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Investigations of Fractal Dimensions in Group-Generated Cellular Automata

Abstract:

This research investigates the realm of cellular automata (CA), discrete, algorithmic structures distinguished by their minimalistic yet elaborate framework. Within these structures, each cell transitions through states derived from an established alphabet, evolving through discrete time increments through the application of a neighborhood-dependent rule that determines each cell's subsequent state. Utilizing the analytical capabilities of the PascGaloisJE software for data collection, our research aims to methodically examine and model multidimensional CAs, with a particular emphasis on their ability to contextualize and analyze fractal-dimensioned patterns. Our exploration involves the enumeration of states within the CA patterns, focusing mainly on the generation of sub-triangles with respect to group theory. By counting the states and calculating the fractal dimension based on the density of zero to nonzero elements within the CA patterns, we can develop a quantitative measure of the complexity, compositional arrangements, and fractal properties of sub-triangles. By conducting computational simulations and analyzing fractal dimensions, we intend to decode the complexity inherent in CA-derived patterns.

Introduction:

Cellular automata (CA), conceptualized in the 1940s by John Von Neumann and Stanislaw Ulam, offer a discrete, rule-based approach to modeling complex systems. Von Neumann and Ulam sought a mathematical computational model that encodes self-replicating behavior, similar to certain biological processes. CA are defined as lattices of cells, each takes on values from a finite set called the alphabet. The cells are updated in discrete time steps based on a local rule, where a given cell is derived from its neighboring cell values at the previous time step [10]. A state in the system occurs when each cell has a unique value from the alphabet. This defines a time evolution from the set of all states to itself. These evolutions can be visualized through stacking the states as they evolve, more formally known as a space-time diagram. Applications in other domains include the modeling of dendritic crystal growth [12], spread of wildfires [2], and the complexities of traffic flow [6]. CA offer insights into certain mechanisms behind numerous complex systems [4]. In particular, these systems often display high levels of pattern formation and self-organization.

Cellular automata are a type of discrete systems that, in some cases, exhibit fractal behaviors analogous to the structures found in the continuous realm of mathematics. Well-known examples of such structures are Koch's Curve (Fig. 1) and the Sierpiński Gasket (Fig. 2). Koch's Curve, constructed by Helge von Koch in 1904, offers a glimpse into the infinite complexity that can emerge from simple, iterative rules. It exists in a two-dimensional space, where each iteration of its generation replaces each parent line segment with four new segments (N = 4), with every new segment being ⅓ of the original (r = 3). This results in a figure with an area of zero but an infinite length, highlighting the fractal's capacity to bridge the gap between the finite and the infinite realms.

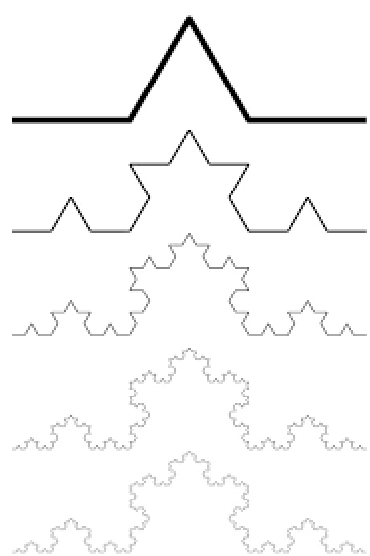
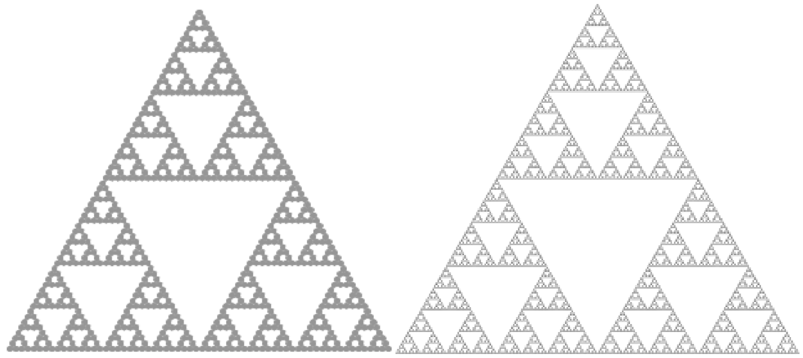


Figure 1 *(First 4 Iterations of Koch's Curve)*

Figure 2 *(Left: Pascal's Triangle Modulo 2 | Right: Sierpiński Gasket)*



We quantify this behavior using the well-known fractal dimension formula $D = \frac{log(N)}{log(r)}$ where N represents the number of self-similar pieces at the $\frac{1}{r}$ scaling factor [8]. For the Koch Curve $D = \frac{log(4)}{log(3)} \approx 1.26$.

