Restriction of Scalars Chabauty and $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

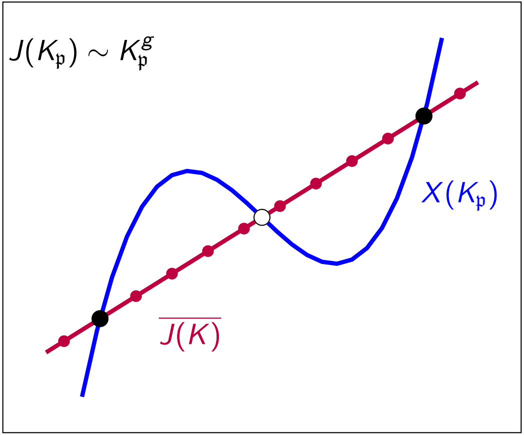
Nicholas Triantafillou

University of Georgia

20th June 2020
Chicago Number Theory Days

Chabauty-Coleman Method to Compute $X(K)$

- X/K = 'nice' curve, genus $g \geq 2$.
- $\mathfrak{p}$ = prime of good reduction.
- J = Jacobian of X.
- r = rank $J(K)$.



$$X(K) \subset \overline{J(K)} \cap X(K_\mathfrak{p})$$

From Chabauty ...

Observation (Chabauty 1941):

$$X(K) \subset \overline{J(K)} \cap X(K_{\mathfrak{p}}).$$

	$\overline{J(K)}$	$X(K_{\mathfrak{p}})$	$J(K_{\mathfrak{p}})$	$\overline{J(K)} \cap X(K_{\mathfrak{p}})$
Exp. Dim.	r	1	g	$r-(g-1)$

Theorem (Chabauty)

If $r \leq g-1$, R.H.S. is finite.

… to Restriction of Scalars Chabauty [RoS Chabauty]

Observation (Wetherell 2000):

$$X(K) = (\mathrm{Res}_{K/\mathbb{Q}} X)(\mathbb{Q})$$

... to Restriction of Scalars Chabauty [RoS Chabauty]

Observation (Wetherell 2000):

$$X(K) = (\mathrm{Res}_{K/\mathbb{Q}} X)(\mathbb{Q}) \subset \overline{(\mathrm{Res}_{K/\mathbb{Q}} J)(\mathbb{Q})} \cap (\mathrm{Res}_{K/\mathbb{Q}} X)(\mathbb{Q}_p).$$

... to Restriction of Scalars Chabauty [RoS Chabauty]

Observation (Wetherell 2000):

$$X(K) = (\mathrm{Res}_{K/\mathbb{Q}} X)(\mathbb{Q}) \subset \overline{(\mathrm{Res}_{K/\mathbb{Q}} J)(\mathbb{Q})} \cap (\mathrm{Res}_{K/\mathbb{Q}} X)(\mathbb{Q}_p).$$

	$\overline{\mathrm{Res}\, J(\mathbb{Q})}$	$\mathrm{Res}\, X(\mathbb{Q}_p)$	$\mathrm{Res}\, J(\mathbb{Q}_p)$	$\overline{\mathrm{Res}\, J(\mathbb{Q})} \cap \mathrm{Res}\, X(\mathbb{Q}_p)$
Exp. Dim.	r	d	dg	$r - d(g-1)$

Expectation/Hope:

If $r \leq d(g-1)$, then R.H.S. is finite.

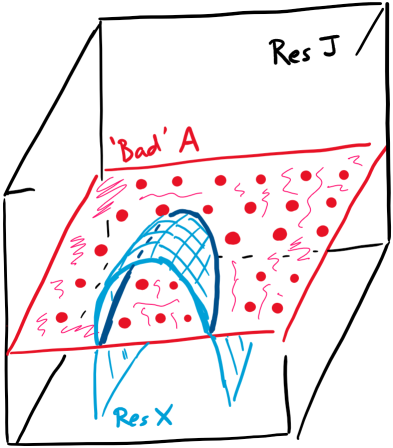
Obstruction: Unlikely Intersections

Hope: $\overline{(\mathrm{Res}_{K/\mathbb{Q}} J)(\mathbb{Q})} \cap (\mathrm{Res}_{K/\mathbb{Q}} X)(\mathbb{Q}_p)$ is finite.

