

1 Classification of Singularities

Many functions have singularities at $z = 0$, but not all singularities are equal. For example, $(\exp(z) - 1)/z$, z^{-4} , and $\exp(1/2z)$ all behave differently near $z = 0$. We will frequently consider functions in this chapter that are holomorphic in a disk except at its center (usually because that's where a singularity lies), and it will be handy to define the **punctured disk** with center z_0 and radius R ,

$$D_{\times}[z_0, R] := \{z \in \mathbb{C} : 0 < |z - z_0| < R\} = D[z_0, R] \setminus \{z_0\}$$

Definition 1.1: If f is holomorphic in the punctured disk $D_{\times}[z_0, R]$ for some $R > 0$ but is not holomorphic at $z = z_0$, then z_0 is an **isolated singularity** of f . We say that the singularity z_0 is

- (a) **removeable** if there exists a function g holomorphic in $D[z_0, R]$ such that $f = g$ in $D_{\times}[z_0, R]$,
- (b) a **pole** if $\lim_{z \rightarrow z_0} = \infty$,
- (c) **essential** if neither a pole or removeable

Example 1.1: Let $f : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$ be given by $f(z) = (\exp(z) - 1)/z$. Since

$$\exp(z) - 1 = \sum_{k \geq 1} \frac{z^k}{k!},$$

the function $g : \mathbb{C} \rightarrow \mathbb{C}$ is defined by

$$g(z) = \sum_{k \geq 0} \frac{z^k}{(k+1)!},$$

which is entire (because the power series converges in \mathbb{C}) agrees with f in $\mathbb{C} \setminus \{0\}$. Thus f has a singularity at 0.

?? 1.1 is pretty neat.