RESTRICTION OF SCALARS AND THE METHOD OF CHABAUTY-COLEMAN FOR $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$

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ABSTRACT. We extend Siksek's development of Chabauty's method for computing the set of S-integral points on restrictions of scalars of curves over $\mathcal{O}_{K,S}$ to curves which are not necessarily complete. When the curve is $\mathcal{C} = \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$, we show that after replacing the curve with a suitable descent set of covers, the method has no base change obstructions. We also use the method to prove that $\mathcal{C}(\mathcal{O}_K)$ is finite for several classes of fields K, including that $\mathcal{C}(\mathcal{O}_K) = \emptyset$ when 3 splits completely in K. This represents the first infinite class of cases where Chabauty's method for restrictions of scalars is proved to succeed where the classical Chabauty's method does not.

1. INTRODUCTION

For a number field K, a smooth projective curve C/K of genus $g \geq 2$ with Jacobian J, and place $\mathfrak{p}$ of K, Chabauty's method proves that $C(K)$ is finite when rank $J(K) \leq g - 1$ [Cha41]. Chabauty's method, based on an technique of Skolem [Sko34] for computing integral points on punctured genus 0 curves, represented the first known cases of the Mordell conjecture, and essentially the only known cases until it was settled completely by Faltings [Fal83]. Although there are now several proofs of Faltings's theorem [Fal83, Voj91, LV18], none of these proofs are effective, in the sense that they do not lead to an algorithm for *computing* the set of rational points on a given curve. On the other hand, with his theory of p-adic integration, Coleman turned Chabauty's method into an algorithm which produces an explicit finite set X of $K_{\mathfrak{p}}$-points of C which contains $C(K)$ [Col85].

Since Coleman's innovation, arithmetic geometers have put a great deal of work into developing variants and improvements to Chabauty's method, both to shrink the set X further, and to apply similar ideas to curves where rank $J(K) > g - 1$. Among many other examples, we mention the method of descent [CTS87, Sto07, HS13], the Mordell-Weil sieve [BS10], elliptic curve Chabauty [FW99, Bru03], and Kim's nonabelian Chabauty [Kim05, CK10, DCW15, BD17, BD18b, BD18a, BDS+17, EH17]. Other work has focused on giving uniform bounds for the number of points on curves [LT02, Sto06, KRZB16].

Particularly relevant to the methods in this paper, Siksek, Park, and Gunther-Morrow have also developed related methods for certain higher dimensional varieties – symmetric powers of C – proving bounds on the number of (unexpected) points over finite extensions of fixed degree [Sik09, Par16, GM17]. In this document, we analyze another version of Chabauty's method for higher dimensional varieties, which involves a Chabauty-like computation for $\mathrm{Res}_{K/\mathbb{Q}} C$ as a way of computing $C(K)$. The method, introduced by Siksek and inspired by ideas of Wetherell [Sik13],

can succeed in proving $\#C(K) < \infty$ for curves with rank $J(K)$ as large as $[K : \mathbb{Q}](g-1)$. Unlike the classical method, Chabauty's method for restrictions of scalars (*RoS Chabauty*) can fail even when this condition is satisfied, making a general analysis much more subtle.

We study the analogue of RoS Chabauty in combination with the method of descent for computing S-integral points on a punctured $\mathbb{P}^1$, focusing in particular on the case of $(\mathbb{P}^1_{\mathcal{O}_{K,S}} \smallsetminus \{0, 1, \infty\})(\mathcal{O}_{K,S})$. This case corresponds to computing/bounding the set of solutions to the S-*unit equation*, i.e. the set

$$\{(x, y) \in \mathcal{O}_{K,S}^\times \times \mathcal{O}_{K,S}^\times : x + y = 1\}.$$

Work of Mahler [Mah33] using diophantine approximation in the style of Seigel and Thue gave the first proof that this set is finite. The theory of linear forms in logarithms can be used to give explicit bounds on the size of the set. The current state-of-the-art upper bound, due to Evertse [Eve84] is $3 \cdot 7^{[K:\mathbb{Q}]+2\#S}$.

Although the S-unit equation has been well-studied, the RoS Chabauty perspective leads us to a result which we have not seen in the literature:

Theorem 5.14. *Suppose that* 3 *splits completely in* K. *Then there is no pair* $x, y \in \mathcal{O}_K^\times$ *such that* $x + y = 1$. *Equivalently,*

$$(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_K) = \emptyset.$$

We also show that RoS Chabauty recovers several well-known results on finiteness of solutions to unit equations over number fields of small degree. In Corollaries/Propositions 5.7, 5.9, 5.10, and 5.13, we show that if K is a real quadratic, mixed cubic, or totally complex quartic field, or a mixed quartic field with a totally real subfield, then

(1.1) $$\#(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_K) \leq 3^d - 2.$$

We strengthen (1.1) further if 3 is not inert in K. This result provides the first infinite collection of examples of fields for which RoS Chabauty has been proven to show finiteness of integral or rational points on a curve but a straightforward application of classical Chabauty cannot prove finiteness.

For quadratic fields, the proof is sufficiently uniform in the field that it allows us to recover the well-known fact that there are exactly 8 solutions to the unit equation in the union of all quadratic fields. Notably, Dan-Cohen and Wewers [DCW15] also recover this result when developing explicit algorithms for computing S-unit equations based on Kim's nonabelian Chabauty.

Although it is difficult to prove in full generality when RoS Chabauty will suffice to prove finiteness of rational points, in all *known* cases where the method fails, the failure can be attributed to a *base change obstruction* (see Section 2.2.1). So long as K has at least one real embedding, when bounding $\#(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_{K,S})$, we show that this obstruction can be avoided using descent by proving

Theorem 4.14. *The punctured curve* $\mathbb{P}^1_{\mathcal{O}_{K,S}} \smallsetminus \{0, 1, \infty\}$ *has a descent set* $\mathscr{D}$ *consisting of punctured genus* 0 *curves* $\mathcal{D} = \mathbb{P}^1_{\mathcal{O}_{K,S}} \smallsetminus \Gamma_{\mathcal{D}}$ *such that*

(4.14) $$\mathrm{rank}\, \mathcal{J}_{\mathcal{D}}(\mathcal{O}_{K,S}) \leq d(\#\Gamma_{\mathcal{D}}(\overline{K}) - 2).$$

If K *has at least one real place, then* $\mathscr{D}$ *can be chosen so that there is no base change obstruction to RoS Chabauty for* $(\mathcal{D}, \mathcal{O}_{K,S})$ *for any* $\mathcal{D} \in \mathscr{D}$.

While the latter two results may not give much new information *about solutions to the S-unit equation*, we emphasize that our primary goal in this paper is not to prove new bounds on the number of solutions the S-unit equation, but rather to understand the power of RoS Chabauty in combination with descent. We focus on $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$ because studying the ranks of the Jacobians in descent sets is possible in this setting. Unfortunately, understanding ranks of Jacobians of curves under descent seems well beyond the range of current mathematical technology – even heuristics for these ranks are lacking. Our work also helps to clarify what can be said about the S-unit equation using purely p-adic methods. It would be interesting to combine our ideas with stronger results from the theory of linear forms in p-adic logarithms to understand this more completely.

The remainder of the paper is organized as follows:

In Section 2, we recall the classical setup for Chabauty's method and present a parallel exposition of RoS Chabauty. Our treatment differs from Siksek's [Sik13] in two main ways. For one, we develop the theory for the more general setting of integral points on curves. Second, we consider a larger set of p-adic points, which allows us to simplify formulas for regular differentials substantially, at the cost of working over larger fields. This section also explains how base change obstructions to RoS Chabauty arise.

In Section 3, we recall definitions and results from the theory of multivariate Newton polygons that we use to give explicit bounds on the number of zero-dimensional components of the common vanishing loci of certain multivariate p-adic analytic functions.

In Section 4, we study the power of classical and RoS Chabauty together with Galois descent for bounding $(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_{K,S})$. We show that classical Chabauty and descent only succeeds in very special cases, but that all known obstacles to RoS Chabauty can be avoided using descent, so long as K has at least one real place.

In Section 5, we give several extended examples using RoS Chabauty to prove that $(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_K)$ is finite, covering the cases where K is a quadratic field, a mixed cubic field, a totally complex quartic field, a mixed quartic field with a real quadratic subfield, and where 3 splits completely in K. The last two cases involve some particularly interesting applications of the method. The proof in the mixed quartic case incorporates some basic results from the theory of linear forms in p-adic logarithms. The proof when 3 splits completely exploits the fact that S_3 acts on solutions to the unit equation and runs Chabauty-like arguments on a infinite collection of subgroups of the $\mathcal{O}_K$ points of the Jacobian, rather than applying Chabauty to the full group $\mathcal{J}(\mathcal{O}_K)$.

2. Classical Chabauty and Chabauty for Restrictions of Scalars

This section aims to give an overview of Chabauty's method for restrictions of scalars of curves (or *RoS Chabauty*, for short). Our description differs from Siksek [Sik13] in two main ways. First, with an eye towards testing the theory on the unit equation, we develop the method for S-integral points on models of curves which may not be complete. Second, after taking restrictions of scalars, we consider points valued in the Galois closure of the field that we start with. Although we would not recommend this approach for computer computation, where the size of the fields in question are likely to play a major role in the computational complexity, this approach is useful for our applications, because it combines well with results of Park

[Par16] from the theory of Newton polygons when proving general results about the power of RoS Chabauty.

We will use the following notation and assumptions throughout:

- K is a number field with $r_1(K)$ real places and $r_2(K)$ complex places.
- $d = [K : \mathbb{Q}]$ is the degree of d over $\mathbb{Q}$.
- $\mathcal{O}_K$ is the ring of integers of K.
- S is a finite set of *finite* places of K. Let S' be the set of primes of $\mathbb{Q}$ lying under primes in S. We assume that if $q \in S'$, then $\mathfrak{q} \in S$ for *all* primes of K which lie over q.
- $\mathcal{O}_{K,S}$ denotes the S-integers of K. When $K = \mathbb{Q}$, we write $\mathbb{Z}_S$ instead. Our assumption on S guarantees that $\mathcal{O}_{K,S}$ is free and module-finite over $\mathbb{Z}_{S'}$.
- Given a prime $\mathfrak{p}$ of K, let $K_{\mathfrak{p}}$ denote the completion of K with respect to the valuation corresponding to $\mathfrak{p}$ and let $\mathcal{O}_{K_{\mathfrak{p}}}$ denote its ring of integers.
- L is the Galois closure of K over $\mathbb{Q}$.
- S_L is the set of primes of L lying over primes in S.
- We (attempt to) reserve lower-case Fraktur characters (e.g. $\mathfrak{p}, \mathfrak{q}$) for primes of K and upper-case Fraktur characters (e.g. $\mathfrak{P}, \mathfrak{Q}$) for primes of L.
- $\overline{C}$ is a smooth, proper, geometrically integral curve over K of genus g.
- $\overline{\mathcal{C}}$ is a regular model for $\overline{C}$ over $\mathcal{O}_{K,S}$. In particular, the structure map $\overline{\mathcal{C}} \to \mathrm{Spec}(\mathcal{O}_{K,S})$ is a surjection, all fibers have dimension one, and the fiber over $\mathrm{Spec}\, K$ is isomorphic to $\overline{C}$.
- Γ is a horizontal divisor on $\overline{\mathcal{C}}$. That is, $\Gamma \subset \overline{\mathcal{C}}$ is flat (but not necessarily étale) over $\mathrm{Spec}(\mathcal{O}_{K,S})$ of relative dimension zero. We assume throughout that Γ_K, the restriction of Γ to the generic fiber, is étale over $\mathrm{Spec}\, K$.
- γ' denotes the degree of the divisor Γ_K. Since we assume this divisor is étale, this is equal to $\#\Gamma_K(\overline{K})$.
- $\gamma := \max(0, \gamma' - 1)$.
- $\mathcal{C} := \overline{\mathcal{C}} \smallsetminus \Gamma$ is a relative curve over $\mathcal{O}_{K,S}$, with generic fiber C. We assume throughout that the fibers of $\mathcal{C}$ are geometrically connected and that C is a hyperbolic curve, i.e.

$$\gamma' \geq \begin{cases} 3, & \text{if } g = 0, \\ 1, & \text{if } g = 1, \\ 0, & \text{if } g \geq 2. \end{cases}$$

- J_C is the generalized Jacobian of C with modulus Γ_K.
- $\mathcal{J}_{\mathcal{C}}$, or $\mathcal{J}$ when there is sufficient context, is the (fiberwise) connected component of the identity of the Néron model of J_C. By the Néron property and connectedness of $\mathcal{C}$, it is equipped with an Abel-Jacobi map $j : \mathcal{C} \to \mathcal{J}$ which is a locally closed embedding on the generic fiber. We abuse notation slightly and refer to $\mathcal{J}$ as *the generalized Jacobian* of $\mathcal{C}$. In particular, we *do not* use the term generalized Jacobian to mean *relative Picard scheme/algebraic space with modulus* Γ.
- For any scheme $\mathcal{X}/\mathcal{O}_{K,S}$, the dimension $\dim \mathcal{X}$ refers to the *relative dimension* of $\mathcal{X}$. In all cases we consider, this will be equal to the dimension of the generic fiber $\mathcal{X}_K$ as a K-scheme, or $\mathcal{X}(\mathbb{Z}_p)$ as a p-adic analytic variety.

2.1. **Recap of (Classical) Chabauty's Method.** Our goal in this subsection is to recap the classical Chabauty's method in order to motivate and clarify the construction of RoS Chabauty in the next subsection. For a more comprehensive to Chabauty's method, we enthusiastically recommend the expository article of McCallum and Poonen [MP12]. The reader who is familiar with Chabauty's method can skip this subsection and refer back as needed.

For convenience, assume that $\mathcal{C}$ has good reduction at all primes $\mathfrak{p}$ lying over p.

Let $\mathcal{J}$ be the (generalized) Jacobian of $\mathcal{C}$. Then, $\mathcal{J}$ is an $\mathcal{O}_{K,S}$-integral model of a semiabelian variety of dimension $\gamma + g$ over $\mathcal{O}_{K,S}$ with toric part of dimension γ and abelian variety part of dimension g. We know $\mathcal{J}$ has good reduction at all primes $\mathfrak{p}$ of K lying over p. Moreover, $\mathcal{J}(\mathcal{O}_{K,S})$ is a finitely-generated abelian group. Set

$$r := \operatorname{rank} \mathcal{J}(\mathcal{O}_{K,S}).$$

Fix an Abel-Jacobi map, i.e. an embedding

$$j : \mathcal{C} \hookrightarrow \mathcal{J}.$$

If $\mathcal{C}$ is complete, we assume that $\mathcal{C}(\mathcal{O}_{K,S}) \neq \emptyset$ and choose a base-point. The assumption can sometimes be dropped if $\mathcal{C}$ is not complete, using a points of Γ in place of the base point. In any case, two choices of Abel-Jacobi map only differ by a translation.

Now, fix a prime $\mathfrak{p}$ of K lying over p. The Abel-Jacobi map commutes with completion at $\mathfrak{p}$, so there is a commutative diagram:

$$\begin{array}{ccc} \mathcal{C}(\mathcal{O}_{K,S}) & \longhookrightarrow & \mathcal{C}(\mathcal{O}_{K_\mathfrak{p}}) \\ \downarrow & & \downarrow \\ \mathcal{J}(\mathcal{O}_{K,S}) & \longhookrightarrow & \mathcal{J}(\mathcal{O}_{K_\mathfrak{p}}) \end{array}$$

We make several observations:

(1) $\mathcal{J}(\mathcal{O}_{K_\mathfrak{p}})$ is a p-adic Lie group of dimension $\gamma + g$ over $\mathcal{O}_{K_\mathfrak{p}}$.
(2) The closure of $\mathcal{J}(\mathcal{O}_{K,S}) \subset \mathcal{J}(\mathcal{O}_{K_\mathfrak{p}})$ in the p-adic topology is a p-adic Lie group of dimension ρ for some $\rho \leq r$.
(3) Inside of $\mathcal{J}(\mathcal{O}_{K_\mathfrak{p}})$, we have the inclusion

$$\mathcal{C}(\mathcal{O}_{K,S}) \subseteq \mathcal{C}(\mathcal{O}_{K_\mathfrak{p}}) \cap \overline{\mathcal{J}(\mathcal{O}_{K,S})}. \tag{2.1}$$

If $\operatorname{codim}(\mathcal{C}, \mathcal{J}) + \operatorname{codim}(\mathcal{J}(\mathcal{O}_{K,S})) \geq \dim \mathcal{J}$, or equivalently if

$$\rho \leq \dim \mathcal{J} - 1, \tag{2.2}$$

the method of Chabauty-Coleman proves that the right-hand side of (2.1) is the vanishing set of a function given locally by p-adic power series. Hence, it is finite, and so $\mathcal{C}(\mathcal{O}_{K,S})$ is finite as well.