Each iteration of Sierpiński's Gasket involves removing the central triangle from every existing triangle, dividing the gasket into 3 self-similar pieces (N = 3) that are each half the size of their predecessor (r = 2). Placing these values in our formula give us $D = \frac{log(3)}{log(2)} \approx 1.58$. A well-known CA, Pascal's Triangle mod 2, generated by applying modular arithmetic to the traditional Pascal's Triangle, is very similar to the Sierpiński's Gasket and provides a unique perspective on fractal geometry, particularly when considering the concepts of "zoom-out" versus "zoom-in" self-similarity [9]. The mod 2 triangle exhibits patterns that can be similarly broken down into self-similar structures. Assuming a similar division into 3 self-similar parts (N = 3) as we double the number of rows in the discrete lattice (r = 2), an analogous fractal dimension, called the *growth rate dimension* [7], is $D = \frac{log(3)}{log(2)}$. We observe that the mod 2 triangle reveals aspects of self-similarity when viewed from a "zoom-out" perspective, where its repetitive fractal pattern becomes clearer from afar. Meanwhile, Sierpiński's Gasket demonstrates the "zoom-in" aspect of self-similarity, revealing smaller self-similar shapes upon closer examination.

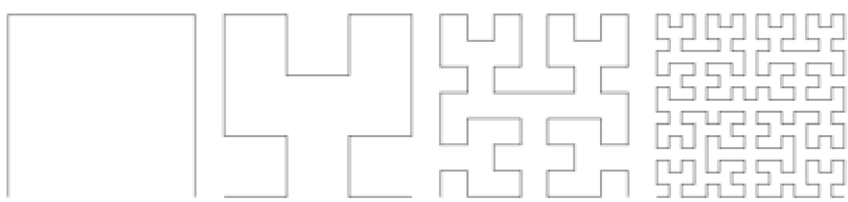


Figure 3 *(Above: Variants of the Hilbert Curve)*

A unique type of a fractal construction is the space-falling curve. These structures, like the Hilbert Curve (Fig. 3) or the Peano Curve, defy concepts of dimensionality by describing a process by which an infinite one-dimensional line can fill a two-dimensional space, in contrast to the fractals previously discussed. By drawing parallels and exploring contrasts between these continuous and discrete systems, we hope to shine a light on the underlying principles that govern certain fractal geometries and their representation across different mathematical settings.

Methods:

To motivate growth rate dimensions in the discrete setting, we consider the well-known concept of a mod p triangle, where p is prime. This triangle, generated through modular arithmetic within a discrete mathematical framework, has been proven to display perfect self-similarity [3, 11] as the numbers of rows are scaled by powers of p. As an example, consider Pascal's

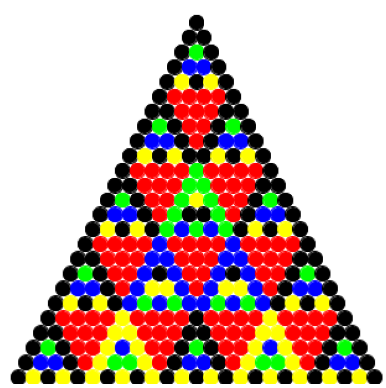


Figure 4 *(1st 25 rows of $\mathbb{Z}_5$)*

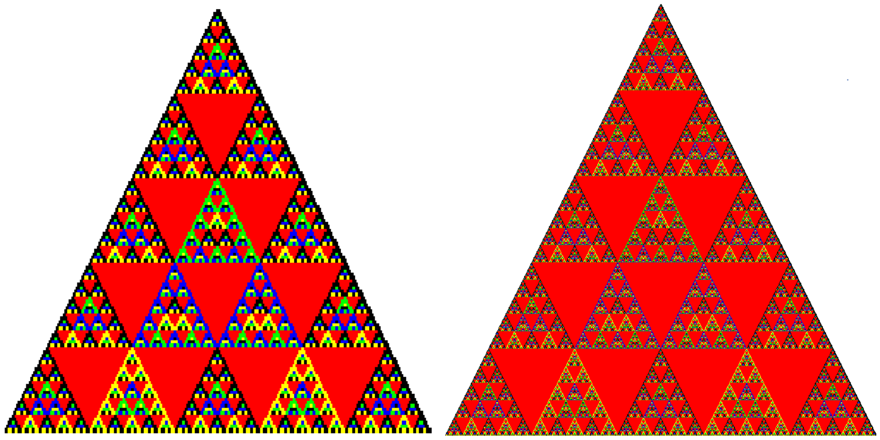


Figure 5 *(1st 125 & 625 rows of* $\mathbb{Z}_5$*)*

Table 1: Calculations of $\mathbb{Z}_5$ at Different Scales		
Row	Non-Zero Elements	Ratio of Non-Zero Elements
5	15	n/a
25	225	15
125	3,375	15
625	50,625	15
3125	759,375	15

