Semi-mixture Regression Model for Incomplete Data

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**Abstract:**

The regression expectation maximization (REM) algorithm, which is a variant of expectation maximization (EM) algorithm, uses parallelly a long regression model and many short regression models to solve the problem of incomplete data. Experimental results proved resistance of REM to incomplete data, in which accuracy of REM decreases insignificantly when data sample is made sparse with loss ratios up to 80%. However, the convergence speed of REM can be decreased if there are many independent variables. In this research, we use mixture model to decompose REM into many partial regression models. These partial regression models are then unified in the so-called semi-mixture regression model. Our proposed algorithm is called semi-mixture regression expectation maximization (SREM) algorithm because it is combination of mixture model and REM algorithm, but it does not implement totally the mixture model. In other words, only mixture coefficients in SREM are estimated according to mixture model whereas regression coefficients are estimated by REM. The experimental results show that SREM converges faster than REM does although the accuracy of SREM is not better than the accuracy of REM in fair tests.

**Keywords:**

Regression Model, Mixture Regression Model, Expectation Maximization Algorithm, Incomplete Data

1. **Introduction**

***1.1. Main work***

As a convention, regression model is a linear regression function *Z = α*0 *+ α*1*X*1 *+ α*2*X*2 *+ … + αnXn* in which variable *Z* is called response variable or dependent variable whereas each *Xi* is called regression variable, regressor, predictor, regression variable, or independent variable. Each *αi* is called regression coefficient. The essence of regression analysis is to calculate regression coefficients from data sample. When sample is complete, these coefficients are determined by least squares method [1, pp. 452-458]. When sample is incomplete, there are some approximation approaches to estimate regression coefficients such as complete case method, ad-hoc method, multiple imputation, maximum likelihood, weighting method, and Bayesian method [2]. We focus on applying expectation maximization (EM) algorithm into constructing regression model in case of missing data with note that EM algorithm belongs to maximum likelihood approach. In previous research [3], we proposed a so-called Regression Expectation Maximization (REM) algorithm to learn linear regression function from incomplete data in which some values of *Z* and *Xi* are missing. REM is a variant of EM algorithm, which is used to estimate regression coefficients. Experimental results in previous research [3] proved that accuracy of REM decreases insignificantly whereas loss ratios increase significantly. We hope that REM will be accepted as a new standard method for regression analysis in case of missing data when there are currently 6 standard approaches such as complete case method, ad-hoc method, multiple imputation, maximum likelihood, weighting method, and Bayesian method [2]. Here we combine REM and mixture model to improve convergence speed of REM. Our proposed algorithm is called Semi-mixture Regression Expectation Maximization (SREM) algorithm. Experimental results mentioned later show that SREM converges faster than REM although it is not as accurate as REM. Because this research is the successive one after our previous research [3], they share some common contents related to research survey and experimental design, but we confirm that their methods are not coincide although SREM is derived from REM.

Because SREM is the combination of REM and mixture model whereas REM is a variant of EM algorithm, we need to survey some works related to application of EM algorithm to regression analysis. Kokic [4] proposed an excellent method to calculate expectation of errors for estimating coefficients of multivariate linear regression model. In Kokic’s method, response variable *Z* has missing values. Ghitany, Karlis, Al-Mutairi, and Al-Awadhi [5] calculated the expectation of function of mixture random variable in expectation step of EM algorithm and then used such expectation for estimating parameters of multivariate mixed Poisson regression model in the maximization step. Anderson and Hardin [6] used reject inference technique to estimate coefficients of logistic regression model when response variable *Z* is missing but characteristic variables (regressors *Xi*) are fully observed. Anderson and Hardin replaced missing *Z* by its conditional expectation on regressors *Xi* where such expectation is logistic function. Zhang, Deng, and Su [7] used EM algorithm to build up linear regression model for studying glycosylated hemoglobin from partial missing data. In other words, Zhang, Deng, and Su [7] aim to discover relationship between independent variables (predictors) and diabetes.

Besides EM algorithm, there are other approaches to solve the problem of incomplete data in regression analysis. Haitovsky [8] stated that there are two main approaches to solve such problem. The first approach is to ignore missing data and to apply the least squares method into observations. The second approach is to calculate covariance matrix of regressors and then to apply such covariance matrix into constructing the system of normal equations. Robins, Rotnitzki, and Zhao [9] proposed a class of inverse probability of censoring weighted estimators for estimating coefficients of regression model. Their approach is based on the dependency of mean vector of response variable *Z* on vector of regressors *Xi* when *Z* has missing values. Robins, Rotnitzki, and Zhao [9] assumed that the probability *λit*(*α*) of existence of *Z* at time point *t* is dependent on existence of *Z* at previous time point *t*–1 but independent from *Z*. Even though *Z* is missing, the probability *λit*(*α*) is also determined and so regression coefficients are calculated based on the inverse of *λit*(*α*) and *Xi*. The inverse of *λit*(*α*) is considered as weight for complete case. Robins, Rotnitzki, and Zhao used additional time-dependent covariates *Vit* to determine *λit*(*α*).

In the article “Much ado about nothing: A comparison of missing data methods and software to fit incomplete data regression models”, Horton and Kleinman [2] classified 6 methods of regression analysis in case of missing data such as complete case method, ad-hoc method, multiple imputation, maximum likelihood, weighting method, and Bayesian method. EM algorithm belongs to maximum likelihood method. According to complete case method, regression model is learned from only non-missing values of incomplete data [2, p. 3]. The ad-hoc method refers missing values to some common value, creates an indicator of missingness as new variable, and finally builds regression model from both existent variables and such new variable [2, p. 3]. Multiple imputation method has three steps. Firstly, missing values are replaced by possible values. The replacement is repeated until getting an enough number of complete datasets. Secondly, some regression models are learned from these complete datasets as usual [2, p. 4]. Finally, these regression models are aggregated together. The maximum likelihood method aims to construct regression model by maximizing likelihood function. EM algorithm is a variant of maximum likelihood method, which has two steps such as expectation step (E-step) and maximization step (M-step). In E-step, multiple entries are created in an augmented dataset for each observation of missing values and then probability of the observation is estimated based on current parameter [2, p. 6]. In M-step, regression model is built from augmented dataset. The REM algorithm proposed in this research is different from the traditional EM for regression analysis because we replace missing values in E-step by expectation of sufficient statistics via mutual balance process instead of estimating the probability of observation. The weighting method determines the probability of missingness and then uses such probability as weight for the complete case. The aforementioned research of Robins, Rotnitzki, and Zhao [9] belongs to the weighting approach. Instead of replacing missing values by possible values like imputation method does, the Bayesian method imputes missing values by the estimation with a prior distribution on the covariates and the close relationship between the Bayesian approach and maximum likelihood method [2, p. 7].

