted the LSMC outputs at different and with the color bar centered around the BBSR benchmark value, then charted the BBSR outputs evolution at different . Visualizations were implemented in RStudio via packages ‘tidyverse’ and ‘ggplot2’ [4,5].

# Results and Discussion

## Grid comparison for pricing parameters

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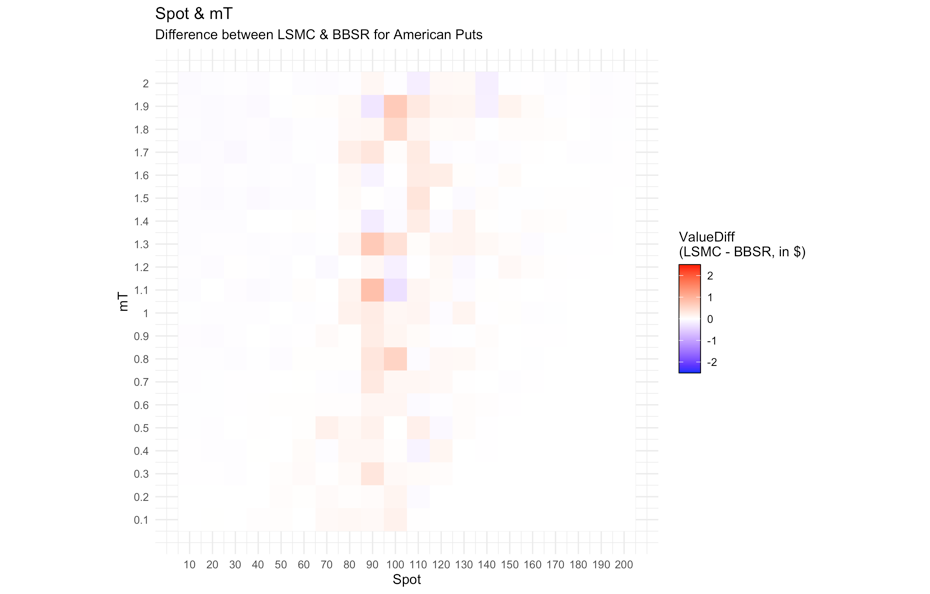
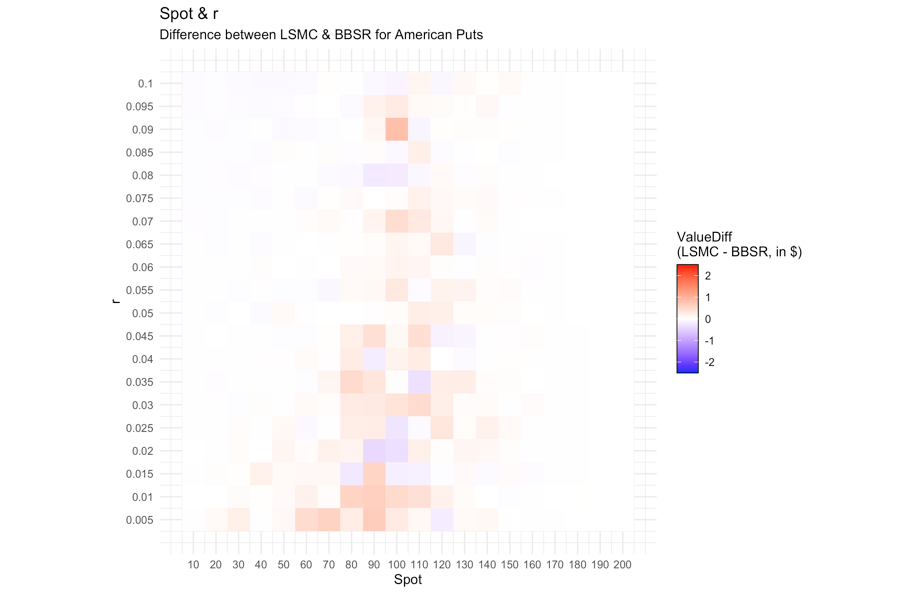
(a) (b)

Figure 1. The differences between LSMC and BBSR among various

(a) spot and strike, as well as (b) spot and volatility ().

