RESTRICTION OF SCALARS CHABAUTY AND THE S-UNIT EQUATION

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ABSTRACT. Given a smooth, proper, geometrically integral curve X of genus g with Jacobian J over a number field K, Chabauty's method is a p-adic technique to bound $\#X(K)$ when $\operatorname{rank} J(K) < g$. In this paper, we study limitations of a variant of this approach which we call 'Restriction of Scalars Chabauty' (RoS Chabauty). RoS Chabauty typically bounds $\#X(K)$ when $\operatorname{rank} J(K) \leq [K : \mathbb{Q}](g-1)$, but can fail when a high-rank subgroup scheme of $\operatorname{Res}_{\mathbb{Q}}^K J$ intersects the image of $\operatorname{Res}_{\mathbb{Q}}^K X$ in higher-than-expected dimension. We define BCP-obstructions, which are subgroup schemes of $\operatorname{Res}_{\mathbb{Q}}^K J$ arising from the geometry of X. BCP-obstructions explain all known examples where RoS Chabauty fails to bound $\#X(K)$. We also extend RoS Chabauty to compute S-integral points on affine curves.

Suppose now that K does not contain a CM-subfield. As an application of this theory, we provide evidence that the combination of RoS Chabauty and descent gives an effective, elementary p-adic method to bound the number of solutions to the S-unit equation in K. More precisely, we produce a set $\{X_{\alpha,q}\}$ of genus 0 sub-covers of $\mathbb{P}^1_{\mathcal{O}_{K,S}} \smallsetminus \{0, 1, \infty\}$ over $\mathcal{O}_{K,S}$ such that (1) each solution to the S-unit equation comes from an $\mathcal{O}_{K,S}$-point on some $X_{\alpha,q}$ and (2) there is no BCP-obstruction to RoS Chabauty for any $X_{\alpha,q}$, which suggests that RoS Chabauty is likely to bound the number of $\mathcal{O}_{K,S}$-points. In contrast, under a generalized Leopoldt conjecture, we show that classical Chabauty and descent by genus 0 covers cannot bound the number of solutions to the S-unit equation when $[K : \mathbb{Q}] \geq 4$.

1. INTRODUCTION

1.1. **Background.** Let X be a smooth, proper, geometrically irreducible curve of genus ≥ 2 over a number field K.[1] By Faltings' Theorem, $X(K)$ is finite [Fal83]. Unfortunately, the known proofs do not give a strategy to compute $X(K)$. Let $\mathfrak{p}$ be a prime of K and let $j : X \to J$ be a finite map from X to its Jacobian. Under the additional assumption

$$\operatorname{rank} J(K) < \dim J = \operatorname{genus}(X)\,, \tag{1}$$

Chabauty's method is a $\mathfrak{p}$-adic technique which produces an explicit finite subset of $X(K_{\mathfrak{p}})$ which contains $X(K)$, namely the intersection of $X(K_{\mathfrak{p}})$ with the p-adic closure of $J(K)$ inside of $J(K_{\mathfrak{p}})$. When (1) holds, one can frequently compute $X(K)$ exactly

[1] We will often take X to be a suitable S-integral model of a smooth, geometrically irreducible affine curve instead of a smooth proper curve over a number field K of genus ≥ 2.

by combining Chabauty's method with other techniques like the Mordell-Weil sieve. Given a finite set S of primes of K, Chabauty's method can be modified to compute the S-integral points on a (suitable) model of an affine curve X under a similar rank versus dimension hypothesis for the generalized Jacobian of X. To extend Chabauty's method to this case, we develop technical tools to deal with integral points on locally of finite type Néron models of generalized Jacobians. We explain this in more detail in Sections 2.1 and 2.2.

Several strategies have been proposed to augment Chabauty's method to compute $X(K)$ when $\dim J \leq \operatorname{rank} J(K)$. In this article, we focus on Restriction of Scalars Chabauty (RoS Chabauty) and descent. We follow two main directions. Our first aim is to develop a framework for understanding obstructions that prevent RoS Chabauty from proving finiteness of rational/S-integral points on curves in particular examples. Our second aim is to apply RoS Chabauty and descent towards the study (by elementary p-adic methods) of S-integral points on $\mathbb{P}^1_{\mathcal{O}_{K,S}} \smallsetminus \{0, 1, \infty\}$, or equivalently the set

$$E_{K,S} := \{(x, y) \in (\mathcal{O}_{K,S}^\times)^2 : x + y = 1\}$$

of solutions to the S-unit equation. This approach has already inspired new results on unit equations in [Tri20], which proves that the unit equation (the S-unit equation with $S = \emptyset$) has no solutions in number fields of degree prime to 3 where 3 splits completely.

1.2. **Development of RoS Chabauty.** One strategy for extending Chabauty's method beyond the bound (1) is to replace X and J with the restrictions of scalars $\operatorname{Res}^K_{\mathbb{Q}} X$ and $\operatorname{Res}^K_{\mathbb{Q}} J$. This approach, which we call RoS Chabauty, first appeared in print in [Sik13], which attributes the idea to [Wet00]. An expected dimension calculation suggests that the intersection of $(\operatorname{Res}^K_{\mathbb{Q}} X)(\mathbb{Q}_p)$ and the p-adic closure of $(\operatorname{Res}^K_{\mathbb{Q}} J)(\mathbb{Q})$ in $(\operatorname{Res}^K_{\mathbb{Q}} J)(\mathbb{Q}_p)$ is finite when

$$\operatorname{rank} J(K) \leq [K : \mathbb{Q}] \cdot (\dim J - 1) . \tag{2}$$

If this intersection is finite, then $X(K)$ is finite as well. However, if a high-rank subgroup scheme of $\operatorname{Res}^K_{\mathbb{Q}} J$ has a translate which intersects $\operatorname{Res}^K_{\mathbb{Q}} X$ in positive dimension, RoS Chabauty typically fails to prove $X(K)$ is finite even when (2) holds. Recent works of Dogra and Hast [Dog19, Has19] make progress towards proving that this is the only reason why RoS Chabauty can fail to produce a finite set containing $X(K)$.[2] So, understanding RoS Chabauty comes down to understanding the arithmetic of subgroup schemes of $\operatorname{Res}^K_{\mathbb{Q}} J$ and the geometry of their intersection with $\operatorname{Res}^K_{\mathbb{Q}} X$.

[2] In fact, [Dog19] and [Has19] work in a much broader context combining ideas of RoS Chabauty with Kim's nonabelian Chabauty program.

1.2.1. ***Contributions of this article.*** Let S_0 be a finite set of finite places of $\mathbb{Q}$. Let S be the set of places of K lying above S_0. Let $R = \mathcal{O}_{K,S}$ be the ring of S-integers.

In Section 2, we extend RoS Chabauty to bound R-points on suitable[3] R-models of affine curves and develop a framework for understanding obstructions that can prevent RoS Chabauty from proving that the set of R-points on a given curve is finite.

To develop our framework for understanding obstructions to RoS Chabauty, we define the class of BCP subgroup[4] schemes of $\operatorname{Res}_{\mathbb{Q}}^{K} J$, where J is the generalized Jacobian of X. BCP subgroups are built out of restrictions of scalars of generalized Jacobians of curves which admit a non-constant map from X after a suitable base change and generalized Prym varieties of morphisms between such curves. By construction, nontrivial BCP subgroups always intersect $\operatorname{Res}_{\mathbb{Q}}^{K} X$ in larger-than-expected dimension. This makes them candidates to obstruct RoS Chabauty. Roughly speaking, we say that a BCP subgroup T is a BCP obstruction if $\operatorname{rank} T(\mathbb{Q})$ is large enough to prevent RoS Chabauty from proving that $X(K)$ is finite. We give the precise definitions in Section 3. Although we present these definitions in the setting where J is the generalized Jacobian of an R-model of an affine genus 0 curve, they generalize immediately to the higher genus and/or proper case.

The class of BCP obstructions is broad enough to include all known examples where RoS Chabauty does not prove $X(K)$ is finite. This includes examples where (2) does not hold, examples in Section 2 of [Sik13] where X is the *base change* of a curve for which (2) does not hold, and the more involved example in Section 2.2 of [Dog19]. BCP obstructions are also amenable to study. For any K not containing a CM-subfield, Theorem 3.6 can be used to produce infinitely many curves whose Jacobians have large rank over R, but which have no BCP obstruction to RoS Chabauty. In Section 1.3, we discuss an application of this family to computing solutions to the S-unit equation in K.

Our definition raises the following natural question.

Question 1.1. *Is there a curve X/K such that RoS Chabauty cannot prove $X(K)$ is finite, but there is no BCP obstruction to RoS Chabauty for X?*

An affirmative answer to Question 1.1 would give an arithmetically interesting example of an unlikely intersection between $\operatorname{Res}_{\mathbb{Q}}^{K} X$ and a subgroup scheme of $\operatorname{Res}_{\mathbb{Q}}^{K} J$. A negative answer would give a finite criterion for the success of RoS Chabauty involving only the arithmetic and geometry of X.

[3]Here, 'suitable' means 'equipped with a map to the connected component of the identity of the locally of finite type Néron model of the generalized Jacobian of the generic fiber.

[4]BC stands for base change and P stands for Prym variety

1.3. **Descent, Chabauty variants, and the S-unit equation.** Descent, named for Fermat's infinite descent, is another strategy which can be used to push Chabauty's method beyond the bound (1). Descent replaces the curve X/K with a collection of covers $f_i : X_i \to X$ defined over K with the property that $X(K) = \bigcup_i f_i(X_i(K))$. Since $\mathrm{genus}(X) \geq 2$, the covers X_i will have higher genus than X by Riemann-Hurwitz.[5] One hopes that (1) is satisfied for all of the X_i so that Chabauty's method can be used to compute each $X_i(K)$. Covering collections can be described explicitly. For instance, taking $[n]$ to be the multiplication-by-n map on J, a covering collection can be constructed as translates by representatives for $J(K)/nJ(K)$ of the pullback $[n]^*X$. One can also use descent to study S-integral points on affine curves.

When $K = \mathbb{Q}$, [Poo19] uses the versions of descent and Chabauty's method for S-integral points on genus 0 affine curves to give a new elementary p-adic proof that the set $E_{\mathbb{Q},S}$ of solutions to the S-unit equation is finite. The argument produces a collection of punctured genus 0 covers of $\mathbb{P}^1_{R_0} \smallsetminus \{0, 1, \infty\}$ such that the ranks of the S-integral points on their generalized Jacobians can be bounded explicitly.

1.3.1. ***Contributions of this article.*** Sections 4 and 5 study limitations to the strategy of [Poo19] for proving finiteness of solutions to the S-unit equation when $[K : \mathbb{Q}] \geq 4$ and provide evidence that RoS Chabauty may allow us to overcome these limitations.

Suppose the following generalization of Leopoldt's Conjecture holds.[6]

Conjecture 1.2 (Generalized Leopoldt over K)**.** *Let L be a finite extension of K and let $\mathfrak{p}$ be a prime of K which is large enough that the $\mathfrak{p}$-adic logarithm map*

$$\log : (\mathrm{Res}_{\mathcal{O}_L/\mathcal{O}_K} \mathbb{G}_m)(\mathcal{O}_{K_\mathfrak{p}}) \to K_\mathfrak{p}^{[L:K]}$$

is well-defined. Then,

$$\dim_{K_\mathfrak{p}} \mathrm{Span}_{K_\mathfrak{p}} \log(\mathrm{Res}_{\mathcal{O}_L/\mathcal{O}_K} \mathbb{G}_{m,\mathcal{O}_L})(\mathcal{O}_K) = \min(\mathrm{rank}\, \mathcal{O}_L^\times, [L : K])\,. \tag{3}$$

When $K_\mathfrak{p} = \mathbb{Q}_p$, Conjecture 1.2 says that the closure in the p-adic topology of the integral points on the generalized Jacobian J/R of a genus 0 curve in $J(K_\mathfrak{p})$ is as large as possible given the rank. More generally, Conjecture 1.2 implies that Chabauty's method produces a finite bound on the set of integral points on a genus 0 curve if and only if $\mathrm{rank}\, J(R) < \dim J$ holds.

The main result of Section 4 is Proposition 4.1, which implies that Chabauty's method for curves *cannot* be used to compute the set of S-integral points on an affine genus 0 curve when $[K : \mathbb{Q}] \geq 4$ under the generalized Leopoldt conjecture.