***1.2. Related Studies***

Recall that SREM is the combination of REM and mixture model and so we need to survey other works related to regression model with support of mixture model. As a convention, such regression model is called mixture regression model. In literature, there are two approaches of mixture regression model:

* The first approach is to use logistic function to estimate the mixture coefficients.
* The second approach is to construct a joint probability distribution as product of the probability distribution of response variable *Z* and the probability distribution of independent variables *Xi*.

According to the first approach [10], the mixture probability distribution is formulated as follows:

|  |  |
| --- | --- |
|  | (1) |

Where Θ = (*αk*, *σk*2)*T* is compound parameter whereas *αk* and *σk*2 are regression coefficients and variance of the partial (component) probability distribution *Pk*(*Z*|*αkTX*, *σk*2). Note, mean of *Pk*(*Z*|*αkTX*, *σk*2) is *αkTX* and mixture coefficients are *ck*. In the first approach, regression coefficients *αk* are estimated by least squares method whereas mixture coefficients are estimated by support of logistic function as follows [10, p. 4]:

|  |  |
| --- | --- |
|  | (2) |

The mixture regression model is:

|  |  |
| --- | --- |
|  | (3) |

According to the second approach, the joint distribution is defined as follows [11, p. 4]:

|  |  |
| --- | --- |
|  | (4) |

Where *αk* are regression coefficients and *σk*2 is variance of the conditional probability distribution *Pk*(*Z*|*αkTX*, *σk*2) whereas *μk* and Σ*k* are mean vector and covariance matrix of the prior probability distribution *Pk*(*X*| *μk*, Σ*k*), respectively. The mixture regression model is [11, p. 6]:

|  |  |
| --- | --- |
|  | (5) |

Where,

|  |  |
| --- | --- |
|  | (6) |

The joint probability can be defined by different way as follows [12, p. 21], [13, p. 24], [14, p. 4]:

|  |  |
| --- | --- |
|  | (7) |

Where *mk*(*X*) and *σk*2 are mean and variance of *Z* given the conditional probability distribution *Pk*(*Z*|*mk*(*X*), *σk*2) whereas *μkX* and Σ*kX* are mean vector and covariance matrix of *X* given the prior probability distribution *Pk*(*X*| *μk*, Σ*k*). When *μkX* and Σ*kX* are calculated from data, other parameters *mk*(*X*) and *σk*2 are estimated for each *k*th component as follows [12, p. 23], [13, p. 25], [14, p. 5]:

|  |  |
| --- | --- |
|  | (8) |

For each *k*th component, *μkZ* is sample mean of *Z*, Σ*kZX* is vector of covariances of *Z* and *X*, and Σ*kZZ* is sample variance of *Z*. The mixture regression model becomes [13, p. 25]:

|  |  |
| --- | --- |
|  | (9) |

Where,

|  |  |
| --- | --- |
|  | (10) |

Grün & Leisch [15] mentioned the full application of mixture model into regression model in which regression coefficients are determined by inverse function of mean of conditional probability distribution as follows:

|  |  |
| --- | --- |
|  | (11) |

In general, the ideology of combination of regression analysis and mixture model which produces mixture regression is not new, but our proposed SREM is different from other methods in literature because of followings:

* SREM does not use the joint probability distribution. In other words, SREM does not concern the probability distribution of independent variables *Xi*.
* Variance and mean of the conditional probability *Pk*(*Z*|*αkTX*, *σk*2) in SREM are not estimated by mixture model. They are instead estimated by one-time balance process of REM. SREM also does not use logistic function to estimate mixture coefficients as the first approach does. However, SREM is similar to the first approach most because both SREM and the first approach use the conditional probability distribution to estimate mixture coefficients except that SREM takes advantages of the mean of component probabilities whereas the first approach takes advantages of logistic function.
* SREM does not re-compute mixture coefficients when evaluating regression function.
* Mixture regression models in literature are learned from complete data whereas SREM supports incomplete data.

In general, SREM does not implement totally mixture model because only mixture coefficients in SREM are estimated by the estimation process of mixture model. In this research, we do not compare SREM with other mixture regression methods because the purpose of SREM is different from the purpose of mixture regression model. SREM aims to speed up the convergence of REM in case of missing data whereas mixture regression model aims to improve accuracy of regression analysis in case that data varies complicatedly with many trends. At the first stage of this research, I aim to decompose REM by SREM with hope that SREM is more accurate than REM in fair testing. Unexpectedly, the accuracy of SREM is not better than the accuracy of REM in fair tests but SREM converges faster than REM. Because speed is a significant aspect of an algorithm when data is large, I write this paper as a contribution of SREM. I guesstimate that SREM can be worse than full mixture regression model when data is complete and varies in many trends. On the other hand, full mixture model combined with REM will be better than SREM when data is incomplete and varies in many trends. However, we need an experimental research to assert this assumption. The methodology of SREM is described in section 2. Section 3 focuses on experimental results. Section 4 is the conclusion.

1. **Methodology**

The probabilistic Mixture Regression Model (MRM) is a combination of normal mixture model and linear regression model. In MRM, the probabilistic Entire Regression Model (ERM) is sum of *K* weighted probabilistic Partial Regression Models (PRMs). Equation (12) specifies MRM [16, p. 3].

|  |  |
| --- | --- |
|  | (12) |

Where,

Note, Θ is called entire parameter,

The superscript “*T*” denotes transposition operator in vector and matrix. In equation (12), the probabilistic distribution *P*(*zi*|*Xi*, Θ) represents the ERM where *zi* is the response variable, dependent variable, or outcome variable. The probabilistic distribution *Pk*(*zi*|*Xi*, *αk*, *σk*2) represents the *k*th PRM *zi* = *αk*0 *+ αk*1*xi*1 *+ αk*2*xi*2 *+ … + αknxin* with suppose that each *zi* conforms to normal distribution according to equation (13) with mean *μk* = *αkTXi* and variance *σk*2.

|  |  |
| --- | --- |
|  | (13) |

The parameter *αk* = (*αk*0, *αk*1,…, *αkn*)*T* is called the *k*th Partial Regression Coefficient (PRC) and *Xi* = (1, *xi*1, *xi*2,…, *xin*)*T* is data vector. Each *xij* in every PRM is called a regressor, predictor, or independent variable.