Although is is possible that $\rho < r$, in the typical case, we expect that $r = \rho$. For instance, under Leopoldt's Conjecture, $r = \rho$ whenever $g_\mathcal{C} = 0$.

Definition 2.3. The pair $(\mathcal{C}, \mathcal{O}_{K,S})$ *satisfies the classical Chabauty inequality* if

$$\operatorname{rank} \mathcal{J}_\mathcal{C}(\mathcal{O}_{K,S}) \leq \dim \mathcal{J}_\mathcal{C} - 1. \tag{2.4}$$

The set $\mathcal{C}(\mathcal{O}_{K_\mathfrak{p}}) \cap \overline{\mathcal{J}(\mathcal{O}_{K,S})}$ can be computed/described explicitly using Coleman's theory of p-adic integration. Let $\Omega^1_{\mathcal{A}}$ be the space of translation-invariant

global one-forms on a semi-abelian variety $\mathcal{A}$ over $K_{\mathfrak{p}}$. Coleman's integration theory gives a pairing

$$\Omega^1_{\mathcal{A}} \times \mathcal{A}(\mathcal{O}_{K_{\mathfrak{p}}}) \to K_{\mathfrak{p}}, \quad (\omega, P) \mapsto \int_0^P \omega,$$

which is $K_{\mathfrak{p}}$-bilinear in the first variable and $\mathbb{Z}$-bilinear in the second variable.

Now, suppose that $P_1, \dots, P_r \in \mathcal{J}(\mathcal{O}_{K,S})$ generate a finite index subgroup of $\mathcal{J}(\mathcal{O}_{K,S})$. By the $\mathbb{Z}$-linearity of Coleman integration in the Jacobian,

$$\Omega' := \left\{ \omega \in \Omega^1_{\mathcal{J}} \otimes K_{\mathfrak{p}} : \int_0^{P_i} \omega = 0 \text{ for all } j \in \{1, \dots r\} \right\}$$

$$= \left\{ \omega \in \Omega^1_{\mathcal{J}} \otimes K_{\mathfrak{p}} : \int_0^P \omega = 0 \text{ for all } P \in \mathcal{J}(\mathcal{O}_{K,S}) \right\},$$

is a subspace of $\Omega_{\mathcal{J}} \otimes K_{\mathfrak{p}}$ of codimension at most r. Moreover,

$$\overline{\mathcal{J}(\mathcal{O}_{K,S})} = \left\{ P \in \mathcal{J}(\mathcal{O}_{K_{\mathfrak{p}}}) : \int_0^P \omega = 0 \text{ for all } \omega \in \Omega' \right\}.$$

Crucially for practical computation, in sufficiently small neighborhoods for the p-adic topology, Coleman integrals can be computed by formal integration of p-adic power series. After pulling back to the curve, on sufficiently small residue discs, these integrals can be expressed as convergent single-variable power series in a uniformizer at a point. When (2.4) or (2.2) is satisfied, these power series, which cut out the intersection of $\mathcal{C}(\mathcal{O}_{K_{\mathfrak{p}}}) \cap \overline{\mathcal{J}(\mathcal{O}_{K,S})}$ within the residue disc, are not identically zero. Applying this on each residue disc shows that $\mathcal{C}(\mathcal{O}_{K,S})$ is finite.

In particular, when the classical Chabauty inequality (2.4) is satisfied, classical Chabauty's method will succeed in bounding the size of $\mathcal{C}(\mathcal{O}_{K,S})$.

2.2. Chabauty's Method and Restriction of Scalars.

The classical Chabauty inequality (2.4) is somewhat inefficient in the following sense: it only uses information at one of the primes $\mathfrak{p}$ lying over p. We can relax this inequality somewhat using restriction of scalars, as follows:

Set

$$\mathcal{V} := \operatorname{Res}_{\mathcal{O}_{K,S}/\mathbb{Z}_{S'}} \mathcal{C} \quad \text{and} \quad \mathcal{A} := \operatorname{Res}_{\mathcal{O}_{K,S}/\mathbb{Z}_{S'}} \mathcal{J}.$$

The Abel-Jacobi map j induces a map $j : \mathcal{V} \hookrightarrow \mathcal{A}$, which we can use to identify $\mathcal{V}$ as a subvariety of $\mathcal{A}$. Similarly to classical Chabauty, there is a commutative diagram:

$$
\begin{array}{ccc}
\mathcal{C}(\mathcal{O}_{K,S}) = \mathcal{V}(\mathbb{Z}_{S'}) & \longhookrightarrow & \mathcal{V}(\mathbb{Z}_p) \\
\downarrow & \downarrow & \downarrow \\
\mathcal{J}(\mathcal{O}_{K,S}) = \mathcal{A}(\mathbb{Z}_{S'}) & \longhookrightarrow & \mathcal{A}(\mathbb{Z}_p).
\end{array}
$$

We observe:

(1) $\mathcal{A}(\mathbb{Q}_p)$ is a p-adic Lie group of dimension $d \cdot \dim(\mathcal{J}) = d(\gamma + g)$ over $\mathbb{Z}_p$.
(2) The closure of $\mathcal{A}(\mathbb{Z}_{S'}) \subset \mathcal{A}(\mathbb{Q}_p)$ in the p-adic topology is a p-adic Lie group of dimension ρ for some $\rho \leq r$.
(3) Inside of $\mathcal{J}(\mathcal{O}_{K_{\mathfrak{p}}})$, we have the inclusion

$$(2.5) \qquad\qquad \mathcal{C}(\mathcal{O}_{K,S}) = \mathcal{V}(\mathbb{Z}_{S'}) \subseteq \mathcal{V}(\mathbb{Z}_p) \cap \overline{\mathcal{A}(\mathbb{Z}_{S'})}.$$

If $\operatorname{codim}(\mathcal{V}, \mathcal{A}) + \operatorname{codim}(\overline{\mathcal{A}(\mathbb{Z}_{S'})}, \mathcal{A}(\mathbb{Z}_p)) \geq \dim \mathcal{A}$, or equivalently if

$$(2.6) \qquad \rho \leq d(\dim \mathcal{J}_{\mathcal{C}} - 1),$$

then we might hope that the right-hand side of (2.5) has dimension zero, which would imply that $\mathcal{C}(\mathcal{O}_{K,S})$ is finite.

By analogy with the classical case, we say:

Definition 2.7. The pair $(\mathcal{C}, \mathcal{O}_{K,S})$ *satisfies the RoS Chabauty inequality* if

$$(2.8) \qquad r \leq d(\dim \mathcal{J}_{\mathcal{C}} - 1).$$

Now, $\mathcal{A}$ is a semi-abelian variety. Coleman's theory of p-adic integration still applies. The vector space

$$\Omega' := \left\{ \omega \in \Omega^1_{\mathcal{A}} \otimes \mathbb{Q}_p : \int_0^P \omega = 0 \text{ for all } P \in \mathcal{A}(\mathbb{Z}_p) \right\},$$

has dimension at least $d(g + \gamma) - r$, and

$$\overline{\mathcal{J}(\mathcal{O}_{K,S})} = \overline{\mathcal{A}(\mathbb{Z}_{S'})} = \left\{ P \in \mathcal{A}(\mathbb{Z}_p) : \int_0^P \omega = 0 \text{ for all } \omega \in \Omega' \right\}.$$

When pulling these equations back to $\mathcal{V}(\mathbb{Z}_p)$, the functions defined by Coleman integration will no longer be expressible as single-variable p-adic analytic functions. Instead, on sufficiently small balls, these functions will be given by d-variable power series. Using multivariate Newton polygon techniques developed in Section 3, one can show that the common vanishing locus of these power series, and therefore $\mathcal{V}(\mathbb{Z}_p) \cap \overline{\mathcal{A}(\mathbb{Z}_{S'})}$, will have finitely many zero-dimensional components. However, if we are unlucky (or in a bad situation), these functions could conspire to have a *positive-dimensional* vanishing locus. Unfortunately, such bad situations can occur for a geometric reason.

Before continuing with the development of RoS Chabauty, we discuss this geometric obstruction to the method. This geometric obstruction exists when our curve is the base change of a curve which does not satisfy the RoS Chabauty inequality.
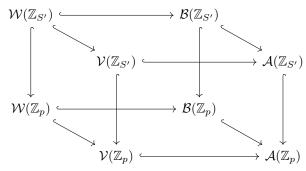
2.2.1. *Base Change Obstructions.* As Siksek notes, $\mathcal{V}(\mathbb{Z}_p) \cap \overline{\mathcal{A}(\mathbb{Z}_{S'})}$ may not be zero-dimensional, even if $(\mathcal{C}, \mathcal{O}_{K,S})$ satisfies the RoS Chabauty inequality [Sik13]. In all known cases where this occurs, the positive-dimensional intersection can be explained by what we call a *base change obstruction*.

Suppose that $k \subset K$ is a subfield and that S_k is the set of primes of k lying under primes in S. Suppose also that $\mathcal{D}/\mathcal{O}_{k,S_k}$ is a curve which becomes isomorphic to $\mathcal{C}$ after base change to $\mathcal{O}_{K,S}$, i.e. a curve such that

$$\mathcal{D}_{\mathcal{O}_{K,S}} \cong \mathcal{C}.$$

We also have $(\mathcal{J}_{\mathcal{D}})_{\mathcal{O}_{K,S}} \cong \mathcal{J}_{\mathcal{C}}$ compatibly with the Abel-Jacobi maps.

Letting $\mathcal{W} := (\operatorname{Res}_{\mathcal{O}_{k,k_S}/\mathbb{Z}_{S'}} \mathcal{D})$ and $\mathcal{B} := (\operatorname{Res}_{\mathcal{O}_{k,k_S}/\mathbb{Z}_{S'}} \mathcal{J}_{\mathcal{D}})$, we have the following commutative diagram:

Clearly, $\mathcal{W}(\mathbb{Z}_p) \cap \overline{\mathcal{B}(\mathbb{Z}_{S'})} \subset \mathcal{V}(\mathbb{Z}_p) \cap \overline{\mathcal{A}(\mathbb{Z}_{S'})}$ inside of $\mathcal{A}(\mathbb{Z}_p)$. If

$$\dim \overline{\mathcal{B}(\mathbb{Z}_{S'})} \geq [k : \mathbb{Q}] \cdot (\dim \mathcal{J}_\mathcal{D} - 1),$$

then $\mathcal{W}(\mathbb{Z}_p) \cap \overline{\mathcal{B}(\mathbb{Z}_{S'})}$ will have positive dimension as a p-adic analytic variety, so $\mathcal{V}(\mathbb{Z}_p) \cap \overline{\mathcal{A}(\mathbb{Z}_{S'})}$ will have postive dimension as a p-adic analytic variety.

Based on this observation, we say

Definition 2.9. A *base change obstruction* to RoS Chabauty for $(\mathcal{C}, \mathcal{O}_{K,S})$ is a pair $(\mathcal{D}, \mathcal{O}_{k,S_k})$ where k is a subfield of K, $\mathcal{D}$ is a curve over $\mathcal{O}_{k,S_k}$, and $\mathcal{D}_{\mathcal{O}_{K,S}} \cong \mathcal{C}$ such that

$$\operatorname{rank} \mathcal{J}_\mathcal{D}(\mathcal{O}_{k,S_k}) > [k : \mathbb{Q}] \cdot (\dim \mathcal{J}_\mathcal{D} - 1).$$

If no such $\mathcal{D}$ exists, we say that $(\mathcal{C}, \mathcal{O}_{K,S})$ *has no base change obstruction to RoS Chabauty*.

Base change obstructions do show up in practice.

Example 2.10. If $\mathcal{C} = \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$, K is a CM sextic field, and k is the totally real cubic subfield of K, then $\mathcal{J} \cong \mathbb{G}_m \times \mathbb{G}_m$ and

$$4 = 2 \cdot \operatorname{rank} \mathcal{O}_K^\times \leq [K : \mathbb{Q}] \cdot 1 = 6,$$

but

$$4 = 2 \cdot \operatorname{rank} \mathcal{O}_k^\times > [k : \mathbb{Q}] \cdot 1 = 3,$$

so $(\mathbb{P}^1_{\mathcal{O}_k} \smallsetminus \{0, 1, \infty\}, \mathcal{O}_k)$ is a base change obstruction to RoS Chabauty for $(\mathbb{P}^1_{\mathcal{O}_K} \smallsetminus \{0, 1, \infty\}, \mathcal{O}_K)$.

Example 2.11. If $\mathcal{C}/\mathbb{Q}$ is a smooth projective curve of genus g and K is a number field with

$$\operatorname{rank} \mathcal{J}_\mathcal{C}(\mathbb{Q}) \geq g,$$

but

$$\operatorname{rank} \mathcal{J}_\mathcal{C}(K) \leq [K : \mathbb{Q}] \cdot (g - 1),$$

then $(\mathcal{C}, \mathbb{Q})$ is a base change obstruction to RoS Chabauty for $(\mathcal{C}, K)$.

Remark 2.12. In the context of smooth proper curves of genus at least 2, Siksek asks the analogue of the open question: If $(\mathcal{C}, \mathcal{O}_{K,S})$ satisfies

$$\operatorname{rank} \mathcal{J}_\mathcal{C}(\mathcal{O}_{K,S}) \leq [K : \mathbb{Q}] \cdot (\dim \mathcal{J}_\mathcal{C} - 1),$$

and has no base change obstruction, is $\mathcal{V}(\mathbb{Z}_p) \cap \overline{\mathcal{A}(\mathbb{Z}_{S'})}$ necessarily 0-dimensional (finite)?

One could ask the somewhat stronger open question: If $(\mathcal{C}, \mathcal{O}_{K,S})$ satisfies

$$\operatorname{rank} \mathcal{J}_\mathcal{C}(\mathcal{O}_{K,S}) \leq [K : \mathbb{Q}] \cdot (\dim \mathcal{J}_\mathcal{C} - 1),$$

are all positive-dimensional analytic subvarieties of $\mathcal{V}(\mathbb{Z}_p) \cap \overline{\mathcal{A}(\mathbb{Z}_{S'})}$ contained in some base-change obstruction?

If the answer to this second, stronger open question is 'yes,' then RoS Chabauty would suffice to prove that the set of points in $\mathcal{C}(\mathcal{O}_{K,S})$ *which do not come from a curve defined over a subvariety* is finite. Although the conclusion holds by Faltings's Theorem, this question seems very difficult in general.

2.3. **RoS Chabauty after Base Change Back Up.** We now return to the development of RoS Chabauty as we will use it in the rest of this paper.

