FE5227 COMMODITIES: FUNDAMENTAL AND MODELLING

Assignment 3

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1 Abstract

Our project models two underlying future contract namely, the WTI Light Sweet Crude Oil Futures and Brent Crude Futures.

$$\frac{dF(t, T_{WTI})}{F(t, T_{WTI})} = \sigma(t) \sum_{i=1}^{2} \sigma_i(T_{WTI} - t) dZ_i(t)$$

$$\frac{dF(t, T_{Brent})}{F(t, T_{Brent})} = \sigma(t) \sum_{i=1}^{2} \sigma_i(T_{Brent} - t) dZ_i(t)$$

$\sum_{i=1}^{2} \sigma_i(T_{WTI} - t)$, $\sum_{i=1}^{2} \sigma_i(T_{Brent} - t)$ come from two seperate PCA. The additional $\sigma(t)$ is calibrated on t=2018/05/16 using WTI European Option to reflect current market volatility.

At the end, we use $F(t, T_{WTI}), F(t, T_{Brent})$ as starting point and find $F(t+1, T_{WTI})$ and $F(t+1, T_{Brent})$ according our two forward curve models and use 3 different methods namely, Monte Carlo, Moment Matching and Kirk's approximation to price a European call spread option whose value is $max(F(t+1, T_{WTI}) - F(t+1, T_{Brent}) - K, 0)$ with expiry at t+1.

2 Data Specs

The following sets of data are used for our models. The attached csv files contain all the raw data.

2.1 Futures

- NYMEX WTI Light Sweet Crude Oil Futures (Ticker Symbol CL)
  The WTI future is one of the world's most liquid and actively traded futures. There are nearly 1.2 million contracts trade daily, with more than 2 million contracts in open interest. It is hence considered as a global benchmark in the crude oil industry. The future stops trading at the end of the designated settlement period on the 4th US business day prior to the 25th calendar day of the month preceding the contract month.

- Brent Crude Futures (Ticker Symbol B)
  Brent Crude Futures is also one of the most popular light sweet crude oil futures worldwide. It is used to price two thirds of the world's internationally traded crude oil supplies. The trading ceases at the end of the designated settlement period on the last business day of the second month preceding the relevant contract month, which is different from that of WTI.

2.2 Options

European options with the above mentioned underlying futures (WTI and Brent) are used for our models. The WTI European options terminates 3 business days before the termination of trading in the underlying futures contract. The Brent European options terminates the 5th last London business day of the month 2 months prior to the contact month. The strike price (K) of the call options are assumed to be 60 USD. $Optionvalue = Max(UnderlyingFuturePrice–K, 0)$.

2.3 Implied Volatilities

The implied volatilities of the European call options in our report are extracted from the $\{OVDV < GO >\}$ function in Bloomberg, which assumes Black-scholes Model. We took 50 Delta implied volatility (ATM) since 60 strike is relative close to the price of the underlying futures.

3 PCA model

All data is assumed to be available on 2018/05/16 to ensure there is enough data for PCA. Forward contract data mentioned above are used in two separate PCA. Δt of both PCA is set to be 1 day. Tau of PCA_{WTI} is set to be $\tau_1 = 5, \tau_2 = 6, \tau_3 = 7, \tau_4 = 8$ days. Tau of PCA_{Brent} is set to be

$\tau_1 = 14, \tau_2 = 15, \tau_3 = 16, \tau_4 = 17$ days. The Tau of PCA_{WTI} is chosen so that $F(te, te + \tau_1)$ can be determined which is later used in calibration.

$$\frac{dF(t,T)}{F(t,T)} = \sum_{i=1}^{n} \sigma_i(T-t)dZ_i(t)$$

$$d\ln F(t,T) = -\frac{1}{2}\sum_{i=1}^{n} \sigma_i(T-t)^2 dt + \sum_{i=1}^{n} \sigma_i(T-t)dZ_i(t)$$

