# Math 11: Discussion 5



**Recap of discussion 5:**

1. Common distributions: their functions, expected values and variances.
2. Calculating the parameters, for Poisson and Exponential distributions in particular.
3. Defining random variables and explicitly writing out their distributions help a lot!

## Contents



### 5.1 Upcoming assignments

| Assignments | Chapters | Deadlines |
|---|---|---|
| Homework | Ch. 15 (2 modules) | Wed. 04/29, Fri. 05/01 |
| Quiz | Ch. 15 | Thu. 04/30 |
| Homework | Ch. 5 | Mon. 05/04 |
| Quiz | Ch. 5 | Tue. 05/05 |

Key concepts (not exhaustive):

1. *Random variables:* expectation and variance
2. *Discrete distributions:* Bernoulli, geometric, binomial, Poisson
3. *Continuous distributions:* uniform, exponential, normal

## 5.2 Random Variables

| | *Discrete r.v.* | *Continuous r.v.* |
|---|---|---|
| $\mathbb{P}(a \leq x \leq b)$ | $\sum_{a \leq x_i \leq b} p_X(x_i)$ | $\int_a^b f_X(x)\,\mathrm{d}x$ |
| $\mathbb{E}[X] = \mu_X$ | $\sum_{x_i \in \Omega} x_i p_X(x_i)$ | $\int_\Omega x f_X(x)\,\mathrm{d}x$ |
| $Var[X] = \sigma_X^2$ | $\sum_{x_i \in \Omega} (x_i - \mu_X)^2 p_X(x_i)$ | $\int_\Omega (x-\mu_X)^2 f_X(x)\,\mathrm{d}x$ |

Similarly, given $Var[X]$, the standard deviation of $X$ is $sd[X] = \sqrt{Var[X]}$. Regarding the *variance*, it is sometimes more convenient to use this alternative formula when computing:

$$\boxed{Var[X] = \mathbb{E}[X^2] - \mathbb{E}^2[X] = \mathbb{E}[X^2] - \mu_X^2}$$

**Properties of Expected Value and Variance**

Suppose $X$ is a random variable (true for both *discrete* and *continuous* cases), and real numbesrs $a, b, c$:

$$\begin{aligned}
\mathbb{E}[X+c] &= \mathbb{E}[X] + c \\
\mathbb{E}[cX] &= c\mathbb{E}[X] \\
\mathbb{E}[X+Y] &= \mathbb{E}[X] + \mathbb{E}[Y] \\
\mathbb{E}[aX+bY+c] &= a\mathbb{E}[X] + b\mathbb{E}[Y] + c && \text{(Linearity of Expectation)}
\end{aligned}$$

Suppose $X$ is a random variable (true for both *discrete* and *continuous* cases), and real numbesrs $a, b, c$:

$$\begin{aligned}
Var[X+c] &= VarE[X] \\
Var[cX] &= c^2 Var[X] && (\implies SD[cX] = |c|SD[X]) \\
Var[X+Y] &= Var[X] + Var[Y] && \text{(if } X \text{ and } Y \text{ are independent)} \\
Var[aX+bY+c] &= a^2 Var[X] + b^2 Var[Y] && \text{(if } X \text{ and } Y \text{ are independent)}
\end{aligned}$$

## 5.3 Models & Distributions

### 5.3.1 Discrete Random Variables

**Example 1.** *Let's describe what these distributions model:*

*(a) Bernoulli distribution:* $Bern(p)$

*(b) Geometric distribution:* $Geom(p)$

*(c) Binomial distribution:* $Bino(n, p)$

*(d) Poisson distribution:* $Pois(\lambda)$

*Solution:*

(a) *Bernoulli* distribution: models a binary event of either success (with probability $p$) and failure otherwise;

(b) *Geometric* distribution: models a sequence of independent *Bernoulli* trials, in which we are interested in the number of trials it takes to get to the $1^{st}$ success;

(c) *Binomial* distribution: models a sequence of independent *Bernoulli* trials, in which we want $k$ success out of a total of $n$ trials ($n \geq k$);

