Restriction of scalars Chabauty applied to cyclic covers of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

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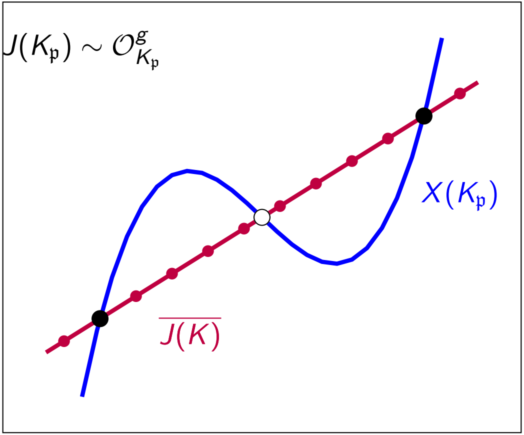
Explicit methods in number theory
Oberwolfach, Germany

Chabauty-Coleman Method to Compute $X(K)$

- X/K = 'nice' curve, genus $g \geq 2$.
- $\mathfrak{p}$ = prime of good reduction.
- J = Jacobian of X.
- r = rank $J(K)$.



$$X(K) \subset \overline{J(K)} \cap X(K_\mathfrak{p})$$

From Chabauty ...

Observation (Chabauty 1941):

$$X(K) \subset \overline{J(K)} \cap X(K_{\mathfrak{p}}).$$

	$\overline{J(K)}$	$X(K_{\mathfrak{p}})$	$J(K_{\mathfrak{p}})$	$\overline{J(K)} \cap X(K_{\mathfrak{p}})$
Exp. Dim.	r	1	g	$r-(g-1)$

Theorem (Chabauty)

If $r \leq g-1$, R.H.S. is finite.

Suffices to have rank less than dimension for any sub-group scheme

... to Restriction of Scalars Chabauty [RoS Chabauty]

Observation (Wetherell 2000, Siksek 2013):

$$X(K) = (\mathrm{Res}_{K/\mathbb{Q}} X)(\mathbb{Q})$$

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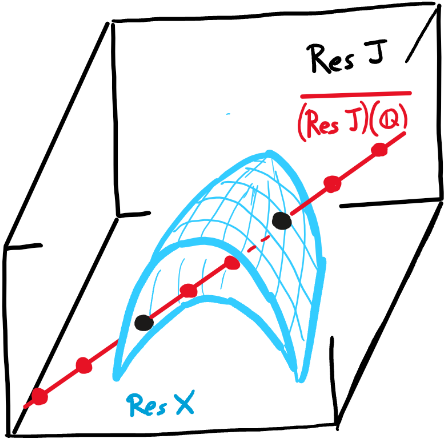

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Exp. Dim.	r	d	dg	$r - d(g-1)$

Expectation/Hope:

If $r \leq d(g-1)$, then R.H.S. is finite.

Subgroup obstructions

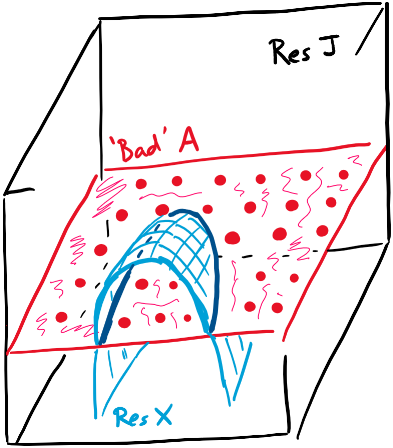
Hope: $\overline{(\mathrm{Res}_{K/\mathbb{Q}} J)(\mathbb{Q})} \cap (\mathrm{Res}_{K/\mathbb{Q}} X)(\mathbb{Q}_p)$ is finite.

Subgroup Obstructions: 'Bad' abelian subvarieties $A \subset \mathrm{Res}_{K/\mathbb{Q}} J$

'Bad' means roughly:

1. rank $A(\mathbb{Q})$ is large, and
2. $\dim(A \cap \mathrm{Res}\, X) > 0$.