Obstructions: Bad abelian subvarieties $A \subset \mathrm{Res}_{K/\mathbb{Q}} J$

'Bad' means that both:

1. $\dim(A \cap \mathrm{Res}\, X) > 0$,
2. $\mathrm{rank}\, A(\mathbb{Q})$ is large.



Large Intersections from Base Change

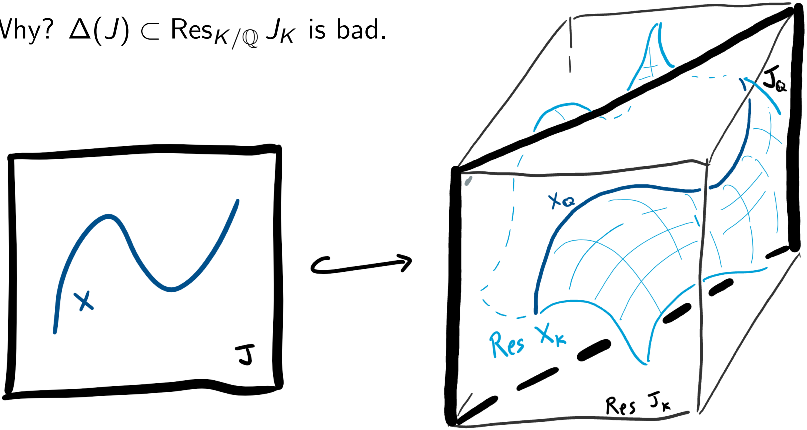
Say $X/\mathbb{Q}$. Try RoS Chabauty for X_K.

$\Delta : J \hookrightarrow \operatorname{Res}_{K/\mathbb{Q}} J_K$ diagonal embedding.

Lemma

If Chabauty fails for X then RoS Chabauty fails for X_K.

Why? $\Delta(J) \subset \operatorname{Res}_{K/\mathbb{Q}} J_K$ is bad.



Large Intersections from Prym varieties

$f : X \to Y$ over K.

$f_* : J_X \to J_Y$

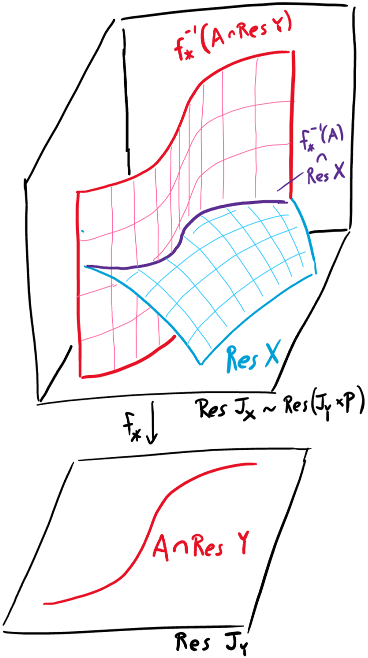
$P := \mathrm{Prym}(X/Y)$.

Lemma

Suppose the following hold:

- $A \subset \mathrm{Res}\, Y$ *is bad,*
- $\mathrm{rank}\, P(K)$ *is large,*
- *technical conditions.*

Then $f_*^{-1}(A) \subset \mathrm{Res}\, X$ *is bad.*



RoS Chabauty and the S-unit equation.

BCP obstruction: Bad subgroup $A \subset \mathrm{Res}_{K/\mathbb{Q}} J$ constructed iteratively as in previous slides.

BCP obstructions explain all known examples where RoS Chabauty fails.

RoS Chabauty and the S-unit equation.

BCP obstruction: Bad subgroup $A \subset \mathrm{Res}_{K/\mathbb{Q}} J$ constructed iteratively as in previous slides.

BCP obstructions explain all known examples where RoS Chabauty fails.

Theorem

- *K does not contain a CM subfield.*
- *S is a set of finite places of K.*
- *q a sufficiently large prime.*

There is no BCP obstruction to RoS Chabuaty for $\mathbb{P}^1_{\mathcal{O}_{K,S}} \smallsetminus \{x : x^q = \alpha\}$.

Corollary

Étale descent + RoS Chabauty 'should' give a p-adic algorithm to compute $(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_{K,S})$ = solutions to S-unit equation.

The unit equation has no solutions in number fields of degree prime to 3 where 3 splits completely.

Theorem

See title of preprint/slide.

Unit equation: $x, y \in \mathcal{O}_K^\times$ such that $x + y = 1$.