**Tutorial on particle swarm optimization and its combinations to other evolutionary algorithms**

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# Abstract

Local optimization with convex function is solved perfectly by traditional mathematical methods such as Newton-Raphson and gradient descent but it is not easy to solve the global optimization with arbitrary function although there are some purely mathematical approaches such as approximation, cutting plane, branch and bound, and interval method which can be impractical because of their complexity and high computation cost. Recently, some evolutional algorithms which are inspired from biological activities are proposed to solve the global optimization by acceptable heuristic level. Among them is particle swarm optimization (PSO) algorithm which is proved as an effective and feasible solution for global optimization in real applications. Although the ideology of PSO is not complicated, it derives many variants, which can make new researchers confused. Therefore, this tutorial focuses on describing, systemizing, and classifying PSO by succinct and straightforward way. Moreover, combinations of PSO and other evolutional algorithms for improving PSO itself or solving other advanced problems are mentioned too.

**Keywords:** particle swarm optimization (PSO), evolutional algorithms, global optimization.

# 1. Introduction to PSO

Particle swarm optimization (PSO) algorithm was developed by James Kennedy (a social psychologist) and Russell C. Eberhart (an electrical engineer). This tutorial is navigated by the article “Particle swarm optimization: An overview” of Riccardo Poli, James Kennedy, and Tim Blackwell. The main idea of PSO is based on social intelligence when it simulates how a flock of birds search for food. Given a target function known as *cost function* *f*(***x***), the optimization problem is to find out the minimum point ***x***\* known as minimizer or optimizer so that *f*(***x***\*) is minimal. In PSO theory, *f*(***x***) is also called *fitness function* and thus, when *f*(***x***) is evaluated at *f*(***x***0) then, *f*(***x***0) is called fitness value which represents the quality of food source for which a flock of birds search. If ***x***\* is an optimizer, *f*(***x***\*) is called optimal value, best value, or best fitness value. As a convention, the optimization problem is global minimization problem when ***x***\* is searched over entire domain of *f*(***x***).

For global maximization, it is simple to change a little bit our viewpoint. Traditional local optimization methods such as Newton-Raphson and gradient descent along with global optimization methods require that *f*(***x***) is differentiable. Alternately, PSO does not require existence of differential. PSO scatters a population of candidate solutions (candidate optimizers) for ***x***\* and such population is called swarm whereas each candidate optimizer is called particle in the swarm. PSO is an iterative algorithm running over many iterations in which every particle is moved at each iteration so that it approaches the global optimizer ***x***\*. Movement of all particles is attracted by ***x***\*. In other words, such movement is attracted by minimizing *f*(***x***) so that *f*(***x***) is small enough. In PSO, ***x*** is considered as position of particle. The movement of each particle is affected by its best position and the best position of the swarm. Note, the closer to ***x***\*, the better the position is.

As a formal definition, let be the swam of particles and let ***x****i* and ***p****i* be current position and best position of particle *i*. Note, ***p****i* is called *local best position*. Moreover, the movement speed of particle *i* is specified by its velocity ***v****i*. Let ***p****g* be the *global best position* of entire swarm. The closer to ***x***\*, the better the positions ***p****i* and ***p****g* are. It is expected that ***p****g* is equals to ***x***\* or is approximated to ***x***\*. The ultimate purpose of PSO is to determine ***p****g*.

Of course, ***x****i*, ***p****i*, and ***p****g* are *n*-dimensional points and ***v****i* is *n*-dimensional vector because *f*(***x***) is from real *n*-dimensional space ***R****n* to real space ***R***. Following is pseudo-code of PSO (Wikipedia, 2017).

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| *Input*: the swam of particles along with their initialized positions and velocities.  *Output*: the global best position ***p****g* of entire swarm with expectation that ***p****g* is equal or approximated to the global minimizer ***x***\*.  Let ***lb*** and ***ub*** be lower bound and upper bound of particles in their search space. They are vectors.  All current positions ***x****i* = (*xi*1, *xi*2,…, *xin*)*T* of all particles are initialized randomly. Moreover, their best positions are set to be their current positions such that ***p****i* = ***x****i*. Note, all particles are randomized in the range [***lb***, ***ub***] as closed sphere.  All current velocities ***vi*** of all particles are initialized randomly. Because each ***vi*** is vector, its elements are randomized in the range [–|***ub*** – ***lb***|, |***ub*** – ***lb***|] from –|***ub*** – ***lb***| to |***ub*** – ***lb***| where the notation |.| denotes distance between two vectors or two points.  The global best position ***p****g* is assigned by the local best position ***p****i* such that *f*(***p****i*) is smallest among particles.  While terminated condition is not met at each current iteration do  For each particle *i* in swarm *S*  Velocity of particle *i* is updated as follows:   |  |  | | --- | --- | |  | (1.1) |   Position of particle *i* is updated as follows:   |  |  | | --- | --- | |  | (1.2) |   If *f*(***x****i*) < *f*(***p****i*) then  The best position of particle *i* is updated: ***p****i* = ***x****i*  If *f*(***p****i*) < *f*(***p****g*) then  The best position of swarm is updated: ***p****g* = ***p****i*  End if  End if  End for  End while |