More precisely:



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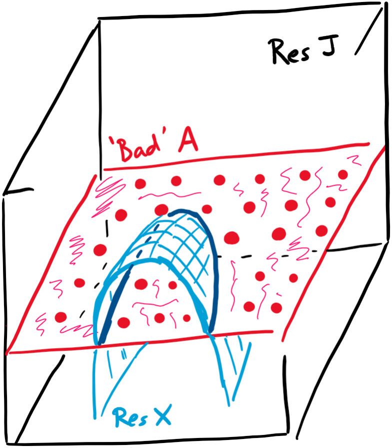
More precisely:

There exists $x \in \mathrm{Res}_{K/\mathbb{Q}} J$
such that for $T = x + A$,
$\dim(T \cap \mathrm{Res}\, X) + \mathrm{rank}\, A(\mathbb{Q}) > \dim A$.



What does (RoS) Chabauty say about $\mathbb{P}^1 \setminus \{0, 1, \infty\}$?

$X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$

$J = \mathbb{G}_m \times \mathbb{G}_m$ is generalized Jacobian of X.

	rank $J(\mathcal{O}_{K,S})$	$\dim J - \dim X$	$d(\dim J - \dim X)$
Exp. Dim.	$d + r_1(K) + 2\#S - 2$	1	d

Unless $r_1(K) + 2\#S \leq 2$, RoS Chabauty won't apply.

Pass to cyclic covers

$\alpha =$ coset representative for $\mathcal{O}_{K,S}^{\times}/\mathcal{O}_{K,S}^{\times q}$.
$q =$ large prime.
$X'_\alpha = \mathbb{P}^1 \setminus (\{0, \infty\} \cup \{z : \alpha z^q = 1\})$.

$$X'_\alpha(\mathcal{O}_{K,S}) \leftrightarrow x \in (\mathbb{P}^1 \setminus \{0, 1, \infty\})(\mathcal{O}_{K,S}) \text{ such that } \frac{x}{\alpha} \text{ is a } q\text{th power}$$

Strategy:
To compute $(\mathbb{P}^1 \setminus \{0, 1, \infty\})(\mathcal{O}_{K,S})$, compute all $X'_\alpha(\mathcal{O}_{K,S})$.

Classical Chabauty and cyclic covers of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$

$$X'_\alpha = \mathbb{P}^1 \setminus (\{0, \infty\} \cup \{z : z^q = \alpha^{-1}\}).$$

If $K = \mathbb{Q}$ (Poonen 2000s):

Can compute finite set containing $X'_\alpha(\mathcal{O}_{K,S})$ using classical Chabauty.

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If $K = \mathbb{Q}$ (Poonen 2000s):
Can compute finite set containing $X'_\alpha(\mathcal{O}_{K,S})$ using classical Chabauty.

Lemma

For any anisotropic torus $T/\mathcal{O}_K$,

$$r_2(K) \dim T \leq \operatorname{rank} T(\mathcal{O}_K) \leq (r_1(K) + r_2(K)) \dim T\,.$$

If $r_2(K) > 0$:

Apply lemma to subtori of Jacobian of X'_α.
Running Chabauty on X'_α is very unlikely to produce a finite set.

RoS Chabauty and cyclic covers of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$

Forget punctures at 0 and ∞.
$X_\alpha = \mathbb{P}^1 \setminus \{z : z^q = \alpha^{-1}\}$
$J_\alpha = (\mathrm{Res}_{\mathcal{O}_K[\sqrt[q]{\alpha^{-1}}]/\mathcal{O}_K} \mathbb{G}_m)/\mathbb{G}_m$ is generalized Jacobian of X_α

Lemma

If α is not a qth power in K, then for any subtorus $T \subset \mathrm{Res}_{\mathcal{O}_K/\mathbb{Z}} J_\alpha$

1. *$q - 1$ divides* $\dim T$,
2. $\operatorname{rank} T(\mathbb{Z}) = \frac{1}{2} \dim T$,
3. $\operatorname{rank} T\left(\mathbb{Z}\left[\frac{1}{S}\right]\right) = \frac{1}{2} \dim T + O\left(\frac{q}{\log q}\right)$.

Proof uses that $\mathrm{Res}_{\mathcal{O}_K/\mathbb{Z}} J_\alpha$ splits over nonabelian extension of K.