Choose a grid for time to maturities $\quad 0 < \tau_1 < \cdots < \tau_n$

$$\Delta \ln F(t, t+\tau_j) = -\frac{1}{2}\sum_{i=1}^{n} \sigma_i(\tau_j)^2 \Delta t + \sum_{i=1}^{n} \sigma_i(\tau_j)\Delta Z_i(t)$$

$$\begin{pmatrix} \Delta \ln F(t, t+\tau_1) \\ \vdots \\ \Delta \ln F(t, t+\tau_j) \\ \vdots \\ \Delta \ln F(t, t+\tau_n) \end{pmatrix} = \overline{\mu}\Delta t + \begin{pmatrix} \sigma_1(\tau_1)\cdots\sigma_n(\tau_1) \\ \vdots \\ \sigma_1(\tau_n)\cdots\sigma_n(\tau_n) \end{pmatrix} \begin{pmatrix} \Delta Z_1(t) \\ \vdots \\ \Delta Z_n(t) \end{pmatrix}$$

Hence:

$$\begin{pmatrix} \Delta \ln F(t, t+\tau_1)/\Delta t \\ \vdots \\ \Delta \ln F(t, t+\tau_j)/\Delta t \\ \vdots \\ \Delta \ln F(t, t+\tau_n)/\Delta t \end{pmatrix} \sim N(\overline{\mu}, C) \qquad (*)$$

where $C = \sum \sum^T$ and $\sum = \begin{pmatrix} \sigma_1(\tau_1)\cdots\sigma_n(\tau_1) \\ \vdots \\ \sigma_1(\tau_n)\cdots\sigma_n(\tau_n) \end{pmatrix}$

Estimate covariance matrix C with historical data of LHS of (*)

Using spectral decomposition $\quad C = \Gamma\Lambda\Gamma^T$

Take

$$\Sigma = \Gamma\Lambda^{\frac{1}{2}}$$

$$\sigma_i(\tau_j) = \Gamma_{j,i}\sqrt{\lambda_i}$$

Dimension reduction and add additional sigma(t) term:

$$\Delta \ln F(t, t+\tau_j) = -\frac{1}{2}\sigma^2(t)\sum_{i=1}^{2} \Gamma_{j,i}^2 \lambda_i \Delta t + \sigma(t)\sum_{i=1}^{2} \Gamma_{j,i}\sqrt{\lambda_i}\Delta Z_i(t)$$

4 Calibration

Table 1: WTI European Option and Corresponding Forward

t	te	K	T	Future Price	Call Option Value	Implied Vol	te-t	T-te
18/05/16	18/05/17	62.22	18/05/22	71.49	9270.00	23.86	1	5

This entry of WTI European option is used to calibrate the additional $\sigma(t)$. As there are no Brent European option who shares the same expiry date with WTI European option, we choose to calibrate againist WTI European option and assume the two models share the same additional $\sigma(t)$.

However, under such setting, model implied volatility won't match market volatility and use similar calculation (total variance divided by $(t_e - t)$ as in calibration, we can derive the model implied volatility for Kirk's for Brent is 0.4965824.

$$\Delta \ln F(t, t+\tau_j) = -\frac{1}{2}\sigma^2(t)\sum_{i=1}^{2}\Gamma_{j,i}^2\lambda_i\Delta t + \sigma(t)\sum_{i=1}^{2}\Gamma_{j,i}\sqrt{\lambda_i}\Delta Z_i(t)$$

$F(t,T)$ is known, $t_e - t = 1$

$$F(t_e, T) = F(t,T)e^{-\frac{1}{2}\sigma^2(t)\sum_{i=1}^{2}\Gamma_{j,i}^2\lambda_i(t_e-t)+\sigma(t)\sum_{i=1}^{2}\Gamma_{j,i}\sqrt{\lambda_i}(Z_i(t_e)-Z_i(t))}$$

Total variance= $\sigma^2(t)\sum_{i=1}^{2}\Gamma_{j,i}^2\lambda_i(t_e - t)$

$$\overline{\sigma_1}^2 = \text{Total variance}/(t_e - t) \Rightarrow \sigma(t) = \frac{\overline{\sigma_1}}{\sqrt{\sum_{i=1}^{2}\Gamma_{j,i}^2\lambda_i}}$$

Note that $\overline{\sigma_1}$ is the known market implied volatility from Black Scholes model with regard to the underlying and our model now completely fits this volatility; therefore, $\overline{\sigma_1}$ will be used for Kirk's approximation for WTI.