**Table 1.1.** Basic particle swarm optimization (PSO) algorithm

If concerning the current *t*th iteration (the *t*th time), equation 1.1 and equation 1.2 are denoted as follows:

Equation 1.1 is the heart of PSO, which is called *velocity update rule*. Equation 1.2 is called *position update rule*. There are two most popular terminated conditions:

1. The cost function at ***p****g* which is evaluated as *f*(***p****g*) is small enough. For example, *f*(***p****g*) is smaller than a small threshold.
2. Or PSO ran over a large enough number of iterations.

Function *U*(0, *ϕ*1) generates a random vector whose elements are random numbers in the range [0, *ϕ*1]. Similarly, function *U*(0, *ϕ*2) generates a random vector whose elements are random numbers in the range [0, *ϕ*2]. For example,

Note, the super script “*T*” indicates transposition operator of vector and matrix. The operator denotes component-wise multiplication of two points (Poli, Kennedy, & Blackwell, 2007, p. 3). For example, given random vector *U*(0, *ϕ*1) = (*r*11, *r*12,…, *r*1*n*)*T* and position ***x****i* = (*xi*1, *xi*2,…, *xin*)*T*, their component-wise multiplication is:

Two components and are considered as attraction forces that push every particle to move. Sources of force and force are the particle *i* itself and its neighbors. Thus, two most important parameters of PSO are *ϕ*1 and *ϕ*2 which represent the two attraction forces. The popular values of them are *ϕ*1 = *ϕ*2 = 1.4962. Parameter *ϕ*1 along with the force express the *exploitation* of PSO whereas parameter *ϕ*2 along with the force express the *exploration* of PSO (Poli, Kennedy, & Blackwell, 2007, p. 4). The larger parameter *ϕ*1 is, the faster PSO converges but it trends to converge at local minimizer. In opposite, if parameter *ϕ*2 is large, convergence to local minimizer will be avoided in order to achieve better global optimizer but convergence speed is decreased. Parameters *ϕ*1 and *ϕ*2 are also called acceleration coefficients or *attraction coefficients*. Especially, *ϕ*1 is called cognitive weight and *ϕ*2 is called social weight because *ϕ*1 reflects thinking of particle itself in moving and *ϕ*2 reflects influence of entire swarm on every particle in moving. In practical, velocity ***v****i* can be bounded in the range [–***v****max*, +***v****max*] in order to avoid out of convergence trajectories but the parameter ***v****max* is not popular because there are some other parameters such as inertial weight and constriction coefficient (mentioned later) which are used to damp the dynamics of particles. Favorite values for the size of swarm (the number of particles) are ranged from 20 to 50.

Because any movement has inertia, inertial force is added to the two attraction forces. Hence, the inertial force is represented by a so-called *inertial weight* *ω* where 0 < *ω* ≤ 1. Equation 1.1 becomes (Poli, Kennedy, & Blackwell, 2007, p. 4):

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|  | (1.3) |

The larger inertial weight *ω* is, the faster particles move because its inertial is high, which leads PSO to explore global optimizer. Note that moving fast does not imply fast convergence. In opposite, the smaller *ω* leads PSO to exploit local optimizer. In general, large *ω* expresses exploration and small *ω* expresses exploitation. The inverse 1–*ω* is known as friction coefficient. The popular value of *ω* is 0.7298.

Pioneers in PSO (Poli, Kennedy, & Blackwell, 2007, p. 5) recognized that if velocities ***v****i* of particles are not restricted, their movements can be out of convergence trajectories at unacceptable levels. Therefore, they proposed a so-called *constriction coefficient* *χ* to damp dynamics of particles. Note, *χ* is also called constriction weightor damping weight where 0 < *χ* ≤ 1. With support of constrictioncoefficient, equation 1.1 becomes (Poli, Kennedy, & Blackwell, 2007, p. 5):

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|  | (1.4) |

It is easy to recognize that equation 1.3 is special case of equation 1.4 when the expression *χ****v****i* is equivalent to the expression *ω****v****i*. The popular value of constriction coefficient is *χ* = 0.7298 given *ϕ*1 = *ϕ*2 = 2.05 and *ω* = 1. Note, inertial weight *ω* is also the parameter that damps dynamics of particles. This is the reason that *ω* = 1 when *χ* ≠ 1 but constriction of *χ* is stronger than *ω* because *χ* affects previous velocity ***v****i* and two attraction forces whereas *ω* affects only previous velocity ***v****i*.