From Fig.1(a), we observed that the LSMC outputs are deviated from BBSR less than $1 in absolute values within the inspected Spot-Strike pairs. The oscillation occurs mainly when the Spot is close to Strike and the differences are mainly positive. There might be more alternating between ITM and OTM if continue to hold for LSMC, in such scenarios upward bias due to the path used to calculate the cash flow also used for regression, more obvious (early exercise by look-ahead OTM); while the binomial trees are well arranged with approximately half leaves OTM and half leaves ITM at expiry.

From Fig.1(b), we observed that the LSMC outputs are deviated from BBSR less than 2$ in absolute values within the inspected Spot- pairs. The oscillation grows with and gets stronger when the Spot is around the Strike, with a slight skew towards higher spot. With higher , the log-Normal distribution at each time step in the GBM paths would have higher mean and lead to stronger payoff asymmetry. In comparison, BBSR only involves sigma in the step.

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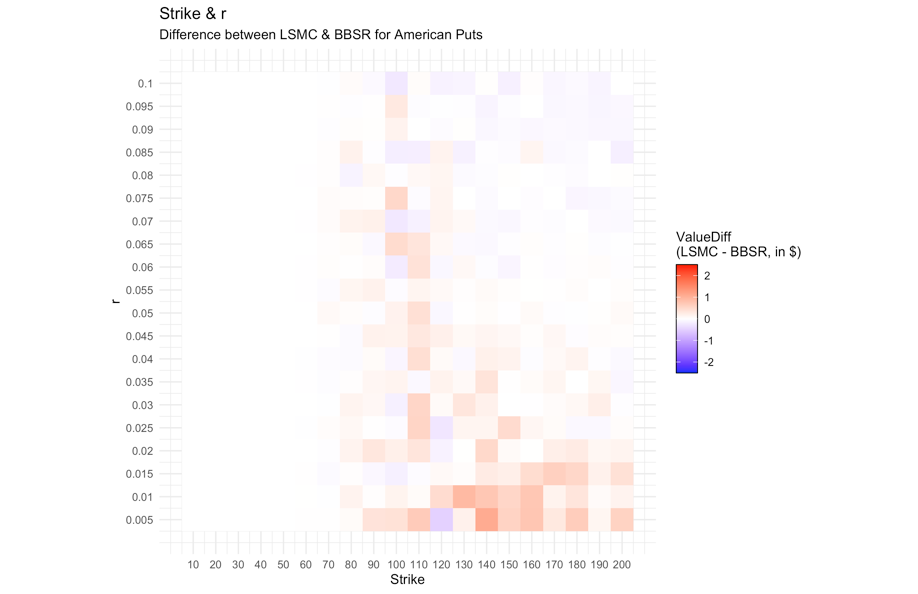
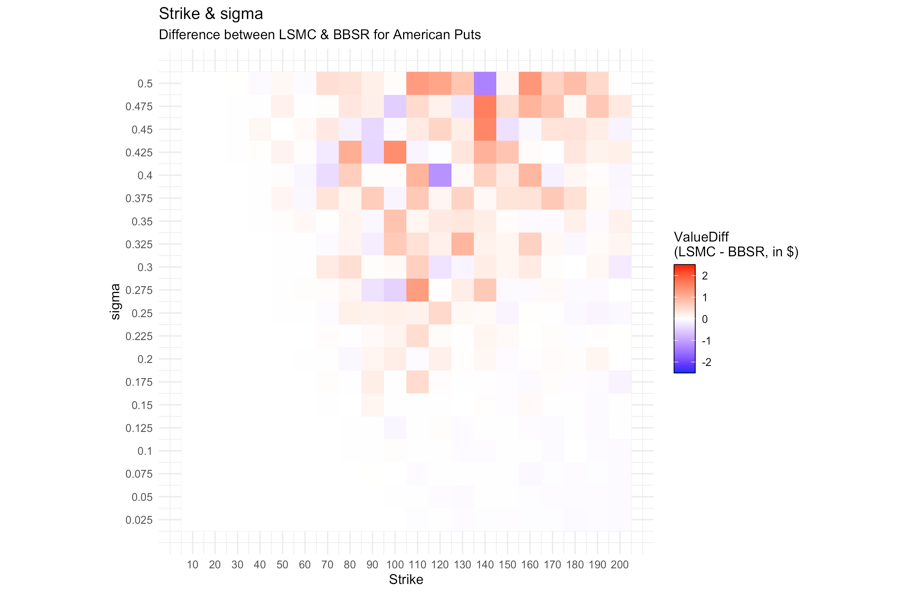
(a) (b)