In equation (12), each mixture coefficient *ck* is the prior probability that any *zi* belongs to the *k*th PRM. Let *Y* be random variable representing PRMs, *Y* = 1, 2,…, *K*. The mixture coefficient *ck* is also called the *k*th weight, which is defined by equation (14). Of course, there are *K* mixture coefficients, *K* PRMs, and *K* PRCs.

|  |  |
| --- | --- |
|  | (14) |

For each *k*th PRM, suppose each has an inverse regression model (IRM) *xij* = *βkj*0 *+ βkj*1*zi*. In other words, *xij* now is considered as the random variable conforming to normal distribution according to equation (15) [17, p. 8].

|  |  |
| --- | --- |
|  | (15) |

Where *βkj* = (*βkj*0, *βkj*1)*T* is an inverse regression coefficient (IRC) and (1, *zi*)*T* becomes an inverse data vector. The mean and variance of each *xij* with regard to the inverse distribution *Pkj*(*xij*|*zi*, *βkj*) are *βkjT*(1, *zi*)*T* and *τkj*2, respectively. Of course, for each *k*th PRM, there are *n* IRMs *Pkj*(*xij*|*zi*, *βkj*) and *n* associated IRCs *βkj*. Totally, there are *n*\**K* IRMs associated with *n*\**K* IRCs.

In this research, we focus on estimating the entire parameter Θ = (*ck*, *αk*, *σk*2, *βkj*)*T* where *k* is from 1 to *K*. In other words, we aim to estimate *ck*, *αk*, *σk*2, and *βkj* for determining the ERM in case of missing data. As a convention, let Θ\* = (*ck*\*, *αk*\*, (*σk*2)\*, *βkj*\*)*T* be the estimate of Θ = (*ck*, *αk*, *σk*2, *βkj*)*T*, respectively. Let ***D*** = (***X***, ***Z***) be collected sample in which ***X*** is a set of regressors and ***Z*** is a set of outcome variables plus values 1, respectively [17, p. 8] with note that both ***X*** and ***Z*** are incomplete. In other words, ***X*** and ***Z*** have missing values. As a convention, let *zi*– and *xij*– denote missing values of ***Z*** and ***X***, respectively.

|  |  |
| --- | --- |
|  | (16) |

The expectation of sufficient statistic *zi* regard to the *k*th PRM *Pk*(*zi*|*Xi*, *αk*, *σk*2) is specified by equation (17) [3].

|  |  |
| --- | --- |
|  | (17) |

Where *xi*0=1 for all *i*. The expectation of the sufficient statistic *xij* with regard to each IRM *Pkj*(*xij*|*zi*, *βj*) of the *k*th PRM *Pk*(*zi*|*Xi*, *αk*, *σk*2) is specified by equation (18) [3].

|  |  |
| --- | --- |
|  | (18) |

Please pay attention to equations (17) and (18) because missing values of data ***X*** and data ***Z*** will be estimated by these expectations later. By applying sample ***D*** into equations (12) and (13) and using maximum likelihood estimation (MLE) method [17, pp. 8-9], we retrieve equation (19) to estimate *αk\**, *βkj\** [1, p. 457], and (*σk*2)\* for each *k*th PRM where ***X***, ***Z***, *Z*, *Xi*, and *Xj* are specified in equation (16). Appendix A1 is the proof of equation (19).

|  |  |
| --- | --- |
|  | (19) |

From sample ***D***, the optimal regression coefficients (*αk\**, (*σk*2)\*) and *βkj\** estimated by equation (19) whereas the optimal mixture coefficient *ck*\* for each *k*th PRM is estimated by equation (20) as follows [16, p. 7]:

|  |  |
| --- | --- |
|  | (20) |

Where [16, p.3],

|  |  |
| --- | --- |
|  | (21) |

Note, each optimal PRM *Pk*(*zi*|*Xi*, *αk*\*, (*σk*2)\*) is determined by equation (13).

Because ***X*** and ***Z*** are incomplete, we apply expectation maximization (EM) algorithm into estimating Θ\* = (*ck*\*, *αk*\*, (*σk*2)\*, *βkj*\*)*T*. According to [18], EM algorithm has many iterations and each iteration has expectation step (E-step) and maximization step (M-step) for estimating parameters. Given current parameter Θ(*t*) = (*ck*(*t*), *αk*(*t*), (*σk*2)(*t*), *βkj*(*t*))*T* at the *t*th iteration, missing values *zi*– and *xij*– are calculated in E-step so that ***X*** and ***Z*** become complete. In M-step, the next parameter Θ(*t*+1) = (*ck*(*t*+1), *αk*(*t*+1), (*σk*2)(*t*+1), *βkj*(*t*+1))*T* is determined by equations (19) and (20) and the complete data ***X*** and ***Z***.

The most important problem in our research is how to estimate missing values *zi*– and *xij*–. Recall that, for each *k*th PRM, every missing value *zi*– is estimated as the expectation based on the current parameter *αk*(*t*), according to equation (17) [3].

Note, *xi*0 = 1. Let *Ui* be a set of indices of missing values *xij*– with fixed *i* for each *k*th PRM. In other words, if then, *xij* is missing. The set *Ui* can be empty. The equation (17) is re-written for each *k*th PRM as follows [3]:

According to equation (18), missing value *xij*– is estimated by [3]:

Combining equation (17) and equation (18), we have [3]:

It implies [3]:

As a result, equation (22) is used to estimate or fulfill missing values for each *k*th PRM [3].

|  |  |
| --- | --- |
|  | (22) |

In previous research, we proposed a so-called Regression Expectation Maximization (REM) which is a variant of EM algorithm for estimating *αk*\* and *βkj*\*. Equation (22) is used in the E-step of REM to fulfill missing values. However, REM does not support mixture model. Here we proposed a so-called Semi-mixture Regression Expectation Maximization (SREM) which is a variant of REM, in which M-step is modified to calculate the optimal mixture coefficient *ck*\*. SREM is described in Table 1. We will explain later why SREM does not conform fully to mixture model although it supports mixture model.