Structure of swarm which is determined by defining neighbors and neighborhood of every particle is called *swarm topology* or *population topology*. Because ***p****g* is the best position of entire swarm, the attraction force indicates the movement of each particle is affected by all other particles, which means that every particle connects to all remaining particles. In other words, neighbors of a particle are all other particles, which is known as fully connected swarm topology. For easily understandable explanation, suppose particles are vertices of a graph, fully connected swarm topology implies that such graph is fully connected, in which all vertices are connected together. Alternately, swarm topology can be defined in different way so that each particle *i* only connects with a limit number *Ki* of other particles. In other words, each particle has only *Ki* neighbors. With custom-defined swarm topology, equation 1.4 is written as follows (Poli, Kennedy, & Blackwell, 2007, p. 6):

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| --- | --- |
|  | (1.5) |

Where ***q****k* is the best position of the *k*th neighbor of particle *i*. Of course, ***q****k* is ***p****j* of some particle *j*.

***q****k* = ***p****j* such that particle *j* is the *k*th neighbor of particle *i*.

Please pay attention that, in equation 1.5, particle *i* is also its neighbor. In other words, in equation 1.5, the set of *Ki* neighbors includes particle *i*. The two parameters *ϕ*1 and *ϕ*2 are reduced into only one parameter *ϕ* > 0, which implies the strengths of all attraction forces from all neighbors on particle *i* are equal. The popular value of *ϕ* is *ϕ* = 4.1 given *χ* = 0.7298. Equation 1.5 is known as Mendes’ fully informed particle swarm (FIPS) method. The topology in the basic PSO specified by equation 1.1, equation 1.3, and equation 1.4 is known *global best topology* because only one best position ***p****g* of entire swarm is kept track. However, equation 1.5 indicates that many best positions from groups implied by neighbors are kept track. Hence, FIPS method specifies a so-called *local best topology*, which converges slowly but avoids converging at local optimizer. In other words, local best topology aims to exploration rather than exploitation.

If we focus on the fact that the attraction force issued by the particle *i* itself is equivalent to the attraction force from the global best position ***p****g* and the other attraction forces from its neighbors ***q****k*, equation 1.5 is modified as follows:

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|  | (1.6) |

In equation 1.6, the set of *Ki* neighbors does not include particle *i* and so, the three parameters *ϕ*1, *ϕ*2, and *ϕ* are co-existent. Inertial weight *ω* is kept intact too. It is easy to recognize that equation 1.6 is the general form of velocity update rule. Equation 1.6 balances local best topology and global best topology with expectation that convergence speed is improved but convergence to local optimizer can be avoided. In other words, equation 1.6 aims to achieve both exploration and exploitation.

In general, the two main aspects of PSO are exploration and exploitation. The exploration aspect aims to avoid premature converging so as to reach global optimizer whereas the exploitation aspect aims to motivate PSO to converge as fast as possible. Besides exploitation property can help PSO to converge more accurately regardless of local optimizer or global optimizer. These two aspects are equally important. Consequently, two problems corresponding to the exploration and exploitation are *premature problem* and *dynamic problem*. Solutions of premature problem are to improve the exploration and solutions of dynamic problem are to improve the exploitation. Inertial weight and constriction coefficient are common solutions for dynamic problem. Currently, solutions of dynamic problem often relate to tuning coefficients which are PSO parameters. Solutions of premature problem relates to increase dynamic ability of particles such as:

* Dynamic topology.
* Change of fitness function.
* Adaptation includes tuning coefficients, adding particles, removing particles, and changing particle properties.
* Diversity control.

*Dynamic topology* is popular in solving premature problem. The topology from equation 1.1, equation 1.3, equation 1.4, equation 1.5, and equation 1.6 is static (Poli, Kennedy, & Blackwell, 2007, p. 6) because it is kept intact over all iterations of PSO. In other words, neighbors and neighborhood in *static topology* are established fixedly. We will later research dynamic topology in which neighbors and neighborhood are changed at each iteration.

# 2. Variants of PSO

The PSO shown in table 1.1 is basic PSO. Recently there are many PSO variants. Some of them aim to improve the basic PSO but the others aim to solve raised problems.

## 2.1. Simplified and improved PSOs

Variants of PSO mentioned in this sub-section aim to simplify or improve basic PSO. In general, these variants focus on modifications of basic PSO. Binary PSO (BPSO) developed by James Kennedy is a simple version of PSO where positions ***x****i* and ***p****i* are binary (0 and 1). After velocity update rule (equation 1.1) is executed, the velocity ***v****i* = (*vi*1, *vi*2,…, *vin*)*T* is in turn squashed into range [0, 1] by squash function (logistic function) as follows (Poli, Kennedy, & Blackwell, 2007, p. 9), (Too, Abdullah, & Saad, 2019, p. 3):

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|  | (2.1.1) |

Where *vij* is the *j*th element of ***v****i* = (*vi*1, *vi*2,…, *vin*)*T*. Of course, the squashed value *s*(*vij*) is in range [0, 1]. The main point of BPSO is to modified position update rule as follows (Too, Abdullah, & Saad, 2019, p. 3):

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|  | (2.1.2) |

Where *xij* is the *j*th element *xij* of ***x****i* = (*xi*1, *xi*2,…, *xin*)*T* and *r* is a random number in range [0, 1]. In general, position update rule in BPSO is specified by equation 2.1.2 instead 1.2. Therefore, the purpose of BPSO is to simplify the basic PSO.

