Stock Price Estimation using Volatility as a Stochastic process

Sankha Narayan Guria

Indian Institute of Technology Jodhpur

sankha@iitj.ac.in

April 15, 2014

Overview

Introduction

We have different probabilistic models for prediction of the stock price at time t. Most of these either take rate of interest as a constant or the risk associated with the investment as a constant.

Different models exist like the Heston's model that consider the volatility as a random function of time. These often require difficult transformations and advanced techniques to solve. In our project we attempt to come to a solution with a simplistic relation between the intial and the final stock prices, considering r and σ as a random variable.

Definitions

- A **random walk** is a mathematical formalization of a path that consists of a succession of random steps.
- The **random walk hypothesis** is a financial theory stating that stock market prices evolve according to a random walk and thus cannot be predicted.
- A **geometric Brownian motion** is a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion with drift.

Limitations of Stock Price Prediction Models

Fixed rate of Interest

Models like Heston's assume a constant rate of interest whereas the interest rates can be changed by policy makers, government, etc. It thus does not take these factors into account.

Constant Volatility

Volatility is in itself a random process thus does cannot stay constant due to the changing conditions of the market. This is a problem in CIR, O-U models, etc.

Fixed Strike Price

The strike price for a call option is always fixed in time in the Black Scholes model. That is rarely true in the real world.

Varying Volatility

In this section we tried to devise a mathematical model that takes into account the random nature of the volatility and tried to incorporate into the stochastic stock price prediction model.

We modeled the risk term as a stochastic process, with the old average risk and the risk of risk being associated with a brownian motion. We use this assumptions to figure out a way to use them in the stochastic stock price prediction.

The Model

Initial Equations

$dS(t) = S(t)\mu\, dt + S(t)\sigma(t)\, dW(t)$ (Stock price equation)
$d\sigma(t) = \sigma(t)\eta\, dt + \sigma(t)\zeta\, dZ(t)$ (Volatility equation)

Here both $W(t)$ and $Z(t)$ are independent Brownian motions.

The Model (contd.)

Solving for $\sigma(t)$

$$d\sigma(t) = \eta\sigma\, dt + \zeta\sigma\, dZ(t)$$
$$df(t, Z(t)) = f_t\, dt + f_Z\, dZ(t) + \frac{1}{2} f_{ZZ}\, dt$$

Comparing with coefficients of dt and dZ(t)

$$f_t\, dt + \frac{1}{2} f_{ZZ}\, dt = \eta\sigma$$
$$f_Z = \zeta\sigma$$
$$\frac{df}{dZ} = f\sigma$$

The Model (contd.)

Solving for $\sigma(t)$

$$f(t, Z) = k(t)e^{\zeta Z(t)}$$

$$\frac{dk(t)}{dt}e^{\zeta Z(t)} + \frac{1}{2}\zeta^2 k(t)e^{\zeta Z(t)} = \eta k(t)e^{\zeta Z(t)}$$

$$\frac{dk(t)}{dt} = k(t)[\eta - \frac{1}{2}\zeta^2]$$

$$k(t) = ce^{\eta - \frac{1}{2}\zeta^2}$$

$$f(t, Z) = ce^{t(\eta - \frac{\zeta^2}{2}) + \zeta Z(t)}$$

The Model (contd.)

Solving for $\sigma(t)$

At time $t = 0, c = \sigma(0)$. Therefore,
$\sigma(t) = \sigma(0)e^{t(\eta - \frac{\zeta^2}{2}) + \zeta Z(t)}$

Solving for $S(t)$

$$df(t, W(t)) = f_t\, dt + f_W\, dW(t) + \frac{1}{2} f_{WW}\, dt$$

$$f_W = f\sigma(t)$$

$$f = k(t)e^{\sigma(t)W(t)}$$

The Model (contd.)

Solving for $S(t)$

$$\mu k(t)e^{\sigma(t)W(t)} = k'(t)e^{\sigma(t)W(t)} + \sigma'(t)k(t)e^{\sigma(t)W(t)} + \frac{1}{2}\sigma(t)^2 k(t)e^{\sigma(t)W(t)}$$

$$\mu k(t) = k'(t) + \sigma(t)'k(t) + \frac{1}{2}\sigma(t)^2 k(t)$$

$$k'(t) = k(t)(\mu - \sigma'(t) - \frac{1}{2}\sigma(t)^2)$$

Hence, solving for $k(t)$ gives,

$$k(t) = C.e^{\mu t - \sigma(t) - \frac{1}{2}\int \sigma^2(t)\,dt}$$

The Model (contd.)