(d) *Poisson* distribution: models the number of occurrences in a time interval given the average number of occurrences in a unit time interval;

| Notation | PMF | $\mathbb{E}[X]$ | $Var[X]$ |
|---|---|---|---|
| Bernoulli $X \sim Bern(p)$ | $\mathbb{P}(X=1) = p$ | $p$ | $p(1-p)$ |
| Geometric $X \sim Geom(p)$ | $\mathbb{P}(X=k) = (1-p)^{k-1}p$ | $\frac{1}{p}$ | $\frac{1-p}{p^2}$ |
| Binomial $X \sim Bino(n,p)$ | $\mathbb{P}(X=k) = \binom{n}{k} p^k(1-p)^{n-k}$ | $np$ | $np(1-p)$ |
| Poisson $X \sim Pois(\lambda)$ | $\mathbb{P}(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}$ | $\lambda$ | $\lambda$ |

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

**Remarks:**

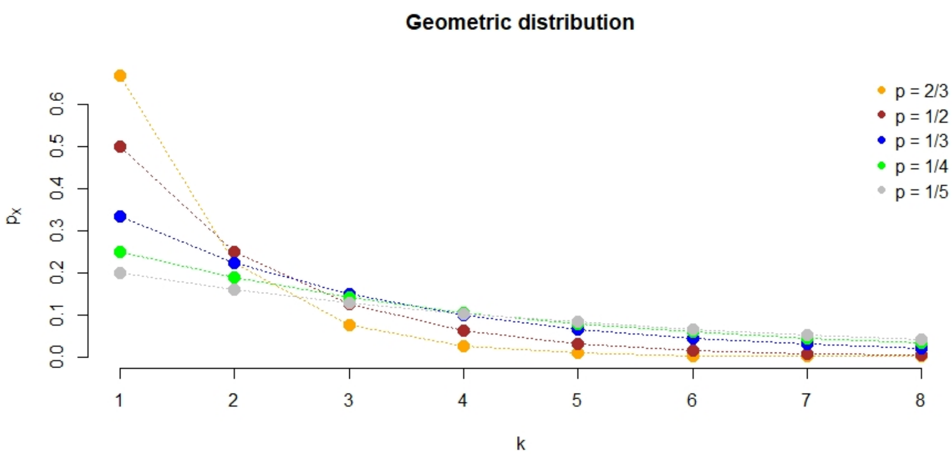
1. Once we know the parameters $p, n, \lambda, \ldots$, we know the distribution completely, i.e. we can compute the probability of any events, the expectation, ...

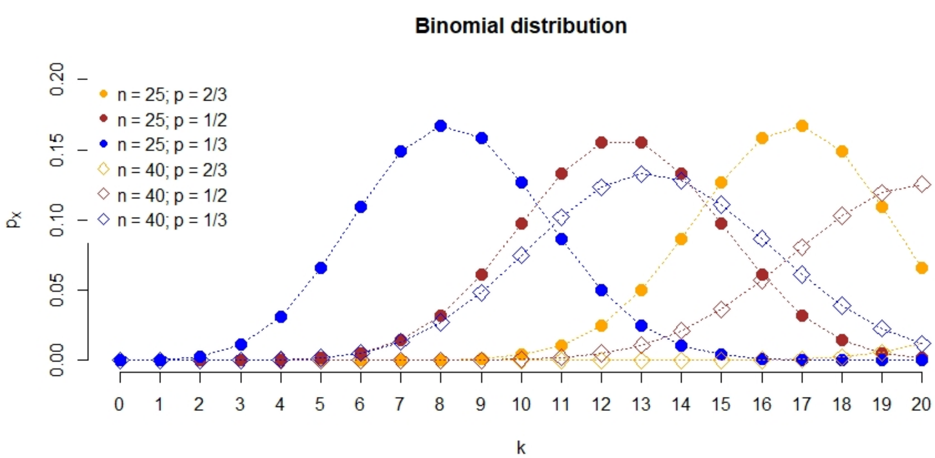
2. When $n$ is large and $p$ is small, then the *Binomial* distribution is well approximated by the *Poisson* distribution:

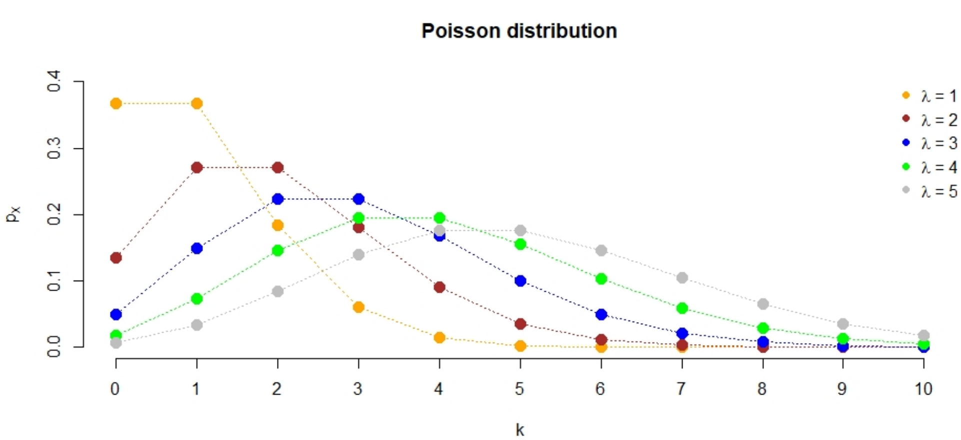
$$\lambda = np$$

an example is when modeling rare events such as the occurrences of earthquakes, tsunami, ...

The following figures give the plots of the density of those distributions, i.e. the *probability mass function* $p_X$. Note that the above distributions take on integer values only, i.e. $k \in \mathbb{Z}^{\geq 0}$, and are not defined for anything in between integers. As such, the dotted lines in the plots are merely to show the "trend" (or behavior) of such distributions.


Geometric distribution
p = 2/3
p = 1/2
p = 1/3
p = 1/4
p = 1/5
$p_X$
k


Binomial distribution
n = 25; p = 2/3
n = 25; p = 1/2
n = 25; p = 1/3
n = 40; p = 2/3
n = 40; p = 1/2
n = 40; p = 1/3
$p_X$
k


Poisson distribution
λ = 1
λ = 2
λ = 3
λ = 4
λ = 5
$p_X$
k

**Example 2.** *Let's consider a 3-sided die: five faces numbered* $1, 2, 3$*, with these probability:*

| *Outcome* | 1 | 2 | 3 |
|---|---|---|---|
| *Probability* | $1/6$ | $2/6$ | $3/6$ |

*Suppose we roll* $n$ *times and sum up the outcomes, denoted by* $S_n$*, for example if we roll* $3$ *times and get* $(1, 2, 3)$*, then* $S_3 = 6$*. Assumme that each roll is independent of each other. Find the probability of these events:*

*(a)* $S_1 \neq 1$.

*(b)* $S_3 = 3$.

*(c)* $S_3 = 9$.

*(d)* $S_3 = 4$.

*(e)* $S_3 = 6$.

*Solution:* Let's first define a random variable to denote the outcome of each roll:

$$X_i : \text{ outcome of roll } i$$

We can now re-express the random variable $S_n$:

$$S_3 \;=\; X_1 + X_2 + X_3$$

Note on notations: suppose we roll 3 times and look at the outcomes:

$$\begin{aligned} (1, 2, 3) &\quad \text{the outcomes are 1, 2, and 3, in this order} \\ \{1, 2, 3\} &\quad \text{the outcomes consist of 1, 2, and 3, in any order} \end{aligned}$$

(a) $S_1 \neq 1$:

$$\mathbb{P}(S_1 \neq 1) = \mathbb{P}(X_1 \neq 1) = 1 - \frac{1}{6} = \frac{5}{6}$$