5 Experiment Setting

Table 2: Settings for Pricing Methods

Input	Name	Description	Value of Input
$F_{t_{WTI}}$	WTI Future Price	WTI May-16 Futures price in USD/bbl;Expiry 22 May 2018;	USD71.49
$F_{t_{Brent}}$	Brent Future Price	Brent May-16 Futures price in USD/bbl;Expiry 31 May 2018;	USD79.28
$\sigma(t)$	Additional term of volatility	Volatility that fits current market volatility	42.10212
$(\Gamma_{WTI,1}, \Gamma_{WTI,2})$	Eigen vector for WTI	The τ of volatility is 5 days	(0.14,0.18)
$(\Gamma_{Brent,1}, \Gamma_{Brent,2})$	Eigen vector for Brent	The τ of volatility is 14 days	(-0.20,0.69)
λ_{WTI_1}	First eigen value of WTI.	The largest eigen value get from PCA	0.00104
λ_{WTI_2}	Second eigen value of WTI.	The second largest eigen value get from PCA	0.00038
λ_{Brent_1}	First eigen value of Brent.	The largest eigen value get from PCA	0.00031
λ_{Brent_2}	Second eigen value of Brent.	The second largest eigen value get from PCA	0.00027
σ_{WTI}	Market implied volatility of WTI.	Market implied volatility on WTI May-16 Futures price	0.2386
σ_{Brent}	Market implied volatility of Brent.	Market implied volatility on Brent May-16 Futures price	0.4965
ρ	Correlation between the price of two futures.	Calculate and Brent future from 2017.05.10 to 2018.05.10.	0.9287
$T - t$	Time to maturity of spread european option.		1/252 in year

6 Monte Carlo

As the payoff of spread option is path dependent, Monte Carlo Method is an ideal way for pricing it. Due to the higher accuracy of Monte Carlo, we set it as a benchmark. The Monte Carlo pricing method can be simply divided into three steps:

step 1 Calculate potential future prices of the underlying assets.
According to the following equation, which is shown in the slides, we select a stochastic model for the time evolution of the underlying assets and then simulate the model through time. We assume Δt is one day and repeat using this equation for 1000 times to generate 1000 different potential paths of two futures.

$$\begin{pmatrix} \Delta \ln F(t, t+\tau_1) \\ \vdots \\ \Delta \ln F(t, t+\tau_j) \\ \vdots \\ \Delta \ln F(t, t+\tau_n) \end{pmatrix} = \overline{\mu}\Delta t + \begin{pmatrix} \sigma_1(\tau_1)\cdots\sigma_n(\tau_1) \\ \vdots \\ \sigma_1(\tau_n)\cdots\sigma_n(\tau_n) \end{pmatrix} \begin{pmatrix} \Delta Z_1(t) \\ \vdots \\ \Delta Z_n(t) \end{pmatrix}$$

step 2 Calculate the payoff of the option for each of the potential underlying price paths according to the following equation:

$$Payoff_{call} = max\{F1 - F2 - K, 0\}$$

step 3 Discount the payoffs back to today and average them to determine the expected price.(For simplicity, we assume the risk free rate is zero, so there's no need for us to discount the payoffs back to today.)

When we choose different strike price from 1 to 10 and try different times of simulation,we get the following results:

Table 3: Result of Monte Carlo for 1000 times simulation

Strike Price	1	2	3	4	5	6	7	8	9	10
Value of Option	7.1044	6.3428	4.9966	3.6001	2.9079	2.7931	1.8703	1.3147	0.8979	0.6906

Table 4: Result of Monte Carlo for 10000 times simulation

Strike Price	1	2	3	4	5	6	7	8	9	10
Value of Option	5.3138	4.5178	3.7989	3.0057	2.4103	1.7887	1.4090	1.0020	0.7145	0.4744

Table 5: Result of Monte Carlo for 50000 times simulation

Strike Price	1	2	3	4	5	6	7	8	9	10
Value of Option	5.3323	4.4675	3.7340	3.0185	2.3815	1.8541	1.3876	1.0123	0.7147	0.4754

Table 6: Result of Monte Carlo for 100000 times simulation

Strike Price	1	2	3	4	5	6	7	8	9	10
Value of Option	5.3374	4.4976	3.7267	3.0327	2.3880	1.8602	1.3983	1.0009	0.6984	0.4813