Figure 2. The differences between LSMC and BBSR among various

(a) spot and interest rate (), as well as (b) spot and time to expiry ().

From Fig.2(a), we observed that the LSMC outputs are deviated from BBSR less than 1$ in absolute values within the inspected Spot-interest rate pairs. The oscillation shrinks with and gets stronger when the Spot is around the Strike. Higher could be bounding the discounted payoff and thus limiting the errors.

From Fig.2(b), we observed that the LSMC outputs are deviated from BBSR less than 1$ in absolute values within the inspected Spot-time to maturity pairs. The oscillation of the differences between two models grows with and gets stronger when the Spot is around the Strike. Longer , given constant time steps , is involving wilder fluctuation per time step in the GBM paths and thus accumulates randomness.

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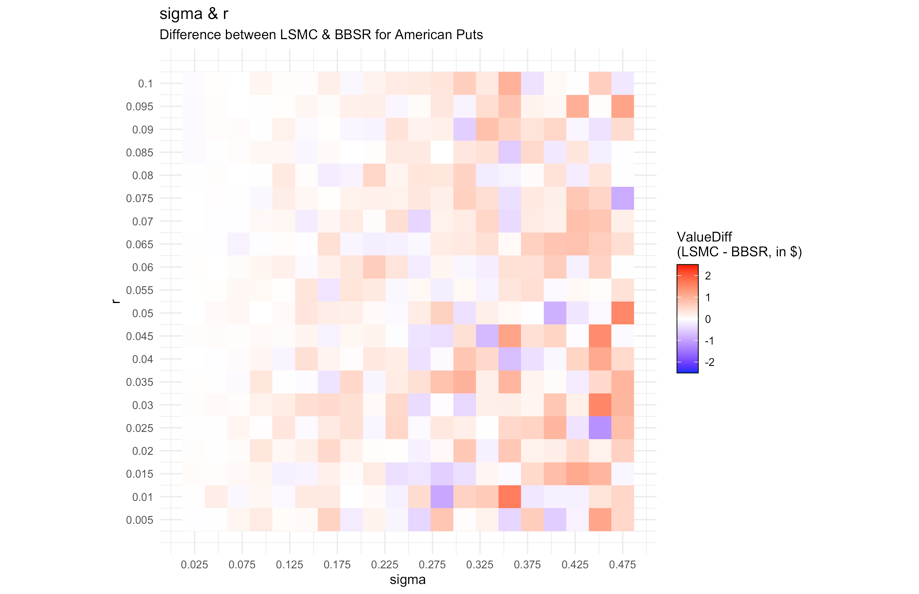
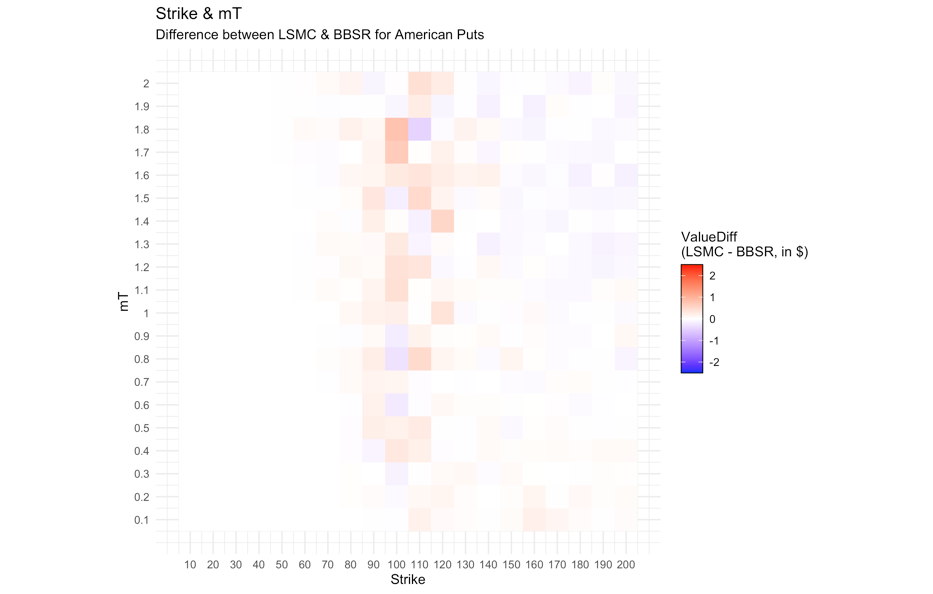
(a) (b)