Even if we understand $\mathcal{C}$ and $\mathcal{J}$ well, $\mathcal{V}$ and $\mathcal{A}$ can be a bit unwieldy. Specifically,

$$(2.13) \qquad \mathcal{V}(\mathbb{Z}_p) \cong \prod_{\mathfrak{p}|p}(\mathrm{Res}_{\mathcal{O}_{K_{\mathfrak{P}}}/\mathbb{Z}_p}\mathcal{C})(\mathbb{Z}_p) \quad \text{and} \quad \mathcal{A}(\mathbb{Z}_p) \cong \prod_{\mathfrak{p}|p}(\mathrm{Res}_{\mathcal{O}_{K_{\mathfrak{P}}}/\mathbb{Z}_p}\mathcal{J})(\mathbb{Z}_p)$$

where the product runs over primes $\mathfrak{p}$ over p without multiplicity. If we take L to be the Galois closure of K, and choose any primes $\mathfrak{P}$ of L lying over p this simplifies to

$$\mathcal{V}(\mathcal{O}_{L_{\mathfrak{P}}}) \cong \mathcal{C}(\mathcal{O}_{L_{\mathfrak{P}}})^d \quad \text{and} \quad \mathcal{A}(\mathcal{O}_{L_{\mathfrak{P}}}) \cong \mathcal{J}(\mathcal{O}_{L_{\mathfrak{P}}})^d.$$

It is slightly more convenient to pass through the $\mathcal{O}_{L,S_L}$ points, since this will give us a better understanding of the Galois action. Let $G = \mathrm{Gal}(L/\mathbb{Q})$, and let a superscript G denote the invariants for the G action. We can extend the RoS Chabauty commutative diagram as follows.

$$(2.14)$$

$$\begin{array}{ccccccccc}
\mathcal{C}(\mathcal{O}_{K,S}) & \!\!=\!\!=\!\! & \mathcal{V}(\mathbb{Z}_{S'}) & \hookrightarrow & (\mathcal{V}(\mathcal{O}_{L,S_L}))^G & \xrightarrow{\cong} & (\mathcal{C}(\mathcal{O}_{L,S_L})^d)^G & \hookrightarrow & \mathcal{C}(\mathcal{O}_{L_{\mathfrak{P}}})^d \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{J}(\mathcal{O}_{K,S}) & \!\!=\!\!=\!\! & \mathcal{A}(\mathbb{Z}_{S'}) & \hookrightarrow & (\mathcal{A}(\mathcal{O}_{L,S_L}))^G & \xrightarrow{\cong} & (\mathcal{J}(\mathcal{O}_{L,S_L})^d)^G & \hookrightarrow & \mathcal{J}(\mathcal{O}_{L_{\mathfrak{P}}})^d.
\end{array}$$

In fact, we even get an action of $\mathrm{Gal}(L/\mathbb{Q})$ on the last column, which combines the action of $\mathrm{Gal}(L_{\mathfrak{P}}/\mathbb{Q}_p)$ and the permutation action of $\mathrm{Gal}(L/\mathbb{Q}_p)$ on the set of primes of L above p.

Let $\sigma_1, \ldots, \sigma_d$ be the embeddings $K \hookrightarrow L$. Up to a permutation of the coordinates, the compositions $\mathcal{C}(\mathcal{O}_{K,S}) \to (\mathcal{C}(\mathcal{O}_{L,S_L})^d)^G$ and $\mathcal{J}(\mathcal{O}_{K,S}) \to (\mathcal{J}(\mathcal{O}_{L,S_L})^d)^G$ are given by

$$P \mapsto (\sigma_1(P), \ldots, \sigma_d(P)).$$

The choice of prime $\mathfrak{P}$ over p changes the map by permutating the coordinates.

The disadvantage of this setup is that we have replaced a $\mathbb{Z}_p$-Lie group with an $\mathcal{O}_{L_{\mathfrak{P}}}$-Lie group. The intersection $\overline{\mathcal{J}(\mathcal{O}_{K,S})} \cap j(\mathcal{C}(\mathcal{O}_{L_{\mathfrak{P}}})) \subset \mathcal{J}(\mathcal{O}_{L_{\mathfrak{P}}})^d$ may well be larger than $\overline{\mathcal{J}(\mathcal{O}_{K,S})} \cap \mathcal{V}(\mathbb{Z}_{S'}) \subset \mathcal{A}(\mathbb{Z}_p)$. This is not a serious problem since we only care about solutions which could come from $\mathcal{J}(\mathcal{O}_{K,S})$. As we will see in Corollary 5.6 this will allow us to rule out many of the extra points that come from taking a larger set. More important for our purposes, it is easier to describe the regular differentials on $(\mathcal{J}_{\mathcal{O}_{L_{\mathfrak{P}}}})^d$ in terms of the regular differentials on $\mathcal{J}$. Namely,

$$\Omega^1_{\mathcal{A}_L} \cong \Omega^1_{\mathcal{J}_L^d} \cong (\Omega^1_{\mathcal{J}} \otimes L)^d \quad \text{and} \quad \Omega^1_{\mathcal{A}_{L_{\mathfrak{P}}}} \cong \Omega^1_{\mathcal{J}_{L_{\mathfrak{P}}}^d} \cong (\Omega^1_{\mathcal{J}} \otimes L_{\mathfrak{P}})^d.$$

Now, for any $\omega \in \Omega^1_{\mathcal{J}_L^d}$, define the function

$$F_\omega : \mathcal{J}(L_{\mathfrak{P}})^d \to L_{\mathfrak{P}}, \quad P \mapsto \int_0^P \omega.$$

On any sufficiently small p-adic disc around any point, the pullback $j^*F_\omega : \mathcal{C}(L_\mathfrak{P})^d \to L_\mathfrak{P}$ can be expressed as a sum

$$j^*F_\omega(t_1, \ldots, t_d) = F_1(t_1) + \cdots F_d(t_d),$$

where each F_j is a single-variable power series in the local coordinate at the point which converges on the disc. As a result,

$$j^{-1}(\overline{\mathcal{O}_{K,S}}) \subset \mathcal{C}(\mathcal{O}_{L_\mathfrak{P}})^d$$

is cut out by power series of this form. We close this section by noting that such power series are particularly amenable to analysis by the Newton polygon methods developed by Park [Par16], which we recall in Section 3.

3. Preliminaries on Newton Polygons

In this section, we recall some notation and statements from the theory of Newton polygons which we will use to bound the number of zero-dimensional components in the simultaneous vanishing locus of several multivariable power series.

The result that we need is a theorem of Park [Par16] (extracted by Gunther and Morrow [GM17]) that builds on Rabinoff's study of tropical deformation and intersection theory [Rab12]. We also state a corollary that applies to the power series that come up in our applications.

Let L be a number field and let $\mathfrak{P}$ be a prime ideal of $\mathcal{O}_L$ lying over p. Throughout, we assume that the valuation $v_\mathfrak{P} : L_\mathfrak{p} \to \mathbb{Q}$ is normalized so that $v_\mathfrak{P}(p) = 1$.

For $m \in \mathbb{Q} \cap (0, \infty)$, let

$$D_m = \{\alpha \in L_\mathfrak{P} : v_\mathfrak{P}(\alpha) \geq m\}$$

be the (closed) p-adic disc of radius p^{-m}. Let D_m^d be its d-fold product, and let $L_\mathfrak{P}\langle D_m^d \rangle$ be the space of overconvergent power series in d variables on D_m^d. For $T \subset \mathbb{R}^d$, let $\mathrm{conv}(T)$ denote its convex hull.

Given a power series F, our first task is to define the *Newton polygon* of F.

Definition 3.1. Fix $m \in \mathbb{Q} \cap (0, \infty)$ and $F(t_1, \ldots, t_d) = \sum_{u \in \mathbb{Z}_{\geq 0}^d} a_u t^u \in L_\mathfrak{P}\langle D_m^d \rangle$.

Say $u = (u_1, \ldots, u_d) \in \mathbb{Z}_{\geq 0}^d$ is minimal for $w = (w_1, \ldots, w_d) \in \mathbb{Q}_{\geq m}^d$ if for all $u' \in \mathbb{Z}_{\geq 0}^d$,

$$v(a_u) + \sum_{j=1}^d w_j u_j \leq v(a_{u'}) + \sum_{j=1}^d w_j u_j'.$$

Say $u = (u_1, \ldots, u_d) \in \mathbb{Z}_{\geq 0}^d$ is non-uniquely minimal for $w = (w_1, \ldots, w_d) \in \mathbb{Q}_{\geq m}^d$ if u is minimal for w and also, there is some $u'' \in \mathbb{Z}_{\geq 0}^d$ such that

$$v(a_u) + \sum_{j=1}^d w_j u_j = v(a_{u''}) + \sum_{j=1}^d w_j u_j''.$$

We define the *Newton polygon of F with respect to m* to be the set

$$\mathrm{New}_m(F) := \mathrm{conv}\left(\left\{u \in \mathbb{Z}_{\geq 0}^d : \exists w \in \mathbb{Q}_{\geq m}^d \text{ with } u \text{ non-uniquely minimal for } w\right\}\right).$$

The set $\mathrm{New}_m(F)$ is the convex hull of the set of indices which could be the indices of terms of minimal valuation at some zero $(t_1, \ldots, t_d)$ of F (with $v_\mathfrak{p}(t_j) \geq m$ for all j) if all we knew about F was the valuation of its coefficients a_u.

Example 3.2. Fix $m \in \mathbb{R}_{>0}$. Suppose that $a_{i,n} \in \mathcal{O}_{L_{\mathfrak{P}}}^{\times}$ for all $n > 1$, $a_0 \in \mathcal{O}_{L_{\mathfrak{P}}}$ and set

$$F(t_1, \ldots, t_d) = a_0 + \sum_{n=1}^{\infty} \sum_{i=1}^{d} \frac{a_{i,n} t_i^n}{n}.$$

Set $M = 0$ if $m > \frac{1}{p-1}$. Otherwise let M be the unique positive integer such that

$$\frac{1}{p^{M-1}(p-1)} > m \geq \frac{1}{p^M(p-1)}.$$

Let e_j be the jth unit vector and

$$\Delta = \mathrm{conv}(\{(0, \ldots, 0)\} \cup \{e_j : j \in \{1, \ldots, d\}\}).$$

Then,

$$\mathrm{New}_m(F) = \begin{cases} p^M \Delta \smallsetminus \{(x_1, \ldots, x_d) : \sum_{j=1}^{d} x_j < 1\}, & \text{if } a_0 = 0, \\ p^M \Delta, & \text{if } a_0 \neq 0. \end{cases}$$

To combine data from multiple power series, we need the notion of *mixed volume*.

Definition 3.3. Given d bounded, convex subsets $C_1, \ldots, C_d$ of $\mathbb{R}^d$ with non-zero volume, the volume of the set

$$\sum_{j=1}^{d} \lambda_j C_j = \left\{ \sum_{j=1}^{d} \lambda_j c_j : c_j \in C_j \right\}$$

is a homogeneous polynomial G of degree d in $\lambda_1, \ldots, \lambda_d$. The *mixed volume*

$$\mathrm{MV}(C_1, \ldots, C_d)$$

of $C_1, \ldots, C_d$ is defined to be the coefficient of $\prod_{j=1}^{d} \lambda_j$ in this polynomial.

If any C_j has volume zero we set $\mathrm{MV}(C_1, \ldots, C_d) = 0$.

Remark 3.4. We follow the conventions from [Par16, GM17]. Some other authors define the mixed volume to be $\frac{1}{n!}$ times our definition of mixed volume.

The following case will most useful for us:

Example 3.5. Suppose that $C_1 = C_2 = \cdots = C_d = C$. Then,

$$\mathrm{MV}(C_1, \ldots, C_d) = d! \mathrm{Vol}(C).$$

In particular, $C = \Delta$, then $\mathrm{MV}(C_1, \ldots, C_d) = 1$.

We will apply these results via the following theorem:

Theorem 3.6 ([Par16], Theorem 4.18 and Proposition 5.7). *. Fix $m \in \mathbb{Q}_{\geq 0}$ and let $F_1, \ldots, F_d \in L_{\mathfrak{P}} \langle D_m^d \rangle$ be power series such that for any $i, j \in \{1, \ldots, d\}$, the power series F_i has a term of the form ct_j^N with $c \neq 0$ and $N > 0$.*

Then, the number of zero-dimensional components of the common zero locus of $F_1, \ldots, F_d$ in $(D_m \cap L_{\mathfrak{P}}^{\times})^d$ is at most

$$\mathrm{MV}(\mathrm{New}_m(F_1), \ldots, \mathrm{New}_m(F_d)).$$

The following corollary will be particularly useful.

Corollary 3.7. *Suppose that* $F_1, \ldots, F_d \in L_{\mathfrak{P}} \langle D_m^d \rangle$ *are power series*

$$F_j = \sum_{n=0}^{\infty} \sum_{i=1}^{d} \frac{a_{i,j,n} t_i^n}{n},$$

for some $a_{i,j,n} \in \mathcal{O}_{L_{\mathfrak{P}}}$. *Suppose also that for some* r,

$$v_{\mathfrak{P}}(a_{i,r,n}) = 0 \text{ for all } i \in \{1, \ldots, d\}, n \in \mathbb{Z}_{\geq 0}.$$

If $m > \frac{1}{p-1}$, *then the common zero locus of* $F_1, \ldots, F_d$ *in* $(D_m \cap L_{\mathfrak{P}}^{\times})^d$ *has at most one zero-dimensional component.*

Proof. For all $j \neq r$, we may replace F_j with $F_j + pF_r$ without changing the common vanishing locus. Applying Example 3.2 to $F_j + pF_r$, we see that for all j,

$$\mathrm{New}_m(F_j + pF_r) \subset \Delta.$$

By Example 3.5,

$$\mathrm{MV}(\mathrm{New}_m(F_1 + pF_r), \ldots, \mathrm{New}_m(F_d + pF_r)) \leq 1.$$

The result is then immediate from Theorem 3.6.   □

4. RoS Chabauty and Genus 0 Descent: When Should It Work?

In this section, we combine Chabauty's method and RoS Chabauty with the method of descent, which reduces the problem of computing $\mathcal{C}(\mathcal{O}_{K,S})$ to computing $\mathcal{D}(\mathcal{O}_{K,S})$ for a finite set of curves which map to $\mathcal{C}$. We will be particularly interested in the case where the $\mathcal{C}$ and $\mathcal{D}$ are all punctured genus zero curves, and the $\mathcal{D}$ are $\mathcal{G}$-torsors over $\mathcal{C}$ for some finite Galois group $\mathcal{G}$. We prove that in most cases, Chabauty's method, together with Galois descent by genus 0 covers *cannot* succeed in proving that $(\mathbb{P}^1_{\mathcal{O}_{K,S}} \smallsetminus \{0, 1, \infty\})(\mathcal{O}_{K,S})$ is finite. In contrast, so long as K has a real place, RoS Chabauty applied to a suitable set of genus zero covers of $(\mathbb{P}^1_{\mathcal{O}_{K,S}} \smallsetminus \{0, 1, \infty\})(\mathcal{O}_{K,S})$ has no base change obstruction.

To begin, we recall some facts about generalized Jacobians of genus zero curves and about Galois descent. We then consider classical Chabauty and descent for $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$, before closing with our study of RoS Chabauty and descent.

4.1. **Generalized Jacobians of Genus Zero Curves.**

In the remainder of this article, we will focus primarily on the case where $\mathcal{C}$ is a punctured curve of genus zero. In preparation, we recall some facts about the structure of the generalized Jacobian of such a curve. Set

$$\mathcal{C} = \mathbb{P}^1_{\mathcal{O}_{K,S}} \smallsetminus \{(x : y) : f(x, y) = 0\}$$

for some homogeneous square-free polynomial $f \in \mathcal{O}_{K,S}[x, y]$.

In this case, the generalized Jacobian of $\mathcal{C}$ can be understood explicitly in terms of norm tori. Write $f = \prod_{i=1}^{c} f_i$ where each f_i is irreducible over $\mathcal{O}_{K,S}$. Set

$$R_i := \begin{cases} \mathcal{O}_{K,S}[x]/f(x, 1) & \text{if } f_i \neq y, \\ \mathcal{O}_{K,S} & \text{if } f_i = y. \end{cases}$$

Let L_i be the fraction field of R_i and let S_i be the set of places of L_i which lie over some prime in S.