(b) $S_3 = 3$:

$$\mathbb{P}(S_3 = 3) = \mathbb{P}((1, 1, 1)) = \frac{1}{6^3} = \frac{1}{216}$$

(c) $S_3 = 9$:

$$\mathbb{P}(S_3 = 9) = \mathbb{P}((3, 3, 3)) = \left(\frac{3}{6}\right)^3 = \frac{1}{8}$$

(d) $S_3 = 4$:

$$\mathbb{P}(S_3 = 4) = \mathbb{P}(\{1, 1, 2\}) = 3\Big(\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{2}{6}\Big) = \frac{1}{36}$$

(e) $S_3 = 6$:

$$\mathbb{P}(S_3 = 6) = \mathbb{P}\Big(\{1, 2, 3\} \cup (2, 2, 2)\Big) = 3!\,\left(\frac{1}{6} \cdot \frac{2}{6} \cdot \frac{3}{6}\right) + \left(\frac{2}{6}\right)^3 = \frac{1}{6} + \frac{1}{27} = \frac{11}{54}$$

**Example 3.** *Going on from the previous example. Here, we are going to roll* 3 *times each time and let* $S_3$ *be the random variable. Here* $S_3$ *can take values from* $3, 4$ *to* 9.

*Consider rolling* 3 *times each. Assume each* 3*-rolls are independent. Find the probability of these events:*

*(a) The first time* $S_3 = 3$ *is the* $5^{th}$ *time.*

*(b) The first time* $S_3 = 6$ *is the* $1^{st}$ *time.*

*(c) Getting* $S_3 = 3$ *for* 3 *times after doing* 10 3*-rolls.*

*Solution:* We first recall that:

$$\mathbb{P}(S_3 = 3) = \frac{1}{216}; \qquad \mathbb{P}(S_3 = 6) = \frac{11}{54};$$

Tips: Think of the event of getting $S_3 = 3$ as a Bernoulli trial. Similarly for getting $S_3 = 6$.

(a) Let $X$ be the number of times until getting the first $S_3 = 3$, then $X \sim Geom(p = \frac{1}{216})$

$$\mathbb{P}(X = 5) = \left(1 - \frac{1}{216}\right)^4 \cdot \frac{1}{216} = \frac{215^4}{216^5}$$

(b) Let $Y$ be the number of times until getting the first $S_3 = 6$, then $Y \sim Geom(p = \frac{11}{54})$

$$\mathbb{P}(Y = 1) = p = \frac{11}{54}$$

(c) Let $X$ be the number of times of getting$S_3 = 3$ after 10 games, then $X \sim Bino(n = 10, p = \frac{1}{216})$:

$$\mathbb{P}(X = 3) = \binom{10}{3}\left(\frac{215}{216}\right)^7\left(\frac{1}{216}\right)^3 = \frac{10!}{(7!)(3!)}\left(\frac{215}{216}\right)^7\left(\frac{1}{216}\right)^3$$

**Example 4.** *Let's look at the uncertainty of catastrophe, including rare but damaging events such as earthquakes, floods, droughts, tsunamis, and pandemics, ...). We are going to group them all together. Suppose each* 100 *years have* 12 *catastrophe on average.*

*Find the probability of these events:*

*(a) There are* 2 *catastrophe in the year* 2021*.*

*(b) There is at least* 1 *catastrophe in the year* 2021*.*

*(c) There is at least* 1 *catastrophe in the years* $2021 - 2025$*.*

*Solution:* Let $X$ be the random variable denoting the event of a catastrophe in any 1 year, then

$$X \sim Pois(\lambda = .12)$$

Similarly, let $Y$ be that in 5 years, then $Y \sim Pois(\lambda = .6)$

(a) There are 2 catastrophe in the year 2021.

$$\mathbb{P}(X = 2) = e^{-.12}\ \frac{.12^2}{2!} = .0064$$

(b) There is at least 1 catastrophe in the year 2021.

$$\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X = 0) = 1 - e^{-.12}\ \frac{.12^0}{0!} = .113$$

(c) There is at least 1 catastrophe in the years $2021 - 2025$.