***Table 1.*** *Semi-mixture Regression Expectation Maximization (SREM) Algorithm.*

|  |
| --- |
| 1. E-step: Missing values *zi*– and *xij*– for each *k*th PRM are fulfilled by equation (22) given current parameter Θ(*t*). 2. M-step: The next parameter Θ(*t*+1) is determined by equations (19), (20), and (21) and the complete data (***X****k*, ***Z****k*) fulfilled in E-step. Please pay attention that each *k*th PRM owns a particular complete data (***X****k*, ***Z****k*). In other words, original sample (***X***, ***Z***) has *K* complete versions (***X****k*, ***Z****k*) fulfilled in E-step for *K* PRMs. Note, such *K* complete versions are changed over each iteration.   Where,  Note that *Zk* is *Z* that belongs to ***Z****k*, *Xki* is *Xi* that belongs to ***X****k*, *Xkj* is *Xj* that belongs to ***X****k*, and *zki* is *zi* that belongs to *Zk*. The next parameter Θ(*t*+1) becomes current parameter in the next iteration. |

EM algorithm stops if at some *t*th iteration, we have Θ(*t*) = Θ(*t*+1) = Θ*\**. At that time, Θ\* = (*ck*\*, *αk*\*, (*σk*2)\*, *βkj*\*) is the optimal estimate of EM algorithm. Note, Θ(1) at the first iteration is initialized arbitrarily. Here SREM stops if ratio deviation between Θ(*t*) and Θ(*t*+1) is smaller than a small enough terminated threshold *ε* > 0 or SREM reaches a large enough number of iterations. The smaller the terminated threshold is, the more accurate SREM is. SREM uses both the terminated threshold *ε* = 0.1% = 0.001 and the maximum number of iterations (10000). The maximum number of iterations prevents SREM from running for a long time.

In traditional Gaussian mixture model, variances (*σk*2)(*t*+1) and means *μk*(*t*+1) are estimated by different way based on *ck*(*t*) and PRMs. Therefore, our model is called semi-mixture regression model when only *ck*(*t*+1) is estimated by PRMs. The reason is that (*σk*2)(*t*+1) and *αk*(*t*+1) were optimized by maximum likelihood estimation (MLE) method and it may be overfitting or redundant to re-estimate (*σk*2)(*t*+1) and *αk*(*t*+1) by Gaussian mixture model. As a result, we save computation cost by estimating (*σk*2)\* and *ck*\* after EM process finished. In other words, (*σk*2)(*t*+1) and *ck*(*t*+1) are not re-computed many times at E-step of every iteration and so (*σk*2)\* and *ck*\* are computed only one time after EM process finished, according to equation (23).

|  |  |
| --- | --- |
|  | (23) |

Note that *Zk* is *Z* that belongs to ***Z****k*, *Xki* is *Xi* that belongs to ***X****k*, and *zki* is *zi* that belongs to *Zk* where (***X****k*, ***Z****k*) is owned by the *k*th PRM, which the *k*th version of the original sample (***X***, ***Z***).

We use the complete case method mentioned in [2, p. 3] to improve the convergence of SREM. The parameters (*αk*(1), *βkj*(1))*T* at the first iteration of EM process are initialized in proper way instead that they are initialized in arbitrary way [19]. Let ***X****k*’ be the complete matrix, which is created by removing all rows whose values are missing from ***X****k*. Similarly, let ***Z****k*’ be the complete matrix, which is created by removing rows whose weights are missing from ***Z****k*. The advanced parameters (*αk*(1), *βkj*(1))*T* are initialized by equation (24).

|  |  |
| --- | --- |
|  | (24) |

Where *Zk*’ is the complete vector of non-missing outcome values for each *k*th PRM and *Xkj*’ is the complete column vector of non-missing regressor values for each *k*th PRM. Equation (24) is a variant of equation (19) where ***X****k*, ***Z****k*, *Xkj*, and *Zk* are replaced by ***X****k*’, ***Z****k*’, *Xkj*’, and *Zk*’.

The evaluation of SREM is different from traditional regression model. It follows mixture model. For example, given input data vector *X*0 = (*x*01, *x*02,…, *x*0*n*), let *z*1, *z*2,…, *zK* are values evaluated from *K* PRMs, we have:

Where *x*00 = 1. The final evaluation *z* is calculated based on mixture coefficients as seen in equation (25).

|  |  |
| --- | --- |
|  | (25) |

Following is the proof of equation (25). From equation (12), let be the estimate of response variable *z*, we have:

Equation (25) is the semi-mixture regression model where mixture coefficients *αkj*\* are resulted from the EM process of SREM shown in Table 1 and *ck*\* is calculated by equation (23). Note, semi-mixture regression model does not re-compute mixture coefficients *ck*\* when evaluating *z* from *X*0. In other words, after SREM finished, *ck*\* are fixed.

1. **Results and Discussions**

We use two data samples for testing SREM. The first one is the gestational dataset of 1027 cases in which each case includes ultrasound measures (regressors) and fetus weight (response variable). Ultrasound measures are bi-parietal diameter (*bpd*), head circumference (*hc*), abdominal circumference (*ac*), and fetal length (*fl*). The unit of *bpd*, *hc*, *ac*, and *fl* is millimeter whereas the unit of fetal weight is gram. Ho and Phan [20], [21] collected the ultrasound measure sample of pregnant women at Vinh Long General Hospital – Vietnam with obeying strictly all medical ethical criteria. These women and their husbands are Vietnamese. Their periods are regular and their last periods are determined. Each of them has only one alive fetus. Fetal age is from 28 weeks to 42 weeks. Delivery time is not over 48 hours since ultrasound scan.

The second sample is the dataset which contains 9568 data points collected from a Combined Cycle Power Plant (CCPP) [22]. Regressors in CCPP dataset are hourly average Ambient Temperature (*AT*), Ambient Pressure (*AP*), Relative Humidity (*RH*) and Exhaust Vacuum (*V*) to predict the net hourly electrical energy output (PE) as response variable. *AT* is in the range 1.81°C and 37.11°C. *AP* is in the range 992.89-1033.30 millibar. *RH* is in the range 25.56% to 100.16%. *V* is in the range 25.36-81.56 cm Hg. *PE* is in the range 420.26-495.76 MW.

