Digital Signal Processing
Self-Test

Coursera - April 2014

The following questions are designed to help you identify potential trouble spots in your math background. This is not a standard "homework," in the sense that the goal is not simply for you to come up with correct results; you should also feel completely confident with your answers and you should research and review the relevant algebra and calculus theory until that is the case. For this reasons, no solutions are provided and you should be the judge of your level of assuredness. The bibliographical references provided in the syllabus should help you find didactic material to brush up.

Complex Numbers

A complex number z admits two canonical representations:

- cartesian form: $z = a + jb$
- polar form: $z = \rho e^{j\theta}$

Please note that, according to usual engineering practice, we use the symbol j to denote the imaginary unit.

Exercise 1. Conversion Between Representations.
Given $z = \rho e^{j\theta}$, express z in cartesian form $a + jb$:

$$\begin{aligned} a &= \\ b &= \end{aligned}$$

Conversely, given $z = a + jb$, express z in polar form $\rho e^{j\theta}$:

$$\begin{aligned} \rho &= \\ \theta &= \end{aligned}$$

Exercise 2. Elementary Operations.
Assume $z = a + jb = \rho e^{j\theta}$ and $w = c + jd = \sigma e^{j\varphi}$. Write the results of the following operations in canonical form (either cartesian or polar) using the one which is most convenient for each case (see the first example, where the

cartesian form is the most practical):

$$\begin{aligned}
z+w &= (a+jb)+(c+jd)=(a+c)+j(b+d)\\
z-w &= \\
z/w &= \\
|z| &= \\
z^* &= \\
z^5 &= \\
\sqrt{z} &=
\end{aligned}$$

Exercise 3. The Complex Plane.
Plot the following complex numbers on the complex plane (mark the points on the plane with the corresponding letter):

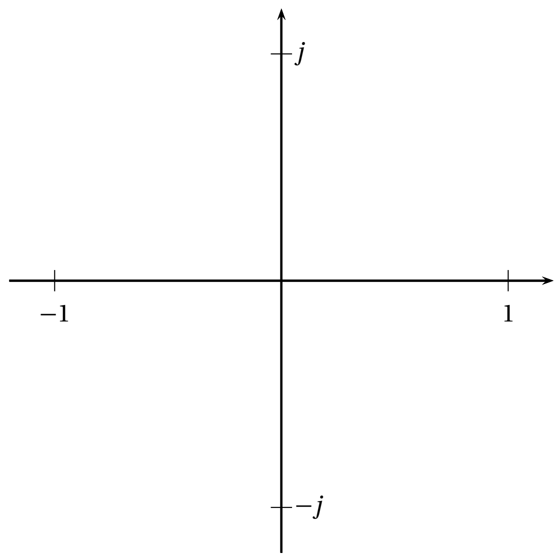
a) 1

b) $1+j$

c) $\sqrt{-1}$

d) $e^{-j\pi/4}$

e) $(1/3)e^{j\pi}$



Exercise 4. Curious, Isn't It?
Compute the following number:

$$j^j =$$

Linear Algebra

N-dimensional complex vectors are by default *column* vectors and are indicated by the notation

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = [x_1\ x_2\ \ldots\ x_N]^T$$

where of course the superscript T denotes transposition. Similarly, an $N \times M$ matrix is represented by

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1M} \\ x_{21} & x_{22} & \dots & x_{2M} \\ & \dots & \dots & \\ x_{N1} & x_{N2} & \dots & x_{NM} \end{bmatrix}$$

Exercise 5. Vectors and Matrices

Assume $\mathbf{x} = [1 \quad 2 \quad 3]^T$ and consider the matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ Compute the following quantities:

$$\mathbf{x}^T\mathbf{x} =$$

$$\mathbf{x}\mathbf{x}^T =$$

$$|\mathbf{x}| =$$

$$\mathbf{A}\mathbf{x} =$$

$$\mathbf{x}\mathbf{A} =$$

$$\mathbf{A}\mathbf{A}^T =$$

Exercise 6. Eigenvalues

Find the eigenvalues of the following matrices:

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \quad :$$

$$\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \quad :$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 8 & 8 \\ -3 & -2 & 0 \end{bmatrix} \quad :$$

Exercise 7. Eigenvectors

For the last matrix in the previous exercise, choose an eigenvalue λ and find its associated eigenvector $\mathbf{x}_\lambda$:

$$\lambda =$$

$$\mathbf{x}_\lambda =$$

Vector spaces

Recall that a vector space is a collection of elements called *vectors*, which may be added together and multiplied by scalars (from a field) to produce another element in the same collection. Although the most common examples

of this involve Euclidean space (in which vectors can be visualized as “arrows” pointing somewhere), the concept of vector space encompasses a much larger set of possible “things”. A vector space is defined by a collection of ten axioms (you can search online for the detailed list) which specify the properties that vector addition and scalar multiplication must possess. As long as those are satisfied, you have a vector space.
For the purpose of our class, it’s very important to embrace this wider view of vector space and consider Euclidean space a special case of a much more encompassing mathematical model.