7 Moment Matching

Spread basket option price a random variable, we don't know its distribution but we can calculate its moments. We can approximate its distribution by lognormal distribution. Then we can use the approximate distribution to price it.

step 1 Moments of Spread Basket Price.
Since our option is a spread option, we assume $\omega_1 = 1, \omega_2 = -1$, correlation for all brownian motion $\rho = 0.8$.
Our basket option can be composed as:

$$B(t) = \sum_{i=1}^{2} \omega_i F(t, T_e)$$

$$F(t, T_e) = F(0, T_e) e^{-\frac{1}{2}\sigma^2(t)\sum_{i=1}^{2}\sigma_i^2(T_e - t)t + \sigma(t)\sum_{i=1}^{2}\sigma_i(T_e - t)Z_{i,e}(t)}$$

with $\sigma(t)$ defined in calibration, $\sigma_i(T_e - t)$ defined in PCA.

$$\sigma_i(x) = \begin{cases} \text{PCA of WTI x=5} \\ \text{PCA of Brent x=14} \end{cases}$$

$$\begin{aligned}
E(B(t))^2 =& E(\sum_{a=1}^{2}\sum_{b=1}^{2} \omega_a\omega_b F(t, T_a)F(t, T_b)) \\
=& E(\sum_{a=1}^{2}\sum_{b=1}^{2} \omega_a\omega_b F(0, T_a)F(0, T_b) exp\{-\frac{1}{2}(\sigma^2(t)\sum_{i=1}^{2}\sigma_i^2(T_a - t) + \sigma^2(t)\sum_{i=1}^{2}\sigma_i^2(T_b - t))t + \\
& \sigma(t)\sum_{i=1}^{2}\sigma_i(T_a - t)Z_{i,a}(t) + \sigma(t)\sum_{i=1}^{2}\sigma_i(T_b - t)Z_ib(t)\}) \\
=& \sum_{a=1}^{2}\sum_{b=1}^{2} \omega_a\omega_b F(0, T_a)F(0, T_b) exp\{-\frac{1}{2}(\sigma^2(t)\sum_{i=1}^{2}\sigma_i^2(T_a - t) + \sigma^2(t)\sum_{i=1}^{2}\sigma_i^2(T_b - t))t + \\
& E(exp\{\sigma(t)\sum_{i=1}^{2}\sigma_i(T_a - t)Z_{i,a}(t) + \sigma(t)\sum_{i=1}^{2}\sigma_i(T_b - t)Z_{i,b}(t)\})
\end{aligned} \tag{1}$$

$$\begin{aligned}
E(\star)^2 =& E(\sigma(t)\sum_{i=1}^{2}\sigma_i(T_a - t)Z_{i,a}(t) + \sigma(t)\sum_{i=1}^{2}\sigma_i(T_b - t)Z_ib(t))^2 \\
=& E(\sigma^2(t)\sum_{i=1}^{2}\sigma_i^2(T_a - t)Z_{i,a}^2(t) + \sigma^2(t)\sum_{i=1}^{2}\sigma_i(T_b - t)Z_{i,b}^2(t) + \sigma^2(t)\prod_{i=1}^{2}\sigma_i(T_a - t)Z_{i,a}(t) + \\
& \sigma^2(t)\prod_{i=1}^{2}\sigma_i(T_b - t)Z_{i,b}(t) + 2\sigma^2(t)\sum_{i=1}^{2}\sum_{j=1}^{2}[\sigma_i(T_a - t)\sigma_j(T_b - t)Z_{i,a}(t)Z_{j,b}(t)]) \\
=& \sigma^2(t)\sum_{i=1}^{2}\sigma_i^2(T_a - t)t + \sigma^2(t)\sum_{i=1}^{2}\sigma_i^2(T_b - t)t + 2\sigma^2(t)\prod_{i=1}^{2}\sigma(i)(T_a - t)\rho_t + 2\sigma^2(t)\prod_{i=1}^{2}\sigma(i)(T_a - t)\rho_t + \\
& 2\sigma^2(t)\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_i(T_a - t)\sigma_j(T_b - t)
\end{aligned} \tag{2}$$