In general, we have two samples such as gestational sample and CCPP sample. The dataset is split separately into one training dataset (50% sample) and one testing dataset (50% sample). Later on, the training dataset is made sparse with loss ratios 10%, 20%, 30%, 40%, 50%, 60%, 70%, 80%, and 90%, which is similar to our previous research [19]. Missing values are made randomly regardless of regressors or response variable. For example, the gestational training dataset (50% gestational sample) has 50%\*1027 ≈ 513 rows and each row has 5 columns (*bpd*, *hc*, *ac*, *fl*, *weight*) and so the training dataset has 513\*5 = 2565 cells. If loss ratio is 10%, there are only 10%\*2565 ≈ 256 missing values which are made randomly among such 2565 cells. In other words, the incomplete training dataset with loss ratio 10% has 2565 – 256 = 2309 non-missing values. Of course, the testing dataset (50% sample) is not made sparse. Each pair of incomplete training dataset and testing dataset is called testing pair. There are ten testing pairs for each sample. As a convention, the origin testing pair which has no missing value in training dataset is the 0th pair. The 0th pair is called complete pair whereas the 1st, 2nd, 3rd, 4th, 5th, 6th, 7th, 8th, and 9th pairs are called incomplete pairs.

Firstly, we test SREM with gestational sample. Table 2 [19] shows ten testing pairs of gestational sample.

***Table 2.*** *Ten testing pairs of gestational sample.*

|  |  |  |  |
| --- | --- | --- | --- |
| Pair | Training dataset | Testing dataset | Loss ratio |
| 0 | Ges.sample.base | Ges.sample.test | 0% |
| 1 | Ges.sample.base.0.1.miss | Ges.sample.test | 10% |
| 2 | Ges.sample.base.0.2.miss | Ges.sample.test | 20% |
| 3 | Ges.sample.base.0.3.miss | Ges.sample.test | 30% |
| 4 | Ges.sample.base.0.4.miss | Ges.sample.test | 40% |
| 5 | Ges.sample.base.0.5.miss | Ges.sample.test | 50% |
| 6 | Ges.sample.base.0.6.miss | Ges.sample.test | 60% |
| 7 | Ges.sample.base.0.7.miss | Ges.sample.test | 70% |
| 8 | Ges.sample.base.0.8.miss | Ges.sample.test | 80% |
| 9 | Ges.sample.base.0.9.miss | Ges.sample.test | 90% |

SREM may be better than REM if SREM has a large enough number of PRMs and each PRM has many enough regressors. Thus, for fair testing, the number of PRMs in SREM is equal to the number of regressors and each PRM has only one regressor. Table 3 shows ten resulted regression models of REM corresponding to ten testing pairs, given gestational sample.

***Table 3.*** *Ten resulted regression models of REM given gestational sample.*

|  |  |
| --- | --- |
| Pair | Regression model |
| 0 | *weight* = -5686.8907 + 46.2369\*(*bpd*) + 1.7148\*(*hc*) + 14.3173\*(*fl*) + 9.3881\*(*ac*) |
| 1 | *weight* = -5685.7848 + 43.1103\*(*bpd*) + 1.4912\*(*hc*) + 17.0387\*(*fl*) + 9.8929\*(*ac*) |
| 2 | *weight* = -5853.1212 + 39.5619\*(*bpd*) + 2.4174\*(*hc*) + 21.7261\*(*fl*) + 9.5005\*(*ac*) |
| 3 | *weight* = -6198.2399 + 44.6901\*(*bpd*) + 5.2472\*(*hc*) + 20.4527\*(*fl*) + 6.6325\*(*ac*) |
| 4 | *weight* = -5941.9821 + 39.9089\*(*bpd*) + 2.6238\*(*hc*) + 23.3260\*(*fl*) + 9.2312\*(*ac*) |
| 5 | *weight* = -6496.2424 + 44.6131\*(*bpd*) + 3.9980\*(*hc*) + 25.8861\*(*fl*) + 7.7759\*(*ac*) |
| 6 | *weight* = -5940.9170 + 31.6952\*(*bpd*) + 2.8293\*(*hc*) + 34.1356\*(*fl*) + 9.0107\*(*ac*) |
| 7 | *weight* = -6296.7603 + 66.8602\*(*bpd*) + 2.7111\*(*hc*) + 16.8848\*(*fl*) + 4.0660\*(*ac*) |
| 8 | *weight* = -5362.1163 + 35.6642\*(*bpd*) + 4.7398\*(*hc*) + 14.8123\*(*fl*) + 8.2385\*(*ac*) |
| 9 | *weight* = -5923.3220 + 87.5165\*(*bpd*) + 3.4471\*(*hc*) - 0.2822\*(*fl*) - 0.0753\*(*ac*) |

Table 4 shows ten resulted semi-mixture regression models of SREM corresponding to ten testing pairs, given gestational sample.

