SUPPORTING DATA FOR TWO RECENT APPROACHES TOWARDS THE MORDELL CONJECTURE

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In this document, we summarize the numerical data related to computations of rational points on the modular curve $X_0(67)^+$. We give a detailed account with some commentary of the computation for one affine patch of $X_0(67)^+$ and summarize the results of the other affine patch.

1. Data for computing rational points on $X_0(67)^+$

We work with the completion $X = X_0(67)^+$ of the curve with affine models

$$Y_{67}\colon y^2 = x^6 + 2x^5 + x^4 - 2x^3 + 2x^2 - 4x + 1 \qquad [\,=: f_{67}(x)\,]\,,$$
$$Y_{67}^*\colon y^2 = x^6 - 4x^5 + 2x^4 - 2x^3 + x^2 + 2x + 1 \qquad [\,=: f_{67}^*(x)\,]\,.$$

The Jacobian J of X has real multiplication by $K := \mathrm{End}_0(J) \otimes \mathbf{Q} = \mathbf{Q}(\sqrt{5})$.

The curve X has 10 known rational points, listed with respect to affine patch Y_{67}:

	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
On Y_{67}	$(0,1)$	$(0,-1)$	$(1,1)$	$(1,-1)$	$(-2,7)$	$(-2,-7)$	$(-1,3)$	$(-1,-3)$	∞^+	∞^-
On Y_{67}^*	∞^+	∞^-	$(1,1)$	$(1,-1)$	$(-1/2,-7/8)$	$(-1/2,7/8)$	$(-1,-3)$	$(-1,3)$	$(0,1)$	$(0,-1)$

We show that this set is precisely $X(\mathbf{Q})$ by computing these points as the zeros of a quadratic Chabauty function for $p = 11$ and ruling out the additional zeros using a Mordell-Weil sieve.

To begin, we note that $X(\mathbf{F}_{11})$ consists of the 12 points

$$(0,1), (0,-1), (1,1), (1,-1), (6,5), (6,-5), (-2,-7), (-2,7), (-1,3), (-1,-3), \infty^+, \infty^-\,.$$

Because of the hyperelliptic involution, we need only consider $\mathbf{Q}_{11}$-points lying in 6 of the 12 residue discs.

1.1. Data for the affine patch Y_{67}.

The affine patch Y_{67} contains all residue discs except near ∞^+ and ∞^-.

We choose differentials

$$\omega_0 = -\frac{dx}{y}, \quad \omega_1 = (-1-x)\cdot\frac{dx}{y}, \quad \omega_2 = (-2+x-x^3-x^4)\cdot\frac{dx}{y},$$
$$\omega_3 = \frac{1}{2}\left(3+x-x^2-x^3\right)\cdot\frac{dx}{y}, \quad \omega_4 = (-x-x^2)\cdot\frac{dx}{y}\,.$$

The basis $(\omega_0, \omega_1, \omega_2, \omega_3)$ for H^1_{dR} induces a K-equivariant splitting of the Hodge filtration. We also choose basepoint $b = (1,1)$.

Remark 1.1. *Note that this is* not *the choice of differentials made in Section 6.3 of* [BBB+19]*. Rather, it is the K-equivariant basis used in Section 6.6. This change explains the small differences in the Hecke operator and p-adic Tate class reported here as compared to* [BBB+19].

In this basis, the action of Frobenius on H^1_{dR} is given (up to an 11^6 term) by the matrix

$$F = \begin{pmatrix} -884191 & 824472 & 776204 & 529507 \\ -824472 & -182336 & 529507 & 812317 \\ -161544 & 558015 & 884194 & 824474 \\ 558015 & 259060 & -824474 & 182333 \end{pmatrix} + O(11^6)\,.$$

The corresponding Hecke operator is given by

$$T_{11} = F^{\mathsf{T}} + 11 \cdot (F^{\mathsf{T}})^{-1} = \begin{pmatrix} 3 & 2 & 0 & 0 \\ -2 & -3 & 0 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 2 & -3 \end{pmatrix}$$

and the corresponding p-adic Tate class is encoded by the matrix

$$Z = \begin{pmatrix} 0 & 0 & 12 & 8 \\ 0 & 0 & -8 & -12 \\ -12 & 8 & 0 & 0 \\ -8 & 12 & 0 & 0 \end{pmatrix}\,.$$

We find that the Hodge filtration of the connection $\mathcal{A}_Z$ is given by

$$\begin{aligned} \eta_Z &= 0 \\ \boldsymbol{\beta}_{\mathrm{Fil}} &= (0,0,0,0)^{\mathsf{T}} \\ \gamma_{\mathrm{Fil}} &= -8x + 8\,. \end{aligned}$$

Next we compute the Frobenius structure. Although we cannot write down the global functions compactly, we give approximations to the local expressions for the remaining terms in the Frobenius structure in Table 1.