Using m.g.f of normal distribution $E(e^{tX}) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$, where $X \sim N(\mu, \sigma^2)$

$$E[e^\star] = E\{exp[\frac{\sigma^2(t)}{2}\sum_{i=1}^{2}\sigma_i(T_a - t) + \frac{\sigma^2(t)}{2}\sum_{i=1}^{2}\sigma_i(T_b - t) + \frac{\text{all term with } \rho}{2}]\}$$

Put it back to (1), we get:

$$\phi(X_1, X_2) = \prod_{i=1}^{2} \sigma_i(T_{X_1} - t) + \prod_{i=1}^{2} \sigma_i(T_{X_2} - t) + \sum_{i=1}^{2}\sum_{j=1}^{2} \sigma_i(T_{X_1} - t)\sigma_i(T_{X_2} - t)$$

$$E[(B(t))^2] = \sum_{a=1}^{2}\sum_{b=1}^{2} \omega_a\omega_b F(0,T_a)F(0,T_b)exp\{\sigma^2(t)\rho_t\phi(a,b)\}$$

Similarly, with all correlation constant

$$E[(B(t))^3] = \sum_{a=1}^{2}\sum_{b=1}^{2}\sum_{c=1}^{2} \omega_a\omega_b\omega_c F(0,T_a)F(0,T_b)F(0,T_c)exp\{\sigma(t)\rho_t(\phi(a,b)+\phi(a,c)+\phi(b,c))\}$$

step 2 Moments of Lognormal Distribution.
For any of the lognormal distribution, we can calculate its moments
For example, for the shifted normal distribution, we have
$T_1(\tau, m, \sigma^2) = \tau + e^{m+\frac{1}{2}\sigma^2}$
$T_2(\tau, m, \sigma^2) = \tau^2 + 2\tau e^{m+\frac{1}{2}\sigma^2} + e^{2m+2\sigma^2}$
$T_3(\tau, m, \sigma^2) = \tau^3 + 3\tau^2 e^{m+\frac{1}{2}\sigma^2} + 3\tau e^{2m+2\sigma^2} + e^{3m+\frac{9}{2}\sigma^2}$
we can easily calculate the moments of lognormal distribution with $\tau = 0$.
Similarly, for the shifted negative lognormal distribution, we have
$T_1(\tau, m, \sigma^2) = \tau - e^{m+\frac{1}{2}\sigma^2}$
$T_2(\tau, m, \sigma^2) = \tau^2 - 2\tau e^{m+\frac{1}{2}\sigma^2} + e^{2m+2\sigma^2}$
$T_3(\tau, m, \sigma^2) = \tau^3 - 3\tau^2 e^{m+\frac{1}{2}\sigma^2} + 3\tau e^{2m+2\sigma^2} - e^{3m+\frac{9}{2}\sigma^2}$
we can easily calculate the moments of negative lognormal distribution with $\tau = 0$.
Then we fit the parameters of the lognormal distribution such its first two or three moments match those of the basket options, i.e.

$$T_1(\tau, m, \sigma^2) = E[(B(t)]$$
$$T_2(\tau, m, \sigma^2) = E[(B(t))^2]$$
$$T_3(\tau, m, \sigma^2) = E[(B(t))^3]$$

We can easily solve τ, m, σ^2 from the system of non-linear equations.

step 3 Option Pricing using Black Scholes Model.
We assume the basket price is shifted lognormal distribution,

$$B(T) \approx e^{\mu+\sigma Y} + \tau \text{ where } Y \sim N(0,1)$$

Note that

$$E(B(t)) = \sum_{i=1}^{n} \omega_i F(0,T) = B(0)$$

From the approximation

$$E(B(t)) = e^{\mu+\frac{1}{2}\sigma^2} + \tau$$

Hence

$$B(0) = e^{\mu+\frac{1}{2}\sigma^2} + \tau$$

We can write B(T) as

$$B(T) = (B(0) - \tau)e^{-\frac{1}{2}\sigma^2+\sigma Y} + \tau$$

Denote $\tilde{B}(T) = B(t) - \tau$, we have

$$\tilde{B}(T) = \tilde{B}(0)e^{-\frac{1}{2}\sigma^2+\sigma Y}$$

We can easily find $\tilde{B}(T)$ is the solution for the GBM process
The call option value at time t=0 is

$$V(0) = E[max\{B(T) - K, 0\}] = E[max\{\tilde{B}(T) - \tilde{K}, 0\}]$$

where $\tilde{K} = K - \tau$ is the adjusted strike.
Finally, we can apply BS formula to an option on $\tilde{B}(T)$ with strike $\tilde{K}$, then we can price the basket option.