Discrete PSO (DPSO) first proposed by Quan-Ke Pan et al. is another simple version of PSO in which positions ***x****i* are discrete or binary, but it is different from BPSO because it eliminates velocity update rule (equation 1.1). Given two positions ***x****i* and ***x****j*­ whose elements are nominal or binary, because no arithmetic operator can be executed on them, DPSO alternately uses genetic operators such as exchange operator, one-cut operator, and two-cut operator to determine the position update rule. Without loss of generality, figure 2.1.1 illustrates exchange operator (Guner & Sevkli, 2008, p. 4).

Table

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**Figure 2.1.1.** Illustration of exchange operator

Let *f*(***x****i*, *ϕ*) represent exchange operator on ***x****i* with parameter *ϕ* as probability threshold. Exactly, when elements of ***x****i* are browsed one by one, at each element *xij* a new random number *r* is created with uniform distribution and then, if *r* ≥ *ϕ*, exchange operator will be executed on group of such elements *xij*. Figure 2.1.2 illustrates one-cut crossover operator (Guner & Sevkli, 2008, p. 5).

A screenshot of a computer

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**Figure 2.1.2.** Illustration of one-cut crossover operator

Let *f*1(***x****i*, ***p****i*, *ϕ*1) represent one-cut crossover operator on ***x****i* and ***p****i* with parameter *ϕ*1 as probability threshold. Exactly, when pairs of ***x****i* and ***p****i* are browsed pair by pair, at each pair (*xij*, *pij*) a new random number *r* is created with uniform distribution and then, if *r* ≥ *ϕ*, one-cut crossover operator will be executed on group of such pairs (*xij*, *pij*). Figure 2.1.3 illustrates two-cut crossover operator. It is easy to recognize that *f*1(***x****i*, ***p****i*, *ϕ*1) plays the role of local attraction force (Guner & Sevkli, 2008, p. 5).



**Figure 2.1.3.** Illustration of two-cut crossover operator

Let *f*2(***x****i*, ***p****g*, *ϕ*2) represent two-cut crossover operator on ***x****i* and ***p****g* with parameter *ϕ*2 as probability threshold. Exactly, when pairs of ***x****i* and ***p****g* are browsed pair by pair, at each pair (*xij*, *pgj*) a new random number *r* is created with uniform distribution and then, if *r* ≥ *ϕ*, two-cut crossover operator will be executed on group of such pairs (*xij*, *pgj*). It is easy to recognize that *f*2(***x****i*, ***p****g*, *ϕ*2) plays the role of global attraction force. Finally, the position update rule in DPSO is defined by combinative function of *f*(***x****i*, *ϕ*), *f*1(***x****i*, ***p****i*, *ϕ*1), and *f*2(***x****i*, ***p****g*, *ϕ*2), as follows (Guner & Sevkli, 2008, p. 4):

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|  | (2.1.3) |

Equation 2.1.3 can be interpreted as an ordered sequence of three equations as follows:

Bare bones PSO (BBPSO) developed by James Kennedy and Russell C. Eberhart is also a simple version of PSO where velocity update rule is eliminated. In other words, positions ***x****i* are updated based on only previous position and previous best position. Given ***x****i* = (*xi*1, *xi*2,…, *xin*)*T*, ***p****i* = (*pi*1, *pi*2,…, *pin*)*T*, and ***p****g* = (*pg*1, *pg*2,…, *pgn*)*T*, BBPSO assumes that the *j*th element *xij* of ***x****i* follows normal distribution with mean (*pij*+*pgj*)/2 and variance (*pij*–*pgj*)2 (Poli, Kennedy, & Blackwell, 2007, p. 13).

|  |  |
| --- | --- |
|  | (2.1.4) |

Note, the sign “~” denotes distribution and denotes normal distribution. Thus, position update rule in BBPSO is modified as follows (Pan, Hu, Eberhart, & Chen, 2008, p. 3), (al-Rifaie & Blackwell, 2012, p. 51):

|  |  |
| --- | --- |
| Every *xij* is randomized according to normal distribution | (2.1.5) |

Obviously, position update rule in BBPSO is specified by equation 2.1.4 instead equation 1.2 and there is no velocity update rule. Therefore, the purpose of BPSO is to simplify the basic PSO.