triangle mod 5 ($\mathbb{Z}_5$) and observe it at different scales—specifically $p = 5^n$ $(n \in \mathbb{Z})$. By focusing on the ratios of non-zero elements at 5, 25, 125 rows (Fig. 4 & 5), and so on, we can observe the repetitive nature of the $\mathbb{Z}_5$ triangle and discern the rate at which complexity increases as the structure expands. This approach is pivotal in understanding the scaling behavior of fractals within discrete mathematical systems, as it provides a direct method to capture the essence of self-similarity that defines fractal structures. When we apply the fractal dimension formula, the convergence to 15 (N = 15) at every scaled iteration (r = 5) can be calculated as $D = \frac{log(15)}{log(5)} \approx 1.68$.

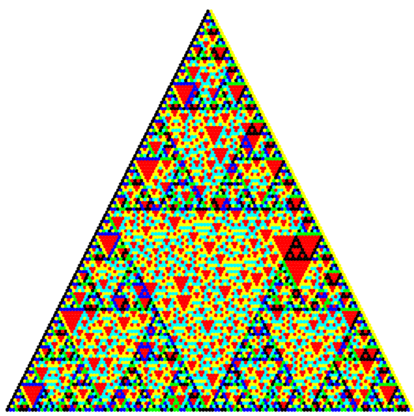
Moving forward, this approach to calculating fractal dimensions mirrors the methodology used for continuous fractals, adjusted for the discrete nature of CA. Through this process, our goal is to derive a precise mathematical representation of the fractal dimensions inherent in certain CA, enhancing our understanding of their geometric and spatial properties. Our investigation focuses on one-dimensional cellular automata generated over non-abelian group alphabets. Within this framework, our study is driven by pivotal questions using permutation groups, dihedral groups, dicyclic groups, and generalized quaternions. The study is propelled by two distinct yet interconnected lines of inquiry:

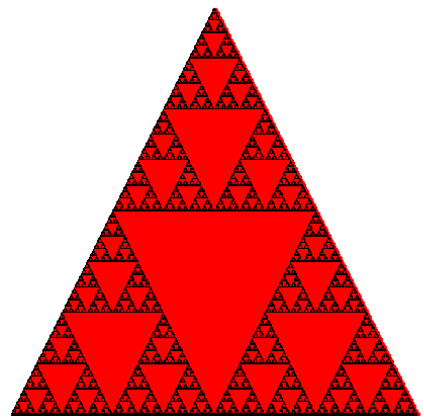
I. Observe how certain parameters shape the complexity and scaling of fractal patterns by manipulating:
   A. The type of non-abelian group and its order,
   B. The initial state of the cellular automata system,
   C. The specific update rule applied to the system
II. Discern the impact on certain closed sub-triangles in space-time diagrams. A significant part of our analysis is dedicated to distinguishing conditions that yield a true fractal dimension versus two-dimensional growth in our space-time diagrams - which is a discrete analog of a space-filling curve in the continuous domain.

In this exploration, we utilize the PascGaloisJE software [5], programmed by Don Spickler. This sophisticated software enables us to simulate cellular automata patterns and expedites our count analysis, which is especially crucial for visualizing system evolution and comprehending fractal dimensions. With this software at our disposal, we delve into the enumeration of the elements of the subgroups triangles within these space-time diagrams. In our exploration of cellular automata (CA), we harness the power of group theory to uncover the intricacies of fractal patterns. Utilizing the analytical tools of subgroups, quotient groups, and the various principles of group theory, we meticulously count states and examine sub-triangles to quantify the complexity of CA patterns. This deep dive into the mathematical structure is crucial for our investigation, as it lays the groundwork for calculating dimensions through the density assessment of zero to nonzero elements within the CA patterns.

By executing a wide range of simulations under varying parameters—including different groups, rule sets, and initial conditions—we systematically micro-analyze the evolution of CA patterns. Each simulation is conducted until a conclusive pattern is observed or for a predetermined number of generations, allowing for documentation of pattern emergence. These simulations are critical for identifying the conditions, which shed light on the mechanisms that drive complexity in discrete systems. As an example of determining the growth-rate dimension of cellular automata generated over non-abelian groups, let us consider a one dimensional system defined over the permutation group S_3.