$$\mathbb{P}(Y \geq 1) = 1 = \mathbb{P}(Y = 0) = 1 - e^{-.6}\ \frac{.6^0}{0!} = .45$$

Let's consider other time intervals.

| Time intervals | 2021 | $2021 - 2025$ | $2021 - 2030$ | 06/2021 |
|---|---|---|---|---|
| Rate | .12 | .6 | 1.2 | .01 |

### 5.3.2 Continuous Random Variables

**Example 5.** *Let's describe what these distributions model:*

*(a) Uniform distribution:* $Unif[a, b]$

*(b) Exponential distribution:* $Exp(\lambda)$

*Solution:*

(a) *Uniform* distribution: models an event which has equal chances over the entire interval;

(b) *Exponential* distribution: models the wait time between 2 consecutive occurrences, given the average number of occurrences in a unit time interval;

(c) *Normal* distribution: also known as the bell curve, which is repeatedly observed in real life;

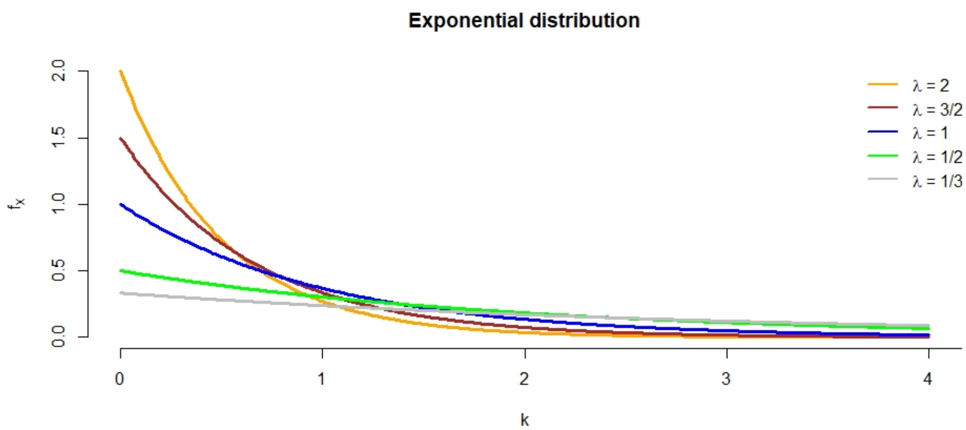
| Notation | PDF | $\mathbb{E}[X]$ | $Var[X]$ |
|---|---|---|---|
| Uniform $X \sim Unif[a, b]$ | $f_X(x) = \frac{1}{b-a}$ | $\frac{1}{2}(a+b)$ | $\frac{1}{12}(b-a)^2$ |
| Exponential $X \sim Exp(\lambda)$ | $f_X(x) = \lambda e^{-\lambda x}$ | $\frac{1}{\lambda}$ | $\frac{1}{\lambda^2}$ |
| Normal $X \sim \mathcal{N}(\mu, \sigma)$ | $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{\frac{-(x-\mu)^2}{2\sigma^2}\right\}$ | $\mu$ | $\sigma^2$ |

