1 Classification of Singularities

Many functions have singularities at $z = 0$, but not all singularities are equal. For example, $(\exp(z) - 1)/z$, z^{-4}, and $\exp(1/2z)$ all behave differently near $z = 0$. We will frequently consider functions in this chapter that are holomorphic in a disk except at its center (usually because thats where a singularity lies), and it will be handy to define the **punctured disk** with center z_0 and radius R,

$$D_\times[z_0, R] := \{z \in \mathbb{C} : 0 < |z - z_0| < R\} = D[z_0, R] \backslash \{z_0\}$$

Definition 1.1: If f is holomorphic in the punctured disk $D_\times[z_0, R]$ for some $R > 0$ but is not holomorpic at $z = z_0$, then z_0 is an **isolated singularity** of f. We say that the singularity z_0 is

(a) **removable** if there exists a function g holomorphic in $D[z_0, R]$ such that $f = g$ in $D_\times[z_0, R]$,

(b) a **pole** if $\lim\limits_{z \to z_0} = \infty$,

(c) **essential** if neither a pole or removable

Proof. If A then B, as seen by Lorem Ipsum Dolor in 1956. □

Example 1.1: Let $f : \mathbb{C}\backslash 0 \to \mathbb{C}$ be given by $f(z) = (\exp(z) - 1)/z$. Since

$$\exp(z) - 1 = \sum_{k \geq 1} \frac{z^k}{k!},$$

the function $g : \mathbb{C} \to \mathbb{C}$ is defined by

$$g(z) = \sum_{k \geq 0} \frac{z^k}{(k+1)!},$$

which is entire (because the power series converges in $\mathbb{C}$) agrees with f in $\mathbb{C}\backslash\{0\}$. Thus f has a singularity at 0.

Definition 1.2: This is the theorem.