Figure 3. The differences between LSMC and BBSR among various

(a) strike and , as well as (b) strike and interest rate.

From Fig.3(a), we observed that the LSMC outputs are deviated from BBSR less than 2$ in absolute values within the inspected Strike- pairs. The oscillation grows with sigma and gets stronger when the Strike is around the Spot, with a slight skew towards higher Strike. With higher , the log-Normal distribution at each time step in the GBM paths would have higher mean and lead to stronger payoff asymmetry. In comparison, BBSR only involves sigma in the step. Higher Strike as allowing more paths to be ITM could be inviting more randomness and thus the asymmetry.

From Fig.3(b), we observed that the LSMC outputs are deviated from BBSR less than 1$ in absolute values within the inspected Strike-interest rate pairs. The oscillation shrinks with and has a skew towards higher Strike. Higher could be bounding the discounted payoff and thus limiting the errors. Higher Strike as allowing more paths to be ITM could be inviting more randomness and thus the asymmetry.

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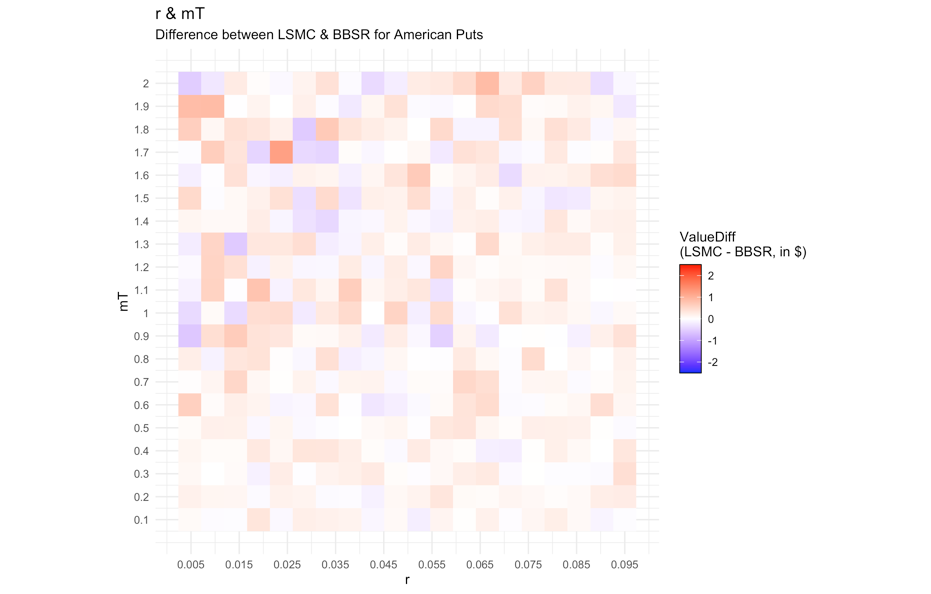
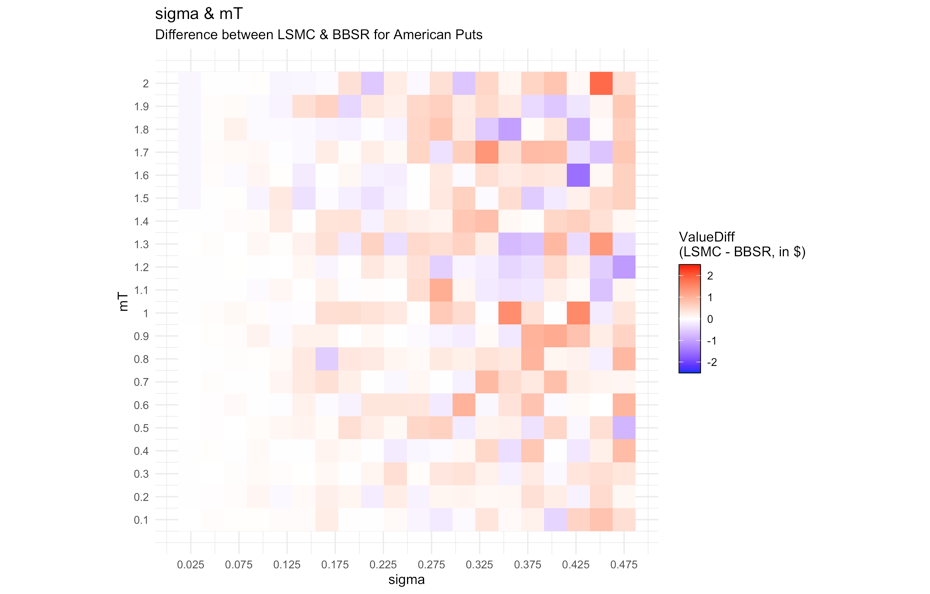
(a) (b)