Each R_i is module-finite over $\mathcal{O}_{K,S}$ and we have

$$(4.1) \qquad \mathcal{J}_{\mathcal{C}} \cong \left(\prod_{i=1}^{c} \mathrm{Res}_{R_i, \mathcal{O}_{K,S}} \mathbb{G}_{m,R_i} \right) / \Delta(\mathbb{G}_{m,\mathcal{O}_{K,S}}),$$

where $\Delta(\mathbb{G}_{m,\mathcal{O}_{K,S}})$ indicates a diagonally embedded copy of $\mathbb{G}_{m,\mathcal{O}_{K,S}}$. If $f_i = y$ for some i, so that $\mathcal{C}$ is a punctured copy of $\mathbb{A}^1$, the formula can be simplified by leaving out the ith component and the quotient. For example, if $f(x,y) = x(x-y)y$, so that $\mathcal{C} = \mathbb{P}^1_{\mathcal{O}_{K,S}} \smallsetminus \{0,1,\infty\}$, then

$$\mathcal{J}_{\mathcal{C}} \cong \mathbb{G}_{m,\mathcal{O}_{K,S}} \times \mathbb{G}_{m,\mathcal{O}_{K,S}}.$$

The expression (4.1) makes it easy to compute the dimension and rank of $\mathcal{J}_{\mathcal{C}}(\mathcal{O}_{K,S})$. Since R_i is finite index in $\mathcal{O}_{L_i,S_i}$ and has the same rank as $\mathcal{O}_{L_i,S_i}$, we have

$$\dim \mathcal{J}_{\mathcal{C}} = \deg(f) - 1 = \sum_{i=1}^{c} \deg(f_i) - 1 = \sum_{i=1}^{c} [L_i : K] - [K : K]$$

and

$$\mathrm{rank}\, \mathcal{J}_{\mathcal{C}}(\mathcal{O}_{K,S}) = \sum_{i=1}^{c} \mathrm{rank}\, R_i^{\times} - \mathrm{rank}\, \mathcal{O}_{K,S}^{\times}$$

$$= \sum_{i=1}^{c} \mathrm{rank}\, \mathcal{O}_{L_i,S_i}^{\times} - \mathrm{rank}\, \mathcal{O}_{K,S}^{\times}$$

$$(4.2) \qquad = \sum_{i=1}^{c} [r_1(L_i) + r_2(L_i) + \#S_i - 1] - [r_1(K) + r_2(K) + \#S - 1].$$

We can also express the rank in terms of the action of the absolute Galois group of K on the set of punctures of our genus zero curve. We state the general result, noting that for the purpose of proving finiteness of integral points, we may assume the curve is a punctured $\mathbb{P}^1$, or else it automatically has no $\mathcal{O}_{K,S}$ points.

Lemma 4.3. *Let $\overline{\mathcal{C}}/\mathcal{O}_{K,S}$ be a regular model of a smooth, projective, genus 0 curve. Let $G = \mathrm{Gal}(\overline{K}/K)$ be the absolute Galois group of K and for a place $\mathfrak{p}$, let $G_{\mathfrak{p}}$ denote the decomposition group of $\mathfrak{p}$. Let Γ be a divisor on $\mathcal{C}$, which we think of as a finite, G-stable suset of $\mathcal{C}(\overline{\mathcal{O}_{K,S}})$. Set*

$$\mathcal{C} = \overline{\mathcal{C}} \smallsetminus \Gamma.$$

Write $G \backslash \Gamma(\overline{K})$ for the set of orbits for the action of G on $\Gamma(\overline{K})$.

   Then, $\mathcal{J}_{\mathcal{C}}$ is a torus of dimension

$$\dim \mathcal{J}_{\mathcal{C}} = \#\Gamma(\overline{K}) - 1,$$

and $\mathcal{J}_{\mathcal{C}}(\mathcal{O}_{K,S})$ is an abelian group of rank

$$\mathrm{rank}\, \mathcal{J}_{\mathcal{C}}(\mathcal{O}_{K,S}) = \sum_{\mathfrak{p} \in S \cup \Sigma_{\infty}} [\#(G_{\mathfrak{p}} \backslash \Gamma(\overline{K})) - 1] - [\#(G \backslash \Gamma(\overline{K})) - 1].$$

We omit the proof, which is a fairly straightforward computation in the character theory of tori. All of the ideas needed can be found, for example, Theorem 8.7.2 of *Cohomology of Number Fields* by Neukirch, Schmidt, and Wingberg [NSW08], or in Chapter 6 of Eisenträger's Ph.D. Thesis [Eis03].

4.2. **Galois Descent (by genus 0 covers).** When the pair $(\mathcal{C}, \mathcal{O}_{K,S})$ does not satisfy the classical Chabauty inequality (2.4) or the RoS Chabauty inequality (2.8), we must use another approach to prove that $\mathcal{C}(\mathcal{O}_{K,S})$ is finite. One possibility is to replace $(\mathcal{C}, \mathcal{O}_{K,S})$ with a finite set of covers of $\mathcal{C}$ such that each point in $\mathcal{C}(\mathcal{O}_{K,S})$ is the image of an integral point on some curve in the set. We define

Definition 4.4. A *descent set* for $(\mathcal{C}, \mathcal{O}_{K,S})$ is a finite set $\mathscr{D}$ of curves $\mathcal{D}$ over $\mathcal{O}_{K,S}$ equipped with maps $f_{\mathcal{D}} : \mathcal{D} \to \mathcal{C}$ such that

$$\mathcal{C}(\mathcal{O}_{K,S}) \subseteq \bigcup_{\mathcal{D} \in \mathscr{D}} f_{\mathcal{D}}(\mathcal{D}(\mathcal{O}_{K,S})).$$

Showing that $f_{\mathcal{D}}(\mathcal{D}(\mathcal{O}_{K,S}))$ is finite for each $\mathcal{D} \in \mathscr{D}$ proves that $\mathcal{C}(\mathcal{O}_{K,S})$ is also finite.

Remark 4.5. If $\mathscr{D}$ is a descent set of $(\mathcal{C}, \mathcal{O}_{K,S})$ and for some $\mathcal{D} \in \mathscr{D}$, we have that $\mathscr{E}$ is a descent set for $(\mathcal{D}, \mathcal{O}_{K,S})$, then $(\mathscr{D} \setminus \{\mathcal{D}\}) \cup \mathscr{E}$ is also a descent set for $(\mathcal{C}, \mathcal{O}_{K,S})$. This iterated descent approach can be used to construct complicated descent sets from relatively simple building blocks.

In certain cases, we can construct descent sets explicitly.

Suppose that $\mathcal{A}$ is an algebraic group over $\mathcal{O}_{K,S}$ and we have an injective map $j : \mathcal{C} \hookrightarrow \mathcal{A}$. Moreover, suppose that $\mathcal{B}$ is another algebraic group over $\mathcal{O}_{K,S}$ equipped with a finite, flat map $\phi : \mathcal{B} \to \mathcal{A}$ of algebraic groups such that $\phi(\mathcal{B}(\mathcal{O}_{K,S}))$ has finite index in $\mathcal{A}(\mathcal{O}_{K,S})$. Choose (right) coset representatives $P_1, \ldots, P_n$ for $\phi(\mathcal{B}(\mathcal{O}_{K,S}))$ in $\mathcal{A}(\mathcal{O}_{K,S})$. For each $i \in \{1, \ldots, n\}$, there is a map of $\mathcal{O}_{K,S}$-varieties

$$\phi_i : \mathcal{B} \to \mathcal{A},$$
$$Q \mapsto \phi(Q) \cdot P_i.$$

Then,

$$(4.6) \qquad \mathcal{A}(\mathcal{O}_{K,S}) = \coprod_{i=1}^{n} \phi(\mathcal{B}(\mathcal{O}_{K,S})) \cdot P_i = \coprod_{i=1}^{n} \phi_i(\mathcal{B}(\mathcal{O}_{K,S})).$$

Let $\mathcal{D}_i$ be the pullback of $\mathcal{C}$ by ϕ_i. From the pullback diagram

$$\begin{array}{ccc} \mathcal{D}_i & \overset{j_{\mathcal{D}}}{\hookrightarrow} & \mathcal{B} \\ {\scriptstyle \phi_i}\downarrow & & \downarrow{\scriptstyle \phi_i} \\ \mathcal{C} & \overset{j}{\hookrightarrow} & \mathcal{A} \end{array}$$

and (4.6), we see that $Q \in \mathcal{C}(\mathcal{O}_{K,S})$ belongs to $\phi_i(\mathcal{D}_i(\mathcal{O}_{K,S}))$ if and only if $Q \in \mathcal{B}(\mathcal{O}_{K,S}) \cdot P_i$. It follows that

$$\mathcal{C}(\mathcal{O}_{K,S}) = \coprod_{i=1}^{n} \phi_i(\mathcal{D}_i(\mathcal{O}_{K,S})).$$

In particular, $\{\mathcal{D}_i : i \in \{1, \ldots, n\}\}$ is a descent set for $(\mathcal{C}, \mathcal{O}_{K,S})$.

In this construction, one can check that each $\mathcal{D}_i$ is a torsor over $\mathcal{C}$ for the finite group scheme $\ker(\phi_i : \mathcal{B} \to \mathcal{A})$. In particular, this construction is an example of a more general technique called Galois descent, where the covers in the descent set are G-torsors for some finite flat group scheme G and are therefore geometrically Galois over $\mathcal{C}$.

A detailed treatment of (fppf) Galois descent can be found in [Sko01] or [Poo17].

Example 4.7. If $\mathcal{C}/\mathcal{O}_{K,S}$ is $\mathbb{P}^1 \smallsetminus (\{0,\infty\} \cup \Gamma')$, then there is a natural inclusion $\mathcal{C} \hookrightarrow \mathbb{G}_{m,\mathcal{O}_{K,S}}$. Taking ϕ to be the nth power map from $\mathbb{G}_m$ to itself, this procedure gives a descent set $\mathscr{D}$ for $(\mathcal{C}, \mathcal{O}_{K,S})$ indexed by elements of $\mathcal{O}_{K,S}^\times / \mathcal{O}_{K,S}^{\times n}$. Moreover, every covering curve in $\mathscr{D}$ is a punctured $\mathbb{P}^1$ for which we can write down explicit equations.

Remark 4.8. We show that Example 4.7 combined with the iterated descent described in Remark is essentially the only way to produce Galois descent sets consisting of punctured genus 0 curves. For $\phi : \mathcal{D} \to \mathcal{C}$ is a geometrically Galois map with $\mathcal{D}$ of genus 0, let B be the set of branch points of ϕ in $\mathcal{C}(\overline{K})$. The Riemann-Hurwitz Theorem implies that

$$2 \deg(\phi) - 2 = \deg(\phi) \cdot \#B - \sum_{P \in B} \#\{Q \in \mathcal{D}(\overline{K}) : \phi(Q) = P\}.$$

For any $P \in B$, we have $\#\{Q \in \mathcal{D}(\overline{K}) : \phi(Q) = P\} \le d/2$, so $1 < \#B < 4$.

If $\#B = 2$, then $\phi : \mathcal{D} \to \mathcal{C}$ is a twist of the nth power map, or equivalently a $(\mathbb{Z}/n\mathbb{Z})$-torsor over $\mathcal{C}$.

If $\#B = 3$, then $\deg(\phi) = 6$ and $\phi : \mathcal{D} \to \mathcal{C}$ is an S_3-torsor over $\mathcal{C}$. By using the structure of S_3 as a solvable group, the corresponding descent set can be constructed via a (two-stage) iterated descent with $\#B = 2$ at each stage.

4.3. **Classical Chabauty and Descent by Genus 0 covers.** Suppose that $\mathcal{C} = \overline{\mathcal{C}} \smallsetminus \Gamma$ is a punctured genus 0 curve over $\mathcal{O}_{K,S}$. The classical Chabauty inequality (2.4) becomes

$$(4.9) \qquad \operatorname{rank} \mathcal{J}(\mathcal{O}_{K,S}) \le (\#\Gamma(\overline{K}) - 1) - 1 = \#\Gamma(\overline{K}) - 2.$$

in this case. We use Lemma 4.3 to study when (4.9) holds (or fails to hold) for $\mathcal{C}$.

As a warm up, note that the classical Chabauty inequality is satisfied for $\mathbb{P}^1_{\mathcal{O}_{K,S}} \smallsetminus \{0,1,\infty\}$ if and only if $\operatorname{rank} \mathcal{O}_{K,S}^\times < 1$, or equivalently $S = \emptyset$ and either $K = \mathbb{Q}$ or K is an imaginary quadratic field.

More generally, we note the following:

Proposition 4.10. *Let K be a number field with absolute Galois group $G = \operatorname{Gal}(\overline{K}/K)$. Suppose that $\mathcal{C} = \mathbb{P}^1_{\mathcal{O}_{K,S}} \smallsetminus \Gamma$ for some divisor Γ. The classical Chabauty-Coleman inequality (4.9) for $\mathcal{C}$ fails if any of the following conditions hold:*

(1) $r_2(K) \ge 1$ and $r_1(K) + r_2(K) + \#S \ge 2$,
(2) $r_1(K) \ge 3$ and $r_1(K) + \#S \ge 4$,
(3) $r_1(K) = 2$ and $r_1(K) + \#S \ge 4$ and $\#(G \backslash \Gamma(\overline{K})) \ge 2$,
(4) $r_1(K) = 3$ and $\Gamma(\overline{K})$ cannot be written as a disjoint union of G-orbits $\{P_i, P_i'\}$ which remain $G_\mathfrak{p}$ orbits for each real place $\mathfrak{p}$. (E.g. The condition on $\Gamma(\overline{K})$ is automatic if $\#\Gamma(\overline{K})$ is odd.)

Of course, the classical Chabauty inequality (4.9) may also fail under many other conditions, especially when S is large. We sketch the proof.

Proof. We use Lemma 4.3 to express $\operatorname{rank} \mathcal{J}_{\mathcal{C}}(\mathcal{O}_{K,S})$ in terms of the action of $\operatorname{Gal}(\overline{K}/K)$ on $\Gamma(\overline{K})$.

The statement follows quickly from three observations:

(1) If $\mathfrak{p} \in \Sigma_\infty$ is a complex place $G_\mathfrak{p} = \{1\}$, so $\#(G_\mathfrak{p} \backslash \Gamma(\overline{K})) - 1 = \#\Gamma(\overline{K}) - 1$.

(2) If $\mathfrak{p} \in \Sigma_\infty$ is a real place $G_\mathfrak{p} = \mathbb{Z}/2\mathbb{Z}$, so $\#(G_\mathfrak{p} \backslash \Gamma(\overline{K})) - 1 \geq \left\lceil \frac{\#\Gamma(\overline{K})}{2} \right\rceil - 1$,
with equality if and only if $\Gamma(\overline{K})$ decomposes into complex conjugate pairs over $K_\mathfrak{p}$.

(3) $[\#(G \backslash \Gamma(\overline{K})) - 1] \leq \min_{\mathfrak{p} \in S \cup \Sigma_\infty} [\#(G_\mathfrak{p} \backslash \Gamma(\overline{K})) - 1]$.

For example, if $r_2(K) \geq 1$ and $r_1(K) + r_2(K) + \#S \geq 2$, let ∞_1 be an infinite place and let $\mathfrak{p} \neq \infty_1$ be another place with $\#(G_\mathfrak{p} \backslash \Gamma(\overline{K}))$ minimal. By Lemma 4.3,

$$\begin{aligned}
\operatorname{rank} \mathcal{J}_\mathcal{C}(\mathcal{O}_{K,S}) &\geq [\#(G_{\infty_1} \backslash \Gamma(\overline{K})) - 1] + [\#(G_\mathfrak{p} \backslash \Gamma(\overline{K})) - 1] - [\#(G \backslash \Gamma(\overline{K})) - 1] \\
&\geq [\#(G_{\infty_1} \backslash \Gamma(\overline{K})) - 1] + [\#(G_\mathfrak{p} \backslash \Gamma(\overline{K})) - 1] - [\#(G_\mathfrak{p} \backslash \Gamma(\overline{K})) - 1] \\
&= \#\Gamma(\overline{K}) - 1 \,.
\end{aligned}$$

The other cases are similar, hinging on the fact that the $-[\#(G \backslash \Gamma(\overline{K})) - 1]$ term in Lemma 4.3 can cancel *at most* the minimal positive term. $\square$

Corollary 4.11. *Suppose that we are* not *in the following situations: (i) $K = \mathbb{Q}$, (ii) K a real quadratic field and $\#S \leq 1$, (iii) K an imaginary quadratic or totally real cubic field and $\#S = 0$.*

Then, the classical Chabauty inequality is not satisfied by any descent set consisting of genus zero covers of $\mathbb{P}^1_{\mathcal{O}_{K,S}} \smallsetminus \{0, 1, \infty\}$. Under Leopoldt's Conjecture, this implies that the combination of classical Chabauty and descent by genus zero covers is insufficient to prove that

$$(\mathbb{P}^1_{\mathcal{O}_{K,S}} \smallsetminus \{0, 1, \infty\})(\mathcal{O}_{K,S})$$

is finite.