Quantum PSO (QPSO) was proposed by Jun Sun, Bin Feng, and Wenbo Xu, in which movement of particles obeys quantum motion instead of Newton motion. Sun et al. assumed that each particle *i* is pulled by its local attractor denoted ***a****i* so that ***a****i* moves on the hyperrectangle formed by the local best position ***p****i* and global best position ***p****g* as follows (Fu, Liu, Zhang, & Deng, 2013, p. 2):

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|  | (2.1.6) |

Where *φ* is the parameter such that 0 < *φ* < 1. According to the stochastic movement of the local attractors, velocity update rule is removed from QPSO and position update rule is changed as follows (Fu, Liu, Zhang, & Deng, 2013, p. 2):

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|  | (2.1.7) |

Where *β* as a parameter is called contraction-expansion coefficient which is used to control the convergence speed. Besides, *h* and *u* are two random numbers which are randomized in runtime process such that 0 < *h* < 1 and 0 < *u* < 1. Moreover, *M* is the mean of all local best positions over the population , as follows (Fu, Liu, Zhang, & Deng, 2013, p. 3):

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|  | (2.1.8) |

Note, is the size of population which is the number of particles in .

Because important parameters known as coefficients such as cognitive weight *ϕ*1, social weight *ϕ*2, inertial weight *ω*, and constriction coefficient *χ* affect on velocity update rule, there are some researches aiming to modifying them at each iteration or each time point so as to improve PSO overtime. If such approach focuses on acceleration coefficients which are cognitive weight *ϕ*1, social weight *ϕ*2, it is called time-varying acceleration coefficient (TVAC) approach or TVAC-PSO. If a method aims to modify other coefficients like *ω* and *χ*, it can be classified into TVAC approach because there is relationship between *ω*, *χ*, *ϕ*1 and *ϕ*2.However, other methods related to adjust other parameters such as population size and topology do not belong to TVAC approach. Here we focus on TVAC-PSOs which can also be solutions for the premature problem and so they can be considered as dynamic PSOs. Shi and Eberhart modified inertial weight *ω* as follows (Ratnaweera, Halgamuge, & Watson, 2004, p. 241):

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|  | (2.1.9) |

Note *t* is current iteration and *T* is the maximum number of iterations in PSO process whereas *ω*1 and *ω*2 are initial value and final value of the inertial weight belonging to the interval (0, 1). Ratnaweera et al. modified acceleration coefficients *φ*1 and *φ*2 as follows (Ratnaweera, Halgamuge, & Watson, 2004, p. 242):

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| --- | --- |
|  | (2.1.10) |

Whereas *φ*1*f*, *φ*1*g*, *φ*2*f* and *φ*2*g* are constants belonging to the interval (0, 1).

~~Modifying coefficients (Bonyadi & Michalewicz, 2017, pp. 24-27)~~

## 2.2. Dynamic PSO

As aforementioned, two main problems of PSO are premature problem and dynamic problem. Solutions of premature problem are to improve the exploration so that PSO is not trapped in local optimizer. Exactly, these solutions relate to increase dynamic ability of particles such as dynamic topology, change of fitness function, adaptation (tuning coefficients, adding particles, removing particles, changing particle properties), and diversity control over iterations. As a convention, the solutions for premature problem derive so-called *dynamic PSOs*. This sub-section list popular dynamic PSOs. Note that QPSO and TVAC-PSO aforementioned can be considered as dynamic PSOs although they are not mentioned here again because their modifications for solving premature problem are not emphasized.

Recall that the topology from equations 1.1, 1.3, 1.4, 1.5, and 1.6 is static topology because it is kept intact over all iterations of PSO. Recall that topology implies groups of particles inside swarm population. For easily understandable explanation, suppose particles are vertices of a graph and a connection between two vertices indicates the interaction or the neighborhood of two particles so that such graph is the easily understandable representation of topology. If the graph is fully connected, the topology is called global topology (gbest topology) which is typically specified by equations 1.1, 1.3, and 1.4. If the graph is separated into many fully connected sub-graphs, the topology is called local topology (lbest topology) which is typically specified by equations 1.5 and 1.6. Here we research dynamic topology in which neighbors and neighborhood are changed at each iteration.

Sugnathan (Poli, Kennedy, & Blackwell, 2007, p. 8) proposed to start PSO with small local best topology with a small number of neighbors and such topology is progressively enlarged with a larger number of neighbors after each iteration until getting the full connected topology known as global best topology (gbest topology). The favorite local best topology is lattice ring.

Peram (Poli, Kennedy, & Blackwell, 2007, p. 8) defined the topology dynamically at each iteration by a so-called fitness distance ratio (FDR). Given target particle *i* and another particle *j*, their FDR is the ratio of the difference between *f*(***x****i*) and *f*(***x****j*) to the Euclidean difference between ***x****i* and ***x****j*.

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|  | (2.1.5) |

Given target particle *i*, if FDR(***x****i*, ***x****j*) is larger than a threshold (> 1), the particle *j* is a neighbor of the target particle *i*. Alternately, top *K* particles whose FDR (s) with ***x****i* are largest are *K* neighbors of particle *i*.