Theorem

For q sufficiently large, if α is not a qth power in K,
there are no subgroup obstructions to RoS Chabauty for $X_\alpha/\mathcal{O}_{K,S}$.

What if $\alpha = 1$?

Forget punctures at $0, 1, \infty$

$X_1 = \mathbb{P}^1 \setminus \{z : z^q = 1, z \neq 1\}$.

If K has a CM-subfield, $\exists$ subgroup obstruction to RoS Chabauty

$k =$ maximal totally real subfield.

X_1 descend to a curve over $\mathcal{O}_k$ with Jacobian J,

$$\operatorname{rank} J(\mathcal{O}_k) = [k : \mathbb{Q}] \dim J$$

BCP (Base Change + Prym) obstructions generalize this construction.

Theorem

If K does not contain a CM subfield, then for q sufficiently large, there are no BCP obstructions to RoS Chabauty for $X_1/\mathcal{O}_{K,S}$.

Summary

Theorem

For q sufficiently large, if α is not a qth power in K, there are no subgroup obstructions to RoS Chabauty for X_α.

Theorem

If K does not contain a CM subfield, then for q sufficiently large, there are no BCP obstructions to RoS Chabauty for X_1.

- 'Sufficiently large' depends on K, S, but not on α.
- 'K does not contain a CM subfield' in second theorem can probably be removed by taking cyclic covers again.

Corollary

Étale descent + RoS Chabauty 'should' give a p-adic algorithm to compute $(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_{K,S})$ = solutions to S-unit equation.

Extra work space

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How to compute ranks of tori

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Goal: Compute rank $T(\mathcal{O}_K)$ using representation theory.

Choose L/K Galois such that T splits over L.

$G = \mathrm{Gal}(L/K)$

$X(T)$ = character lattice of $T/\mathbb{C}$; let χ = associated G-rep.

G acts on $\mathcal{O}_L^\times$; let ψ = associated G-rep.

Then, rank $T(\mathcal{O}_K) = \langle \chi, \psi \rangle_G$.

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Then, rank $T(\mathcal{O}_K) = \langle \chi, \psi \rangle_G$.

Apply to $\mathbb{Z}$-points of subtorus $T \subset (\mathrm{Res}_{\mathcal{O}_K[\sqrt[q]{\alpha^{-1}}]/\mathbb{Z}} \mathbb{G}_m)/(\mathrm{Res}_{\mathcal{O}_K/\mathbb{Z}} \mathbb{G}_m)$.

Lemma

If α is not a qth power in K, then for any subtorus $T \subset \mathrm{Res}_{\mathcal{O}_K/\mathbb{Z}} J_\alpha$

2. rank $T(\mathbb{Z}) = \dfrac{1}{2} \dim T$,

Proof of Lemma

Lemma

If α is not a qth power in K, then for any subtorus $T \subset \mathrm{Res}_{\mathcal{O}_K/\mathbb{Z}} J_\alpha$

2. rank $T(\mathbb{Z}) = (1/2)\dim T$,

$L' =$ Galois closure of K

$L = L'[\sqrt[q]{\alpha^{-1}}, \zeta_q]$

$H =$ subgroup gen. by complex conjugation in $\mathrm{Gal}(L/\mathbb{Q})$.

$T \subset (\mathrm{Res}_{\mathcal{O}_K[\sqrt[q]{\alpha^{-1}}]/\mathbb{Z}} \mathbb{G}_m)/(\mathrm{Res}_{\mathcal{O}_K/\mathbb{Z}} \mathbb{G}_m)$ splits over L.

χ is anisotropic $\quad + \quad \psi \oplus \mathbf{1} \cong \mathrm{Ind}_H^G \mathbf{1} \quad + \quad$ Frob. reciprocity $\quad \Rightarrow$

$$\text{rank } T(\mathbb{Z}) = \langle \chi, \psi \rangle_G = \langle \chi, \psi \oplus \mathbf{1} \rangle_G = \langle \chi, \mathrm{Ind}_H^G \mathbf{1} \rangle_G = \langle \mathrm{Res}_H^G \chi, \mathbf{1} \rangle_H$$

Use $H \ltimes \mathbb{Z}/q\mathbb{Z} \subset \mathrm{Gal}(L/\mathbb{Q})$.