Remark 4.12. If $K = \mathbb{Q}$ the results of the next section show that $(\mathbb{Z}/q\mathbb{Z})$-descent and classical Chabauty suffice to prove finiteness of $(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathbb{Z}_{S'})$ for any set S' of primes. Of course, when K is imaginary quadratic, classical Chabauty without descent trivially suffices to prove that $(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_K)$ is finite.

In the remaining cases, where K is a totally real quadratic or cubic field, one can check that $(\mathbb{Z}/q\mathbb{Z})$-descent and classical Chabauty *is not* sufficient to prove the desired finiteness result. It may be possible to prove a similar finiteness result in these cases using an *iterated* cyclic descent, or a descent-like procedure using covers which are not torsors over the base curve, but this would require a more careful analysis.

4.4. **RoS Chabauty and Descent by Genus** 0 **covers.** For the punctured genus zero curve $\mathcal{C} = \mathbb{P}^1_{\mathcal{O}_{K,S}} \smallsetminus \Gamma$, the condition for RoS Chabauty to apply becomes

(4.13) $$\operatorname{rank} \mathcal{J}_\mathcal{C}(\mathcal{O}_{K,S}) \leq d \left(\#\Gamma(\overline{K}) - 2 \right) \,.$$

The main result of this section is:

Theorem 4.14. *The punctured curve $\mathbb{P}^1_{\mathcal{O}_{K,S}} \smallsetminus \{0, 1, \infty\}$ has a descent set $\mathscr{D}$ consisting of punctured genus 0 curves $\mathcal{D} = \mathbb{P}^1_{\mathcal{O}_{K,S}} \smallsetminus \Gamma_\mathcal{D}$ such that*

(4.15) $$\operatorname{rank} \mathcal{J}_\mathcal{D}(\mathcal{O}_{K,S}) \leq d(\#\Gamma_\mathcal{D}(\overline{K}) - 2).$$

If K has at least one real place, then $\mathscr{D}$ can be chosen so that there is no base change obstruction to RoS Chabauty for $(\mathcal{D}, \mathcal{O}_{K,S})$ for any $\mathcal{D} \in \mathscr{D}$.

Remark 4.16. We saw in Section 4.3 that descent and classical Chabauty had no hope of proving finiteness of $(\mathbb{P}^1_{\mathcal{O}_{K,S}} \smallsetminus \{0, 1, \infty\})(\mathcal{O}_{K,S})$ except in a few special cases with K of low degree.

In Remark 5.1, we will see that RoS Chabauty alone cannot prove finiteness of $(\mathbb{P}^1_{\mathcal{O}_{K,S}} \smallsetminus \{0, 1, \infty\})(\mathcal{O}_{K,S})$ unless every subfield $k \subseteq K$ has at most two real places.

In contrast, Theorem 4.14 says that descent and RoS Chabauty together are likely to be sufficient to prove that $(\mathbb{P}^1_{\mathcal{O}_{K,S}} \smallsetminus \{0, 1, \infty\})(\mathcal{O}_{K,S})$ is finite, so long as K has at least one real embedding. Of course, there are many other ways to prove that this set is finite; the real value of Theorem 4.14 is not to prove this finiteness, but to provide evidence that Chabauty's method could be used to prove finiteness of integral points on a curve over a wide range of number fields. One might hope to extend these ideas in combination with Kim's non-abelian Chabauty to get similar results about higher genus curves over number fields.

Proof of Theorem 4.14. We construct an explicit set $\mathscr{D}$ consisting of covers of twists of the qth power map for q any sufficiently large prime. We determine how large q needs to be later in the proof.

Choose a set U of coset representatives for $\mathcal{O}^\times_{K,S}/\mathcal{O}^{\times q}_{K,S}$ with $1 \in U$.

Let $\mathcal{D}_u$ be the curve $\mathbb{A}^1_{\mathcal{O}_{K,S}} \smallsetminus \{x(x^q - u^{-1}) = 0\}$. If we take Γ_u to be the divisor (defined over K) given by

$$\Gamma_u = \{0, \infty\} \cup \left\{ x \in \overline{K} : ux^q - 1 = 0 \right\},$$

then $\mathcal{D}_u = \mathbb{P}^1_{\mathcal{O}_{K,S}} \smallsetminus \Gamma_u$. The curve $\mathcal{D}_u$ maps to $\mathbb{P}^1_{\mathcal{O}_{K,S}} \smallsetminus \{0, 1, \infty\}$ via

$$\mathbb{A}^1_{\mathcal{O}_{K,S}} \smallsetminus \{x(x^q - u^{-1}) = 0\} \to \mathbb{P}^1_{\mathcal{O}_{K,S}} \smallsetminus \{0, 1, \infty\}$$
$$x \mapsto ux^q \,.$$

Let $\mathscr{D}$ be the set $\{\mathcal{D}_u : u \in U\}$. For each $u \in U$, the image of $\mathcal{D}_u(\mathcal{O}_{K,S})$ consists of those elements of $(\mathbb{P}^1_{\mathcal{O}_{K,S}} \smallsetminus \{0, 1, \infty\})(\mathcal{O}_{K,S})$ which can be expressed in the form uv^q for some $v \in \mathcal{O}^\times_{K,S}$. As a result, $\mathscr{D}$ is a descent set for $\mathbb{P}^1_{\mathcal{O}_{K,S}} \smallsetminus \{0, 1, \infty\}$.

Let $\mathcal{J}_u$ be the generalized Jacobian of $\mathcal{D}_u$. For each $u \in U$, the generic fiber of $\mathcal{D}_u$ is $\mathbb{P}^1$ with a divisor of degree $q + 2$ removed. Hence, $\dim \mathcal{J}_u = q + 1$.

Bounding $\mathrm{rank}\,\mathcal{J}_u$ will require more work. We break into two cases based on whether or not $u = 1$.

Case 1: $u = 1$
Part 1: $(\mathcal{D}_1, \mathcal{O}_{K,S})$ satisfies the classical Chabauty inequality.
We use Lemma 4.3 to compute the rank of $\mathcal{J}_1$ in terms of the action of the $G_\mathfrak{p}$ on $\Gamma_1(\overline{K})$.

By construction,

$$\#(G \backslash \Gamma_1(\overline{K})) = 4 \,.$$

Moreover,

$$\#(G_\mathfrak{p} \backslash \Gamma_1(\overline{K})) = \begin{cases} q + 2 & \text{if } \mathfrak{p} \text{ is a complex place of } K \,, \\ 3 + \frac{q-1}{2} & \text{if } \mathfrak{p} \text{ is a real place of } K \,, \\ 3 + \#\{\text{primes } \mathfrak{P} \text{ of } K[\zeta_p] \text{ above } \mathfrak{p}\} & \text{if } \mathfrak{p} \text{ is a finite place of } K \,. \end{cases}$$

While an exact formula for the number of primes of $K[\zeta_p]$ above $\mathfrak{p}$ is complicated, we can bound it as follows.

let $\kappa_{\mathfrak{p}}$ be the residue field of $K_{\mathfrak{p}}$. There is some minimal $a_{\mathfrak{p}} \in \mathbb{Z}_{>0}$ such that qth roots of unity are defined in the degree $a_{\mathfrak{p}}$ unramified extension of $\kappa_{\mathfrak{p}}$. Alternately, $a_{\mathfrak{p}}$ is characterized as the minimum $a_{\mathfrak{p}} \in \mathbb{Z}_{>0}$ such that q divides $((\#\kappa_{\mathfrak{p}})^{a_{\mathfrak{p}}} - 1)$. Equivalently, $a_{\mathfrak{p}}$ is the order of $\#\kappa_{\mathfrak{p}}$ in $(\mathbb{Z}/q\mathbb{Z})^{\times}$. The number of primes of $K[\zeta_q]$ lying over $\mathfrak{p}$ is $(q-1)/a_{\mathfrak{p}}$.

Using the characterization of $a_{\mathfrak{p}}$ as the order of $\#\kappa_{\mathfrak{p}}$ in $(\mathbb{Z}/q\mathbb{Z})^{\times}$, and using $\log_{\infty}$ to denote the real logarithm, we note that

$$a_{\mathfrak{p}} \geq \left\lceil \frac{\log_{\infty}(q-1)}{\log_{\infty}(\#\kappa_{\mathfrak{p}})} \right\rceil \quad \text{and so} \quad \#\{\text{primes } \mathfrak{P} \text{ of } K[\zeta_p] \text{ above } \mathfrak{p}\} \leq \frac{q-1}{\left\lceil \frac{\log_{\infty}(q-1)}{\log_{\infty}(\#\kappa_{\mathfrak{p}})} \right\rceil}.$$

Given any $\varepsilon > 0$, we may choose q sufficiently large (depending on ε, K, and S) so that

$$\sum_{\mathfrak{p} \in S} \#\{\text{primes } \mathfrak{P} \text{ of } K[\zeta_p] \text{ above } \mathfrak{p}\} \leq \varepsilon(q-1).$$

For such q, Lemma 4.3 implies that

$$\operatorname{rank} \mathcal{J}_1(\mathcal{O}_{K,S}) \leq \left(\frac{q+3}{2} \right) r_1(K) + (q+1)r_2(K) + 2\#S + \varepsilon(q-1) - 3.$$

In comparison,

$$[K : \mathbb{Q}](\dim \mathcal{J}_1 - 1) = (r_1(K) + 2r_2(K))q.$$

Hence

$$\operatorname{rank} \mathcal{J}_1(\mathcal{O}_{K,S}) \leq [K : \mathbb{Q}](\dim \mathcal{J}_1 - 1).$$

In other words, $(\mathcal{D}_1, \mathcal{O}_{K,S})$ satisfies the RoS Chabauty inequality (2.8).

Part 2: $(\mathcal{D}_1, \mathcal{O}_{K,S})$ has no base change obstruction.

We claim that if K has at least one real embedding then the pair $(\mathcal{D}_1, \mathcal{O}_{K,S})$ has no base change obstruction to RoS Chabauty for sufficiently large q.

Let k be a subfield of K, let S_k be the primes lying under primes in S and suppose that $\mathcal{D}/\mathcal{O}_{k,S_k}$ is a curve with $\mathcal{D}_{\mathcal{O}_{K,S}} \cong \mathcal{D}_1$. Let $H = \operatorname{Gal}(\overline{k}/k)$ be the absolute Galois group of k, and given a prime $\mathfrak{l}$ of $\mathcal{O}_k$, let $H_{\mathfrak{l}}$ be the decomposition group at $\mathfrak{l}$. Note that if $\mathfrak{p}$ is a prime of $\mathcal{O}_K$ lying over $\mathfrak{l}$, we can identify $G_{\mathfrak{p}}$ with a subgroup of $H_{\mathfrak{l}}$.

Let $\Gamma_1 = \mathbb{P}^1 \smallsetminus \mathcal{D}_1$ and $\Gamma_{\mathcal{D}} = \overline{\mathcal{D}} \smallsetminus \mathcal{D}$. Since $\Gamma_{\mathcal{D}}$ is an odd degree divisor on $\mathcal{D}$, we may assume that $\mathcal{D} \cong \mathbb{P}^1_{\mathcal{O}_{k,S_k}} \smallsetminus \Gamma_{\mathcal{D}}$.

For any place $\mathfrak{l}$ of k lying under a place $\mathfrak{p}$ of K, we have

$$\#(H_{\mathfrak{l}} \backslash \Gamma_{\mathcal{D}}(\overline{k})) \leq \#(G_{\mathfrak{p}} \backslash \Gamma_{\mathcal{D}}(\overline{K})) = \#(G_{\mathfrak{p}} \backslash \Gamma_1(\overline{K})).$$

For the infinite places, the best bounds this gives us are

$$\#(H_{\mathfrak{l}} \backslash \Gamma_{\mathcal{D}}(\overline{k})) \leq \begin{cases} q+2 & \text{if } \mathfrak{l} \text{ is a complex place of } k, \\ q+2 & \text{if } \mathfrak{l} \text{ is a real place of } k \text{ and every } \mathfrak{p} \text{ above } \mathfrak{l} \text{ is complex}, \\ 3 + \frac{q-1}{2} & \text{if } \mathfrak{l} \text{ is a real place of } k \text{ and some } \mathfrak{p} \text{ above } \mathfrak{l} \text{ is real}. \end{cases}$$

Considering the finite places as a group, we have

$$\sum_{\mathfrak{l} \in S_k} \#(H_{\mathfrak{l}} \backslash \Gamma_{\mathcal{D}}(\overline{k})) \leq \sum_{\mathfrak{p} \in S} \#(G_{\mathfrak{p}} \backslash \Gamma_{\mathcal{D}}(\overline{K})) = \sum_{\mathfrak{p} \in S} \#(G_{\mathfrak{p}} \backslash \Gamma_1(\overline{K})),$$

Let $r_1'(k)$ denote the number of real places $\mathfrak{l}$ of k such that every place above $\mathfrak{l}$ is real and let $r_1''(k) = r_1(k) - r_1'(k)$.

By Lemma 4.3, we have

$$\operatorname{rank} \mathcal{J}_{\mathcal{D}}(\mathcal{O}_{k,S_k}) \leq r_1'(k)\frac{q+3}{2} + r_1''(k)(q+1) + 2r_2(k)\frac{q+1}{2} + 3\#S - \#S_k + \varepsilon(q-1)\,.$$

$$\leq \frac{q([k:\mathbb{Q}] + r_1''(k))}{2} + r_1'(k) + \frac{[k:\mathbb{Q}]}{2} + 3\#S - \#S_k + \varepsilon(q-1)\,.$$

So long as $r_1''(k) \neq [k:\mathbb{Q}]$, (which holds if K has a real place), we may choose ε sufficiently small and q sufficiently large so that

$$\operatorname{rank} \mathcal{J}_{\mathcal{D}}(\mathcal{O}_{k,S_k}) \leq [k:\mathbb{Q}] \cdot q = [k:\mathbb{Q}](\dim \mathcal{J}_{\mathcal{D}} - 1)\,.$$

Case 2: $u \notin \mathcal{O}_{K,S}^{\times q}$
Part 1: $(\mathcal{D}_u, \mathcal{O}_{K,S})$ satisfies the classical Chabauty inequality.
The proof in this case is very similar to the previous case. There is a slight difference in the treatment of the finite places.

$$\#(G_{\mathfrak{p}} \backslash \Gamma_u(\overline{K})) = \begin{cases} q+2 & \text{if } \mathfrak{p} \text{ is a complex place of } K\,, \\ 3 + \frac{q-1}{2} & \text{if } \mathfrak{p} \text{ is a real place of } K\,, \\ 2 + \#\{\text{primes } \mathfrak{P} \text{ of } K[\sqrt[q]{u}] \text{ above } \mathfrak{p}\} & \text{if } \mathfrak{p} \text{ is a finite place of } K\,. \end{cases}$$

With $a_{\mathfrak{p}}$ defined as in the previous case, for a finite place $\mathfrak{p}$ of K,

$$\#\{\text{primes } \mathfrak{P} \text{ of } K[\sqrt[q]{u}] \text{ above } \mathfrak{p}\} = \begin{cases} 1, & \text{if } \overline{u} \notin \kappa_{\mathfrak{p}}^{\times q}\,, \\ 1 + (q-1)/a_{\mathfrak{p}}, & \text{otherwise}\,. \end{cases}$$

For ε as in Part 1, Lemma 4.3 gives

$$\operatorname{rank} \mathcal{J}_u(\mathcal{O}_{K,S}) \leq \left(\frac{q+3}{2}\right) r_1(K) + (q+1)r_2(K) + 2\#S + \varepsilon(q-1) - 2\,.$$

Again, in comparison,

$$[K:\mathbb{Q}](\dim \mathcal{J}_u - 1) = (r_1(K) + 2r_2(K))q\,.$$

Taking q large enough, we have

$$\operatorname{rank} \mathcal{J}_u(\mathcal{O}_{K,S}) \leq [K:\mathbb{Q}](\dim \mathcal{J}_u - 1)\,,$$

so that $(\mathcal{D}_u, \mathcal{O}_{K,S})$ satisfies the RoS Chabauty inequality (2.8).