[5] In the analogous setup to compute integral points on affine curves, either the genus or number of punctures will grow (or both).

[6] The usual Leopoldt conjecture (that the p-adic regulator of L does not vanish) is the $K = \mathbb{Q}$ case of Conjecture 1.2.

In particular, the strategy of [Poo19] to compute solutions to the S-unit equation by computing S-integral points on genus 0 covers of $\mathbb{P}^1 \smallsetminus \{0,1,\infty\}$ will not succeed. The main ingredient in the proof of Proposition 4.1 is a lower bound on the ranks of generalized Jacobians of genus 0 curves.

Section 5 is devoted to proving our main result, Theorem 3.6.

Theorem 3.6. *Let q be a prime number and let K be a number field which does not contain a CM subfield. Let S_0 be a finite set of finite places of $\mathbb{Q}$ and let S be the set of places of K lying above S_0. Let $R = \mathcal{O}_{K,S}$ be the ring of S-integers. For $\alpha \in R^\times$, set*

$$\mathfrak{m}_{\alpha,q} := \begin{cases} \{x \in \overline{K} : x^q - 1 = 0, x \neq 1\} & \text{if } \alpha \in (R^\times)^q\,, \\ \{x \in \overline{K} : x^q - \alpha = 0\} & \text{if } \alpha \notin (R^\times)^q\,, \end{cases} \tag{16}$$

viewed as a divisor on $\mathbb{P}^1_K$. Let $\Gamma_{\alpha,q}$ be the closure of $\operatorname{supp}(\mathfrak{m}_{\alpha,q})$ in $\mathbb{P}^1_R$. Set $X_{\alpha,q} := \mathbb{P}^1_R \smallsetminus \Gamma_{\alpha,q}$.

For q sufficiently large (depending on K and S, but not on α), there are no BCP obstructions to RoS Chabauty for $X_{\alpha,q}$.

For each sufficiently large q, Theorem 3.6 constructs an explicit finite set of affine genus 0 curves $\{X_{\alpha,q}\}_\alpha$ without BCP obstructions to RoS Chabauty. The set $\{X_{\alpha,q}\}_\alpha$ becomes a covering collection $\{X'_{\alpha,q}\}_\alpha$ for $\mathbb{P}^1_R \smallsetminus \{0,1,\infty\}$ after removing 'sections' at 0, ∞, and sometimes 1. Removing these sections can only increase the number of R-points. Hence, Theorem 3.6 suggests that the following elementary p-adic procedure (generalizing the strategy in [Poo19]) is likely to compute the set of solutions to the S-unit equation in K.

(1) Choose a suitable q and compute the curves $X_{\alpha,q}$.
(2) Using RoS Chabauty and the Mordell-Weil sieve, compute each $X_{\alpha,q}(R)$.
(3) Compute $X'_{\alpha,q}(R)$ as a subset of $X_{\alpha,q}(R)$ by throwing out any sections which intersect the removed sections.
(4) Compute $\mathbb{P}^1 \smallsetminus \{0,1,\infty\}(R)$ as the union of the images of the $X'_{\alpha,q}(R)$.

The assumption in Theorem 3.6 that K does not contain a CM-subfield is essential. In Remark 3.8, we show that if K has a CM-subfield, the curve $X_{1,q}$ has a BCP obstruction for any choice of q. We expect that it is possible to overcome this obstacle to computing solutions to the S-unit equation via descent and RoS Chabauty by *iterating* the descent. In other words, we believe it is possible to construct a genus 0 covering collection for $X_{1,q}$ with no BCP obstructions even if K contains a CM-subfield. We do not pursue that line of study in this article.

The main ingredient in the proof of Theorem 3.6 is a bound on the ranks of BCP subtori coming from a careful analysis of the possible positions in the complex plane of the images of the qth roots of a fixed $\alpha \in R$ under an automorphism of $\mathbb{P}^1_K$. Controlling the number of these images which lie on the real axis is particularly important. The restriction that K does not contain a CM subfield arises because fields containing a CM-subfield are characterized by the existence of an automorphism of $\mathbb{P}^1_K$ which maps the complex unit circle to the real axis in every embedding $\iota : K \hookrightarrow \mathbb{C}$.

1.4. **Additional context on the S-unit equation.** The set $E_{K,S}$ of solutions to the S-unit equation is finite, due to [Sie21] in the case $K = \mathbb{Q}$ and [Lan60] for general K, although this case is often attributed to Mahler. The problem of computing/bounding solutions to the S-unit equation has remained of substantial interest because of a wide range of applications in number theory and related fields. In arithmetic geometry, computing the set $E_{K,S}$ is roughly equivalent to computing the set of elliptic curves with good reduction outside a fixed set of primes. In dynamics, studying $E_{K,S}$ is closely related to studying periodic points of odd order in arithmetic dynamical systems. S-unit equations are also important in many other fields. See [EGST88] and [EG15] for a more thorough discussion of these applications.

Until recently, most progress on S-unit equations has involved the theory of linear forms in logarithms. The best general upper bound on $\#E_{K,S}$ is $3 \cdot 7^{2\#S+3r_1+4r_2}$ where r_1 and r_2 are the number of real and complex embeddings respectively of K [Eve84]. The true upper bound is typically expected to be subexponential in $\#S, r_1$, and r_2. Better bounds exist under suitable restrictions on the set S [Győ19]. This strategy has also been developed into an effective algorithm which can be used to compute $E_{K,S}$ exactly in practice, at least when $[K : \mathbb{Q}]$ and $\#S$ are not too large [AKM+18].

Recently, several authors have given new proofs of finiteness of the S-unit equation as a proof-of-concept of strategies to prove an effective version of Faltings' theorem. For instance, the first proof of finiteness by Kim's nonabelian Chabauty showed that $E_{K,S}$ is finite when $K = \mathbb{Q}$ [Kim05]. The method has since been used to prove that $X(\mathbb{Q})$ is finite whenever X of genus ≥ 2 is a solvable cover of $\mathbb{P}^1$ [EH17]. Effective versions of the method have been used to compute the set of rational points on the 'cursed' split Cartan modular curve of level 13 [BDS+17], several other modular curves [BBB+19], and the set $E_{K,S}$ for some $[K : \mathbb{Q}]$ and $\#S$ small [DCW15], among other examples. Similarly, before extending their strategy to give a new proof of Faltings' Theorem, Lawrence and Venkatesh first gave a new proof of the finiteness of $E_{K,S}$ by similar methods [LV18].

Following these lines, we hope this work will eventually lead to a new effective proof of Faltings' Theorem suitable to practical computation.

1.5. **Notation and Conventions.** Throughout, we use the following notation and conventions:

- $\mathbb{Q} \subset K' \subset K$ are number fields.
- $d = [K : \mathbb{Q}]$ and $d' = [K' : \mathbb{Q}]$.

- S_0 is a finite set of finite places of $\mathbb{Q}$.
- S is the set of places of K lying above S_0 and S' is the set of places of K' lying above S_0.
- Σ_∞ is the set of infinite places of K.
- If $\mathfrak{p}$ is a finite place of X then $G_\mathfrak{p}$ is the decomposition group at $\mathfrak{p}$. If $\mathfrak{p} \in \Sigma_\infty$, then $G_\mathfrak{p}$ is either the trivial group if $\mathfrak{p}$ is a complex place and is $\mathbb{Z}/2\mathbb{Z}$ (with non-trivial element corresponding to complex conjugation) if $\mathfrak{p}$ is a real place.
- $R = \mathcal{O}_{K,S}$ and $R' = \mathcal{O}_{K',S'}$ and $R_0 = \mathcal{O}_{\mathbb{Q},S_0}$ are the rings of S, S', and S_0-integers, respectively. Note that conventions around S-integers are not entirely standardized. While we assume S consists only of finite places, other articles may require that S contains all infinite places implicitly in this notation.
- Given a prime $\mathfrak{p}$ of R, the completion of K at $\mathfrak{p}$ is $K_\mathfrak{p}$, the ring of integers in $K_\mathfrak{p}$ is $R_\mathfrak{p}$, and the residue field is $k_\mathfrak{p}$.
- For F a number field, $r_1(F)$ is the number of real embeddings of F and $r_2(F)$ is the number of complex embeddings of F.
- Given a divisor $\mathfrak{m}$ on a curve X/K, we write $\operatorname{supp}(\mathfrak{m})$ for the support of $\mathfrak{m}$.
- The dimension of a flat scheme over R or R_0 refers to the *relative dimension* of the scheme, or equivalently the dimension of the generic fiber.
- If P is a $\overline{\mathbb{Q}}$-point of a curve defined over a number field K, then $K(P)$ is the minimal field of definition of P.
- A CM field is a totally complex number field which is a quadratic extension of a totally real number field.
- RoS Chabauty is shorthand for Chabauty's method applied to a restriction of scalars of a curve X equipped with a map to the corresponding restriction of scalars of the generalized Jacobian J of X.

We also use the following conventions:

Given a set of algebraic S-integers $\{x_1, \ldots, x_n\}$ which is stable under the action of $\operatorname{Gal}(\overline{K}/K)$, let $f \in R[x]$ be the monic square-free polynomial with roots $x_1, \ldots, x_r$. Define

$$\mathbb{P}^1_R \smallsetminus (\{x_1, \ldots, x_r\} \cup \{\infty\}) := \operatorname{Spec} R[x, f(x)^{-1}]\,.$$

If the x_i are S-units, we define $\mathbb{P}^1_R \smallsetminus \{x_1, \ldots, x_r\}$ by gluing $\mathbb{P}^1_R \smallsetminus \{x_1, \ldots, x_r, \infty\}$ and $\mathbb{P}^1_R \smallsetminus \{x_1^{-1}, \ldots, x_r^{-1}, \infty\}$ by identifying the subsets $\mathbb{P}^1_R \smallsetminus \{0, x_1, \ldots, x_r, \infty\}$ and $\mathbb{P}^1_R \smallsetminus \{0, x_1^{-1}, \ldots, x_r^{-1}, \infty\}$ via the morphism $t \mapsto t^{-1}$.

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2. Background on RoS Chabauty

2.1. **Sound thru-hole curves and their generalized Jacobians.**

In order to extend RoS Chabauty to affine curves, we define the class of *sound thru-hole curves* X over R together with a notion of the generalized Jacobian J/R of such curves. Our definitions are chosen so that

(1) J is a commutative group scheme of finite type over R, so that $J(R)$ is a finitely-generated abelian group;
(2) The generic fiber J_K of J is the generalized Jacobian of the generic fiber X_K of X;
(3) An Abel-Jacobi map from the generic fiber X_K to its generalized Jacobian J_K extends to a morphism from X to J whenever it is defined with respect to a divisor of X_K which extends to a divisor of X.

Definition 2.1. *Let X be a flat relative curve over R with smooth generic fiber X_K. Let $\overline{X_K}$ over K be the completion of X_K. Suppose that $\overline{X_K}$ is geometrically integral. The complement of X_K in $\overline{X_K}$ is a finite union of closed points $P_1, \ldots, P_c$. Let $\mathfrak{m}$ be the divisor $P_1 + \cdots + P_c$. If $\overline{X}$ over R is any proper regular model of $\overline{X_K}$, let Γ be the closure of $\{P_1, \ldots, P_c\}$ in $\overline{X}$. We say that X is a* thru-hole *model for X_K if $X = \overline{X} \smallsetminus \Gamma$ for some proper regular model $\overline{X}$ of $\overline{X_K}$.*

If such an X is smooth, separated, and finite-type over R and the fibers of X are geometrically connected, we say that X is a sound *thru-hole model for X_K.*

We call a relative curve X over R a (sound) thru-hole curve *if X is a (sound) thru-hole model for its generic fiber X_K.*

Remark 2.2. A thru-hole is a hole through an object which is large enough to allow the threads of a screw or bolt to pass through but not the head of the screw or bolt. In other words, it is a hole which passes all the way through an object and is exactly as large as it needs to be, but no larger. Carrying on with this analogy, the word *sound* (as in 'structurally sound') is intended to convey that despite the puncture, the fibers remain connected.

Remark 2.3. After possibly enlarging S by a finite set of primes, every curve X_K over K has a sound thru-hole model. When applying Chabauty's method, we generally prefer to take the smallest enlargement of S so that our model becomes a sound thru-hole model.