Figure 4. The differences between LSMC and BBSR among various

1. strike and time to expiry, as well as (b) volatility and interest rate.

From Fig.4(a), we observed that the LSMC outputs are deviated from BBSR less than 1$ in absolute values within the inspected Strike-time to maturity pairs. The oscillation grows with and has a slight skew towards higher Strike. Longer , given constant time steps , is involving wilder fluctuation per time step in the GBM paths and thus accumulates randomness. Higher Strike as allowing more paths to be ITM could be inviting more randomness and thus the asymmetry.

From Fig.4(b), we observed that the LSMC outputs are deviated from BBSR less than 2$ in absolute values within the inspected -interest rate pairs. The oscillation shrinks slightly with increasing but grows significantly with . Higher could be bounding the discounted payoff and thus limiting the errors. With higher , the log-Normal distribution at each time step in the GBM paths would have higher mean and lead to stronger payoff asymmetry. In comparison, BBSR only involves sigma at the step.



(a) (b)

Figure 5. The differences between LSMC and BBSR among various

(a) volatility and time to expiry, as well as (b) interest rate and time to expiry.

From Fig.5(a), we observed that the LSMC outputs are deviated from BBSR less than 2$ in absolute values within the inspected -time to maturity pairs. The oscillation grows gradually with increasing and grows rapidly with increasing . Increments in involves more randomness than increments in .

From Fig.5(b), we observed that the LSMC outputs are deviated from BBSR less than 1$ in absolute values within the inspected interest rate-time to maturity pairs. The oscillation grows with longer but shrinks slowly with increasing . Increments in involves more randomness than decrements in .