Figure 6 (1st 512 rows: Left: S_3 | Right: S_3 mod 2)

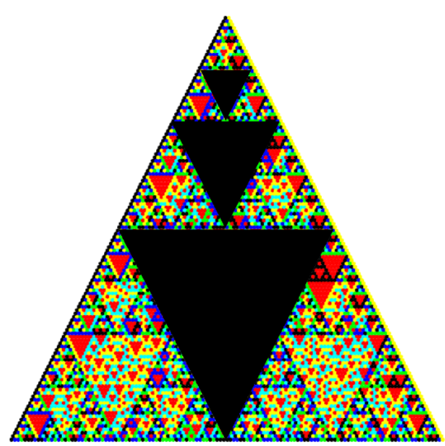




These space-time diagrams of an S_3 triangle is shown in Figure 6, where we generate the structure with a 2-seed consisting of an even and odd permutation. In the inverted subgroup triangles of even permutations, one might initially see a collection of elements that appear random at first glance. The color-coding of red, black, green, dark blue, yellow, and light blue, corresponds to the six elements within S_3: (1), (1 2), (1 3), (2 3), (1 2 3), and (1 3 2), respectively. Despite this color-coding, discerning a clear pattern within the triangle can be challenging. In an attempt to understand this structure's fractal dimension, we apply a reduction technique.

To accomplish this, note that (1 2), (1 3), and (2 3) are odd permutations and (1), (1 2 3), and (1 3 2) are even. We perform a recoloring with these group elements, denoting the odd permutation as black and the even as red. Through this reduction, the complexity of the S_3 triangle is simplified into the well-known Pascal's Triangle mod 2, allowing us to observe the self-similar growth and obtain a type of growth-rate dimension $D = \frac{log(3)}{log(2)} \approx 1.58$ for the entire space-time diagram (Fig. 6).

Figure 7 (Shaded: Sub-triangles of S_3)



We next focus on the inverted subgroup triangles of the even permutations, which are shaded black in Figure 7 to indicate their location. Within these sub-triangles, we count the amount of the non-identity elements: (1 2 3) and (1 3 2). By taking the ratio at each scaled iteration of the sub-triangles, one obtains a quantitative measure of the sub-structure's complexity. Using our collected data (Table 2), we can observe a conjecture convergence to 4 of the density of non-zero elements, produced from scaling the number of rows by a factor of 2. This gives a predicted dimension $D = \frac{log(4)}{log(2)} = 2$. The integer result of 2 indicates that the sub-structure exhibits properties similar to that of a space-filling curve from the continuous realm.

This is just one example using the smallest non-abelian group and a simple 2-seed initial state. Based on the suggestive calculations of the ratios, further exploration of other non-abelian groups, various initial states, and the update rules, is our ultimate goal.

Table 2: Row Summations of Center Grouping & Calculations

Row	(1)	(1 2 3)	(1 3 2)	Total Excluding Identity	Ratio Excluding Identity
7	9	14	5	19	4.75000
15	43	33	44	77	4.05263
31	181	156	159	315	4.09091
. . .					
1023	174354	174777	174645	349422	**4.021198**

Significance:

Our research utilizes group theory within cellular automata (CA) to uncover the mathematical principles guiding pattern formation and evolution, with an emphasis on non-abelian groups and the mathematics of symmetry. Our focus on group theory, acts as a gateway to understanding the mathematical principles of pattern formation and growth in cellular automata. This focus is based on the fundamental understanding that symmetries and group theory provides concepts that have the capacity to describe the patterns of these complex systems. Drawing on the concept of "zoom-out" self-similarity from classical fractal geometry, and inspired by Benoit Mandelbrot's [4] work, we extend our inquiry beyond linear, prime-order groups to the complex properties of nonlinear systems.

The study of prime-order groups offers insights into linear systems, providing an accessible foundation for understanding the generation of patterns. However, our intent is to extend this understanding to nonlinear systems, employing symmetry groups as an alphabet for constructing fractal and space-time diagrams. This approach leverages the insights from Wilson's [7-11] exploration of non-linear pattern formation, particularly through the application of computational simulations, to visualize how these complex systems evolve from simple integer growth to intricate fractal growth. By contextualizing the formation of patterns within CA and employing symmetries as a foundational alphabet, we provide a general understanding of how fractal and space-time diagrams emerge. Our analysis focuses on discerning how these systems transition from integer to fractal growth, conceptualizing the process of pattern formation within CA but also allows us to appreciate the complexity inherent in non-linear systems.

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