A sound thru-hole curve X/R is well-adapted to Chabauty's method because it is equipped with an R-embedding into its *generalized Jacobian*, a commutative group scheme J_X of finite type over R with the property that $J_X(R)$ is a finitely-generated abelian group. See Chapter 5 of [Ser88] for the theory of generalized Jacobians of curves over number fields. We now describe what we mean by the generalized Jacobian of a sound thru-hole curve.

If X is a sound thru-hole curve, its generic fiber X_K has the form $\overline{X_K} \setminus \mathrm{supp}(\mathfrak{m})$ for some reduced divisor $\mathfrak{m}$. The generalized Jacobian J_{X_K} of $\overline{X_K}$ with reduced modulus $\mathfrak{m}$ is a semi-abelian group scheme (an extension of an abelian variety by a torus) over K. Let $\mathcal{J}_X$ be the (locally of finite type) Néron model for J_{X_K} and let J_X be the connected component of the identity of $\mathcal{J}_X$. (See [BLR90] § 10.2 Theorem 2.) We abuse notation slightly and call J_X the *generalized Jacobian* of X.

Lemma 2.4. *For X/R a sound thru-hole curve, $J_X(R)$ is a finitely-generated abelian group.*

Proof. The semi-abelian variety J_{X_K} is an extension of an abelian variety A by a torus T. The group structure on $J_{X_K}(K)$ is abelian, so the same is true for $J_X(R)$.

(Part 1.) We first consider the case where $T \cong \mathbb{G}_{m,K}^t$ is a split torus. Let $\mathcal{A}$ be the Néron model of A. From the proof of [BLR90] §10.1 Prop. 7, J_X is an extension of $\mathcal{A}^\circ$ by $\mathbb{G}_{m,R}^t$. So there is an exact sequence

$$0 \to \mathbb{G}_{m,R}^t(R) \to J_X(R) \to \mathcal{A}^\circ(R) \to 0\,.$$

Now, $\mathcal{A}^\circ(R) \subset A(K)$ is finitely-generated by the Mordell-Weil theorem and $\mathbb{G}_{m,R}^t(R) = (R^\times)^t$ is finitely-generated by Dirichlet's unit theorem, so $J_X(R)$ is finitely-generated in this case.

(Part 2.) Now suppose T is a non-split torus. Let L be a finite extension of K where T splits. Let S_L be a set of places of L lying above S together with any primes which ramify in L/K. Let R_L be the set of S_L-integers in L. Taking l.f.t Néron models commutes with étale base change ([BLR90] §10.1 Prop. 3) so $(\mathcal{J}_X)_L = \mathcal{J}_{X_L}$. The identity component of each fiber of $\mathcal{J}_X$ is smooth, connected, and contains a rational point (the identity element), so each fiber of $\mathcal{J}_X$ is geometrically connected. In particular, $J_{X_{R_L}} = (J_X)_{R_L}$. This allows us to identify $J_X(R)$ as a subgroup of $J_{X_{R_L}}(R_L)$, which is a finitely-generated abelian group by Part 1. Hence, $J_X(R)$ is a finitely-generated abelian group. □

Suppose we are given a point $P \in X(R)$. Then X_K is equipped with an Abel-Jacobi K-morphism $j_K : X_K \hookrightarrow J_{X_K}$ which sends P_K to the identity element of J_{X_K}.[7] By the Néron mapping property, j_K extends to a map $j : X \hookrightarrow \mathcal{J}_X$ which sends P to the identity section of the group scheme J_X. Since the fibers of X are geometrically connected and their image includes the identity, we see that the image of X under j lies in J_X.

In the absence of an integral point $P \in X(R)$, we can define a finite R-morphism $j : X \to J_X$ with respect to any horizontal divisor of X. By a mild abuse of the language we will call any such map an Abel-Jacobi map.

[7]If X_K is proper, this is standard. If X_K is affine, [Ser88] describes a K-morphism $X_K \hookrightarrow J_{X_K}$. One can ensure that P_K maps to the identity element by translating by -1 times the image of P_K.

2.1.1. *Generalized Jacobians of genus* 0 *curves.* For the remainder of this article, we will primarily be concerned with the case where X is a smooth thru-hole curve and $\overline{X_K}$ has genus zero. In preparation, we recall some facts about the structure of the generalized Jacobian of such a curve.

Suppose X/R is a smooth thru-hole curve and $\overline{X_K}$ has genus zero. Define $\mathfrak{m}$ and $P_1, \ldots, P_s$ as in Definition 2.1. For $i \in \{1, \ldots, c\}$, let L_i be the residue field of P_i, let S_i be the set of places of L_i which lie over S, and let R_i be the ring of S_i-integers in L_i. Then, the generalized Jacobian of X_K with modulus $\mathfrak{m}$ is

$$J_{X_K} \cong \left(\prod_{i=1}^{c} \operatorname{Res}_K^{L_i} \mathbb{G}_{m,L_i} \right) / \Delta(\mathbb{G}_{m,K}) \,, \tag{4}$$

where the Δ denotes the diagonal embedding of $\mathbb{G}_{m,K}$ into the product of the Weil restrictions. The generalized Jacobian of X is the connected component of the lft-Néron model of J_{X_K}. Using Proposition 4.2 of [LL01] to compute the quotient, this is

$$J_X \cong \left(\prod_{i=1}^{c} \operatorname{Res}_R^{R_i} \mathbb{G}_{m,R_i} \right) / \Delta(\mathbb{G}_{m,R}) \,, \tag{5}$$

where $\Delta(\mathbb{G}_{m,R})$ indicates a diagonally embedded copy of $\mathbb{G}_{m,R}$. If $L_i = K$ for some i, the formula can be simplified by leaving out the ith component and the quotient.

The expression (5) makes it easy to compute the dimension of J_X and rank of $J_X(R)$. Since $R_i = \mathcal{O}_{L_i,S_i}$, we have

$$\dim J_X = \deg(f) - 1 = \sum_{i=1}^{c} \deg(f_i) - 1 = \sum_{i=1}^{c} [L_i : K] - [K : K]$$

and

$$\begin{aligned} \operatorname{rank} J_X(R) &= \sum_{i=1}^{c} \operatorname{rank} \mathcal{O}_{L_i,S_i}^{\times} - \operatorname{rank} R^{\times} \\ &= \sum_{i=1}^{c} [r_1(L_i) + r_2(L_i) + \#S_i - 1] - [r_1(K) + r_2(K) + \#S - 1] \,. \end{aligned} \tag{6}$$

We can also express the rank in terms of the action of the absolute Galois group of K on the set of punctures of our genus zero curve. We state the general result.

Lemma 2.5. *Suppose that X/R is a smooth thru-hole model of a genus 0 curve over K. Define $\mathfrak{m}$ and Γ as in Definition 2.1. Then, $\Gamma(\overline{K}) = \operatorname{supp}(\mathfrak{m})$, viewed as a set of $\overline{K}$-points on the generic fiber X_K. Let $G = Gal(\overline{K}/K)$ be the absolute Galois group of K. Let Σ_∞ be the set of infinite places of K. For each finite place $\mathfrak{p}$, let $G_\mathfrak{p}$ denote the decomposition group of $\mathfrak{p}$. For $\mathfrak{p} \in \Sigma_\infty$, let $G_\mathfrak{p}$ be the trivial group if $\mathfrak{p}$ is a complex place and $\mathbb{Z}/2\mathbb{Z}$ (corresponding to complex conjugation) if K is a real place.*

Then, the generic fiber J_{X_K} of J_X is a torus,

$$\dim J_X = \#\Gamma(\overline{K}) - 1,$$

and $J_X(R)$ is an abelian group of rank

$$\operatorname{rank} J_X(R) = \sum_{\mathfrak{p}\in S\cup\Sigma_\infty} \left[\#(G_{\mathfrak{p}}\backslash\Gamma(\overline{K})) - 1\right] - \left[\#(G\backslash\Gamma(\overline{K})) - 1\right].$$

We omit the proof, which is a fairly straightforward computation in the character theory of tori. The rank computation can be found, for example, Theorem 8.7.2 of *Cohomology of Number Fields* by Neukirch, Schmidt, and Wingberg [NSW08], or in Chapter 6 of Eisenträger's Ph.D. Thesis [Eis03].

When X is a subscheme of $\mathbb{A}^1_R$, it is possible to define an Abel-Jacobi map $j : X \to J$ without reference to an integral point on X. We give two examples since we will not need this fact going forward and the general case is notationally burdensome.

Example 2.6. If $X = \mathbb{P}^1_R \smallsetminus \{0, 1, \infty\} = \mathbb{A}^1_R \smallsetminus \{0, 1\}$, then $J_{X_K} \cong \mathbb{G}_{m,K} \times \mathbb{G}_{m,K}$. Then, $J_X \cong \mathbb{G}_{m,R} \times \mathbb{G}_{m,R}$ and the Abel-Jacobi map is

$$\begin{aligned} j : X &\to J_X \\ x &\mapsto (x, x-1), \end{aligned}$$

where we view $X \subset \mathbb{A}^1_R$ and use the simplified form for the generalized Jacobian.

Example 2.7. Suppose $\sqrt{2} \notin K$, set $L = K(\sqrt{2})$, and let R' be the integral closure of R in L. Suppose also that $\{1, \sqrt{2}\}$ is an integral basis for R' over R.

If $X = \mathbb{P}^1_R \smallsetminus \{\pm\sqrt{2}, \infty\} = \mathbb{A}^1_R \smallsetminus \{\pm\sqrt{2}\}$, then $J_{X_K} \cong \operatorname{Res}^L_K \mathbb{G}_{m,L}$ and

$$\begin{aligned} J_X &\cong \operatorname{Res}^{R'}_R \mathbb{G}_{m,R'} \\ &= \operatorname{Spec} R[x_1, x_2, y_1, y_2]/(x_1y_1 + 2x_2y_2 - 1, x_1y_2 + x_2y_1). \end{aligned}$$

Then, we have the Abel-Jacobi map

$$\begin{aligned} j : X &\to J_X \\ x &\mapsto \left(x, -1, \frac{x}{x^2-2}, \frac{1}{x^2-2}\right) \text{“} = x - \sqrt{2}\,.\text{”} \end{aligned}$$

2.2. **Chabauty's method for curves.** Before discussing the higher-dimensional case, we first recall Chabuaty's p-adic method for computing the set of R-points on a sound thru-hole curve X with generalized Jacobian J. In modern language, Chabauty and Skolem proved that if $\operatorname{rank} J(R) \leq \dim J - 1$, then $X(R)$ is finite. We now summarize their argument.

Suppose there is some $P_0 \in X(R)$. (Otherwise, $X(R)$ is empty and therefore finite.) Let $j : X \to J$ be the Abel-Jacobi map with respect to P_0. Set $g = \dim J$. Fix a prime $\mathfrak{p}$ of R of good reduction for X. Say $\mathfrak{p}$ lies over the rational prime p. Then, $J(K_\mathfrak{p})$ in a p-adic Lie group, so it is equipped with a *logarithm* map to its Lie algebra, which is isomorphic to $K_\mathfrak{p}^g$. Concretely, if we fix a basis $(\omega_1, \dots, \omega_g)$ for $H^0(J_K, \Omega^1)$, the map is given in coordinates by the abelian integrals

$$\begin{aligned} \log : J(K_\mathfrak{p}) &\to K_\mathfrak{p}^g\,, \\ Q &\mapsto \left(\int_O^Q \omega_i\right)_i\,. \end{aligned}$$

The kernel of log is isomorphic to $J(k_\mathfrak{p})$. In particular, the fibers are finite. There is a commutative diagram

$$\begin{tikzcd} X(R) \arrow[r, hook] \arrow[d, "j"] & X(K_\mathfrak{p}) \arrow[d, "j"] \arrow[dr] & \\ J(R) \arrow[r, hook] & J(K_\mathfrak{p}) \arrow[r, "\log"] & K_\mathfrak{p}^g\,. \end{tikzcd} \tag{7}$$

The image of $J(K)$ in $K_\mathfrak{p}^g$ is contained in a $K_\mathfrak{p}$-linear subspace of dimension at most $\operatorname{rank} J(K)$. On the other hand, Chabauty proved that $j(X(K_\mathfrak{p}))$ is Zariski-dense in $K_\mathfrak{p}^g$. In particular, if the classical Chabauty inequality

$$\operatorname{rank} J(R) \leq \dim J - 1 \tag{8}$$

is satisfied, then $\log X(K_\mathfrak{p})$ is not contained in $K_\mathfrak{p} \otimes \log J(R)$. Since $\log X(K_\mathfrak{p})$ is a $\mathfrak{p}$-adic analytic curve, the intersection $K_\mathfrak{p} \otimes \log J(K) \cap \log X(K_\mathfrak{p})$ in $K_\mathfrak{p}^g$ is finite. Since $X(R) \subset J(K) \cap X(K_\mathfrak{p}) \subset J(K_\mathfrak{p})$ and log has finite kernel, $X(R)$ is finite as well.