Clerc (Poli, Kennedy, & Blackwell, 2007, p. 8) proposed a so-called TRIBES algorithm which is a dynamic topology PSO. TRIBES divides the entire swarm into sub-populations called “tribes”. A good tribe has good particles whose best values (best fitness values) *f*(***p****i*) are small enough whereas a bad tribe has bad particles whose best values (best fitness values) *f*(***p****i*) are not small enough. After some iterations, a good tribe can remove particles that are not good enough in order to maintains its preeminence and a bad tribe can add more particles to increase its possibility of improvement. Because TRIBES adds and removes dynamically particles, it can be classified into adaptation solution for dynamic problem.

Recall that premature problem is solved by many solutions such as dynamic topology, change of fitness function, adaptation (tuning coefficients, adding particles, removing particles, changing particle properties), and diversity control over iterations. Here we research the ways which change fitness function evaluation. Noisy adder is an interesting way to change how to evaluate fitness function, but fitness function is indeed not changed. For instance, any time at any iteration when *f*(***x***) is evaluated, a random noise is added into the evaluated result.

The noise *ε* often conforms normal distribution. When ***x****i* is evaluated more than one time, evaluated results can be different. Consequently, it is possible to avoid converging local optimizer while the effectiveness in convergence is kept (Poli, Kennedy, & Blackwell, 2007, p. 10).

As aforementioned, *diversity control* is a solution of premature problem to prevent PSO from converging to local optimizer because a reason of local trapping is that many particles are clustered too tight into one region. Hence, we research here the diversity control.

(Poli, Kennedy, & Blackwell, 2007, p. 13), (Bonyadi & Michalewicz, 2017, p. 7)

Chaos PSO (Zhang, Wang, & Ji, 2015, p. 5)

## 2.3. Multi-objective PSO

(Bonyadi & Michalewicz, 2017, p. 6), (Zhang, Wang, & Ji, 2015, p. 12)

## 2.4. Constrained PSO

(Zhang, Wang, & Ji, 2015, p. 12), (Bonyadi & Michalewicz, 2017, p. 40)

Recall that two main aspects of PSO are exploration and exploitation. Exploitation is as important as exploration because it asserts success of convergence and speed of convergence. Some methods like BPSO and BBPSO improve the exploitation whereas the others like FDR and TRIBES improve the exploration with notice that the exploration for global convergence seemingly receives more concerns. Thus, some algorithms mentioned in next section which are combinations of PSO and other evolutionary algorithms aim to improve the exploitation.

# 3. PSO and other evolutionary algorithms

Evolutionary algorithms (EA) are the ones which simulate natural activities in biological world. PSO is an EA because it simulates how a flock of birds search for food and so its combination to other EA is natural. Some combinations are so tight that it is possible to form hybrid PSOs which are considered as variants of PSO mentioned in previous section.

## 3.1. PSO and artificial bee colony algorithm

Ant bee colony (ABC) algorithm proposed by Derviş Karaboğa simulates how a colony of bees search for food sources (Wikipedia, Artificial bee colony algorithm, 2007). There are three types of bees such as employed bees, onlooker bees and scout bees. Each employed bee is associated with its position ***x****i* = (*xi*1, *xi*2,…, *xin*)*T* which in turn represents a food source as a candidate solution for optimizing the target function. The nectar amount of a food source ***x****i* is called fitness value which can be is evaluated as the inverse of target function as = –*f*(***x****i*) in minimization. After finding food sources, all employed bees share their nectar amounts with onlooker bees so that onlookers select the best food source – the best position ***x***g whose fitness value is best. The selection strategy can be a roulette wheel selection mechanism which is based on the largest solution probability of nectar amounts. An employed bee becomes a scout if it cannot find out a new source after some unsuccessful trials of finding food sources; at that time such scout will reset its position. In general, ABC which is an iterative algorithm has many iterations in which each iteration has three phase such as employed phase, onlooker phase, and scout phase. The terminated condition can be that the overall fitness value is good value, or the number of iterations is large enough. Following is the pseudo-code of ABC (Karaboga & Akay, 2009, pp. 113-114):

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| The *N*-size population of employed bees and onlooker bees is initialized. Each employed bee has a random position ***x****i*.  Repeat  *Employed phase*: each employed bee looks a new position ***x****i*’ regarding other employed bees.   |  |  | | --- | --- | |  | (3.1.1) |   If the new fitness value (nectar amount) is larger than the old one , such employed bee moves to ***v****i*.   |  |  | | --- | --- | |  | (3.1.2) |   *Onlooker phase*: onlooker bees select the *ith* food source which is the *ith* solution based on nectar amounts which are modeled as selection probability *Pi* as follows:   |  |  | | --- | --- | |  | (3.1.3) |   Therefore, which *gth* food source has the largest *Pg* is selected. Of course, the fitness value of the *gth* food source is .  *Scout phase*: If an employed bee cannot find out better food source, for example, *f*(***x****i*) > after some iterations then, it becomes a scout. At that time, such scout reset its position as follows:   |  |  | | --- | --- | |  | (3.1.4) |   Until some terminated condition is met. |

**Table 3.1.1.** Ant bee colony (ABC) algorithm

In equation 3.1.1, indices *k* and *j* are randomized whereas *r*1 is a random number in interval [–1, 1]. In equation 3.1.4, ***lb*** and ***ub*** are lower bound and upper bound of search space, respectively whereas *r*2 is a random number in interval [0, 1].

