Numerical Solution of an Eigenvalue Problem
Due: Wednesday, November 29.

Consider the *Legendre Equation*:

$$\left(1-x^2\right)y'' - 2xy' + \ell(\ell+1)y = 0,$$

expressed in the dimensionless variables x and y, which is applicable in a wide variety of physical systems in spherical coordinates, and for which $\ell \geq 0$. We wish to find the numerical solution to this equation on $-1 \leq x \leq 1$, subject to the boundary conditions $y(-1) = \pm 1$ and $y(1) = 1$, for various $y'(-1)$.

- Write a `Matlab` function similar to `caseI.m` that can be used to set the values of $\vec{f}$ from Eq. 5.1. In your function, treat ℓ as the eigenvalue, and equate that variable with `w(3)`.
- Write a `Matlab` function similar to `BC_caseI.m` that uses `rk2.m` to calculate $y(1) - 1$, in which you activate separately (*by commenting and uncommenting as necessary, or by having four separate functions, whichever you find more convenient*) the following sets of boundary conditions:

$$\begin{array}{lll} \textbf{Case A}: & y(-1) = 1 & y'(-1) = 0 \\ \textbf{Case B}: & y(-1) = -1 & y'(-1) = 1 \\ \textbf{Case C}: & y(-1) = 1 & y'(-1) = -3 \\ \textbf{Case D}: & y(-1) = 1 & y'(-1) = -21 \end{array}$$

  To avoid the singularity in y'' at $|x| = 1$, consider only the region of x between ± 0.999999. And remember that `bisect.m` works by finding a zero, which is why we look at $y(1) - 1$.
- For each case, make a plot of $y(1) - 1$ as a function of ℓ for $\ell \geq 0$, and indicate the point where $y(1) - 1 = 0$ for the smallest value of ℓ that is approximately an integer (<u>not</u> the smallest value of ℓ rounded to an integer). Overlay these four on one plot, clearly distinguishing each of the four cases with a legend.
- For each of the four cases, determine the smallest integer value of ℓ that satisfies the boundary condition at $x = 1$ using `bisect.m` similar to what is done in the text, and plot $y(x)$ for that ℓ value. Again, overlay these four on one plot. *Step sizes that are too large will not reveal values of ℓ that satisfy $y(1) = 1$ in the $y(x)$ plots.*

To submit HW12 to D2L for grading:

1. Deposit a copy of your functions and the plots you generated (in JPEG format, with axes labeled) in your HW12 Assignments Submission Folder. There is no need to submit `rk2.m` or `bisect.m`.
2. Complete the HW12 Survey.

This homework is worth 25 points. Your Introduction to Theoretical Physics textbook has a thorough treatment of the Legendre Equation.