Coleman [Col85] showed how to use this strategy to show that $\#X(K)$ is contained in an explicitly-computable subset of $X(K_\mathfrak{p})$ of size at most $X(k_\mathfrak{p}) + 2g - 2$. See the excellent survey paper of McCallum-Poonen [MP12] for more detail.

Remark 2.8. When $K \neq \mathbb{Q}$, the rank bound in Chabauty's method is inefficient in two ways.

(1) (If $[K_\mathfrak{p} : \mathbb{Q}_p] > 1$.) While $K_\mathfrak{p} \otimes \log J(K)$ is the smallest $K_\mathfrak{p}$-subspace of $K_\mathfrak{p}$ containing $J(K)$, it is typically larger than $\overline{\log J(K)} = \mathbb{Q}_p \otimes \log J(K)$, the closure of $\log J(K)$ in the p-adic topology on $K_\mathfrak{p}^g$.
(2) (If $[K_\mathfrak{p} : \mathbb{Q}_p] < [K : \mathbb{Q}]$.) Using only the $K_\mathfrak{p}$-points for Chabauty's method ignores all information coming from other primes above p. Combining Chabauty's method at all primes above p has the potential to compute $X(K)$ when $J(K)$ has larger rank.

The solution to both inefficiencies is to apply an analogue of Chabauty's method to $\operatorname{Res}_{K/\mathbb{Q}} X$ instead. Calculating expected dimensions suggests that Chabauty's method applies to $\operatorname{Res}_{K/\mathbb{Q}} X$ would suffice to compute $X(K)$ whenever $\operatorname{rank} J(K) \leq [K : \mathbb{Q}] \cdot (g - 1)$. Although we expect this to hold in generic examples, it is not sufficient in several particular examples.

2.3. **Chabauty's method for restrictions of scalars.** Chabauty's method extends to a strategy which can be used to compute integral points on higher-dimensional schemes in some cases. Suppose that Z is a scheme over R_0 equipped with a morphism j to its Albanese A (or any commutative group scheme over R_0 such that $\log(j(Z(\mathbb{Q}_p))$ is Zariski dense in $\log(Z(\mathbb{Q}_p))$.) By replacing X with Z and J with A in (7), we get the diagram

$$\begin{tikzcd} Z(R_0) \arrow[r, hook] \arrow[d, "j"] & Z(\mathbb{Q}_p) \arrow[d, "j"] \arrow[dr] & \\ A(R_0) \arrow[r, hook] & A(\mathbb{Q}_p) \arrow[r, "\log"] & \mathbb{Q}_p^{\dim A}\,. \end{tikzcd} \tag{9}$$

If the intersection of $\mathbb{Q}_p \otimes \log A(R_0)$ and $\log \circ j(X(\mathbb{Q}_p))$ is finite, then $Z(R_0)$ is contained in an explicitly computable finite set of fibers of j. When $\operatorname{rank} A(R_0) \leq \dim A_{\mathbb{Q}} - \dim Z_{\mathbb{Q}}$, the expected dimension of $\mathbb{Q}_p \otimes \log A(R_0) \cap \log j(Z)$ is zero, so we 'expect' that this intersection is finite.

Remark 2.9. Since we are now working over subrings of $\mathbb{Q}_p$, we can simplify notation somewhat by working directly in the Jacobian rather than in its Lie algebra. Let $\overline{A(R_0)}$ denote the saturation of the closure of $A(R_0)$ in $A(\mathbb{Q}_p)$ in the p-adic topology. Equivalently, $\overline{A(R_0)}$ is the preimage of $\mathbb{Q}_p \otimes \log A(R_0)$ in $A(\mathbb{Q}_p)$. Then, $\mathbb{Q}_p \otimes \log A(R_0) \cap \log j(Z)$ is finite if and only if $\overline{A(R_0)} \cap j(Z)$ is finite. For compactness of notation, we switch to this perspective going forward.

We can use this strategy to compute $X(R)$ by taking $Z = \operatorname{Res}_{R_0}^R X$ and $A = \operatorname{Res}_{R/R_0} J$. We have

$$\begin{tikzcd} X(R) \arrow[r, equal] \arrow[d, "j"] & (\operatorname{Res}_{R_0}^R X)(R_0) \arrow[r, hook] \arrow[d, "j"] & (\operatorname{Res}_{R_0}^R X)(\mathbb{Q}_p) \arrow[d, "j"] \\ J(R) \arrow[r, equal] & (\operatorname{Res}_{R_0}^R J)(R_0) \arrow[r, hook] & (\operatorname{Res}_{R_0}^R J)(\mathbb{Q}_p)\,. \end{tikzcd} \tag{10}$$

In this case, we 'expect' the intersection $I := \overline{(\operatorname{Res}_{R_0}^R J)(R_0)} \cap j(\operatorname{Res}_{R_0}^R X)(\mathbb{Q}_p)$ will be finite when the RoS Chabauty inequality

$$\operatorname{rank} J(R) \leq d(\dim J - 1) \tag{11}$$

holds. We call this strategy *Chabauty's method for restrictions of scalars* or *RoS Chabauty* for short. We say that RoS Chabauty produces a finite set if I is finite.

[Dog19] shows that $X(R)$ is not Zariski-dense in $(\operatorname{Res}_{R_0}^R X)(\mathbb{Q}_p)$ when (11) holds. Unfortunately, this is no longer enough to conclude that RoS Chabauty produces a finite set when $\dim Z = d > 1$.

In fact, I can be infinite even when (11) is satisfied! Let $T \subset A$ be a subgroup scheme, $x \in A(R_0)$ be an integral point, and let $B = x \cdot T$ be the translate of T by x. We have

$$\overline{T(R_0)} \cap (B \cap j(Z))(\mathbb{Q}_p) \subseteq \overline{A(R_0)} \cap j(Z)(\mathbb{Q}_p)\,. \tag{12}$$

If also

$$\dim \overline{T(R_0)} \geq \dim B - \dim(B \cap j(Z)) \tag{13}$$

then the left-hand side of (12) will be infinite, so the right-hand side will as well. In particular, Chabauty's method for Z fails to produce a finite set.

The point is that even when rank $J(R)$ is small, RoS Chabauty may not produce a finite set of p-adic points if the rank is concentrated in a subgroup scheme of $\operatorname{Res}^R_{R_0} J$ which intersects $\operatorname{Res}^R_{R_0} X$ with larger than expected dimension.

Proposition 2.10, due to [Wet00] and first appearing in print in [Sik13] explains one source of translates B of subgroup schemes of $\operatorname{Res}^R_{R_0} J$ with $\dim(B \cap j(\operatorname{Res}^R_{R_0} X))$ larger than expected which can cause RoS Chabauty to produce an infinite set.

Recall that by convention, K' is a subfield of K and $R' = K' \cap R$ is the set of S' integers in K' for S' the set of primes lying under S.

Proposition 2.10. *Let X/R' be a sound thru-hole curve with generalized Jacobian J. Suppose that X_R is also a sound thru-hole curve. If RoS Chabauty applied to $j : X \hookrightarrow J$ fails to produce a finite set, then RoS Chabauty applied to $j : X_R \hookrightarrow J_R$ also fails to produce a finite set.*

Proof. The maps

$$\begin{aligned} j &: \operatorname{Res}_{R'/R_0} X \to \operatorname{Res}_{R'/R_0} J \\ j &: \operatorname{Res}_{R/R_0} X_R \to \operatorname{Res}_{R/\mathbb{Q}} J_R \end{aligned}$$

induced by the Abel-Jacobi map $j : X \to J$ commute with the diagonal embeddings

$$\begin{aligned} \iota &: \operatorname{Res}_{R'/R_0} X \hookrightarrow \operatorname{Res}_{R/R_0} X_R \\ \iota &: \operatorname{Res}_{R'/R_0} J \hookrightarrow \operatorname{Res}_{R/R_0} J_R \end{aligned}$$

In particular, the embeddings ι induce

$$\overline{(\operatorname{Res}^{R'}_{R_0} J)(R_0)} \cap j(\operatorname{Res}^{R'}_{R_0} X)(\mathbb{Q}_p) \subseteq \overline{(\operatorname{Res}^{R}_{R_0} J_R)(R_0)} \cap j(\operatorname{Res}^{R}_{R_0} X_R)(\mathbb{Q}_p)\,.$$

The left-hand side is infinite by assumption, so the right-hand side must be infinite as well. □

Remark 2.11. It is easy to find examples of curves satisfying the conditions of Proposition 2.10 by brute-force search, first for smooth proper curves $X/\mathbb{Q}$ with rank $J(\mathbb{Q}) <$ genus(X) and then for number fields K such that rank $J(K)$ is not much larger than rank $J(\mathbb{Q})$. See [Sik13] for an explicit example.

However, Proposition 2.10 does not explain all failures of RoS Chabauty to compute R-points.

Proposition 2.12. *Let $f : X \to Y$ be a morphism of sound thru-hole curves over R. Let $f_* : \operatorname{Res}^R_{R_0} J_X \to \operatorname{Res}^R_{R_0} J_Y$ be the induced pushforward map between the restrictions of scalars of the Jacobians of X and Y. Let $P = f_*^{-1}(O)$ be the preimage of the origin on J_Y. (P is a generalized Prym variety associated to f.)*

Suppose that $\overline{P(R_0)} = P(\mathbb{Q}_p)$. Suppose also that there is some $x \in (\operatorname{Res}^R_{R_0} X)(\mathbb{Q}_p)$ such that the image $f_*(j(x))$ lies in $I_Y := \overline{(\operatorname{Res}^R_{R_0} J_Y)(R_0)} \cap j(\operatorname{Res}^R_{R_0} Y_R)(\mathbb{Q}_p)$ and is not isolated in I_Y in the p-adic topology. (In particular, RoS Chabauty applied to $j : Y \to J_Y$ fails to produce a finite set.) Then, RoS Chabauty applied to $j : X \to J_X$ fails to produce a finite set.

Proof. First, we claim that

$$f_*^{-1}(\overline{J_Y(R_0)}) \subset \overline{J_X(R_0)}\,. \tag{14}$$

Indeed, for any $y \in \overline{J_Y(R_0)}$, we have $(f_*^{-1}(y))(\mathbb{Q}_p) = f^*(y) \cdot P(\mathbb{Q}_p)$. Claim (14) follows since $f^*(y) \in \overline{J_X(R_0)}$ and $P(\mathbb{Q}_p) = \overline{P(R_0)} \subset \overline{J_X(R_0)}$.

Now, (14) implies that if $x' \in (\operatorname{Res}^R_{R_0} X)(\mathbb{Q}_p)$ and $j(f(x')) \in I_Y$, then $j(x') \in I_X := \overline{(\operatorname{Res}^R_{R_0} J_X)(R_0)} \cap j(\operatorname{Res}^R_{R_0} X_R)(\mathbb{Q}_p)$. To conclude, we observe by Krasner's Lemma that if $y \in I_Y$ is sufficiently close p-adically to $j(f(x)) = f_*(j(x))$, then there is some $x' \in (\operatorname{Res}^R_{R_0} X)(\mathbb{Q}_p)$ such that $y = j(f(x'))$. In particular, $j(x') \in I_X$, so I_X is infinite. □

Example 2.13. In [Dog19], Dogra gives an example where both (11) and the conditions of Proposition 2.12 are satisfied (at least up to finite index.) In Dogra's example, K is a quadratic field, $Y/\mathbb{Q}$ is a genus 2 curve with rank $J_Y(\mathbb{Q}) = \operatorname{rank} J_Y(K) = 2$, X/K is a genus 3 curve with rank $J_X(K) = 4$, and $f : X \to Y_K$ is an unramified morphism of curves.

In Dogra's example, $\overline{J_Y(\mathbb{Q})} = J_Y(\mathbb{Q}_p)$, so RoS Chabauty for $j : Y \to J_Y$ fails to produce a finite set. By Proposition 2.10, RoS Chabauty for $j : Y_K \to J_{Y_K}$ also fails to produce a finite set.