There are some researches which combined PSO and ABC. Sharma et al. (Sharma, Pant, & Abraham, 2013) proposed a so-called Local Global variant Artificial Bee Colony (LGABC) that puts PSO information into ABC, which aims to balance the exploration and exploitation in ABC when ABC achieves higher exploitation than PSO because ABC updates new positions in employed phase more randomly than PSO whereas PSO achieves higher exploration than ABC because PSO updates new positions based on best local positions and best global positions. Concretely, Sharma et al. proposed a controlled parameter called modification rate (*MR*) and modified the position update equation 3.1.1 in employed phase as follows (Sharma, Pant, & Abraham, 2013, pp. 2-3):

|  |  |
| --- | --- |
|  | (3.1.5) |

Where ***v****i* is PSO velocity update rule specified by equation 1.1:

Note, *r* is random number in interval [0, 1] and the parameter *MR* is predefined in open interval (0, 1). Simply, the ABC position update equation 3.1.5 becomes PSO position update equation 1.2 if *r* > *MR*.

ABC and DE (Zhang, Wang, & Ji, 2015, p. 11)

## 3.2. PSO and genetic algorithm

Genetic algorithm (GA) is classified into evolutionary approach when it simulates biological activities including natural hybridization and selection. In other words, GA considers computational problems as biological activities and then, solves such problems by imitating biological activities too. For instance, given a problem whose candidate solutions compose a population, GA codes each solution candidate as a chromosome which consists of genes. Concretely, a chromosome represents a solution candidate, and a gene represents a property of such solution candidate with note that any solution has a set of properties. GA will perform so-call biological operators on chromosomes in order to reach a good enough solution. These biological operators simulating biological activities includes crossover operator, mutation operator, etc. In GA literature, a chromosome is also considered as an individual of population; exactly, each individual owns a chromosome. Essentially, given an initialized population of individuals, GA, which is an iterative algorithm, will produce a generation of new individuals at each iteration by performing biological operators on current individuals of the population. The generation of new individuals will be added to the population at each iteration so that the population get larger and larger. Another important thing in GA is to define the fitness of a given individual (chromosome) because GA performs biological operators only on the individuals whose fitness values are good enough. In order words, GA selects good enough individuals for making biological operators. How to define fitness function depends on concrete applications. GA goes through many iterations until best solution is reached, population is large enough, or fitness level (maybe fitness average) over entire population is good enough. Following is pseudo code of GA.

|  |
| --- |
| Initialize the population of individuals with note that chromosome of each individual is coded according to properties of solution candidate.  Repeat  Compute fitness values for individuals on populations.  Select individuals whose fitness values are good enough.  Perform biological operators (crossover, mutation, etc.) on the selected individuals to produce new individuals called offspring(s).  Set of the offspring(s) called generation is added to the entire population.  Util termination conditions are met. |

**Table 3.2.1.** General GA

Recall that the termination conditions can be that best solution is reached, the population is large enough, or fitness average over entire population is good enough. Biological operators like crossover, mutation, etc. are executed on selected individuals but exactly, such operators are executed on selected chromosomes because each individual has its own chromosome. Recall that each chromosome is a set of genes where each gene represents a property of solution. For easy explanation, a chromosome here is coded by an array of bit (0 and 1) and so each bit codes a gene.

**Figure 3.2.1.** Simple coding chromosome

Because crossover and mutation are two most important operators, they are described shortly here. Crossover, which is binary operator, exchanges mutually genes of two origin individuals called parents to create a new offspring whose chromosome inherits genes of its both parents. Task of exchanging genes occurs at a so-called crossover point.

**Figure 3.2.2.** Crossover operator

Mutation, which is simpler unary operator, changes randomly some genes of one individual so as to create a new offspring.

**Figure 3.2.3.** Mutation operator

Recall that there are two most important points in GA, in which the first one is to code and perform biological operators on chromosomes whereas the second one is to define the fitness function. Exactly, fitness function implies the way to calculate fitness values of chromosomes (individuals) and so maybe it is not algebraic function. In other words, the fitness implies evaluation of solution. For easy explanation, given optimization problem as follows:

Fitness function can be inverse of target function –*f*(***x***).