Solving for $S(t)$

$$S(t) = C.e^{\mu t - \sigma(t) - \frac{1}{2}\int \sigma^2(t)\,dt + \sigma W(t)}$$

At $t = 0,\ C = S(0)$,

$$S(t) = S(0).e^{\mu t - \sigma(t) - \frac{1}{2}\int \sigma^2(t)\,dt + \sigma W(t)}$$

$$\sigma(t) = \sigma(0)e^{t(\eta - \frac{\zeta^2}{2}) + \zeta Z(t)}$$

This gives us the model for a stock price prediction where we account for the randomness in the volatility in the form of a brownian motion.

Varying Rate of Interest

In this section we come up with a mathematical model that takes into account the random nature of the rate and tried to incorporate into the stochastic stock price prediction model.

We can use any of the form of $r(t)$ given by the well established models like O-U, CIR, Vasicek into these and get the corresponding stock price at time t.

The Model (contd.)

Taking r as a random function

Certain models such as CIR, OU, etc. propose the rate of interest as a random function, so let us assume it to be $r(t)$.

$$dS(t) = S(t)r(t)\,dt + S(t)\sigma\,dW(t)$$

$$df(t, W(t)) = f_t\,dt + f_W\,dW(t) + \frac{1}{2}f_{WW}\,dt$$

$$f_W = f\sigma$$

$$f(t, W(t)) = k(t)e^{\sigma W(t)}$$

$$\therefore\ k'(t)e^{\sigma W(t)} + \frac{1}{2}\sigma^2 k(t)e^{\sigma W(t)} = r(t)e^{\sigma W(t)}$$

The Model (contd.)

Taking r as a random function

This gives us a differential equation:

$$k'(t) + \frac{1}{2}\sigma^2 k(t) = r(t)$$

To integrate this equation we need an integrating factor.

$$I.F. = e^{\frac{1}{2}\int \sigma^2\, dt} = e^{\frac{1}{2}\sigma^2}$$

$$k(t) = \frac{\int r(t) e^{\frac{1}{2}\sigma^2 t}\, dt}{\frac{1}{2}\sigma^2 t}$$

$$f(t, W(t)) = C.\frac{\int r(t) e^{\frac{1}{2}\sigma^2 t}\, dt}{\frac{1}{2}\sigma^2 t}.e^{\sigma W(t)}$$

Taking r as a random function

Thus we can substitute any one of the $r(t)$ as given by one of models into this equation and we will get the expected stock price at time t.

$$S(t) = S(0)\frac{\int r(t)e^{\frac{1}{2}\sigma^2 t}\, dt}{\frac{1}{2}\sigma^2 t}.e^{\sigma W(t)}$$

The Model (contd.)

Combining both models

We can now combine the two separate random sources into a single model so that it can better account for randomness in the financial markets.

$$dS(t) = S(t)r(t)\,dt + S(t)\sigma(t)\,dW(t)$$

$$df(t, W(t)) = f_t\,dt + f_W\,dW(t) + \frac{1}{2}f_{WW}\,dt$$

$$f_W = f\sigma(t)$$

$$f(t, W(t)) = k(t)e^{\sigma(t)W(t)}$$

$$\therefore\ k'(t)e^{\sigma(t)W(t)} + \frac{1}{2}\sigma^2(t)k(t)e^{\sigma(t)W(t)} = r(t)e^{\sigma(t)W(t)}$$

The Model (contd.)

Combining both models

This gives us a differential equation:

$$k'(t) + \frac{1}{2}\sigma^2(t)k(t) = r(t)$$

To integrate this equation we need an integrating factor.

$$I.F. = e^{\frac{1}{2}\int \sigma^2(t)\,dt}$$

$$k(t) = \frac{\int r(t)e^{\frac{1}{2}\int \sigma^2(t)\,dt}\,dt}{\frac{1}{2}\int \sigma^2(t)\,dt}$$

$$f(t, W(t)) = C.\frac{\int r(t)e^{\frac{1}{2}\int \sigma^2(t)\,dt}\,dt}{\frac{1}{2}\int \sigma^2(t)\,dt}.e^{\sigma(t)W(t)}$$

The Model (contd.)

Combining both models

Thus we can substitute any one of the $r(t)$ as given by one of models and the value of $\sigma(t)$ into this equation and we will get the expected stock price at time t.

$$S(t) = S(0)\frac{\int r(t)e^{\frac{1}{2}\int \sigma^2(t)\,dt}\,dt}{\frac{1}{2}\int \sigma^2(t)\,dt}.e^{\sigma(t)W(t)}$$

Conclusion

The equations derived in our project show the relation between the initial stock price value and the expected final stock price value under the conditions:

- $\sigma(t)$ is a random process following brownian motion, with mean η and volatility of volatility ζ.
- $r(t)$ is a given by any of the established models and we use them for our stock price prediction model.
- We combine both the above conditions to get a final formula that can predict the stock price at time t given that σ and r are random processes.

We can extend our work to test this model on real world data.

Thank You!