## Grid comparison for numerical parameters

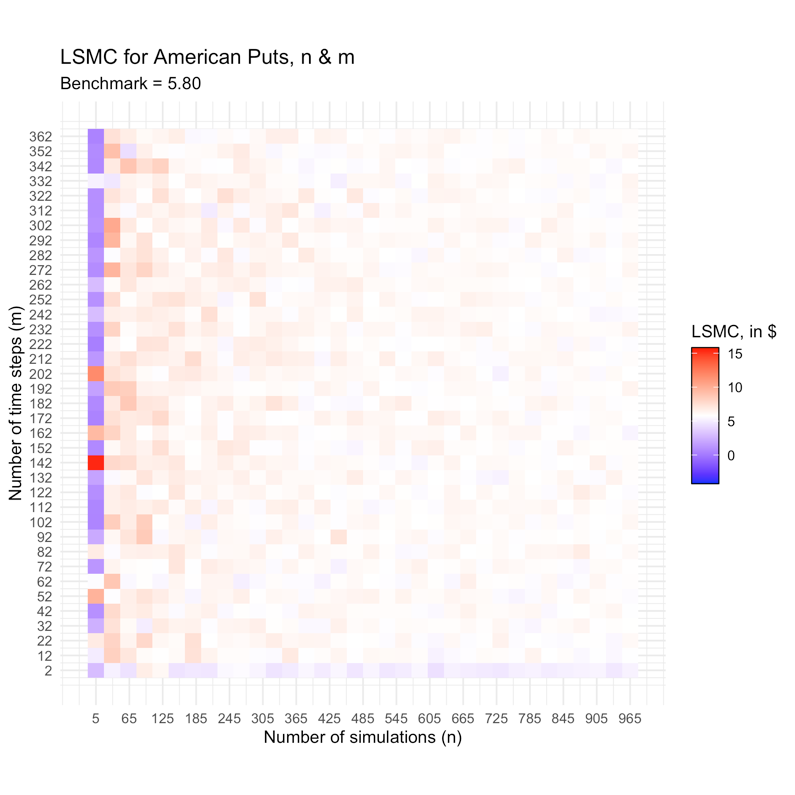


Figure 6. The LSMC results among various numbers of Monte Carlo simulations () and numbers of time steps (). We are assuming Spot = 100, Strike = 100, = 0.2, = 0.06, and =1. The white stripe in the color bar is set at the BBSR benchmark value $ 5.80.

From Fig.6, we noticed that the increase in number of simulations () until around 300 is significantly improving LSMC results towards the benchmark value, while the effect from the increase in number of time steps (m) after 2 steps might be negligible in comparison. It is likely that BBSR has not yet converged at = 2. To make a trade-off between time steps and simulation paths to price with satisfactory accuracy and acceptable computational complexity, we may constrain from 5 to 10 and maximize within acceptable computation time.