***Table 4.*** *Ten resulted semi-mixture regression models of SREM given gestational sample.*

|  |  |
| --- | --- |
| Pair | Semi-mixture regression model |
| 0 | {*weight* = -6651.5534 + 108.5531\*(*bpd*): coeff=0.2721, var=113888.6649},  {*weight* = -4986.7292 + 24.6736\*(*hc*): coeff=0.2041, var=188973.6069},  {*weight* = -4505.6926 + 109.6790\*(*fl*): coeff=0.2450, var=119971.2307},  {*weight* = -3385.5925 + 19.4249\*(*ac*): coeff=0.2788, var=97458.0445} |
| 1 | {*weight* = -6802.9586 + 110.3231\*(*bpd*): coeff=0.2700, var=98865.5195},  {*weight* = -5089.2989 + 25.0105\*(*hc*): coeff=0.2012, var=163173.8380},  {*weight* = -4744.7739 + 113.4482\*(*fl*): coeff=0.2426, var=103291.1775},  {*weight* = -3515.6183 + 19.8394\*(*ac*): coeff=0.2862, var=77538.8650} |
| 2 | {*weight* = -6977.8017 + 112.5302\*(*bpd*): coeff=0.2628, var=86798.8614},  {*weight* = -5312.2016 + 25.7209\*(*hc*): coeff=0.2000, var=138635.9025},  {*weight* = -4866.0218 + 115.1670\*(*fl*): coeff=0.2514, var=78089.6011},  {*weight* = -3615.3155 + 20.1265\*(*ac*): coeff=0.2858, var=61080.1223} |
| 3 | {*weight* = -7044.9992 + 113.0877\*(*bpd*): coeff=0.2850, var=49040.4530},  {*weight* = -5765.6434 + 27.0933\*(*hc*): coeff=0.2164, var=74811.7832},  {*weight* = -4850.8460 + 114.8885\*(*fl*): coeff=0.2321, var=60022.4336},  {*weight* = -3639.0104 + 20.2068\*(*ac*): coeff=0.2665, var=47595.5654} |
| 4 | {*weight* = -7176.0173 + 115.2133\*(*bpd*): coeff=0.2716, var=38495.8850},  {*weight* = -5580.7794 + 26.5639\*(*hc*): coeff=0.1939, var=71575.3627},  {*weight* = -5143.9012 + 119.9590\*(*fl*): coeff=0.2319, var=46756.6663},  {*weight* = -3824.3390 + 20.8679\*(*ac*): coeff=0.3026, var=29724.0337} |
| 5 | {*weight* = -7660.3431 + 120.8204\*(*bpd*): coeff=0.2693, var=30819.8738},  {*weight* = -6110.0704 + 28.2196\*(*hc*): coeff=0.2138, var=48373.8766},  {*weight* = -5331.1994 + 122.4455\*(*fl*): coeff=0.2369, var=36807.6503},  {*weight* = -3967.5178 + 21.3295\*(*ac*): coeff=0.2800, var=27240.5556} |
| 6 | {*weight* = -8097.3745 + 125.7068\*(*bpd*): coeff=0.2302, var=22289.0842},  {*weight* = -7015.6149 + 31.3566\*(*hc*): coeff=0.2103, var=28635.3775},  {*weight* = -5480.6406 + 125.3284\*(*fl*): coeff=0.2674, var=13952.3164},  {*weight* = -3676.3555 + 20.3238\*(*ac*): coeff=0.2920, var=11540.1306} |
| 7 | {*weight* = -7076.9202 + 112.8536\*(*bpd*): coeff=0.3705, var=3375.2380},  {*weight* = -5497.9202 + 26.2185\*(*hc*): coeff=0.1612, var=18787.3282},  {*weight* = -4947.5898 + 117.8865\*(*fl*): coeff=0.2113, var=9967.0914},  {*weight* = -3653.8140 + 20.3827\*(*ac*): coeff=0.2569, var=8618.4241} |
| 8 | {*weight* = -7018.2030 + 112.6524\*(*bpd*): coeff=0.2678, var=3654.3436},  {*weight* = -5235.5481 + 25.2899\*(*hc*): coeff=0.2162, var=5459.5803},  {*weight* = -5647.3688 + 127.7972\*(*fl*): coeff=0.2054, var=5974.4689},  {*weight* = -3285.2965 + 19.3967\*(*ac*): coeff=0.3106, var=2526.1926} |
| 9 | {*weight* = -6350.5284 + 104.5601\*(*bpd*): coeff=0.1787, var=204.7618},  {*weight* = -5140.6601 + 24.4881\*(*hc*): coeff=0.0745, var=1245.6621},  {*weight* = -6791.1342 + 152.4635\*(*fl*): coeff=0.3553, var=68.6443},  {*weight* = -3831.9687 + 21.4992\*(*ac*): coeff=0.3915, var=53.0970} |

In Table 4, each PRM is wrapped in two brackets “{.}”. Notation “coeff” denotes mixture coefficient and notation “var” denotes the variance of a PRM. For explanation, the 1th regression model is interpreted according to equation (25) as follows: *weight* = 0.2700\*(-6802.9586 + 110.3231\*(*bpd*)) + 0.2012\*(-5089.2989 + 25.0105\*(*hc*)) + 0.2426\*(-4744.7739 + 113.4482\*(*fl*)) + 0.2862\*(-3515.6183 + 19.8394\*(*ac*)) = -5018.02 + 5.6780(*ac*) + 29.7872(*bpd*) + 27.5225(*fl*) + 5.0321(*hc*).

Given gestational sample, we compare SREM with REM given with regard to the ratio mean absolute error (*RMAE*) and the number *t* of iterations. The number *t* reflects speed of an algorithm. The smaller the number *t* is, the faster the algorithm is. Let *W* = {*w*1, *w*2,…, *wK*} and *V* = {*v*1, *v*2,…, *vK*} be sets of actual weights and estimated weights, respectively. Equation (26) specifies the *RMAE* metric [23, p. 814].

|  |  |
| --- | --- |
|  | (26) |

The smaller the *RMAE* is, the more accurate the algorithm is. Table 5 is the comparison of REM and SREM with regard to *RMAE* and *t* given gestational sample.

***Table 5.*** *Comparison of REM and SREM regarding RMAE and t, given gestational sample.*

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Pair | *RMAE*  (REM) | *RMAE*  (SREM) | *t*  (REM) | *t*  (SREM) |
| 0 | 0.0711 | 0.0786 | 1 | 2 |
| 1 | 0.0722 | 0.0759 | 4 | 4 |
| 2 | 0.0739 | 0.0738 | 6 | 4 |
| 3 | 0.0724 | 0.0720 | 7 | 4 |
| 4 | 0.0746 | 0.0727 | 11 | 5 |
| 5 | 0.0780 | 0.0721 | 18 | 5 |
| 8 | 0.0777 | 0.0745 | 22 | 4 |
| 7 | 0.0709 | 0.0706 | 37 | 5 |
| 8 | 0.0729 | 0.0752 | 112 | 4 |
| 9 | 0.0853 | 0.1147 | 444 | 4 |
| Average | 0.0749 | 0.0780 | 66.2 | 4.1 |

From Table 5, given gestational sample, SREM is faster than REM according to *t* but the accuracy of REM is better than the accuracy of SREM according to *RMAE*. Note [19], values of paired t-test statistic *t*0 [1, p. 376] of *RMAE* for REM and SREM are 5.3294 and 6.4541, respectively. Because all these values are larger than the percentage point *t*0.05,8 = 1.860 [1, p. 711] given significant level 95%, the resistance of REM and SREM to missing values given gestational sample is proved.