Let $P = f_*^{-1}(O)$ as in Proposition 2.12. Then,

$$\operatorname{rank} P(K) = \operatorname{rank} J_X(K) - \operatorname{rank} J_Y(K) = 4 - 2 = 2 = \dim P\,.$$

With a bit more work, one can check that $\overline{P(R_0)} = P(\mathbb{Q}_p)$ and that there is some $x \in (\operatorname{Res}^R_{R_0} X)(\mathbb{Q}_p)$ such that the image $f_*(j(x))$ in I_Y is not isolated. By Proposition 2.12, RoS Chabauty for $j : X \to J_X$ fails to produce a finite set.

On the other hand,

$$\operatorname{rank} J_X(K) = 4 \le 4 = 2(3-1) = [K : \mathbb{Q}](\dim J - 1)\,, \tag{15}$$

so the RoS Chabauty inequality (11) is satisfied. Moreover, Dogra's X is not the base change of any curve defined over $\mathbb{Q}$, so the failure of RoS Chabauty for $j : X \to J_X$ to produce a finite set cannot be explained by Proposition 2.10 alone.

Remark 2.14. In the setup of Proposition 2.12 and its proof, it is possible for the geometric structure to cause RoS Chabauty applied to $j : X \to J_X$ to produce an infinite set even if $\overline{P(R_0)}$ is not finite index in $P(\mathbb{Q}_p)$. In fact, an expected dimension calculation suggests that in 'typical' cases

$$\dim I_X \geq \dim I_Y - (\dim P - \dim \overline{P(R_0)})$$

so long as the preimage of some component of I_Y (as a naïve p-adic analytic variety in the sense of Serre) of maximal dimension has nonempty preimage under f_* in $(\mathrm{Res}^R_{R_0} X)(\mathbb{Q}_p)$.

3. Main Definitions and Statement of Main Result

For simplicity of notation, we now restrict to the case that X/R is a genus 0 sound thru-hole curve.

In Propositions 2.10 and 2.12, the failure of RoS Chabauty applied to $j : X \to J$ to produce a finite set is explained by the existence of a subgroup scheme of $\mathrm{Res}_{R/R_0} J$ which intersects $\mathrm{Res}_{R/R_0} X$ in larger than expected dimension because of the geometry of the curve X. Motivated by these propositions, we now define the class of BCP-subtori of $\mathrm{Res}^R_{R_0} J$ for J the generalized Jacobian of a genus 0 sound thru-hole curve X/R. As Definitions 3.1, 3.2, and 3.3 make precise, BCP subtori are the subtori which which can be built inductively by iteratively (1) taking diagonal embeddings from curves defined over subrings which become isomorphic to X after base change to R (Proposition 2.10) and (2) adding Prym varieties (Proposition 2.12 and Remark 2.14). By replacing each instance of 'subtorus' with 'subgroup scheme,' these definitions generalize in a natural way to restrictions of scalars of generalized Jacobians of higher genus curves.

Definition 3.1. *Let Y be a genus 0 sound thru-hole curve over R' and let $T' \subset \mathrm{Res}_{R'/R_0} J_Y$ be a subtorus of the restriction of scalars of the generalized Jacobian of Y. Suppose that $X \cong Y_R$ is the base change of Y to R. The isomorphism induces a natural inclusion*

$$\mathrm{Res}_{R'/R_0} J_Y \hookrightarrow \mathrm{Res}_{R/R_0} J_X$$

of restrictions of scalars of generalized Jacobians. Let $T \subset \mathrm{Res}_{R/R_0} J_X$ be the image of T' under this inclusion. We call the pair (X, T) a BC-successor *of the pair (Y, T').*

Definition 3.2. *Let $f : X \to Y$ be a morphism of genus 0 sound thru-hole curves over R and let $T' \subset \mathrm{Res}_{R'/R_0} J_Y$ be a subtorus of the restriction of scalars of the generalized Jacobian of Y. The map f induces a push-forward map*

$$f_* : \mathrm{Res}_{R/R_0} J_X \to \mathrm{Res}_{R'/R_0} J_Y\,.$$

The connected component $T = f_*^{-1}(T')^\circ$ of the preimage of T' under f_* is a subtorus of $\operatorname{Res}_{R/R_0} J_X$. We call the pair (X,T) a P-successor *of (Y,T').*

Definition 3.3. *Let X be a genus 0 sound thru-hole curve over R. We say that a torus $T \subset \operatorname{Res}_{R/R_0} J_X$ is a* 0-BCP torus for X *if $T = \operatorname{Res}_{R/R_0} J_X$.*

For $n \geq 1$, we say a torus $T \subset \operatorname{Res}_{R/R_0} J_X$ is an n-BCP torus for X *if the pair (X,T) is either a BC-successor or a P-successor of some pair (Y,T') where T' is an $(n-1)$-BCP torus for Y.*

We say a torus $T \subset \operatorname{Res}_{R/R_0} J_X$ is a BCP torus for X *if it is an n-BCP torus for X for some n.*

Remark 3.4. Arguing by induction, one sees that if $T \subset \operatorname{Res}_{R/R_0} J_X$ is a BCP torus, then the intersection

$$T \cap j(\operatorname{Res}_{R/R_0} X) \subset \operatorname{Res}_{R/R_0} J_X .$$

has positive dimension. More precisely, if $(X,T)/R$ is a 0-BCP torus,

$$\dim T \cap j(\operatorname{Res}_{R/R_0} X) = d .$$

If $(X,T)/R$ is the BC-/P-successor of the pair $(Y,T')/R'$, then

$$\dim T \cap j(\operatorname{Res}_{R/R_0} X) = \dim T' \cap j(\operatorname{Res}_{R'/R_0} Y) .$$

Definition 3.5. *Let X be a sound thru-hole curve over R. A* BCP obstruction *to RoS Chabauty for X is a BCP torus T for X such that*

$$\dim(T \cap j(\operatorname{Res}_{R/R_0} X)) > \dim T - \dim \overline{T(R_0)} .$$

Given a BCP obstruction T to RoS Chabauty for X, an expected dimension calculation suggests that $\overline{T(R_0)} \cap (j(\operatorname{Res}_{R/R_0} X)(\mathbb{Q}_p)$ is infinite. Our main result shows that there are no BCP obstructions to RoS Chabauty for a large collection of subcovers of $\mathbb{P}^1 \smallsetminus \{0, 1\infty\}$.

Theorem 3.6. *Let q be a prime number and let K be a number field which does not contain a CM subfield. Let S_0 be a finite set of finite places of $\mathbb{Q}$ and let S be the set of places of K lying above S_0. Let $R = \mathcal{O}_{K,S}$ be the ring of S-integers. For $\alpha \in R^\times$, set*

$$\mathfrak{m}_{\alpha,q} := \begin{cases} \{x \in \overline{K} : x^q - 1 = 0, x \neq 1\} & \text{if } \alpha \in (R^\times)^q , \\ \{x \in \overline{K} : x^q - \alpha = 0\} & \text{if } \alpha \notin (R^\times)^q , \end{cases} \tag{16}$$

viewed as a divisor on $\mathbb{P}^1_K$. Let $\Gamma_{\alpha,q}$ be the closure of $\operatorname{supp}(\mathfrak{m}_{\alpha,q})$ in $\mathbb{P}^1_R$. Set $X_{\alpha,q} := \mathbb{P}^1_R \smallsetminus \Gamma_{\alpha,q}$.

For q sufficiently large (depending on K and S, but not on α), there are no BCP obstructions to RoS Chabauty for $X_{\alpha,q}$.

Remark 3.7. Recall that $R = \mathcal{O}_{K,S}$. For any prime q and $\alpha \in R^\times$, set $X'_{\alpha,q} = P^1_R \smallsetminus (\{x : x^q = \alpha\} \cup \{0,\infty\})$.

There is a natural inclusion map $X'_{\alpha,q} \to X_{\alpha,q}$, so to compute $X'_{\alpha,q}(R)$, it suffices to compute $X_{\alpha,q}(R)$. Moreover, elements $X'_{\alpha,q}(R)$ correspond to elements $x \in \mathbb{P}^1_R \smallsetminus \{0,1,\infty\}(R)$ such that x/α is a qth power in $R^\times$. Since $R^\times$ is a finitely-generated abelian group, this means that if one can compute $X'_{\alpha,q}(R)$ for α in a (finite) set of coset representative of $R^\times/(R^\times)^q$, then one can compute $\mathbb{P}^1_R \smallsetminus \{0,1,\infty\}(R)$.

In particular, Theorem 3.6 suggest that it should be possible to compute the set

$$\mathbb{P}^1_R \smallsetminus \{0,1,\infty\}(R) = \{(x,y) \in R^\times \times R^\times : x+y=1\}$$

of solutions to the S-unit equation in K using RoS Chabauty applied to the curves $X_{\alpha,q}$.

Remark 3.8. The assumption in Theorem 3.6 that K does not contain a CM-field is necessary under the generalized Leopoldt Conjecture 1.2 on p-adic linear independence of logarithms of algebraic numbers. In this remark, we exhibit a BCP obstruction to RoS Chabauty for $X_{1,q} = \mathbb{P}^1_R \smallsetminus \{x \in \overline{K}, x \neq 1 : x^q - 1 = 0\}$ under generalized Leopoldt for a single totally real number field.

Suppose K is a CM field. We can write K as $K'(\beta)$ for K' the totally real subfield with ring of S-integers R' and $\beta^2 \in R'$ negative in all real embeddings $K' \hookrightarrow \mathbb{R}$. Let $\overline{\beta}$ be the Galois conjugate of β under the $\mathrm{Gal}(K/K')$ action. Then, $\overline{\beta}$ is the complex conjugate of β for every embedding $K \hookrightarrow \mathbb{C}$. Define the fractional linear transformation:

$$f : \mathbb{P}^1 \to \mathbb{P}^1\,,$$
$$x \mapsto \frac{\beta x - \overline{\beta}}{x-1}\,.$$

Set $\Gamma := \{f(x) : \{x \in \overline{K}, x \neq 1 : x^q - 1 = 0\}$. The divisor Γ is stable under the action of $\mathrm{Gal}(K(\zeta_q)/K')$ so we may define $Y := \mathbb{P}^1_{R'} \smallsetminus \Gamma$. By construction, $X_{1,q} \cong Y_R$.

Set $T = \mathrm{Res}^{R'}_{R_0} J_Y$. Assuming generalized Leopoldt, $\dim \overline{T(R_0)} = \min(\dim T, \operatorname{rank} T(R_0))$. If this holds, we claim that the 1-BCP torus T is a BCP obstruction to RoS Chabauty for $X_{1,q}$.

In every complex embedding, the map f takes the unit circle to the real axis, so the field $K'(\Gamma)$ is totally real. Applying (6) gives

$$\operatorname{rank}(T(R_0)) = \operatorname{rank} J_Y(R') \geq [K' : \mathbb{Q}](q-2) = \dim \mathrm{Res}^{R'}_{R_0} J_Y\,,$$

with equality if and only if every prime in S' splits completely in $K'(\Gamma)$. Generalized Leopoldt implies $\dim \overline{T(R_0)} = \dim T$. Since

$$\dim(T \cap j(\mathrm{Res}^R_{R_0} X_{1,q})) = \dim \mathrm{Res}^{R'}_{R_0} Y = [K' : \mathbb{Q}] > 0\,,$$

we see that T is indeed a BCP obstruction to RoS Chabauty for $X_{1,q}$.

Of course, if K is not CM, but contains a CM field, we may apply the same argument to the CM subfield to construct a base change obstruction to RoS Chabauty for $X_{1,q}$.

4. Classical Chabauty and genus 0 descent.

Suppose that $X = \overline{X} \smallsetminus \Gamma$ is a sound, genus 0 clearance hole curve over $R = \mathcal{O}_{K,S}$. Let J be the generalized Jacobian of X. The classical Chabauty inequality (8) becomes

$$\operatorname{rank} J(R) \leq (\#\Gamma(\overline{K}) - 1) - 1 = \#\Gamma(\overline{K}) - 2\,. \tag{17}$$

in this case. We use Lemma 2.5 to study when (17) holds (or fails to hold) for X.

As a warm up, note that the classical Chabauty inequality is satisfied for $\mathbb{P}^1_R \smallsetminus \{0, 1, \infty\}$ if and only if $\operatorname{rank} R^\times < 1$, or equivalently $S = \emptyset$ and either $K = \mathbb{Q}$ or K is an imaginary quadratic field.