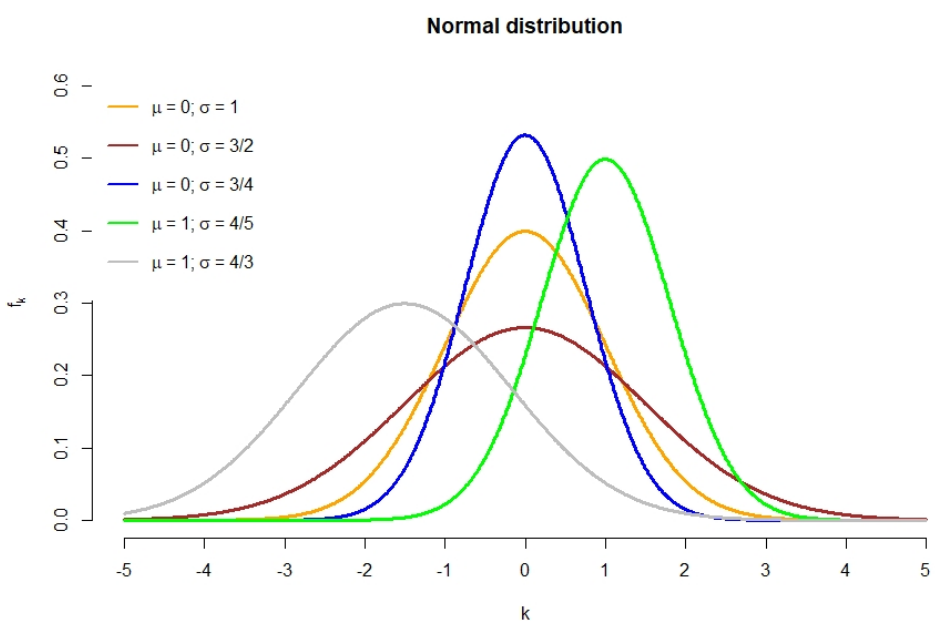
**Remarks:**

1. *Memorylessness property* in the *Exponential* distribution: if $X \sim Exp(\lambda)$, then
$$\mathbb{P}(X \geq t + s | X \geq s) = \mathbb{P}(X \geq t) = \int_t^\infty \lambda e^{-\lambda x}\, \mathrm{d}x = e^{-\lambda t}$$

2. A *standard Normal* distribution is when $\mu = 0, \sigma = 1$, in particular, if $Z \sim \mathcal{N}(0, 1)$, then
$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

3. If $X \sim \mathcal{N}(\mu, \sigma)$, then $Z = \dfrac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$.


Exponential distribution
λ = 2
λ = 3/2
λ = 1
λ = 1/2
λ = 1/3
$f_x$
0.0
0.5
1.0
1.5
2.0
0
1
2
3
4
k


Normal distribution
μ = 0; σ = 1
μ = 0; σ = 3/2
μ = 0; σ = 3/4
μ = 1; σ = 4/5
μ = 1; σ = 4/3
$f_k$
0.0
0.1
0.2
0.3
0.4
0.5
0.6
-5
-4
-3
-2
-1
0
1
2
3
4
5
k

**Example 6.** *Let's take a second look at the uncertainty of catastrophe. This time, instead of looking at the catastrophe, we are going to look at the waiting time between catastrophe.*

*Find the probability of these events:*

*(a) There is no catastrophe for* 1 *year.*

*(b) The next catastrophe is after* 10 *years.*

*(c) The next catastrophe is after* 10 *years assuming that there is no catastrophe in the next* 5 *years.*

*(d) The expected time until the next catastrophe, and the next* 3 *catastrophe.*

*Solution:* Let $X$ be the random variable denoting the waiting time until the next catastrophe, then

$$X \sim Exp(\lambda = .12)$$

(a) There is no catastrophe for 1 year:

$$\mathbb{P}(X \geq 1) = \int_1^\infty \lambda e^{-\lambda x}\, \mathrm{d}x = e^{-\lambda} = e^{(-.12)(1)} = .89$$

(b) The next catastrophe is after 10 years:

$$\mathbb{P}(X \geq 10) = e^{-10\lambda} = e^{-1.2} = .30$$

(c) The next catastrophe is after 10 years assuming that there is no catastrophe in the next 5 years:

$$\mathbb{P}(X \geq 10 | X \geq 5) = \mathbb{P}(X \geq 5) = e^{-5\lambda} = e^{-.6} = .55$$

(d) The expected time until the next catastrophe:

$$\mathbb{E}[X] = \frac{1}{\lambda} = \frac{25}{3} \approx 8.3 \text{ years}$$

Let $X_1$, $X_2$ and $X_3$ be the waiting time until the $1^{st}$, between the $1^{st}$ and $2^{nd}$, and the $2^{nd}$ and the $3^{rd}$ catastrophe respectively, the expected time until the next 3 catastrophe:

$$\mathbb{E}[X_1 + X_2 + X_3] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3] = 25 \text{ years}$$