We now choose auxiliary points $x_1 = (-2, 7)$ and $x_2 = (-1, 3)$.

Remark 1.2. *In the following, we again make a slight departure from the notation of* [BBB+19]*. As in* [BBB+19], *we set $K_p = \mathbf{Q}(\sqrt{5}) \otimes \mathbf{Q}_p = \mathit{End}_0(J(67)^+) \otimes \mathbf{Q}_p$. However, we use a different basis to define the projections ψ_1 and ψ_2 from K_p to $\mathbf{Q}_p$. Let τ satisfy $\tau^2 + 3\tau + 1 = 0$ corresponding to the Hecke operator T_3. Then, we choose ψ_1 and ψ_2 to satisfy*

$$x = \tau \cdot \psi_1(x) + 1 \cdot \psi_2(x)\,.$$

It is straightforward to recover the values listed in [BBB+19] *from the values in these coordinates. We also set*

$$Q(x) = \det \begin{pmatrix} h_p(\mathrm{per}_p(x)) & \psi_1(\pi(\rho(x))) & \psi_2(\pi(\rho(x))) \\ h_p(\mathrm{per}_p(x_1)) & \psi_1(\pi(\rho(x_1))) & \psi_2(\pi(\rho(x_1))) \\ h_p(\mathrm{per}_p(x_2)) & \psi_1(\pi(\rho(x_2))) & \psi_2(\pi(\rho(x_2))) \end{pmatrix}\,,$$

which differs from the definition of Q in [BBB+19] *only by a multiplicative constant. In particular, the zeros of these p-adic analytic functions will be the same.*