Part 2: $(\mathcal{D}_u, \mathcal{O}_{K,S})$ has no base change obstruction.
The proof that $\mathcal{D}_u$ has no base change obstruction so long as K has a real place is essentially identical to the proof for $\mathcal{D}_1$, so we omit it. $\qquad\square$

Remark 4.17. In contrast, if K is a CM-field, we will show that there *is* a base change obstruction to RoS Chabauty for $(\mathcal{D}_1, \mathcal{O}_{K,S})$ for any large q, at least when $S_K \geq 2$.

Suppose K is CM, with RM field k and that $K = k[\alpha]$ for some $\alpha \in \mathcal{O}_K$. Let $\overline{\alpha}$ be the Galois conjugate of α under the $\operatorname{Gal}(K/k)$ action. Then, $\overline{\alpha}$ is the complex conjugate of α for every embedding $K \hookrightarrow \mathbb{C}$.

Consider the fractional linear transformation:

$$f : \mathbb{P}^1 \to \mathbb{P}^1\,,$$

$$x \mapsto \frac{\alpha x - \overline{\alpha}}{x - 1}\,.$$

For any embedding of $K[\zeta_q] \hookrightarrow \mathbb{C}$, and any qth root of unity ζ_q, we have

$$\overline{\left(\frac{\alpha\zeta_q - \overline{\alpha}}{\zeta_q - 1}\right)} = \frac{\overline{\alpha}\zeta_q^{-1} - \alpha}{\zeta_q^{-1} - 1} = \frac{\alpha\zeta_q - \overline{\alpha}}{\zeta_q - 1}\,.$$

Also, $f(1) = \infty$ and

$$f(\{0, \infty\}) = \{\alpha, \overline{\alpha}\}\,,$$

is defined over k. Now,

$$f^{-1}(y) = \frac{y - \overline{\alpha}}{y - \alpha}\,,$$

So the images of the qth roots of unity are $f(1) = \infty$ and the roots of

$$(4.18) \quad g(y) = \frac{1}{\overline{\alpha} - \alpha}\left((y - \overline{\alpha})^q - (y - \alpha)^q\right) = \sum_{i=0}^{q-1} y^i \binom{q}{i} (-1)^{q-i} \frac{\overline{\alpha}^{q-i} - \alpha^{q-i}}{\overline{\alpha} - \alpha}\,.$$

Since (4.18) has coefficients in k, the set $\Gamma_{\mathcal{D}} := f(\{x : x^q - 1 = 0\} \cup \{0, \infty\})$ is defined over k. Moreover, for each infinite place of k, we have

$$\#(H_{\mathfrak{l}} \backslash \Gamma_{\mathcal{D}}(\overline{k})) = q + 1\,.$$

For any finite place $\mathfrak{l}$ of k,

$$\#(H_{\mathfrak{l}} \backslash \Gamma_{\mathcal{D}}(\overline{k})) \geq \#(H \backslash \Gamma_{\mathcal{D}}(\overline{k})) = 3\,,$$

with equality if and only if $\mathfrak{l}$ is inert in both K and $k[y]/g(y)$. Set $\mathcal{D} = \mathbb{P}^1_{\mathcal{O}_{k,S_k}} \smallsetminus \Gamma_{\mathcal{D}}$ and let $\mathcal{J}_{\mathcal{D}}$ be its Jacobian. Then, Lemma 4.3 implies that

$$\operatorname{rank} \mathcal{J}_{\mathcal{D}}(\mathcal{O}_{k,S_k}) \geq [k : \mathbb{Q}](\dim \mathcal{J}_{\mathcal{D}} - 1) + 2\#S_k - 2\,,$$

with equality if and only if every $\mathfrak{l} \in S_k$ is inert in both K and $k[y]/g(y)$. In particular, $(\mathcal{D}, \mathcal{O}_{k,S_k})$ is a base change obstruction to RoS Chabauty for $(\mathcal{D}_1, \mathcal{O}_{K,S})$ if $\#S_K \geq 2$.

Of course, if K is not CM, but contains a CM field, we may apply the same argument to the CM subfield to construct a base change obstruction.

Remark 4.19. The obstruction constructed in Remark 4.17 is essentially the only way that $(\mathcal{D}_1, \mathcal{O}_{K,S})$ can have a base change obstruction.

Refining the proof of Theorem 4.14, one can check that if there is a base change obstruction to $(\mathcal{D}_u, \mathcal{O}_{K,S})$ for all large q, then there must be a fractional linear transformation f defined over K which maps the unit circle to the real line in *every* embedding $K \hookrightarrow \mathbb{C}$. Such f are characterized by mapping 0 and ∞ to complex conjugate values and -1 and 1 to real values under *every* embedding $K \hookrightarrow \mathbb{C}$.

Then, K has CM subfield $\mathbb{Q}[f(0), f(\infty)]$ with RM subfield $\mathbb{Q}[f(0)+f(\infty), f(0)f(\infty)]$.

Without additional data about the primes in S, the most *precise* general condition for the $(\mathcal{D}_u, \mathcal{O}_{K,S})$ to have no base change obstruction is that K does not contain a CM-field (i.e. that neither K nor any of its totally complex subfields have an index 2 totally real subfield).

Remark 4.20. If K is a CM-field (or contains a CM-field), it should still be possible to find a descent set consisting of genus zero curves which have no base change obstruction as follows. We briefly outline a construction and give an idea for the proof.

Base change $\mathcal{D}_u$ to a larger ring of S-integers R so that all of the punctures are rational. Over R, there is a fractional linear transformation which takes the circle of radius $|\sqrt[q]{u}|$ centered at the origin to the real line in every embedding

$\mathrm{Frac}(R) \hookrightarrow \mathbb{C}$ and which takes two of the punctures in Γ_u to 0 and ∞. Let $\mathcal{D}'_u$ be the image of $\mathcal{D}_u$ under this fractional linear transformation. Take q' to be a large auxiliary prime, and let v range over $R^\times / R^{\times q'}$. The preimages $\mathcal{E}_{u,v}$ of $\mathcal{D}'_u$ under $\mathbb{G}_m \to \mathbb{G}_m : x \mapsto vx^{q'}$ form a descent set of $\mathcal{D}'_u$ and therefore also form a descent set for $\mathcal{D}_u$. Together, across all u and v, they give a descent set for $\mathcal{C}$.

As in Remark 4.19, base change obstructions must come from fractional linear transformations which take all but finitely many (independent of q') punctures of $\mathcal{E}_{u,v}$ to the real line over every embedding of $\mathrm{Frac}(R) \hookrightarrow \mathbb{C}$. In this case, this is impossible, because at most $\max(q, q')$ of the punctures lie on any line or circle.

We plan to explain and simplify this construction in further detail in a future update to this note.

5. RoS Chabauty and $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$: Computations

In this section, we study the application of restriction of scalars and Chabauty's method to compute solutions to the unit equation (without using descent). Most of the section is an extended series of examples of RoS Chabauty for $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$. The final subsection uses a slight variant of this setup to prove that when 3 splits completely in K, $(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_K) = \emptyset$.

5.1. **RoS Chabauty and $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$: Generalities.** We make explicit the theory of RoS Chabauty from Sections 2.3 and 2.2 in the context of $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$. From here on, we drop subscripts signifying the base ring to ease notation if we think it is unlikely to cause confusion.

Remark 5.1. If $\mathcal{C} = \mathbb{P}^1_{\mathcal{O}_{K,S}} \smallsetminus \{0, 1, \infty\}$ then $\mathcal{J}_{\mathcal{C}} \cong \mathbb{G}_{m, \mathcal{O}_{K,S}} \times \mathbb{G}_{m, \mathcal{O}_{K,S}}$. Let r_1 and r_2 be the number of real and complex embeddings of K, so $d = r_1 + 2r_2$. We have $\dim \mathcal{J}_{\mathcal{C}} = 2$ and

$$\mathrm{rank}\, \mathcal{J}(\mathcal{O}_{K,S}) = 2\, \mathrm{rank}\, \mathcal{O}_{K,S}^\times = 2(r_1 + r_2 + \#S - 1)\,.$$

In this case, the RoS Chabauty inequality (2.8) becomes:

$$2(r_1 + r_2 + \#S - 1) \leq (r_1 + 2r_2)(2 - 1)\,,$$

or equivalently

$$r_1 + 2\#S \leq 2\,.$$

Hence, $(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}, \mathcal{O}_{K,S})$ satisfies the RoS Chabauty inequality if and only if either $r_1 = 0$ and $\#S \leq 1$, $r_1 = 1$ and $\#S = 0$, or $r_1 = 2$ and $\#S = 0$.

If $\#S > 0$, $(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}, \mathbb{Z}_{S'})$ will not satisfy the RoS Chabauty inequality and will therefore represent a base change obstruction to RoS Chabauty for $(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}, \mathcal{O}_{K,S})$. So, we do not lose much generality if we restrict to the case $S = \emptyset$ and we will do so for the rest of the section to ease notation.

Throughout, we fix the Abel-Jacobi map:

$$j : \mathbb{P}^1 \smallsetminus \{0, 1, \infty\} \hookrightarrow \mathbb{G}_m \times \mathbb{G}_m$$
$$x \mapsto (x, x - 1)\,.$$

The outside of the diagram (2.14) becomes

$$
\begin{array}{ccc}
(\mathbb{P}^1 \smallsetminus \{0,1,\infty\})(\mathcal{O}_K) & \longrightarrow & (\mathbb{P}^1 \smallsetminus \{0,1,\infty\})(\mathcal{O}_{L_{\mathfrak{P}}})^d \\
\downarrow & & \downarrow \\
\mathbb{G}_m(\mathcal{O}_K) \times \mathbb{G}_m(\mathcal{O}_K) & \longrightarrow & \mathbb{G}_m(\mathcal{O}_{L_{\mathfrak{P}}})^d \times \mathbb{G}_m(\mathcal{O}_{L_{\mathfrak{P}}})^d \,.
\end{array}
$$
(5.2)

Let $\sigma_1, \ldots, \sigma_d : K \to L$ be the embeddings of K into L. The bottom horizontal map in (5.2) is given by

$$
(x, y) \mapsto ((\sigma_1(x), \ldots, \sigma_d(x)), (\sigma_1(y), \ldots, \sigma_d(y))) \,,
$$

while the right-most vertical map in (5.2) is given by

$$
(u_1, \ldots, u_d) \mapsto ((u_1, \ldots, u_d), (u_1 - 1, \ldots, u_d - 1)) \,.
$$

Let $x_1, \ldots, x_d, y_1, \ldots, y_d$ be the local coordinates on $\mathbb{G}_m^d \times \mathbb{G}_m^d$. We have

$$
\Omega^1_{\mathbb{G}_m^d \times \mathbb{G}_m^d} = \mathrm{Span}\left\{ \frac{dx_1}{x_1}, \ldots, \frac{dx_d}{x_d}, \frac{dy_1}{y_1}, \ldots, \frac{dy_d}{y_d} \right\} \,.
$$

In these coordinates, the p-adic integration theory has a particularly simple interpretation: For $i \in \{0,1\}$ set

$$
Q_i = ((x_1^{(i)}, \ldots, x_d^{(i)}), (y_1^{(i)}, \ldots, y_d^{(i)})) \,.
$$

Let $\log$ denote the p-adic logarithm. Then,

$$
\int_{Q_0}^{Q_1} \sum_{j=1}^d a_j \frac{dx_j}{x_j} + b_j \frac{dy_j}{y_j} = \sum_{j=1}^d a_j \log \frac{x_j^{(1)}}{x_j^{(0)}} + b_j \log \frac{y_j^{(1)}}{y_j^{(0)}} \,.
$$

Since we will only ever evaluate at points where the $x_j^{(i)}$ and $y_j^{(i)}$ are units, we do not need to specify the value of $\log p$.

Now, we must gather information about the space of a_j and b_j such that for any

$$
Q = ((x_1', \ldots, x_d'), (y_1', \ldots, y_d')) \in \mathbb{G}_m(\mathcal{O}_K) \times \mathbb{G}_m(\mathcal{O}_K) \,,
$$

we have

$$
\int_{(1,1)}^Q \sum_{j=1}^d a_j \frac{dx_j}{x_j} + b_j \frac{dy_j}{y_j} = \sum_{j=1}^d a_j \log x_j' + b_j \log y_j' = 0 \,.
$$

Let $\mathscr{A} \subset L_{\mathfrak{p}}^d$ be the subspace of $(a_1, \ldots, a_d)$ such that

$$
\sum_{j=1}^d a_j \log \sigma_j(x) = 0
$$

for all $x \in \mathbb{G}_m(\mathcal{O}_K)$. If $r' = \mathrm{rank}\, \mathbb{G}_m(\mathcal{O}_K)$, then $\dim \mathscr{A} \geq d - r'$ with equality if Leopoldt's Conjecture holds.

For any $x \in \mathcal{O}_K$, we have $\prod_{j=1}^d \sigma(x) = \pm 1 \in \mathbb{G}_m(\mathbb{Z})$. In particular,

$$
\sum_{j=1}^d \log \sigma_j(x) = 0 \,,
$$

so $(1, \ldots, 1) \in \mathscr{A}$ regardless of K.

Then, $\overline{\mathbb{G}_m(\mathcal{O}_K) \times \mathbb{G}_m(\mathcal{O}_K)}$ is cut out by the vanishing of the $2(d-r)$-dimensional vector space of functions:

$$\left\{ \sum_{j=1}^{d} a_j \log x_j : (a_1, \ldots, a_d) \in \mathscr{A} \right\} \oplus \left\{ \sum_{j=1}^{d} a_j \log y_j : (a_1, \ldots, a_d) \in \mathscr{A} \right\}.$$

Since the image of $(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})^d(\mathcal{O}_{L_{\mathfrak{P}}})$ is cut out by the equations

(5.3)
$$\{x_j - y_j = 1 : j \in [1, \ldots, d]\},$$

the intersection

$$\overline{\mathbb{G}_m(\mathcal{O}_K) \times \mathbb{G}_m(\mathcal{O}_K)} \cap (\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})^d(\mathcal{O}_{L_{\mathfrak{P}}}) \subset \mathbb{G}_m(\mathcal{O}_{L_{\mathfrak{P}}}) \times \mathbb{G}_m(\mathcal{O}_{L_{\mathfrak{P}}})$$

is an analytic subvariety of a $2d$-dimensional p-adic analytic variety cut out by at least $2d - 2r + d$ equations.

Pulling back to $(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})^d(\mathcal{O}_{L_{\mathfrak{P}}})$, any $\mathcal{O}_K$-points must be contained in the common vanishing locus of the p-adic analytic functions in

$$\mathcal{F} := \left\{ \sum_{j=1}^{d} a_j \log x_j : (a_1, \ldots, a_d) \in \mathscr{A} \right\} \oplus \left\{ \sum_{j=1}^{d} a_j \log(x_j - 1) : (a_1, \ldots, a_d) \in \mathscr{A} \right\}.$$

The space $\mathscr{A}$ has a basis $\mathscr{B}$ including $(1, \ldots, 1)$ and such that for all $(a_1, \ldots, a_d) \in \mathscr{A}$, we have $a_j \in \mathcal{O}_K$. Then, $\mathcal{F}$ has a basis

(5.4)
$$\mathcal{G} := \left\{ \sum_{j=1}^{d} a_j \log x_j : (a_1, \ldots, a_d) \in \mathscr{B} \right\} \cup \left\{ \sum_{j=1}^{d} a_j \log(x_j - 1) : (a_1, \ldots, a_d) \in \mathscr{B} \right\}.$$

For $Q = (x_1', \ldots, x_d') \in (\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_{L_{\mathfrak{P}}})^d$, and $m > \frac{1}{p-1}$, let

$$B_m(Q) = \{(x_1'', \ldots, x_d'') \in (\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_{L_{\mathfrak{P}}})^d : v_p(x_j' - x_j'') \geq m \text{ for all } j \in \{1, \ldots, d\}\}.$$

In a neighborhood of Q, the functions in $\mathcal{G}$ can be expressed as d-variable p-adic power series. Moreover, the functions in $\mathcal{G}$ satisfy the hypothesis of Corollary 3.7. Hence, the common vanishing locus of the functions in $\mathcal{F}$ has at most one zero-dimensional component contained in $B_m(Q)$ which does not share a coordinate with Q. The same is true for any point $Q' \in B_m(Q)$. As a result, we have

Proposition 5.5. *Choose* $m \in \mathbb{Q}$ *with* $m > \frac{1}{p-1}$. *For any point* $Q \in (\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_{L_{\mathfrak{P}}})^d$, *the set*

$$j^{-1}(\overline{\mathbb{G}_m(\mathcal{O}_K) \times \mathbb{G}_m(\mathcal{O}_K)}) \subset (\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})^d(\mathcal{O}_{L_{\mathfrak{P}}})$$

has at most one zero-dimensional component in $B_m(Q)$.