$\mathrm{Res}_{\mathbb{Z}/q\mathbb{Z}}^G \chi$ has no trivial subrep. $\quad + \quad$ rep. theory of $H \ltimes \mathbb{Z}/q\mathbb{Z} \quad \Rightarrow$

$$\mathrm{Res}_H^G \chi = (\mathbf{1} \oplus \chi_{\text{nontriv.}})^{\frac{1}{2} \dim T}$$

Defining BCP Obstructions

Large Intersections from Base Change

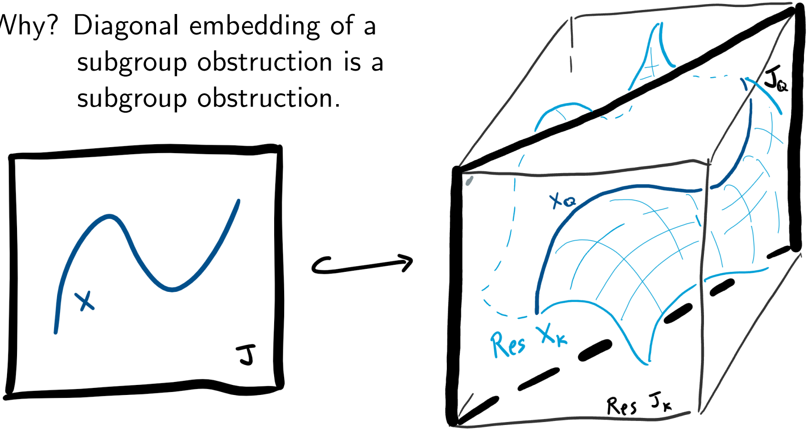
Say $X/\mathbb{Q}$. Try RoS Chabauty for X_K.

$\Delta : J \hookrightarrow \mathrm{Res}_{K/\mathbb{Q}} J_K$ diagonal embedding.

Lemma

If Chabauty fails for X then RoS Chabauty fails for X_K.

Why? Diagonal embedding of a subgroup obstruction is a subgroup obstruction.



Large Intersections from Prym varieties

$f : X \to Y$ over K.

$f_* : J_X \to J_Y$

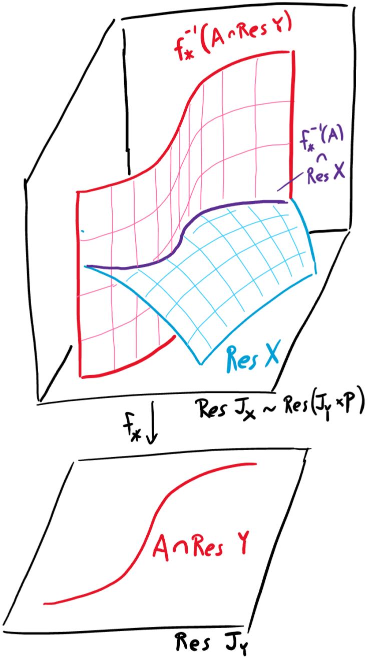
$P := \mathrm{Prym}(X/Y)$.

Lemma

Suppose the following hold:

- $A \subset \mathrm{Res}\, Y$ *is bad,*
- rank $P(K)$ *is large,*
- *technical conditions.*

Then $f_*^{-1}(A) \subset \mathrm{Res}\, X$ *is bad.*



BCP obstructions

BCP obstruction: Bad subgroup $A \subset \operatorname{Res}_{K/\mathbb{Q}} J$ constructed iteratively as in previous slides. Possibilities include:

- High rank Jacobian
- Diagonal embedding (from base change) of a BCP obstruction.
- Direct sum of a BCP obstruction and a high rank Prym variety.

BCP obstructions explain all known examples where RoS Chabauty fails.

Other results inspired by RoS Chabauty

The unit equation has no solutions in number fields of degree prime to 3 where 3 splits completely.

Theorem

See title of preprint/slide.

Unit equation: $x, y \in \mathcal{O}_K^\times$ such that $x + y = 1$.