We continue to test SREM with CCPP sample. Table 6 shows ten testing pairs of CCPP sample.

***Table 6.*** *Ten testing pairs of CCPP sample.*

|  |  |  |  |
| --- | --- | --- | --- |
| Pair | Training dataset | Testing dataset | Loss ratio |
| 0 | CCPP.sample.base | CCPP.sample.test | 0% |
| 1 | CCPP.sample.base.0.1.miss | CCPP.sample.test | 10% |
| 2 | CCPP.sample.base.0.2.miss | CCPP.sample.test | 20% |
| 3 | CCPP.sample.base.0.3.miss | CCPP.sample.test | 30% |
| 4 | CCPP.sample.base.0.4.miss | CCPP.sample.test | 40% |
| 5 | CCPP.sample.base.0.5.miss | CCPP.sample.test | 50% |
| 6 | CCPP.sample.base.0.6.miss | CCPP.sample.test | 60% |
| 7 | CCPP.sample.base.0.7.miss | CCPP.sample.test | 70% |
| 8 | CCPP.sample.base.0.8.miss | CCPP.sample.test | 80% |
| 9 | CCPP.sample.base.0.9.miss | CCPP.sample.test | 90% |

Table 7 shows ten resulted regression models of REM corresponding to ten testing pairs, given CCPP sample.

***Table 7.*** *Ten resulted regression models of REM given CCPP sample.*

|  |  |
| --- | --- |
| Pair | Regression model |
| 0 | *PE* = 469.7296 - 1.9885\*(*AT*) - 0.2332\*(*V*) + 0.0474\*(*AP*) - 0.1602\*(*RH*) |
| 1 | *PE* = 415.9687 - 1.9131\*(*AT*) - 0.2579\*(*V*) + 0.0979\*(*AP*) - 0.1272\*(*RH*) |
| 2 | *PE* = 416.5671 - 1.8401\*(*AT*) - 0.2940\*(*V*) + 0.0963\*(*AP*) - 0.1047\*(*RH*) |
| 3 | *PE* = 401.8042 - 1.8324\*(*AT*) - 0.2999\*(*V*) + 0.1099\*(*AP*) - 0.0869\*(*RH*) |
| 4 | *PE* = 369.4165 - 1.7559\*(*AT*) - 0.3281\*(*V*) + 0.1410\*(*AP*) - 0.0789\*(*RH*) |
| 5 | *PE* = 346.6202 - 1.7208\*(*AT*) - 0.3237\*(*V*) + 0.1615\*(*AP*) - 0.0633\*(*RH*) |
| 6 | *PE* = 341.1562 - 1.6900\*(*AT*) - 0.3300\*(*V*) + 0.1647\*(*AP*) - 0.0383\*(*RH*) |
| 7 | *PE* = 346.4257 - 1.6501\*(*AT*) - 0.3776\*(*V*) + 0.1618\*(*AP*) - 0.0467\*(*RH*) |
| 8 | *PE* = 302.7665 - 1.5758\*(*AT*) - 0.3174\*(*V*) + 0.1942\*(*AP*) + 0.0391\*(*RH*) |
| 9 | *PE* = 564.1434 - 2.1327\*(*AT*) + 0.0188\*(*V*) - 0.0684\*(*AP*) + 0.0205\*(*RH*) |

Table 8 shows ten resulted semi-mixture regression models of SREM corresponding to ten testing pairs, given CCPP sample.

***Table 8.*** *Ten resulted semi-mixture regression models of SREM given CCPP sample.*

|  |  |
| --- | --- |
| Pair | Semi-mixture regression model |
| 0 | {*PE* = 497.0645 - 2.1763\*(*AT*): coeff=0.4227, var=29.6573},  {*PE* = 517.8105 - 1.1672\*(*V*): coeff=0.2769, var=71.6045},  {*PE* = -1058.5211 + 1.4933\*(*AP*): coeff=0.1597, var=211.5011},  {*PE* = 421.6716 + 0.4486\*(*RH*): coeff=0.1406, var=248.4670} |
| 1 | {*PE* = 497.4977 - 2.1979\*(*AT*): coeff=0.4280, var=24.2516},  {*PE* = 519.3656 - 1.1965\*(*V*): coeff=0.2768, var=60.5493},  {*PE* = -1214.8271 + 1.6475\*(*AP*): coeff=0.1584, var=180.1064},  {*PE* = 417.8420 + 0.5020\*(*RH*): coeff=0.1368, var=217.2895} |
| 2 | {*PE* = 497.6871 - 2.2081\*(*AT*): coeff=0.4291, var=20.0817},  {*PE* = 520.7027 - 1.2180\*(*V*): coeff=0.2841, var=48.7344},  {*PE* = -1304.3280 + 1.7359\*(*AP*): coeff=0.1541, var=157.6827},  {*PE* = 413.7453 + 0.5563\*(*RH*): coeff=0.1327, var=189.0623} |
| 3 | {*PE* = 498.5778 - 2.2479\*(*AT*): coeff=0.4453, var=13.8541},  {*PE* = 522.2781 - 1.2467\*(*V*): coeff=0.2830, var=37.3203},  {*PE* = -1512.8163 + 1.9414\*(*AP*): coeff=0.1472, var=128.9238},  {*PE* = 405.7745 + 0.6610\*(*RH*): coeff=0.1245, var=156.3326} |
| 4 | {*PE* = 498.5320 - 2.2546\*(*AT*): coeff=0.4335, var=10.5627},  {*PE* = 523.8185 - 1.2793\*(*V*): coeff=0.2961, var=26.0347},  {*PE* = -1714.4568 + 2.1407\*(*AP*): coeff=0.1511, var=91.7893},  {*PE* = 401.0777 + 0.7325\*(*RH*): coeff=0.1192, var=123.9264} |
| 5 | {*PE* = 498.4271 - 2.2470\*(*AT*): coeff=0.4353, var=7.9534},  {*PE* = 523.2183 - 1.2717\*(*V*): coeff=0.2939, var=19.5630},  {*PE* = -1857.9068 + 2.2820\*(*AP*): coeff=0.1528, var=67.4559},  {*PE* = 392.9270 + 0.8393\*(*RH*): coeff=0.1181, var=90.1255} |
| 6 | {*PE* = 498.0319 - 2.2315\*(*AT*): coeff=0.4395, var=5.0596},  {*PE* = 524.2077 - 1.2912\*(*V*): coeff=0.2861, var=13.3621},  {*PE* = -1963.7000 + 2.3864\*(*AP*): coeff=0.1552, var=42.8344},  {*PE* = 387.3950 + 0.9189\*(*RH*): coeff=0.1192, var=60.0808} |
| 7 | {*PE* = 498.3792 - 2.2522\*(*AT*): coeff=0.4358, var=2.9110},  {*PE* = 525.1901 - 1.3086\*(*V*): coeff=0.2879, var=7.2515},  {*PE* = -2134.9587 + 2.5554\*(*AP*): coeff=0.1520, var=23.8247},  {*PE* = 381.4177 + 0.9984\*(*RH*): coeff=0.1243, var=29.7061} |
| 8 | {*PE* = 496.4571 - 2.1705\*(*AT*): coeff=0.4590, var=1.1633},  {*PE* = 524.3892 - 1.2790\*(*V*): coeff=0.2884, var=3.3270},  {*PE* = -2349.5928 + 2.7669\*(*AP*): coeff=0.1334, var=12.5737},  {*PE* = 369.4027 + 1.1507\*(*RH*): coeff=0.1192, var=16.3293} |
| 9 | {*PE* = 497.3288 - 2.1356\*(*AT*): coeff=0.5466, var=0.1691},  {*PE* = 532.0547 - 1.4489\*(*V*): coeff=0.2210, var=1.0673},  {*PE* = -2537.2255 + 2.9526\*(*AP*): coeff=0.1349, var=2.7906},  {*PE* = 369.2398 + 1.1183\*(*RH*): coeff=0.0975, var=4.1247} |