Since the image of $(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_K)$ must lie in the image of

$$\prod_{\mathfrak{p}|p} (\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_{K_{\mathfrak{p}}}) \subset \mathcal{O}_{L_{\mathfrak{P}}}^d,$$

under the identification in (2.13), and since the image of $\mathcal{O}_K$ points must be Galois-invariant, the image of $(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_K)$ can be covered by

$$\prod_{\mathfrak{p}|p} (p^{f_{\mathfrak{p}}} - 2) \, p^{f_{\mathfrak{p}} \cdot \left\lfloor \frac{e_{\mathfrak{p}}}{p-1} \right\rfloor}$$

balls $B_m(Q)$, taking the product without multiplicity. The -2 in each factor comes from the fact that units in $\mathcal{O}_{K_\mathfrak{p}}$ cannot be congruent to 0 or 1 modulo a uniformizer for $\mathcal{O}_{K_\mathfrak{p}}$. This gives an explicit bound on the number $\mathcal{O}_K$ points which are contained in zero-dimensional components of the vanishing locus.

Corollary 5.6. *The zero-dimensional components of*

$$j^{-1}(\overline{\mathbb{G}_m(\mathcal{O}_K) \times \mathbb{G}_m(\mathcal{O}_K)}) \subset (\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_{L_\mathfrak{P}})^d$$

contain at most

$$\prod_{\mathfrak{p}|p} \left(p^{f_\mathfrak{p}} - 2\right) p^{f_\mathfrak{p} \cdot \left\lfloor \frac{e_\mathfrak{p}}{p-1} \right\rfloor}$$

points of $(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_K)$, where the product is over all primes $\mathfrak{p}$ of $\mathcal{O}_K$ lying over p.

If we could show that

$$j^{-1}(\overline{\mathbb{G}_m(\mathcal{O}_K) \times \mathbb{G}_m(\mathcal{O}_K)})$$

is zero-dimensional, this would give the explicit bound

$$(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_K) < p^d \,.$$

For $p = 3$ or 5, this would improve Evertse's upper bound of $3 \cdot 7^d$ [Eve84] by an exponential factor in fields where the method applies.

5.2. **Ruling Out Positive-Dimensional Components.** In this section, we focus on several particular examples, proving that $j^{-1}(\overline{\mathbb{G}_m(\mathcal{O}_K) \times \mathbb{G}_m(\mathcal{O}_K)})$ is zero-dimensional when K is a quadratic field, mixed cubic field, or complex quartic field. When K is a mixed quartic field with a totally real quadratic subfield, we prove the slightly weaker statement that $(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_K)$ is contained in the zero-dimensional components of $j^{-1}(\overline{\mathbb{G}_m(\mathcal{O}_K) \times \mathbb{G}_m(\mathcal{O}_K)})$. In all cases, this bounds the size of $(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_K)$.

5.2.1. *Real Quadratic Fields.* Suppose that K is a real quadratic field. In this case, rank $\mathcal{O}_K^\times = 1$ and $[K : \mathbb{Q}] = 2$, so $\mathscr{A}$ is 1-dimensional. Thus, $\mathscr{A}$ is spanned by $(1, 1)$.

We wish to show that the analytic subvariety X of $\mathcal{O}_{L_\mathfrak{P}}^4$ cut out by the equations

$$\begin{aligned} \log x_1 + \log x_2 &= 0 \,, \\ \log y_1 + \log y_2 &= 0 \,, \\ x_1 - y_1 - 1 &= 0 \,, \\ x_2 - y_2 - 1 &= 0 \,, \end{aligned}$$

has no positive-dimensional components. To do this, we check that the tangent space of X is zero-dimensional, or equivalently that the matrix of derivatives at $x = (x_1, x_2, y_1, y_2) \in X$,

$$M_x := \begin{pmatrix} \frac{1}{x_1} & \frac{1}{x_2} & 0 & 0 \\ 0 & 0 & \frac{1}{y_1} & \frac{1}{y_2} \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

has rank 4 outside of a zero-dimensional subvariety of X. After some elementary column/row operations and substituting $y_j = x_j - 1$, the condition rank $M_x = 4$ is equivalent to

$$\mathrm{rank}\left(\begin{array}{cc} x_1 - 1 & x_2 - 1 \\ x_1 & x_2 \end{array}\right) = 2\,.$$

This holds unless $x_1 = x_2$. In that case, $\log x_1 = \log x_2 = 0$. Hence, the set of points of X where the rank drops is zero-dimensional. It follows that every component of X is zero-dimensional. Corollary 5.6 gives

Corollary 5.7. *Let K be a real quadratic field. Let $N = \#(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_K)$. Then,*

$$N \leq 1 \text{ if } 3 \text{ splits completely in } K\,,$$
$$N \leq 7 \text{ if } 3 \text{ is inert in } K\,,$$
$$N \leq 3 \text{ if } 3 \text{ ramifies in } K\,.$$

Remark 5.8. We can strengthen Corollary 5.7 with two observations:

(1) $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$ has a natural S_3 action by fractional linear transformations which permute $0, 1$, and ∞. For any K and S, these transformations preserve $(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_{K,S})$. With the exception of the five numbers:

$$\frac{1 + \sqrt{-3}}{2}, \frac{1 - \sqrt{-3}}{2}, \frac{1}{2}, 2, \text{ and } -1$$

which have non-trivial stabilizer for this action, points of $(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_K)$ must come in groups of 6. Since none of these special points are units in real quadratic fields, we may reduce the upper bounds on N from 1 to 0, 7 to 6 and 3 to 0 in the statement of Corollary 5.7.

(2) Since the functions defining the intersection did not depend on the field, this argument is uniform over all quadratic fields K in the following sense. The bounds in Corollary 5.7 hold for

$$N = \# \bigcup_{K:\, \text{s.t.}\ 3\ \left\{\begin{smallmatrix}\text{splits completely}\\ \text{is inert}\\ \text{ramifies}\end{smallmatrix}\right\}\ \text{in } K} (\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_K)\,.$$

In other words, the first bound in Corollary 5.7 (improved as in (1)) holds not just over a single field, but over the union of all quadratic fields where 3 splits completely, and similarly for the other two bounds.

Combining these observations, we recover the well-known result that the only solutions to the unit equation in real quadratic fields are

$$\frac{1 + \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2}, \frac{3 + \sqrt{5}}{2}, \frac{-1 - \sqrt{5}}{2}, \frac{3 - \sqrt{5}}{2}, \text{ and } \frac{-1 + \sqrt{5}}{2}\,.$$

In [DCW15], Dan-Cohen and Wewers develop an explicit version Kim's of non-abelian Chabauty to study points on $(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_K)$. Their computation cuts out solutions using single-variable polylogarithms instead of sums of logarithms in different variables. In the case where K is a quadratic field, their method is *also* uniform in the field, giving yet another proof that these are the only solutions to the unit equation in quadratic fields. Their proof needs some information about the splitting of 11 in the field, but does not appear to use the extra information that the solutions to the unit equation (typically) come in groups of 6.

As an amusing digression, we note a quick geometric proof of the same result.

$$\mathrm{Sym}^2(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}) \cong \mathbb{G}_m \times \mathbb{G}_m \,,$$

$$\{x, y\} \mapsto (xy, (x - 1)(y - 1)) \,.$$

Since $\#(\mathbb{G}_m \times \mathbb{G}_m)(\mathbb{Z}) = 4$ and each corresponds to an unordered pair of solutions, there are exactly 8 solutions to the unit equation in quadratic fields.

5.2.2. *Complex Cubic Fields.*

Suppose now that K is a complex field. In this case, $\mathrm{rank}\,\mathcal{O}_K^\times = 1$ and $[K : \mathbb{Q}] = 3$, so $\mathscr{A}$ is 2-dimensional. Thus, $\mathscr{A}$ is spanned by $(1, 1, 1)$ and one other vector, call it (a_1, a_2, a_3).

We wish to show that the analytic subvariety X of $\mathcal{O}_{L_\mathfrak{P}}^6$ cut out by the equations

$$\log x_1 + \log x_2 + \log x_3 = 0 \,,$$
$$a_1 \log x_1 + a_2 \log x_2 + a_3 \log x_3 = 0 \,,$$
$$\log y_1 + \log y_2 + \log y_3 = 0 \,,$$
$$a_1 \log y_1 + a_2 \log y_2 + a_3 \log y_3 = 0 \,,$$
$$x_1 - y_1 - 1 = 0 \,,$$
$$x_2 - y_2 - 1 = 0 \,,$$
$$x_3 - y_3 - 1 = 0 \,,$$

is zero-dimensional. To do this, we will check that outside of a zero-dimensional subvariety, the tangent space of X is zero-dimensional. Equivalently, we will show that the matrix of derivatives at $x = (x_1, x_2, x_3, y_1, y_2, y_3) \in X$,

$$M_x := \begin{pmatrix} \frac{1}{x_1} & \frac{1}{x_2} & \frac{1}{x_3} & 0 & 0 & 0 \\ \frac{a_1}{x_1} & \frac{a_2}{x_2} & \frac{a_3}{x_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{y_1} & \frac{1}{y_2} & \frac{1}{y_3} \\ 0 & 0 & 0 & \frac{a_1}{y_1} & \frac{a_1}{y_2} & \frac{a_1}{y_3} \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \,,$$

has rank 6 outside of a zero-dimensional subvariety of X. We may assume without loss of generality that $a_1 = 0$. Substituting $y_j = x_j - 1$, applying elementary row and column operations, the condition $\mathrm{rank}\,M_x = 6$ is equivalent to

$$\mathrm{rank} \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ 0 & a_2 & a_3 \\ 0 & a_2 x_2 & a_3 x_3 \end{pmatrix} = 3 \,.$$

If this fails, then $(0, a_2, a_3)$ is proportional to $(0, a_2 x_2, a_3 x_3)$. If these vectors are proportional, then either (i) $a_2 = 0$, (ii) $a_3 = 0$, or (iii) $x_2 = x_3$.

If we are in case (i), we may replace $(0, 0, a_3)$ with $(-a_3, -a_3, 0)$ and $(0, 0, a_3 x_3)$ with $(-a_3 x_1, -a_3 x_2, 0)$ and reindex the variables to reduce to case (iii). We can do the same in case (ii). Making a similar transformation, we may assume that $a_2 + a_3 \neq 0$. So, when $\mathrm{rank}\,M_x \neq 6$, we must have $x_2 = x_3$ and $y_2 = y_3$.

When we plug this into the equations defining X, we get $(a_2 + a_3) \log x_2 = 0$. So, if $\mathrm{rank}\,M_x \neq 6$, then $\log x_2 = 0$. It follows quickly that if $\mathrm{rank}\,M_x \neq 6$,

$$\log x_1 = \log x_2 = \log x_3 = \log y_1 = \log y_2 = \log y_3 = 0 \,,$$

so the dimension of the tangent space of X is zero-dimensional except perhaps at a finite set of points. Therefore, X is zero-dimensional.

Corollary 5.9. *Let K be a complex cubic field. Let $N = \#\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}(\mathcal{O}_K)$. Then,*

$$N \leq 6 \left\lfloor \frac{1}{6} \right\rfloor = 0 \ \text{if } 3 = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3 \ \text{in } K \,,$$

$$N \leq 6 \left\lfloor \frac{7}{6} \right\rfloor = 6 \ \text{if } 3 = \mathfrak{p}_1\mathfrak{p}_2 \ \text{in } K \,,$$

$$N \leq 6 \left\lfloor \frac{25}{6} \right\rfloor = 24 \ \text{if } 3 \ \text{is inert in } K \,,$$

$$N \leq 6 \left\lfloor \frac{3}{6} \right\rfloor = 0 \ \text{if } 3 = \mathfrak{p}_1\mathfrak{p}_2^2 \ \text{in } K \,,$$

$$N \leq 6 \left\lfloor \frac{3}{6} \right\rfloor = 0 \ \text{if } 3 = \mathfrak{p}^3 \ \text{in } K \,.$$

5.2.3. *Totally Complex Quartic Fields.* Let K be a totally complex quartic field. As in the previous cases, let X be the analytic variety defined by the equations in (5.3) and (5.4). At $x \in X$ checking that the intersection of the tangent spaces of the defining equations is zero-dimensional is equivalent to checking that the matrix

$$M_x := \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ 0 & 1 & a_3 & a_4 \\ 0 & x_2 & a_3x_3 & a_4x_4 \\ 0 & 0 & 1 & b_4 \\ 0 & 0 & x_3 & b_4x_4 \end{pmatrix}$$

has rank 4. If this fails, then either (i) $x_3 = x_4$ or (ii) $b_4 = 0$. If $x_3 = x_4$ and rank $M_x < 4$, we must have

$$\mathrm{rank} \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_4 \\ 0 & 1 & (a_3 + a_4)/2 \\ 0 & x_2 & x_4(a_3 + a_4)/2 \end{pmatrix} < 3 \,.$$

If $b_4 = 0$ and rank $M_x < 4$, we must have

$$\mathrm{rank} \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_4 \\ 0 & 1 & a_4 \\ 0 & x_2 & a_4x_4 \end{pmatrix} < 3 \,.$$

In both cases, the proof that the rank drops at only finitely many points $x \in X$ is essentially the same as in the cubic field case.

Proposition 5.10. *If K is a totally complex quartic field, then the intersection*

$$(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_{L_\mathfrak{P}})^d \cap \overline{\mathbb{G}_m(\mathcal{O}_K) \times \mathbb{G}_m(\mathcal{O}_K)} \subset \mathbb{G}_m(\mathcal{O}_{L_\mathfrak{P}})^d \times \mathbb{G}_m(\mathcal{O}_{L_\mathfrak{P}})^d$$

is zero-dimensional. Thus, for any p,

$$\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}(\mathcal{O}_K) \leq \prod_{\mathfrak{p}|p} \left(p^{f_\mathfrak{p}} - 2\right) p^{f_\mathfrak{p} \cdot \left\lfloor \frac{e_\mathfrak{p}}{p-1} \right\rfloor} \,.$$

5.3. **Mixed Quartic Fields with a Totally Real Subfield.** Suppose now that K is a mixed quartic field, i.e. that $r_1(K) = 2$ and $r_2(K) = 1$. In this case, rank $\mathcal{O}_K^\times = 2$ and $[K : \mathbb{Q}] = 4$, so $\mathscr{A}$ is 2-dimensional. Thus, $\mathscr{A}$ is spanned by $(1, 1, 1, 1)$ and one other vector, call it (a_1, a_2, a_3, a_4).

Suppose also that K has a totally real subfield k. The field k has two embeddings $\tau_1, \tau_2 \hookrightarrow L$. Say that σ_1 and σ_2 extend τ_1 and that σ_3 and σ_4 extend τ_2. We do not yet declare which pair are the real embeddings and which are the complex embeddings. In our chosen coordinates, we can identify $\left(\mathrm{Res}_{\mathcal{O}_k/\mathbb{Z}}\mathbb{G}_m\right)_L$ inside of $\left(\mathrm{Res}_{\mathcal{O}_k/\mathbb{Z}}\mathbb{G}_m\right)_L$ as the space $x_1 = x_2$, $x_3 = x_4$.