More generally, we note the following:

Proposition 4.1. *Let K be a number field with absolute Galois group $G = Gal(\overline{K}/K)$. Suppose that Γ is a flat horizontal divisor on $\mathbb{P}^1_R$ and that $X = \mathbb{P}^1_R \smallsetminus \Gamma$ is a sound thru-hole curve. The classical Chabauty-Coleman inequality (8) for X fails if any of the following conditions hold:*

(1) $r_2(K) \geq 1$ and $r_1(K) + r_2(K) + \#S \geq 2$,
(2) $r_1(K) \geq 3$ and $r_1(K) + \#S \geq 4$,
(3) $r_1(K) = 2$ and $r_1(K) + \#S \geq 4$ and $\#(G\backslash\Gamma(\overline{K})) \geq 2$,
(4) $r_1(K) = 3$ and $\Gamma(\overline{K})$ cannot be written as a disjoint union of G-orbits $\{P_i, P_i'\}$ which remain $G_{\mathfrak{p}}$ orbits for each real place $\mathfrak{p}$.[8]

Of course, the classical Chabauty inequality (17) may also fail under many other conditions, especially when S is large. We sketch the proof.

Proof. We use Lemma 2.5 to express $\operatorname{rank} J_X(R)$ in terms of the action of $\operatorname{Gal}(\overline{K}/K)$ on $\Gamma(\overline{K})$.

The statement follows quickly from three observations:

(1) If $\mathfrak{p} \in \Sigma_\infty$ is a complex place $G_{\mathfrak{p}} = \{1\}$, so $\#(G_{\mathfrak{p}}\backslash\Gamma(\overline{K})) - 1 = \#\Gamma(\overline{K}) - 1$.
(2) If $\mathfrak{p} \in \Sigma_\infty$ is a real place $G_{\mathfrak{p}} = \mathbb{Z}/2\mathbb{Z}$, so $\#(G_{\mathfrak{p}}\backslash\Gamma(\overline{K})) - 1 \geq \left\lceil \frac{\#\Gamma(\overline{K})}{2} \right\rceil - 1$, with equality if and only if $\Gamma(\overline{K})$ decomposes into complex conjugate pairs over $K_{\mathfrak{p}}$.
(3) $[\#(G\backslash\Gamma(\overline{K})) - 1] \leq \min_{\mathfrak{p} \in S \cup \Sigma_\infty} [\#(G_{\mathfrak{p}}\backslash\Gamma(\overline{K})) - 1]$.

[8] This condition on $\Gamma(\overline{K})$ is automatic if $\#\Gamma(\overline{K})$ is odd.

For example, if we are in Case (1): $r_2(K) \geq 1$ and $r_1(K) + r_2(K) + \#S \geq 2$, let ∞_1 be an infinite place and let $\mathfrak{p} \neq \infty_1$ be another place with $\#(G_{\mathfrak{p}} \backslash \Gamma(\overline{K}))$ minimal. By Lemma 2.5,

$$\begin{aligned} \operatorname{rank} J_X(R) &\geq [\#(G_{\infty_1} \backslash \Gamma(\overline{K})) - 1] + [\#(G_{\mathfrak{p}} \backslash \Gamma(\overline{K})) - 1] - [\#(G \backslash \Gamma(\overline{K})) - 1] \\ &\geq [\#(G_{\infty_1} \backslash \Gamma(\overline{K})) - 1] + [\#(G_{\mathfrak{p}} \backslash \Gamma(\overline{K})) - 1] - [\#(G_{\mathfrak{p}} \backslash \Gamma(\overline{K})) - 1] \\ &= \#\Gamma(\overline{K}) - 1\,. \end{aligned}$$

The other cases are similar, hinging on the fact that the $-[\#(G \backslash \Gamma(\overline{K})) - 1]$ term in Lemma 2.5 can cancel *at most* the minimal positive term. $\square$

Corollary 4.2. *Suppose that we are* not *in the following situations: (i) $K = \mathbb{Q}$, (ii) K a real quadratic field and $\#S \leq 1$, (iii) K an imaginary quadratic or totally real cubic field and $\#S = 0$.*

Then, the classical Chabauty inequality is not satisfied by any descent set consisting of genus zero covers of $\mathbb{P}^1_R \smallsetminus \{0, 1, \infty\}$. If the generalized Leopoldt Conjecture 1.2 holds over K then the combination of classical Chabauty and descent by genus zero covers is insufficient to prove that the set

$$(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathcal{O}_{K,S})$$

of solutions to the S-unit equation over K is finite.

Remark 4.3. If $K = \mathbb{Q}$ or is imaginary quadratic, [Poo19] shows that $(\mathbb{Z}/q\mathbb{Z})$-descent and classical Chabauty suffice to prove finiteness of $(\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})(\mathbb{Z}_{S_0})$ for any choice of primes S_0.

In the remaining cases, where K is a totally real quadratic or cubic field, one can check that $(\mathbb{Z}/q\mathbb{Z})$-descent and classical Chabauty *is not* sufficient to prove the desired finiteness result. It may be possible to prove a finiteness result in these cases using an *iterated* cyclic descent, or a descent-like procedure using covers which are not torsors over the base curve, but this would require a more careful analysis.

5. Covering collections for $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$ with no BCP obstructions.

In this section, we show there are no BCP obstructions to RoS Chabauty applied to certain families of curves obtained by filling in punctures of cyclic covers of $\mathbb{P}^1_R \smallsetminus \{0, 1, \infty\}$. Our main goal is to prove Theorem 3.6.

Before we begin a series of technical lemmas, we sketch the idea of the proof. Given a BCP torus T, Lemma 5.1 allows us to bound $\dim T - \dim \overline{T(R_0)}$ from below in terms of ranks and dimensions of the generalized Jacobians of certain curves. Theorem 5.6 shows that if X/R' is any curve which becomes isomorphic to $X_{\alpha,q}$ after base change to R, then the rank of the R' points of the generalized Jacobian J_X of X is 'small'. More precisely, for large q, there is a constant $c > 0$ such that

$\operatorname{rank} J_X(R') + cq < \dim J_X$. Lemma 5.7 shows that if the punctures of X form a single Galois orbit, (X,T) is a P-successor of some (Y,T'), and $\operatorname{rank} J_X(R)$ is 'small' in the sense above, then for large q the difference $(\operatorname{rank} J_X(R') - \operatorname{rank} J_Y(R'))$ is 'small' compared to $[K' : \mathbb{Q}](\dim J_X - \dim J_Y)$. Combining these lemmas, any BCP torus T for $X_{\alpha,q}$ satisfies $\dim T - \dim \overline{T(R_0)} > \dim \operatorname{Res}_{R/R_0} X_{\alpha,q}$, so there is no BCP obstruction to RoS Chabauty for $X_{\alpha,q}$.

Lemma 5.1. *Suppose that T is an n-BCP torus for X/R. Set $T_n = T, R'_n = R$, and $X_n = X$. For each $i \in \{0,\ldots,n-1\}$ there is a ring of S-integers R'_i with fraction field K_i, a curve X_i/R'_i with generalized Jacobian J_i, and an i-BCP torus T_i such that T_0 is a 0-BCP torus for X_0 and (X_{i+1},T_{i+1}) is either a BC-successor or a P-successor of (X_i,T_i).*

Set

$$\delta_i = \begin{cases} [K_0 : \mathbb{Q}] \cdot \dim J_0 - \operatorname{rank} J_0(R'_0) & \text{if } i = 0\,, \\ 0 & \text{if } (X_i,T_i) \text{ is a BC-successor of } (X_{i-1},T_{i-1})\,, \\ [K_i : \mathbb{Q}] \cdot (\dim J_i - \dim J_{i-1}) & \text{if } (X_i,T_i) \text{ is a P-successor of } (X_{i-1},T_{i-1})\,. \\ \quad - (\operatorname{rank} J_i(R_i) - \operatorname{rank} J_{i-1}(R_{i-1})) & \end{cases}$$

Then,

$$\dim T - \dim \overline{T(R_0)} \geq \sum_{i=0}^{n} \max(0,\delta_i)\,.$$

Proof. The proof is by induction on n.

For $n = 0$, the claim is equivalent to $\operatorname{rank} T_0(R_0) \geq \dim \overline{T_0(R_0)}$, which holds because the log map respects the p-adic topology.

Suppose the claim holds for (X_{n-1},T_{n-1}). If (X_n,T_n) is the BC-successor of (X_{n-1},T_{n-1}) then $T_n = T_{n-1}$ and $\delta_n = 0$, so the claim is immediate.

Otherwise, (X_n,T_n) is the P-successor of (X_{n-1},T_{n-1}), so there is a map $f : X_n \to X_{n-1}$ and $R_n = R_{n-1}$. Set $P = f_*^{-1}(O)$ for f_* as in Definition 3.2. Up to isogeny, $T_n \sim P \times T_{n-1}$ and $\operatorname{Res}_{R_0}^{R'_n} J_n \sim P \times \operatorname{Res}_{R_0}^{R'_{n-1}} J_n$. So,

$$\begin{aligned} \dim T_n - \dim \overline{T_n(R_0)} &= \Big(\dim T_{n-1} - \dim \overline{T_{n-1}(R_0)}\Big) + \Big(\dim P - \dim \overline{P(R_0)}\Big) \\ &\geq \sum_{i=0}^{n-1} \max(0,\delta_i) + \max(0, \dim P - \operatorname{rank} P(R_0)) \\ &= \sum_{i=0}^{n-1} \max(0,\delta_i) + \max(0,\delta_n)\,. \end{aligned}$$

$\square$

We now begin a series of lemmas aimed at bounding the ranks of the generalized Jacobians of certain genus 0 curves, culminating in Theorem 5.6.

Note that if $\mathbb{P}^1_R \smallsetminus \Gamma_1$ and $\mathbb{P}^1_R \smallsetminus \Gamma_2$ are isomorphic, then there is some fractional linear transformation $\phi : x \mapsto \frac{ax+b}{cx+d}$ with $a, b, c, d \in K$ such that $\phi(\Gamma_1(\overline{K})) = \Gamma_2(\overline{K})$. To bound the rank of the generalized Jacobians of $\mathbb{P}^1_R \smallsetminus \Gamma_1$ we will need to understand the number of real and complex embeddings of the fields generated by the Galois orbits in Γ_1, as well as the splitting of the primes in S in these fields.

Lemma 5.2. *Suppose that $K' \subset K$ are number fields. Let X be a genus 0 curve defined over K' equipped with an isomorphism $\phi : \mathbb{P}^1_K \to X_K$. Fix $P \in \mathbb{P}^1_K(\overline{\mathbb{Q}})$. Suppose also that $[K'(\phi(P)) : K'] = [K(P) : K]$. Given $r \in \mathbb{R}_{>0}$, let*

$$U_r := \{x \in \mathbb{C} : |x| = r\}$$

be the circle of radius r in $\mathbb{C} \subset \mathbb{P}^1(\mathbb{C})$.

Let I be a set of embeddings $\iota : K \to \mathbb{C}$. For each $\iota \in I$, fix some $r_\iota \in \mathbb{R}_{>0}$. Suppose that for all embeddings $\iota' : K(P) \hookrightarrow \mathbb{C}$ extending ι, we have $\iota'(P) \in U_{r_\iota}$. Set

$$K_{bad} := \{x \in \mathbb{P}^1(K) : \iota(x) \in U_{r_\iota} \text{ for all } \iota \in I\}.$$

If K_{bad} is finite and either

(1) $K_{bad} \neq \emptyset$ or
(2) $X \cong \mathbb{P}^1_{K'}$

then for some $\iota \in I$ there are at most 2 real embeddings $K'(\phi_\iota(P)) \hookrightarrow \mathbb{R}$ extending $\iota|_{K'}$.

Proof of Lemma 5.2. Observe first that $[K'(\phi(P)) : K'] = [K(P) : K]$ implies that embeddings $K'(\phi(P)) \hookrightarrow \mathbb{C}$ extending $\iota|_{K'}$ correspond exactly to embeddings $\iota' : K(P) \hookrightarrow \mathbb{C}$ extending ι.

We now prove the contrapositive. Suppose that for each $\iota \in I$ there are at least three $\iota' : K(P) \to \mathbb{C}$ so that $\iota'(K'(\phi(P))) \subset \mathbb{R}$.