(Zhang, Wang, & Ji, 2015, p. 9)

## 3.3. PSO and artificial neural network

(Zhang, Wang, & Ji, 2015, p. 9), (Bonyadi & Michalewicz, 2017, p. 7),

## 3.4. PSO and machine learning algorithms

SA (Zhang, Wang, & Ji, 2015, p. 10)

Fuzzy PSO (Zhang, Wang, & Ji, 2015, p. 5)

# 4. Theoretical analysis of PSO

PSO is the simple but effective method to solve the optimization problem. Its ideology and practice are reasonable and interesting but there are not fully comprehensive mathematical models for PSO yet (Poli, Kennedy, & Blackwell, 2007, p. 14). However, it is necessary to survey these models to understand PSO and its preeminent features.

(Bonyadi & Michalewicz, 2017, p. 8)deterministic model stability analysis, first-order stability analysis, and second-order stability analysis

Garcia-Gonzalo and Fernandez-Martinez (Bonyadi & Michalewicz, 2017, p. 12)

# 5. Discussions

## 5.1. Limitations and open questions

## 5.2. Applications of PSO

(Zhang, Wang, & Ji, 2015, p. 16)

# References

al-Rifaie, M. M., & Blackwell, T. (2012). Bare Bones Particle Swarms with Jumps. In M. Dorigo, M. Birattari, C. Blum, A. L. Christensen, A. P. Engelbrecht, R. Groß, & T. Stützle (Ed.), *International Conference on Swarm Intelligence.* *Lecture Notes in Computer Science 7461*, pp. 49-60. Brussels: Springer Berlin. doi:10.1007/978-3-642-32650-9\_5

Bonyadi, M. R., & Michalewicz, Z. (2017, March 2). Particle Swarm Optimization for Single Objective Continuous Space Problems: A Review. (N. Lindsay, Ed.) *Evolutionary Computation, 25*(1), 1-54. doi:10.1162/EVCO\_r\_00180

Fu, X., Liu, W., Zhang, B., & Deng, H. (2013, October 24). Quantum Behaved Particle Swarm Optimization with Neighborhood Search for Numerical Optimization. *Mathematical Problems in Engineering, 2013*. doi:10.1155/2013/469723

Guner, A. R., & Sevkli, M. (2008, April 8). A Discrete Particle Swarm Optimization Algorithm for Uncapacitated Facility Location Problem. *Journal of Artificial Evolution and Applications, 2008*, 1-9. doi:10.1155/2008/861512

Karaboga, D., & Akay, B. (2009, April 11). A comparative study of Artificial Bee Colony algorithm. *Applied Mathematics and Computation, 214*(1), 108-132. doi:10.1016/j.amc.2009.03.090

Pan, F., Hu, X., Eberhart, R., & Chen, Y. (2008, September 21). An Analysis of Bare Bones Particle Swarm. *IEEE Swarm Intelligence Symposium 2008 (SIS 2008)* (pp. 1-5). St. Louis, MO, US: IEEE. doi:10.1109/SIS.2008.4668301

Poli, R., Kennedy, J., & Blackwell, T. (2007, June). Particle swarm optimization. (M. Dorigo, Ed.) *Swarm Intelligence, 1*(1), 33-57. doi:10.1007/s11721-007-0002-0

Ratnaweera, A., Halgamuge, S. K., & Watson, H. C. (2004, June 14). Self-organizing hierarchical particle swarm optimizer with time-varying acceleration coefficients. (R. C. Eberhart, & Y. Shi, Eds.) *IEEE Transactions on Evolutionary Computation, 8*(3), 240-255. doi:10.1109/TEVC.2004.826071

Sharma, T. K., Pant, M., & Abraham, A. (2013). Blend of Local and Global Variant of PSO in ABC. *The 2013 World Congress on Nature and Biologically Inspired Computing.* IEEE. doi:10.1109/NaBIC.2013.6617848

Too, J., Abdullah, A. R., & Saad, N. M. (2019, May 8). A New Co-Evolution Binary Particle Swarm Optimization with Multiple Inertia Weight Strategy for Feature Selection. (A. Bryant, R. Theron, K. Sedig, & D. J. Lizotte, Eds.) *informatics, 6*(2), 1-14. doi:10.3390/informatics6020021

Wikipedia. (2007, November 23). *Artificial bee colony algorithm*. (Wikimedia Foundation) Retrieved from Wikipedia website: https://en.wikipedia.org/wiki/Artificial\_bee\_colony\_algorithm

Wikipedia. (2017, March 7). *Particle swarm optimization*. (Wikimedia Foundation) Retrieved April 8, 2017, from Wikipedia website: https://en.wikipedia.org/wiki/Particle\_swarm\_optimization

Zhang, Y., Wang, S., & Ji, G. (2015, October 7). A Comprehensive Survey on Particle Swarm Optimization Algorithm and Its Applications. (G. Xie, Ed.) *Mathematical Problems in Engineering, 2015*(Special Issue: Artificial Intelligence and Its Applications 2014), 1-38. doi:10.1155/2015/931256

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