Exercise 8. Identifying vector spaces

Which of the following collections of elements form vector spaces over the field of real numbers?

- The set of real numbers, $\mathbb{R}$.
- The set of all non-negative real numbers.
- The set of all vectors in the two-dimensional Euclidean space $\mathbb{R}^2$.
- The set of all vectors in the two-dimensional Euclidean space $\mathbb{R}^2$ with magnitude less than or equal to 1.
- The collection of vectors $[x, y]^T$ in the two-dimensional Euclidean space $\mathbb{R}^2$ satisfying $x = y$.
- The collection of all real-valued continuous functions defined on the interval $[0,1]$ on the real line.

Exercise 9. Euclidean vector spaces

Assume $\mathbf{v_1}$ and $\mathbf{v_2}$ are vectors in the three-dimensional Euclidean space $\mathbb{R}^3$ given by $\mathbf{v_1} = [1,0,1]^T$ and $\mathbf{v_2} = [1,0,0]^T$.

- Compute the Euclidean distance between the points whose coordinates are given by $\mathbf{v_1}$ and $\mathbf{v_2}$.
- Compute the angle between the vectors $\mathbf{v_1}$ and $\mathbf{v_2}$.
- Identify a vector $\mathbf{v_3}$ of unit-length that is orthogonal to both $\mathbf{v_1}$ and $\mathbf{v_2}$.
- What is the dimension of the vector space spanned by $\mathbf{v_1}$, $\mathbf{v_2}$ and $\mathbf{v_3}$? Identify an orthonormal basis for the space.
- Let $\mathbf{v_4} = [2,0,3]$. What is the dimension of the vector space spanned by $\mathbf{v_1}$, $\mathbf{v_2}$ and $\mathbf{v_4}$? Identify an orthonormal basis for the space.

Exercise 10. Bases

Describe a basis for each of the following vectors spaces defined over the field of reals equipped with the standard addition and multiplication operations:

- Space of all 2×3 matrices with real entries.
- Space of all polynomials with degree less than or equal to 3.

Exercise 11. Spin-offs

Let $\mathbf{P} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Let $\mathbf{x}$ denote a 2-dimensional vector with real entries, and $\mathbf{y} = \mathbf{P}x$.

- Describe $\mathbf{y}$ in terms of $\mathbf{x}$.
- If $\mathbf{x} = [2,1]$ and $\theta = \frac{\pi}{2}$, plot $\mathbf{x}$ and $\mathbf{y}$ on a 2-dimensional plane.

Calculus

Exercise 12. Polynomials

Find the roots of the following polynomials (remember that, in the complex field, a polynomial of degree N has N roots!):

a) $p(x) = 2x^2 - 5x + 2$;

b) $p(x) = x^3 + 1$;

c) $p(x) = x^4 + 3x^2 + 2$;

Exercise 13. Series

Compute the following sums:

$$\sum_{n=0}^{17} n =$$

$$\sum_{n=0}^{N} a^n =$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} =$$

$$\sum_{n=1}^{\infty} \frac{1}{n} =$$

Exercise 14. Limits

Compute the following limits:

$$\lim_{x\to\infty} \frac{\log x}{x} =$$

$$\lim_{x\to 0} \frac{\sin x}{x} =$$

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x =$$

Exercise 15. Integrals

Compute the following integrals:

$$\int_{-\pi}^{\pi} \sin x \, dx =$$

$$\int e^{\alpha x} \, dx =$$

$$\int x \cos nx \, dx =$$

Probability theory

Exercise 16. Mean and variance

Assume a is a discrete random variable assuming values in $\{-1,2\}$ with probabilities $P[a=-1]=0.3$ and $P[a=2]=0.7$.

- compute the mean of a
- compute the variance of a

Exercise 17. Probability distribution function and expectation

Assume x is a uniformly distributed random variable over the interval $[1,3]$. Compute the following values

$$\begin{aligned} E[x] &= \\ E[(x-2)^2] &= \\ E[1/x] &= \end{aligned}$$