Then, each $\iota|_{K'} : K' \hookrightarrow \mathbb{C}$ is a real embedding. Given $\iota : K \hookrightarrow \mathbb{C}$, let X_ι be the base change of X from K' to $\mathbb{R}$ along ι. Let ϕ_ι denote the induced map $\mathbb{P}^1_{\mathbb{C}} \to X_{\iota,\mathbb{C}}$.

If $X \cong \mathbb{P}^1_{K'}$, let $\psi : X \to \mathbb{P}^1_{K'}$ be the identity map. The induced isomorphisms $\psi_\iota : X_\iota \to \mathbb{P}^1_{\mathbb{R}}$ and $\psi_\iota : X_{\iota,\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ are again the identity.

Otherwise, X can be written as a conic in $\mathbb{P}^2_{K'}$. Choose some $Q \in K_{\text{bad}}$ and let $\psi : X_K \to \mathbb{P}^1_K$ be projection from $\phi(Q)$ to a line. Since $\iota(\phi(Q)) \in \mathbb{R}$ for all $\iota \in I$, there are again induced isomorphisms $\psi_\iota : X_\iota \to \mathbb{P}^1_{\mathbb{R}}$ and $\psi_\iota : X_{\iota,\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ given by projection from $\iota(\phi(Q))$.

In either case, $\psi_{\iota,\mathbb{C}}$ maps $X_\iota(\mathbb{R})$ isomorphically to the real axis in $\mathbb{P}^1(\mathbb{C})$.

Now, by assumption, for any $\iota \in I$, we can choose distinct $\iota_1, \iota_2, \iota_3$ extending ι to $K(P)$ such that $\phi_\iota(\iota_j(P)) \in X_\iota(\mathbb{R})$ for $j \in \{1, 2, 3\}$. Then, $(\psi_\iota \circ \phi_\iota)(\iota_j(P))$ lies on the real axis in $\mathbb{P}^1(\mathbb{C})$ for $j \in \{1,2,3\}$. Since any automorphism of $\mathbb{P}^1(\mathbb{C})$ which maps three points on a circle to a (real) line induces a bijective map between the circle and the line, we see that the composition $\psi_\iota \circ \phi_\iota$ maps U_{r_ι} bijectively to the real axis in $\mathbb{P}^1(\mathbb{C})$. Hence, ϕ_ι maps U_{r_ι} bijectively to $X_\iota(\mathbb{R})$.

Now, consider the map $\psi \circ \phi : \mathbb{P}^1_K \to \mathbb{P}^1_K$. Since, $\mathbb{P}^1$ is definable over $\mathbb{Q}$, it makes sense to consider $\mathbb{P}^1(\mathbb{Q})$ as a set inside $\mathbb{P}^1(K)$. Moreover, for each $\iota \in I$, the set $\mathbb{P}^1(\mathbb{Q})$ is contained in the real axis of $\mathbb{P}^1(\mathbb{C})$. In particular, for all $x \in \mathbb{P}^1(\mathbb{Q})$ and all $\iota \in I$, we have

$$\iota((\psi \circ \phi)^{-1}(x)) \in U_{r_\iota}\,.$$

Hence,

$$(\psi \circ \phi)^{-1}(\mathbb{P}^1(\mathbb{Q})) \subset K_{\mathrm{bad}}\,.$$

so K_{bad} is infinite. □

Recall that a CM field is a totally complex field which is a degree 2 extension of a totally real field. We state and prove a well-known fact characterizing fields containing CM subfields.

Lemma 5.3. *Let K be a number field and fix $\alpha \in K$. Then, K contains a CM subfield if and only if the set*

$$C_\alpha := \{x \in K : |\iota(x)| = |\iota(\alpha)| \text{ for all } \iota : K \hookrightarrow \mathbb{C}\}$$

is infinite, or equivalently, if $\#C_\alpha \geq 3$.)

Proof of Lemma 5.3. Dividing by α if necessary, it suffices to prove the claim for $\alpha = 1$.

If $\#C_1 \geq 3$ then there is some $\beta \neq \pm 1$ on the unit circle in every complex embedding. The same is true of β^{-1}, so $\mathbb{Q}(\beta)$ is a totally complex degree two extension of $\mathbb{Q}\left(\frac{\beta+\beta^{-1}}{2}\right)$.

If K contains a CM subfield, suppose $K'(\beta)$ is a CM subfield containing the totally real subfield K'. Let $\sigma \in \mathrm{Gal}(K'(\beta)/K')$ be the nontrivial automorphism, which corresponds to complex conjugation. Then, $\beta - \sigma(\beta) \neq 0$ lies on the imaginary axis in every embedding $K \hookrightarrow \mathbb{C}$. Finally, the fractional linear transformation $x \mapsto \frac{x+1}{x-1}$ maps the imaginary axis to the unit circle in every embedding. Hence,

$$C_1 \supset \left\{\frac{n\beta - n\sigma(\beta) + 1}{n\beta - n\sigma(\beta) - 1} : n \in \mathbb{Q}\right\}$$

so C_1 is infinite. □

Lemma 5.4. *Suppose that $[K : \mathbb{Q}]$ is odd. Fix an $\iota : K \hookrightarrow \mathbb{C}$ and an $r \in \mathbb{R}_{>0}$. Then,*

$$\#\{x \in K : |\iota(x)| = r\} \leq 2\,.$$

Proof of Lemma 5.4. Suppose not. Then there are $\alpha, \beta \in K$ with $\iota(\alpha) = \iota(\beta) = r$ and $\alpha \neq \pm\beta$. Set $x = \alpha/\beta$, so $|\iota(x)| = 1$. Then, $\iota(x) \notin \mathbb{R}$ but $\iota(x) + \iota(x)^{-1} \in \mathbb{R}$. Hence, $[\mathbb{Q}(x) : \mathbb{Q}(x + x^{-1})] = 2$. But then $[K : \mathbb{Q}] = [K : \mathbb{Q}(x)] \cdot [\mathbb{Q}(x) : \mathbb{Q}]$ is even. □

Lemma 5.5. *Suppose that $K' \subset K$ are number fields. Let q be a prime. Let X be a genus 0 curve defined over K' equipped with an isomorphism $\phi : \mathbb{P}^1_K \to X_K$.*

(1) Suppose that the set $Z_q := \left\{\zeta_q^j : j \in \{1, \ldots, q-1\}\right\}$ of primitive qth roots of unity defines an irreducible degree $q-1$ point of $\mathbb{P}^1_K$ and that $\{\phi(x) : x \in Z_q\}$ descends to an irreducible degree $q-1$ point of X. Then,

a) If $[K : \mathbb{Q}]$ is odd,

$$r_1(K'(\phi(\zeta_q))) \leq 2r_1(K')\,.$$

b) If K does not contain a CM subfield,

$$r_1(K'(\phi(\zeta_q))) \leq [K' : \mathbb{Q}](q-1) - (q-3)\,.$$

(2) Fix $\alpha \in K^\times$ not a qth power. Suppose that the set $Z_{q,\alpha} := \left\{\zeta_q^j \sqrt[q]{\alpha} : j \in \{0, \ldots, q-1\}\right\}$ of qth roots of α defines an irreducible degree q point of $\mathbb{P}^1_K$ and that $\{\phi(x) : x \in Z_{q,\alpha}\}$ descends to an irreducible degree q point of X. Then,

a) If $[K : \mathbb{Q}]$ is odd,

$$r_1(K'(\phi(\sqrt[q]{\alpha}))) \leq 2r_1(K')\,.$$

b) If K does not contain a CM subfield,

$$r_1(K'(\phi(\sqrt[q]{\alpha}))) \leq [K' : \mathbb{Q}]q - (q-2)\,.$$

Proof of Lemma 5.5. The assumptions that we have irreducible points imply that

$$[K(\zeta_q) : K] = [K'(\phi(\zeta_q)) : K'] = q-1 \text{ and } [K(\sqrt[q]{\alpha}) : K] = [K'(\phi(\sqrt[q]{\alpha})) : K'] = q\,.$$

(1a) and (2a): Suppose $[K : \mathbb{Q}]$ is odd. Fix an $\iota : K \hookrightarrow \mathbb{C}$ and set $I = \{\iota\}$. In case (1), set $r_\iota = 1$. In case (2), set $r_\iota = \sqrt[q]{|\iota(\alpha)|}$. Define K_{bad} as in Lemma 5.2. By Lemma 5.4, $\#K_{\text{bad}} \leq 2$.

Moreover, in case (1), we have $\pm 1 \in K_{\text{bad}}$, while in case (2), we are given an odd degree divisor on X, so $X \cong \mathbb{P}^1_{K'}$. Then, Lemma 5.2 says there are at most 2 real embeddings of $K'(\phi(\zeta_q))$ (respectively $K'(\sqrt[q]{\alpha})$) extending $\iota|_{K'}$. Of course, if $\iota|_{K'}$ is not real, there are no such embeddings. We conclude

$$r_1(K'(\phi(\zeta_q))) \leq 2r_1(K')\,,$$
$$r_1(K'(\phi(\sqrt[q]{\alpha}))) \leq 2r_1(K')\,.$$

(1b) and (2b): Suppose K does not contain a CM subfield. Let I be the set of all embeddings $\iota : K \hookrightarrow \mathbb{C}$. In case (1), set $r_\iota = 1$ for all ι. In case (2), set $r_\iota = \sqrt[q]{|\iota(\alpha)|}$ for all ι. Define K_{bad} as in Lemma 5.2. By Lemma 5.3, $\#K_{\text{bad}} \leq 2$.

In case (1), we again have $\pm 1 \in K_{\mathrm{bad}}$, while in case (2) we have $X \cong \mathbb{P}^1_{K'}$. By Lemma 5.2,

$$r_1(K'(\phi(\zeta_q))) \leq ([K':\mathbb{Q}]-1)(q-1)+2 = [K':\mathbb{Q}](q-1)-(q-3)\,,$$
$$r_1(K'(\phi(\sqrt[q]{\alpha}))) \leq ([K':\mathbb{Q}]-1)q+2 = [K':\mathbb{Q}]q-(q-2)\,.$$

□

Theorem 5.6. *Let K be a number field which does not contain a CM subfield and let S be a finite set of finite places. For any $\varepsilon > 0$, there exists $C \in \mathbb{R}$ such that for all primes $q > C$ the following holds:*

Let K' be any subfield of K. Let S' be places of K' lying under S and let $R' = \mathcal{O}_{K',S'}$ be the ring of S'-integers of K'. Choose any $\alpha \in R^\times$ and let Γ be the divisor on $\mathbb{P}^1_R$ given by spreading out the divisor $\{x \in \overline{R} \smallsetminus R : x^q - \alpha = 0\}$ on $\mathbb{P}^1_K$.

Let X/R' be any curve such that there is an isomorphism $\phi : \mathbb{P}^1_R \smallsetminus \Gamma \to X_R$. Then,

$$\operatorname{rank} J_X(R') \leq \left([K':\mathbb{Q}] - \frac{1}{2} + \varepsilon\right)(q-2)\,.$$

Proof of Theorem 5.6. Case 1: $\alpha \in R^{\times q}$. We may assume $\alpha = 1$.

Note that $\mathbb{P}^1_R \smallsetminus \Gamma \cong \mathbb{P}^1_R \smallsetminus \{\zeta_q^j : j \in \{1,\dots,q-1\}\}$. By choosing C sufficiently large, we may assume that $\mathbb{Q}(\zeta_q)$ is linearly disjoint from K. Then,

$$[K(\zeta_q):K] = [K(\phi(\zeta_q)):K] \leq [K'(\phi(\zeta_q)):K']\,,$$

so the punctures of $X_{K'}$ form an irreducible degree $q-1$ point on $\overline{X_{K'}}$.

Using Lemma 2.5 or equation (6) to bound the rank of $J_X(R')$, it will suffice to bound the number of infinite places of $K'(\phi(\zeta_q))$ and the number of finite places of $K'(\phi(\zeta_q))$ above S'.

We start with the infinite places. By 1b of Lemma 5.5,

$$r_1(K'(\phi(\zeta_q))) \leq [K':\mathbb{Q}](q-1)-(q-3)\,.$$

Also,

$$r_1(K'(\phi(\zeta_q))) + 2r_2(K'(\phi(\zeta_q))) = [K':\mathbb{Q}](q-1)\,,$$

so

$$r_1(K'(\phi(\zeta_q))) + r_2(K'(\phi(\zeta_q))) \leq [K':\mathbb{Q}](q-1) - \frac{q-3}{2}\,. \tag{18}$$

To address the finite places, we note that the total number of places of $K'(\phi(\zeta_q))$ above S' is at most the total number of places of $K(\zeta_q)$ above S.