In Table 8, each PRM is wrapped in two brackets “{.}”. Notation “coeff” denotes mixture coefficient and notation “var” denotes the variance of a PRM.

Table 9 is the comparison of REM and SREM with regard to *RMAE* and *t* given CCPP sample.

***Table 9.*** *Comparison of REM and SREM regarding RMAE and t, given CCPP sample.*

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Pair | *RMAE*  (REM) | *RMAE*  (SREM) | *t*  (REM) | *t*  (SREM) |
| 0 | 0.0081 | 0.0123 | 1 | 2 |
| 1 | 0.0081 | 0.0119 | 5 | 5 |
| 2 | 0.0081 | 0.0116 | 10 | 7 |
| 3 | 0.0082 | 0.0111 | 27 | 8 |
| 4 | 0.0082 | 0.0109 | 23 | 10 |
| 5 | 0.0083 | 0.0109 | 68 | 10 |
| 8 | 0.0084 | 0.0109 | 994 | 9 |
| 7 | 0.0084 | 0.0109 | 47 | 8 |
| 8 | 0.0089 | 0.0110 | 90 | 13 |
| 9 | 0.0101 | 0.0104 | 1780 | 23 |
| Average | 0.0085 | 0.0112 | 304.5 | 9.5 |

From Table 9, given CCPP sample, SREM is faster than REM according to *t* but the accuracy of REM is better than the accuracy of SREM according to *RMAE*. Note [19], values of paired t-test statistic *t*0 [1, p. 376] of *RMAE* for REM and SREM are 6.1786 and 5.9070, respectively. Because all these values are larger than the percentage point *t*0.05,8 = 1.860 [1, p. 711] given significant level 95%, the resistance of REM and SREM to missing values given CCPP sample is proved.

From experimental results of both gestational sample and CCPP sample, the convergence of SREM is always faster than the convergence of REM because SREM decomposes a long regression model into many shorter regression models. In optimization process of SREM, of course each short model with only one independent variable in two-dimension space will converge faster than the long model because the long model needs much more iterations to reach and balance the optimal point (optimizer) in multi-dimension space with many independent variables.

1. **Conclusions**

From the number of iterations, we conclude that SREM converges faster than REM does. According to *RMAE* metric, the accuracy of REM is better than the accuracy of REM but their distance in accuracy is not large. Moreover, the number of PRMs in fair tests is equal to the number of regressors and each PRM has only one regressor. If the number of PRMs is large enough and each PRM has many enough regressors with some combination of regressors, SREM may be better than REM. Note, Bayesian Information Criterion (BIC) was proposed to estimate the number of PRMs in [11, p. 5]. This may be true but finding the optimal number of PRMs and regressors for SREM is not a methodological ideology because the essence of SREM is decomposition of REM. Therefore, for further research, we will modify SREM so that it implements fully mixture model in which both mixture coefficients *ck* and regression coefficients *αk* are estimated by normal mixture model and balance process (estimation of missing values) of REM. We expect that taking advantages of both mixture model and REM via iterative process will result out better estimation at least in the case that incomplete data varies in many trends.

In general, the combination of REM and mixture model like SREM is potential. The website of REM and SREM is http://rem.locnguyen.net.

**Appendix**

***A1. Proof of equation (19)***

The joint probability of data ***X*** and data *Z* for each *k*th PRM is defined as follows:

When ***X***, *Xj*, and *Z* are specified in equation (16), we have:

(Because all *zi* are mutually independent given *Xi* and all *xij* with fixed *j* are mutually independent given *zi*)

(Due to equations (13) and (15))

The log-likelihood function is natural logarithm of the joint probability *Pk*(***X***, *Z*|*αk*, *σk*2, *βkj*) as follows:

The optimal estimate (*αk*\*, (*σk*2)\*, *βkj*\*)*T* is a maximizer of *L*(*αk*, *σk*2, *βkj*) [17, p. 9].

By taking first-order partial derivatives of *L*(*αk*, *σk*2, *βkj*) with regard to *αk*, *σk*2, and *βkj*, we obtain [24, p. 34]:

When first-order partial derivatives of *L*(*αk*, *σk*2, *βkj*) are equal to zero, it gets local maximal. In other words, (*αk*\*, (*σk*2)\*, *βkj*\*)*T* is solution of the following system of linear equations:

The notation **0** = (0, 0,…, 0)*T* denotes zero vector. Solution of the system of linear equations above is [1, p. 457]:

Where ***X***, *Xi*, *Xj*, and ***Z*** are specified by equation (16). Therefore, the equation (19) is established■

**Conflicts of Interest**

The authors declare that there is no conflict of interest regarding the publication of this article.

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