If we restrict the functions in $\mathcal{G}$ (5.4) to this subspace, then they must vanish on the points corresponding to elements of $\mathcal{O}_k^\times$. Taking parameters x_1 and x_3, the space of functions of the form $b_1 \log x_1 + b_3 \log x_3$ on

$$\left(\mathrm{Res}_{\mathcal{O}_k/\mathbb{Z}}\mathbb{G}_m\right)_L (L_\mathfrak{P})$$

which vanish on the image of $\mathbb{G}_m(\mathcal{O}_k)$ is one-dimensional vector space spanned by $\log x_1 + \log x_3$. This implies that $a_1 + a_2 = a_3 + a_4$. So, we may assume that

$$a_1 = 0, a_2 = 1, a_3 = b, a_4 = 1 - b \,.$$

Substituting for the y_i using the (5.3), we would like to show that the analytic subvariety X of $\mathcal{O}_{L_\mathfrak{P}}^4$ cut out by the equations

$$\log x_1 + \log x_2 + \log x_3 + \log x_4 = 0 \,,$$
$$\log x_2 + b \log x_3 + (1 - b) \log x_4 = 0 \,,$$
$$\log(1 - x_1) + \log(1 - x_2) + \log(1 - x_3) + \log(1 - x_4) = 0 \,,$$
$$\log(1 - x_2) + b \log(1 - x_3) + (1 - b) \log(1 - x_4) = 0 \,.$$

is zero-dimensional. Instead, we will show that if X has a positive-dimensional component Y, then Y is disjoint from $(\mathbb{P}^1_{\mathcal{O}_K} \smallsetminus \{0, 1, \infty\})(\mathcal{O}_K)$.

For $x = (x_1, \ldots, x_4) \in X$, let

$$M_x := \begin{pmatrix} \frac{1}{x_1} & \frac{1}{x_2} & \frac{1}{x_3} & \frac{1}{x_4} \\ 0 & \frac{1}{x_2} & \frac{b}{x_3} & \frac{1-b}{x_4} \\ \frac{1}{x_1-1} & \frac{1}{x_2-1} & \frac{1}{x_3-1} & \frac{1}{x_4-1} \\ 0 & \frac{1}{x_2-1} & \frac{b}{x_3-1} & \frac{1-b}{x_4-1} \end{pmatrix} \,.$$

If $x \in Y$, then $\mathrm{rank}(M_x) < 4$. If $\mathrm{rank}(M_x) < 4$, then

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ 0 & 1 & b & 1 - b \\ 0 & x_2 & bx_3 & (1 - b)x_4 \end{pmatrix} = 0 \,.$$

So, $\mathrm{rank}(M_x) < 4$ if and only if

$$\frac{(x_1 - x_3)(x_2 - x_4)}{(x_2 - x_3)(x_1 - x_4)} = \frac{(0 - b)(1 - (1 - b))}{(1 - b)(0 - (1 - b))} = \frac{b^2}{(1 - b)^2} \,.$$

In particular, if some point $x \in (\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_K) \subset X$ lies on a positive-dimensional component Y, then b must be algebraic.

We will prove by contradiction that b cannot be algebraic. The key tool is a p-adic analogue due to Brumer [Bru67] of Baker's Theorem on the linear independence of logarithms of algebraic numbers.

Theorem 5.11 (Brumer, [Bru67]). *Let $\alpha_1, \ldots, \alpha_n$ be elements of the completion of the algebraic closure of $\mathbb{Q}_p$ which are algebraic over the rationals $\mathbb{Q}$ and whose p-adic logarithms are linearly independent over $\mathbb{Q}$. These logarithms are then linearly independent over the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{Q}_p$.*

Choose generators u_1, u_2 for a finite index subgroup of $\mathcal{O}_K^\times$ with $u_1 \in \mathcal{O}_k^\times$.

If b is algebraic, the contrapositive of Brumer's theorem implies that there are integers $c_2', c_3', c_4' \in \mathbb{Z}$, not all zero, such that

$$c_2' \log \sigma_2(u_2) + c_3' \log \sigma_3(u_2) + c_4' \log \sigma_4(u_2) = 0 \,,$$

or equivalently that there are $c_2, c_3, c_4 \in \mathbb{Z}$, not all zero, such that

$$\sigma_2(u_2)^{c_2} \sigma_3(u_2)^{c_3} \sigma_4(u_2)^{c_4} = 1 \,.$$

Since k is a quadratic field, $\tau_1(k) = \tau_2(k) \subset L$ *as subsets* of L. Moreover, since K is a quadratic extension of k, we have $\sigma_1(K) = \sigma_2(K) \subset L$ and $\sigma_3(K) = \sigma_4(K) \subset L$ as subsets of L. On the other hand, $\sigma_2(K) \cap \sigma_3(K) = \tau_1(k)$ as subsets of L.

Then, if $\sigma_2(u_2)^{c_2} \sigma_3(u_2)^{c_3} \sigma_4(u_2)^{c_4} = 1$, we must have $\sigma_2(u_2)^{c_2} \in \tau_1(k) \subset L$. Hence, there are nonzero $c, d \in \mathbb{Z}$ such that $\sigma_2(u_2)^{c \cdot c_2} = \sigma_2(u_1)^d$. But this is only possible if $c_2 = 0$, since u_1 and u_2 generate a rank 2 group and are therefore multiplicatively independent.

Hence, we have

$$\sigma_3(u_2)^{c_3} \sigma_4(u_2)^{c_4} = 1 \,.$$

By the same argument applied to $u_1 u_2$ in place of u_2, there are non-zero $d_3, d_4 \in \mathbb{Z}$ such that

$$(5.12) \qquad\qquad \sigma_3(u_2 u_1)^{d_3} \sigma_4(u_2 u_1)^{d_4} = 1 \,.$$

So, we have

$$\sigma_3(u_2)^{c_3 d_3} \sigma_4(u_2)^{c_4 d_3} = 1 \,,$$

$$\sigma_3(u_2 u_1)^{c_3 d_3} \sigma_4(u_2 u_1)^{c_3 d_4} = 1 \,.$$

This implies that $\sigma_4(u_2)^{c_4 d_3 - c_3 d_4} = \sigma_3(u_1)^{c_3 d_3} \sigma_4(u_1)^{c_3 d_4} \in \tau_2(k)$. So, $c_4 d_3 = c_3 d_4$. Then,

$$(\sigma_3(u_1)^{d_3} \sigma_4(u_1)^{d_4})^{c_3} = 1 \,,$$

so $\sigma_3(u_1)^{d_3} \sigma_4(u_1)^{d_4} = \tau_2(u_1)^{d_3 + d_4}$ is a root of unity. But if $u_1^{d_3 + d_4}$ is a root of unity, $d_3 = -d_4$, since u_1 generates the free part of $\mathcal{O}_k$.

Then, 5.12 becomes $\sigma_3(u_2)^{d_3} = \sigma_4(u_2)^{d_3}$, which implies that $\sigma_3(u_2)^{d_3} \in \tau_2(k)$. Since d_3 and d_4 are non-zero and $u_2 \notin k$, this is a contradiction. This proves that b cannot be algebraic.

As a consequence, if Y is a positive-dimensional component of X, then Y is disjoint from $(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_K)$.

We conclude:

Proposition 5.13. *If K is a mixed quartic field with a totally real quadratic subfield, then for any p,*

$$(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_K) \le \prod_{\mathfrak{p} | p} \left(p^{f_\mathfrak{p}} - 2\right) p^{f_\mathfrak{p} \cdot \left\lfloor \frac{e_\mathfrak{p}}{p - 1} \right\rfloor} \,,$$

where the product runs over primes of K lying over p.

5.4. **Fields where 3 splits completely.**

Theorem 5.14. *Suppose that* 3 *splits completely in* K. *Then there is no pair* $x, y \in \mathcal{O}_K^\times$ *such that* $x + y = 1$. *Equivalently,*

$$(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_K) = \emptyset.$$

Remark 5.15. Before we begin the proof, we note that the corresponding result for 2 is trivial. In fact, if there is *any* prime $\mathfrak{p}$ above 2 such that $K_\mathfrak{p}$ is a totally ramified extension of $\mathbb{Z}_2$ and $\mathfrak{p} \notin S$ then there are no $x, y \in \mathcal{O}_{K,S}^\times$ such that $x + y = 1$. In that setup, the proof is very short — there is a local obstruction at $\mathfrak{p}$. Solutions to the unit equation cannot be congruent to 0 or 1 modulo $\mathfrak{p}$. On the other hand, if $K_\mathfrak{p}$ is totally ramified over $\mathbb{Z}_2$, then every element of $\mathcal{O}_{K,S}$ is either congruent to 0 or 1 modulo $\mathfrak{p}$. The corresponding result for 3 will not be as simple, since even if 3 splits completely, there is a residue class (consisting of elements of $\mathcal{O}_K$ which are congruent to 2 modulo $\mathfrak{p}$ for all $\mathfrak{p}$ above 3) which *in principle* could contain solutions to the unit equation.

As in (1) of Remark 5.8, the key to proving Theorem 5.14, is the observation that 'solutions to the unit equation are never alone.' More precisely, if (x, y) is a solution to the unit equation, then so are $(x^{-1}, -yx^{-1}), (y, x), (y^{-1}, -xy^{-1}), (-yx^{-1}, x^{-1})$, and $(-xy^{-1}, y^{-1})$. In contrast, a Newton polygon computation will show that if 3 unramified in K, then $(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_K)$ has at most one point in each residue disc. When 3 splits completely, there is only one residue disc which *could* contain solutions. Together, these imply that there is at most one solution to the unit equation in $\mathcal{O}_K$. Since 'solutions to the unit equation are never alone,' we conclude that in fact there are no solutions.

In the proof, the assumption that $S = \emptyset$ is used to get control of the Newton polygons of multivariate power series vanishing on certain subsets of the units. The assumption that 3 is unramified turns the control of the Newton polygons into the strict bound of 1 solution in a residue disc. It may be possible to weaken this assumption slightly (e.g. by imposing congruence conditions on the primes in S,) but it clearly cannot be removed completely, since there are solutions to the S-unit equation, even in $\mathbb{Q}$, for many sets of primes S.

Proof. We assume $3 = \prod_{j=1}^d \mathfrak{p}_j$ splits completely in K. Let $\sigma_j : K \to K_{\mathfrak{p}_j}$ be the standard embedding. There is an inclusion

$$\mathcal{O}_K \hookrightarrow \prod_{i=1}^d \mathcal{O}_{K_{\mathfrak{p}_i}} = \mathbb{Z}_3^d,$$

$$x \mapsto (\sigma_1(x), \dots, \sigma_d(x)).$$

We will identify elements of $\mathcal{O}_K$ with their images in $\mathbb{Z}_3^d$ without comment in the remainder of the proof.

Step 1: Reduce to a subgroup of the Jacobian.
We will show that for any $\alpha \in \mathcal{O}_K^\times$, the intersection $X \subset (\mathbb{Z}_3^\times)^d \times (\mathbb{Z}_3^\times)^d$ of the sets

$$\{(x_1, \dots, x_d, y_1, \dots, y_d) : x_j + y_j = 1\}$$

and

$$Y_\alpha := \overline{\{\alpha^n : n \in \mathbb{Z}\}} \times \overline{\mathcal{O}_K^\times}$$

is empty, where the x_i are our parameters on the first $(\mathbb{Z}_3^\times)^d$ and the y_i are the parameters on the second copy of $(\mathbb{Z}_3^\times)^d$. Since every solution (x, y) to the unit equation in $\mathcal{O}_K$ must live in some such set (e.g. taking $\alpha = x$), this will suffice to prove Theorem 5.14.

In fact, we consider a slightly larger variety. Let log denote the 3-adic logarithm. Reindexing if necessary, we may assume that that $v_3(\log \sigma_j(\alpha))$ is minimal for $j = d$. Then, there exist $a_1, \cdots a_{d-1} \in \mathbb{Z}_3$ such that Y is contained in the common vanishing locus Z_α of the equations

$$\log(x_j) + a_j \log(x_d) = 0, \text{ for } j = 1, \ldots, d-1,$$
$$\log(y_1) + \cdots + \log(y_d) = 0.$$

Moreover, $\sum_{j=1}^{d-1} a_j = 1$. If K is totally real, $Z_\alpha = Y_\alpha$. Otherwise, Y_α is an analytic subvariety of Z_α cut out by additional linear forms in the logarithms of the y_j. To show that there are no solutions to the unit equation in $\mathcal{O}_K$, it will suffice to check that Z_α does not contain any integral points.

Step 2: Check that Z_α has at most one zero-dimensional component.
Using the relations $x_j + y_j = 1$, we now forget about the x_j variables and focus on counting common solutions to

$$\log(1 - y_j) + a_j \log(1 - y_d) = 0, \text{ for } j = 1, \ldots, d-1,$$
$$\log(y_1) + \cdots + \log(y_d) = 0.$$

We use a Newton polygon argument.

Let $(z_1, \ldots, z_d)$ be any point in the residue disc $(2, \ldots, 2) + (3\mathbb{Z}_3)$. This is the only residue disc which could contain a solution to the unit equation. At $(z_1 + t_1, \ldots, z_d + t_d)$, we can express the p-adic analytic functions as power series in the t_j:

$$F_j := \log(1 - z_j) + a_j \log(1 - z_d) - \sum_{n=1}^\infty \left(\frac{t_j^n}{n(1 - z_j)^n} + \frac{a_j t_d^n}{n(1 - z_d)^n} \right), \text{ for } j = 1, \ldots, d-1,$$

$$F := \sum_{j=1}^d \left(\log(z_j) + \sum_{n=1}^\infty \frac{(-t_j)^n}{n z_j^n} \right),$$

By Corollary 3.7, Z has at most one zero-dimensional component in the disc $(2, \ldots, 2) + (3\mathbb{Z}_3)^d$ away from $\bigcup_{j=1}^d \{y_j = z_j\}$. This is true for any $(z_1, \ldots, z_d)$ in the disc. Hence, Z has at most one zero-dimensional component.

Step 3: Check that Z_α cannot have exactly one zero-dimensional component (and therefore has no zero-dimensional components.)
The map

$$((x_j), (y_j)) \mapsto ((x_j^{-1}), (-x_j^{-1} y_j))$$

(or $(y_j) \mapsto (-(1 - y_j)^{-1} y_j)$, having forgotten the x_j) induces a automorphism of Z (for the p-adic topology). In particular, the map takes zero-dimensional components to zero-dimensional components.

So if (y_j) is a zero-dimensional component of Z_α, we must have $y_j = \frac{-y_j}{1 - y_j}$. Since $y_j \neq 0$, this implies that $y_j = 2$ for all j, which is impossible since $d \log(2) \neq 0$.

The intersection contains at most one zero-dimensional component, and not exactly one, so it contains no zero-dimensional components.

We conclude that any solutions to the unit equation must lie in a positive-dimensional component of Z_α for some α.

Step 4: Check that the Z_α is zero-dimensional.
Note that log defines an isomorphism from $1 + 3\mathbb{Z}_3$ to $3\mathbb{Z}_3$, with inverse exp. Since $a_j \in \mathbb{Z}_3$ for all j, the equation

$$\log(1 - y_j) + a_j \log(1 - y_d) = 0\,,$$

can be used to realize y_j as a function of y_d on the dis $(2, \dots, 2) + (3\mathbb{Z}_3)^d$, namely:

$$y_j = 1 + \exp(-a_j \log(y_d - 1))\,.$$

When we plug this formula into

$$\log(y_1) + \cdots + \log(y_d)$$

we get an single-variable p-adic analytic function F which vanishes on the y_d-coordinates of points in Z_α. Since the other coordinates y_j are expressible as functions of y_d, if we can prove that F (representable as a single-variable power series) is not uniformly zero, we will be done.

Indeed, taking $y_d = 2$, we have $\log(y_d - 1) = \log(1) = 0$, so $\log(y_j - 1) = 0$, whence $y_j = 2$ for all j. But then, at $y_d = 2$, we have

$$\sum_{j=1}^d \log(y_j) = d \log 2 \neq 0\,.$$

Hence, Z_α is zero-dimensional, which completes the proof.

For good measure, we record an explicit power series for F (although we do not actually need to compute it for the proof.) Taking $y_d = 2 - t$ for $t \in 3\mathbb{Z}_3$,

$$\log(y_j) = -\sum_{m=1}^\infty \frac{1}{m} \left(3 + \sum_{\ell=1}^\infty \frac{1}{\ell!} \left(a_j \sum_{n=1}^\infty \frac{t^n}{n} \right)^\ell \right)^m\,, \text{ and } \log(y_d) = -\sum_{m=1}^\infty \frac{(3-t)^m}{m}\,.$$

Then $F(t) = \sum_{j=1}^d \log(y_j)$. $\square$

6. ACKNOWLEDGMENTS

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