Fix a prime $\mathfrak{p} \in S$. Let $\kappa_\mathfrak{p}$ be the residue field of $K_\mathfrak{p}$. Let $a_\mathfrak{p} = [\kappa_\mathfrak{p}(\zeta_q) : \kappa_\mathfrak{p}]$. Then, $a_\mathfrak{p}$ is the order of $\#\kappa_\mathfrak{p}$ in $(\mathbb{Z}/q\,\mathbb{Z})^\times$, so

$$a_\mathfrak{p} \geq \left\lceil \frac{\ln(q-1)}{\ln(\#\kappa_\mathfrak{p})} \right\rceil\,.$$

Also,

$$\#\{\text{primes } \mathfrak{P} \text{ of } K(\zeta_q) \text{ above } \mathfrak{p}\} = \frac{q-1}{a_{\mathfrak{p}}} \leq \frac{q-1}{\left\lceil \frac{\ln(q-1)}{\ln(\#\kappa_{\mathfrak{p}})} \right\rceil}.$$

The set S is finite and the denominators shrink as q grows, so there is some C (depending only on ε, K, and S) such that for $q > C$ we have

$$\sum_{\mathfrak{p}\in S} \#\{\text{primes } \mathfrak{P} \text{ of } K'(\phi(\zeta_p)) \text{ above } \mathfrak{p}\} \leq \frac{\varepsilon}{2}(q-1). \tag{19}$$

Combining (18) and (19) in (6) gives

$$\operatorname{rank} J_X(R') \leq \left([K':\mathbb{Q}] - \frac{1}{2} + \frac{\varepsilon}{2}\right)(q-1) + \frac{3}{2}.$$

Increasing C if necessary, we may assume $\varepsilon q/2$ is larger than any given constant. We find

$$\operatorname{rank} J_X(R') \leq \left([K':\mathbb{Q}] - \frac{1}{2} + \varepsilon\right)(q-2).$$

Case 2: $\alpha \notin R^{\times q}$.

Since $\sqrt[q]{\alpha} \notin K$, we have

$$[K(\sqrt[q]{\alpha}):K] = [K(\phi(\sqrt[q]{\alpha})):K] \leq [K'(\phi(\sqrt[q]{\alpha})):K'],$$

so the punctures of $X_{K'}$ form an irreducible degree q point on $\overline{X_{K'}}$. As in case 1, using (6) we can bound the rank of $J_X(R')$ by bounding the number of infinite places of $K'(\phi(\sqrt[q]{\alpha}))$ and the number of finite places of $K'(\phi(\sqrt[q]{\alpha}))$ above S'.

From 2b of Lemma 5.5, we see

$$r_1(K'(\phi(\sqrt[q]{\alpha}))) + r_2(K'(\phi(\sqrt[q]{\alpha}))) \leq [K':\mathbb{Q}]q - \frac{q-2}{2}. \tag{20}$$

As in Case 1, we bound the total number of finite places of $K'(\phi(\sqrt[q]{\alpha}))$ above S' from above by the total number of finite places of $K(\sqrt[q]{\alpha})$ above S.

Define $a_{\mathfrak{p}}$ the same as in Case 1. For a finite place $\mathfrak{p}$ of K not lying over q,

$$\#\{\text{primes } \mathfrak{P} \text{ of } K(\sqrt[q]{\alpha}) \text{ above } \mathfrak{p}\} = \begin{cases} 1, & \text{if } \overline{\alpha} \notin \kappa_{\mathfrak{p}}^{\times q}, \\ 1 + (q-1)/a_{\mathfrak{p}}, & \text{otherwise} . \end{cases}$$

As in Case 1, there is some C (depending only on ε, K, and S) such that for $q > C$ we have

$$\sum_{\mathfrak{p}\in S} \#\{\text{primes } \mathfrak{P} \text{ of } K'(\phi(\sqrt[q]{\alpha})) \text{ above } \mathfrak{p}\} \leq \#S + \frac{\varepsilon}{2}(q-1). \tag{21}$$

Combining (20) and (21) in (6) gives

$$\operatorname{rank} J_X(R') \leq \left([K':\mathbb{Q}] - \frac{1}{2} + \frac{\varepsilon}{2}\right) q + 1.$$

Again, increasing C if necessary, for $q > C$ we have

$$\operatorname{rank} J_X(R') \leq \left([K' : \mathbb{Q}] - \frac{1}{2} + \varepsilon \right) (q-2) .$$

□

Lemma 5.7. *Let $X_1 := \overline{X_1} \setminus \Gamma_1$ and $X_2 := \overline{X_2} \setminus \Gamma_2$ be sound thru-hole curves over R with generalized Jacobians J_1 and J_2. Let $G = Gal(\overline{K}/K)$. Suppose we are given a morphism $\varphi : X_1 \to X_2$ such that the induced morphism $\varphi : \overline{X_1} \to \overline{X_2}$ satisfies $\varphi(\Gamma_1(\overline{K})) = \Gamma_2(\overline{K})$. Suppose also that $\#(G \backslash \Gamma_2(\overline{K})) = 1$ and set $\delta = \#\Gamma_1(\overline{K})/\#\Gamma_2(\overline{K})$. Set*

$$\Delta := [K : \mathbb{Q}] \cdot \#\Gamma_1(\overline{K}) - (\operatorname{rank} \mathcal{J}_1(R) + \operatorname{rank} \mathbb{G}_m(R) + 1) .$$

Then

$$[K : \mathbb{Q}](\dim J_1 - \dim J_2) - (\operatorname{rank} J_1(R) - \operatorname{rank} J_2(R)) \geq \left(\frac{\delta - 1}{\delta} \right) \Delta . \tag{22}$$

Proof of Lemma 5.7. Let $n = \#(G \backslash \Gamma_1(\overline{K}))$. Since $\#(G \backslash \Gamma_2(\overline{K})) = 1$, every point in $\#\Gamma_2(\overline{K})$ has exactly δ preimages in $\Gamma_1(\overline{K})$. Each orbit of $G \setminus \Gamma_1(\overline{K})$ contains at least one preimage of each point of $\Gamma_2(\overline{K})$, so $n \leq \delta$. Similarly, for each $\mathfrak{p} \in S \cup \Sigma_\infty$, we have $\#(G_\mathfrak{p} \backslash \Gamma_1(\overline{K})) \leq \delta \#(G_\mathfrak{p} \backslash \Gamma_2(\overline{K}))$. Adding $\operatorname{rank} \mathbb{G}_m(\mathcal{O}_{K,S}) = -1 + \sum_{\mathfrak{p} \in S \cup \Sigma_\infty} 1$ to the formula from Lemma 2.5 gives

$$\operatorname{rank} \mathcal{J}_1(\mathcal{O}_{K,S}) + \operatorname{rank} \mathbb{G}_m(\mathcal{O}_{K,S}) + n = \sum_{\mathfrak{p} \in S \cup \Sigma_\infty} \#(G_\mathfrak{p} \backslash \Gamma_1(\overline{K})) , \text{ and} \tag{23}$$

$$\operatorname{rank} \mathcal{J}_2(\mathcal{O}_{K,S}) + \operatorname{rank} \mathbb{G}_m(\mathcal{O}_{K,S}) + 1 = \sum_{\mathfrak{p} \in S \cup \Sigma_\infty} \#(G_\mathfrak{p} \backslash \Gamma_2(\overline{K})) . \tag{24}$$

Subtracting δ times (24) from (23) gives

$$(\delta - (\delta - 1)) \operatorname{rank} \mathcal{J}_1(R) - \delta \operatorname{rank} \mathcal{J}_2(R) - (\delta - 1) \operatorname{rank} \mathbb{G}_m(R) - (\delta - n) \leq 0 .$$

Since $n \geq 1$, the inequality is true after replacing n with 1. Rearranging gives

$$\operatorname{rank} \mathcal{J}_1(R) - \operatorname{rank} \mathcal{J}_2(R) \leq \frac{\delta - 1}{\delta} (\operatorname{rank} \mathcal{J}_1(R) + \operatorname{rank} \mathbb{G}_m(R) + 1) . \tag{25}$$

We also compute

$$\dim \mathcal{J}_1 - \dim \mathcal{J}_2 = (\#\Gamma_1(\overline{K}) - 1) - (\#\Gamma_2(\overline{K}) - 1) = \frac{\delta - 1}{\delta} \#\Gamma_1(\overline{K}) . \tag{26}$$

Subtracting (25) from $[K : \mathbb{Q}]$ times (26), we conclude that (22) holds. □

After unpacking some definitions, Theorem 3.6 is almost a corollary of Theorem 5.6 and Lemma 5.7.

Proof of Theorem 3.6. Suppose that T is an n-BCP torus for $X_{\alpha,q}$. Choose R'_i, X_i, and T_i and define δ_i as in Lemma 5.1.

If $n = 0$ so that $(X, T) = (X_0, T_0)$ or if $n = 1$ and $(X_{\alpha,q}, T)$ is a BC-successor of (X_0, T_0) for X_0/R'_0 then in the notation of Theorem 5.6, we have $R' = R'_0$ and $T_0 = \operatorname{Res}_{R'_0/R_0} J_X$. Applying Theorem 5.6, we have

$$\operatorname{rank} T_0(R_0) \leq \left([K' : \mathbb{Q}] - \frac{1}{2} + \varepsilon\right)(q-2)\,,$$
$$\dim T_0 \geq [K' : \mathbb{Q}](q-2)\,.$$

Choosing any $\varepsilon < 1/2$ and q sufficiently large, we may arrange that

$$\sum_{i=0}^{n} \max(0, \delta_i) \geq \delta_0 \geq \left(\frac{1}{2} - \varepsilon\right)(q-2) \geq [K' : \mathbb{Q}] \geq \dim(T \cap j(\operatorname{Res}_{R/R_0} X))\,.$$

Thus, (X, T) is not a BCP obstruction.

Otherwise, we may write (X, T) as a BC-successor of some (X_{n-1}, T_{n-1}) (possibly equal to (X, T)) which is a P-successor of some (X_{n-2}, T_{n-2}) (and where $X_{n-1} \to X_{n-2}$ is not an isomorphism.) For ease of notation, say X_{n-1} and X_{n-2} are defined over R', the ring of S-integers in K'.

By Theorem 5.6, taking $\varepsilon < 1/4$ and q sufficiently large, we may assume

$$\operatorname{rank} J_{n-1}(R') \leq \left([K' : \mathbb{Q}] - \frac{1}{2} + \varepsilon\right)(q-2) \leq \left([K' : \mathbb{Q}] - \frac{1}{4}\right)q - \operatorname{rank} \mathbb{G}_m(R') - 1\,.$$

In particular,

$$[K' : \mathbb{Q}]q - (\operatorname{rank} J_{n-1}(R') + \operatorname{rank} \mathbb{G}_m(R') + 1) \geq \frac{q}{4}\,.$$

By construction, the punctures of X_{n-1} form a single $\operatorname{Gal}(\overline{K'}/K')$-orbit so the same is true of X_{n-2}. We claim that the number of punctures of $X_{n-1}(\overline{K'})$ and $X_{n-2}(\overline{K'})$ cannot be equal. If they were, then the map $\overline{X_{n-1}} \to \overline{X_{n-2}}$ would be totally ramified at every puncture. But by the Riemann-Hurwitz formula, a map between genus zero curves is totally ramified at at most 2 points. Applying Lemma 5.7, we have $\delta \geq 2$ and so

$$[K' : \mathbb{Q}](\dim J_{n-1} - \dim J_{n-2}) - (\operatorname{rank} J_{n-1}(R') - \operatorname{rank} J_{n-2}(R')) \geq \left(\frac{\delta - 1}{\delta}\right)\frac{q}{4} \geq \frac{q}{8}\,.$$

Taking q sufficiently large, we may arrange that

$$\sum_{i=0}^{n} \max(0, \delta_i) \geq \delta_{n-1} \geq \frac{q}{8} \geq [K : \mathbb{Q}] = \dim\, j(\operatorname{Res}_{R/R_0} X) \geq \dim\, T \cap j(\operatorname{Res}_{R/R_0} X)\,.$$

Again, we see that (X, T) is not a BCP obstruction.

To complete the proof, we note that at each step the bounds on q depend only on K and S. In particular, they can be chosen uniformly for all α. $\square$

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