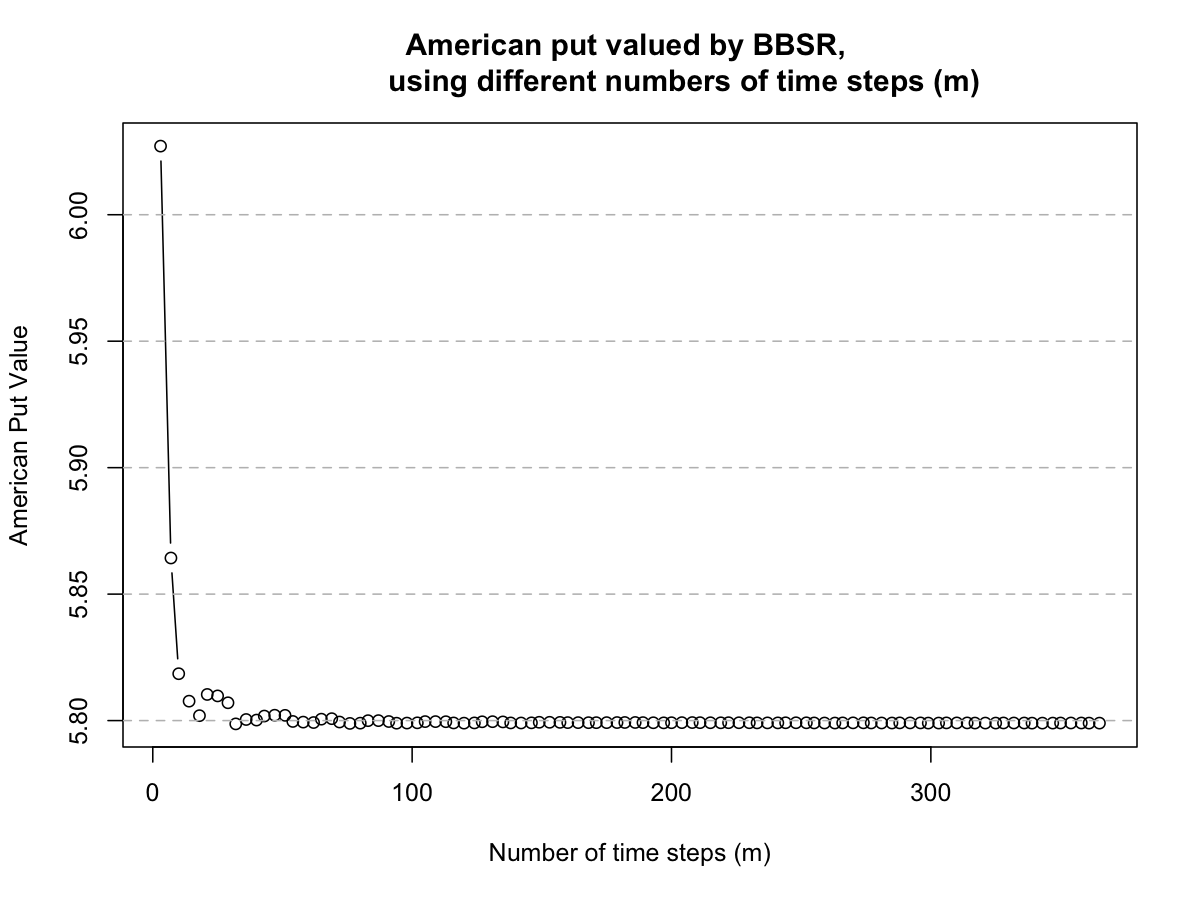


Figure 7. The BBSR results among various numbers of time steps ().

We are assuming Spot = , Strike = , = , = , = .

From Fig.7 we noticed that BBSR converges efficiently within time steps, and remains stable afterward. It is serving as a good benchmark for vanilla American option pricing.

## Conclusion and Future Research

In this project we investigated and implemented LSMC together with BBSR on American put options pricing, then further compared their performances in scenarios with various pricing parameters and numerical parameters. BBSR as a binomial-tree-based model showed efficient convergence within time steps and served as a stable benchmark. LSMC as a simulation-based model showed flexibility and good convergence using more than time steps and sample paths. The difference between LSMC and BBSR oscillates stronger when the Spot is around the Strike. With higher , the log-Normal distribution at each time step in the GBM paths would have higher mean and lead to stronger payoff asymmetry. In comparison, BBSR only involves sigma in the step. The oscillation shrinks with and gets stronger when the Spot is around the Strike. Higher could be bounding the discounted payoff and thus limiting the errors. Longer , given constant time steps , is involving wilder fluctuation per time step in the GBM paths and thus accumulates randomness. Higher Strike as allowing more paths to be ITM could be inviting more randomness and thus the asymmetry. Increments in involves more randomness than increments in . Increments in involves more randomness than decrements in .

For future research, we would like to experiment different basis functions in the regression, and apply LSMC on other path-dependent options. Their performances would be compared with other tree-based models.

# References

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3. Longstaff, F.A., and Schwartz, E.S., 2001. Valuing American options by simulation: a simple least-squares approach. The review of financial studies, 14(1), pp.113-147.
4. Wickham, H., Averick, M., Bryan, J., Chang, W., McGowan, L.D.A., François, R., Grolemund, G., Hayes, A., Henry, L., Hester, J. and Kuhn, M., 2019. Welcome to the Tidyverse. Journal of Open Source Software, 4(43), p.1686.
5. Wickham, H., 2016. ggplot2: elegant graphics for data analysis. springer.
6. Wilmott, P., 2007. Paul Wilmott introduces quantitative finance. John Wiley & Sons.

# Appendix:

1. LSMC in Python Jupyter notebook, by Ho Ngok Chao
2. BBSR in Python Jupyter notebook, by Gao Jichen
3. Comparison & Visualization in R Codes, by Cheng Tuoyuan