$$e^{-rT}(\tilde{B}N(\frac{\ln(\tilde{B}/K) + \frac{1}{2}\sigma^2}{\sigma}) - \tilde{K}N(\frac{\ln(\tilde{B}/K) - \frac{1}{2}\sigma^2}{\sigma}))$$

Table 7: Result of Moment Matching

Strike Price	1	2	3	4	5	6	7	8	9	10
Value of Option	5.1650	4.8871	4.6229	4.3718	4.1333	3.9068	3.6919	3.4880	3.2946	3.1114

8 Kirk's Approximation

Kirk's approximation represents a closed form approximation formula to price a European spread option. This numerical method is commonly used in energy and commodity markets.
A European spread option can be valued using the standard Black Scholes model(1973) by performing the following transformation, as originally shown by Kirk(1995):

$$c = max(S_1 - S_2 - X, 0) = max(\frac{S_1}{S_2 + X} - 1, 0) \times (S_2 + X)$$

Looking a pricing a call option on the Brent – WTI spread, it therefore follows:

$$c \approx (Q_2 S_2 e^{(b_2 - r)T} + Xe^{-rT})[SN(d_1) - N(d_2)]$$

where

$$d_1 = \frac{\ln(S) + (\sigma^2 \diagup 2)T}{\sigma\sqrt{T}}, d_2 = d_1 - \sigma\sqrt{T}$$

$$S = \frac{Q_1 S_1 e^{(b_1 - r)T}}{Q_2 S_2 e^{(b_2 - r)T} + Xe^{-rT}}$$

and the volatility can be approximated by

$$\sigma \approx \sqrt{\sigma_1^2 + (\sigma_2 F)^2 - 2\rho\sigma_1\sigma_2 F},$$

where

$$F = \frac{Q_2 S_2 e^{(b_2 - r)T}}{Q_2 S_2 e^{(b_2 - r)T} + Xe^{-rT}}$$

where

$$S_1 = \text{Price on asset one.}$$

$$S_2 = \text{Price on asset two.}$$

$$Q_1 = \text{Quantity of asset one.}$$

$$Q_2 = \text{Quantity of asset two.}$$

$$X = \text{Strike price.}$$

$$T = \text{Time to expiration of the option in years.}$$

$$b_1 = \text{Cost-of-carry asset 1.}$$

$$b_2 = \text{Cost-of-carry asset 2.}$$

$$r = \text{Risk-free interest rate.}$$

$$\sigma_1 = \text{Volatility of asset 1.}$$

$$\sigma_2 = \text{Volatility of asset 2.}$$

$$\rho = \text{Correlation between the two assets.}$$

Therefore, Kirk's approximation was used to price a call option on the following spread For simplicity, we assume Q_1 and Q_2 as one and b_1, b_2, r as zero.
When we choose different strike price from 1 to 10, we get the following results:

Table 8: Result of Kirk's Approximation

Strike Price	1	2	3	4	5	6	7	8	9	10
Value of Option	6.7900	5.7900	4.7900	3.7900	2.7900	1.7900	0.7900	0	0	0

9 Comparison of Different Pricing Method

9.1 Monte Carlo and Moment Matching

When comparing Monte Carlo and Moment matching, we can see notable price difference. It should be noted that it is only when there are "enough" samples should Monte Carlo converge to true distribution of our models, therefore it could be the first reason of error. The second error may come from the assumption of Moment matching which is we are trying to fit our model with lognormal distribution which may not be a perfect fit.

9.2 Monte Carlo and Kirk's Approximation

The result we got by Kirk's approximation is a little bit different from that we got by Monte Carlo. We believe that this is mainly because of the assumption we have made when doing Kirk's approximation.
As we haven't considered the cost of carry asset and risk free interest rate when we calibrate the forward curve model, we assume all of them are zero in the Kirk's approximation. However, the result shows that without taking this three elements into consideration, we may not find the most fit forward curve model for Kirk's approximation.