Residue Disc	Uniformizer		Expansion
$](0,1)[$	$t = x$	$\boldsymbol{\alpha}_\phi$	$6823 \cdot 11^1 + 161050 \cdot t^1 + 161050 \cdot t^2 + 53682 \cdot t^3 + 40259 \cdot t^4 + O(t^5 + p^5)$
			$13066 \cdot 11^1 + 161050 \cdot t^1 + 80524 \cdot t^2 + 107365 \cdot t^3 + 161046 \cdot t^4 + O(t^5 + p^5)$
			$111171 + 161049 \cdot t^1 + 80524 \cdot t^2 + 53681 \cdot t^3 + 80519 \cdot t^4 + O(t^5 + p^5)$
			$57682 + 80527 \cdot t^1 + 2 \cdot t^2 + 3 \cdot t^3 + 80532 \cdot t^4 + O(t^5 + p^5)$
		$\boldsymbol{\beta}_\phi$	$145399 + 12 \cdot t^1 + 2 \cdot t^2 + 8 \cdot t^3 + 26 \cdot t^4 + O(t^5 + p^5)$
			$93662 + 2 \cdot t^1 + 12 \cdot t^2 + 9762 \cdot 11^1 \cdot t^3 + 26 \cdot t^4 + O(t^5 + p^5)$
			$4975 \cdot 11^1 + 161047 \cdot t^1 + 107366 \cdot t^3 + 161046 \cdot t^4 + O(t^5 + p^5)$
			$1109 \cdot 11^1 + 4 \cdot t^1 + 10 \cdot t^2 + 9762 \cdot 11^1 \cdot t^3 + 30 \cdot t^4 + O(t^5 + p^5)$
		$\boldsymbol{\gamma}_\phi$	$73972 + 7826 \cdot t^1 + 35873 \cdot t^2 + 77793 \cdot t^3 + 98718 \cdot t^4 + O(t^5 + p^5)$
$](1,1)[$	$t = x - 1$	$\boldsymbol{\alpha}_\phi$	$161050 \cdot t^1 + 80529 \cdot t^2 + 9759 \cdot 11^1 \cdot t^3 + 80648 \cdot t^4 + O(t^5 + p^5)$
			$161049 \cdot t^1 + 80532 \cdot t^2 + 107333 \cdot t^3 + 40494 \cdot t^4 + O(t^5 + p^5)$
			$161048 \cdot t^1 + 80533 \cdot t^2 + 14637 \cdot 11^1 \cdot t^3 + 80825 \cdot t^4 + O(t^5 + p^5)$
			$80527 \cdot t^1 + 161045 \cdot t^2 + 53714 \cdot t^3 + 160850 \cdot t^4 + O(t^5 + p^5)$
		$\boldsymbol{\beta}_\phi$	$8295 + 24 \cdot t^1 + 161009 \cdot t^2 + 53969 \cdot t^3 + 159065 \cdot t^4 + O(t^5 + p^5)$
			$122620 + 161045 \cdot t^1 + 161039 \cdot t^2 + 12 \cdot t^3 + 161035 \cdot t^4 + O(t^5 + p^5)$
			$12986 \cdot 11^1 + 4 \cdot t^1 + 161041 \cdot t^2 + 107422 \cdot t^3 + 160671 \cdot t^4 + O(t^5 + p^5)$
			$830 \cdot 11^1 + 16 \cdot t^1 + 161001 \cdot t^2 + 53949 \cdot t^3 + 159256 \cdot t^4 + O(t^5 + p^5)$
		$\boldsymbol{\gamma}_\phi$	$7 \cdot t^3 + 53646 \cdot t^4 + O(t^5 + p^5)$
$](6,5)[$	$t = x - 6$	$\boldsymbol{\alpha}_\phi$	$802 \cdot 11^1 + 89707 \cdot t^1 + 134093 \cdot t^2 + 103093 \cdot t^3 + 57868 \cdot t^4 + O(t^5 + p^5)$
			$6212 \cdot 11^1 + 144796 \cdot t^1 + 8884 \cdot 11^1 \cdot t^2 + 59475 \cdot t^3 + 120031 \cdot t^4 + O(t^5 + p^5)$
			$2001 + 156367 \cdot t^1 + 81692 \cdot t^2 + 64678 \cdot t^3 + 147463 \cdot t^4 + O(t^5 + p^5)$
			$69670 + 81439 \cdot t^1 + 15512 \cdot t^2 + 112991 \cdot t^3 + 129473 \cdot t^4 + O(t^5 + p^5)$
		$\boldsymbol{\beta}_\phi$	$71127 + 48900 \cdot t^1 + 2087 \cdot 11^1 \cdot t^2 + 91497 \cdot t^3 + 93578 \cdot t^4 + O(t^5 + p^5)$
			$8362 + 12231 \cdot 11^1 \cdot t^1 + 34425 \cdot t^2 + 101755 \cdot t^3 + 156564 \cdot t^4 + O(t^5 + p^5)$
			$2196 \cdot 11^1 + 7197 \cdot 11^1 \cdot t^1 + 22069 \cdot t^2 + 117112 \cdot t^3 + 56270 \cdot t^4 + O(t^5 + p^5)$
			$537 \cdot 11^2 + 107461 \cdot t^1 + 61107 \cdot t^2 + 111044 \cdot t^3 + 149929 \cdot t^4 + O(t^5 + p^5)$
		$\boldsymbol{\gamma}_\phi$	$44279 + 90566 \cdot t^1 + 7031 \cdot 11^1 \cdot t^2 + 160236 \cdot t^3 + 25413 \cdot t^4 + O(t^5 + p^5)$
$](-2,7)[$	$t = x + 2$	$\boldsymbol{\alpha}_\phi$	$5538 \cdot 11^1 + 92029 \cdot t^1 + 939 \cdot t^2 + 4558 \cdot t^3 + 145334 \cdot t^4 + O(t^5 + p^5)$
			$5717 \cdot 11^1 + 69022 \cdot t^1 + 125601 \cdot t^2 + 157119 \cdot t^3 + 99661 \cdot t^4 + O(t^5 + p^5)$
			$72714 + 138042 \cdot t^1 + 8345 \cdot 11^1 \cdot t^2 + 110571 \cdot t^3 + 2718 \cdot 11^1 \cdot t^4 + O(t^5 + p^5)$
			$96420 + 103533 \cdot t^1 + 136635 \cdot t^2 + 106243 \cdot t^3 + 9505 \cdot 11^1 \cdot t^4 + O(t^5 + p^5)$
		$\boldsymbol{\beta}_\phi$	$135928 + 8368 \cdot 11^1 \cdot t^1 + 60094 \cdot t^2 + 77918 \cdot t^3 + 8472 \cdot 11^1 \cdot t^4 + O(t^5 + p^5)$
			$89811 + 92018 \cdot t^1 + 119266 \cdot t^2 + 65821 \cdot t^3 + 4035 \cdot 11^1 \cdot t^4 + O(t^5 + p^5)$
			$4424 \cdot 11^1 + 69019 \cdot t^1 + 133817 \cdot t^2 + 712 \cdot 11^2 \cdot t^3 + 141465 \cdot t^4 + O(t^5 + p^5)$
			$5812 \cdot 11^1 + 69019 \cdot t^1 + 110810 \cdot t^2 + 83648 \cdot t^3 + 127791 \cdot t^4 + O(t^5 + p^5)$
		$\boldsymbol{\gamma}_\phi$	$140769 + 79795 \cdot t^1 + 97712 \cdot t^2 + 54233 \cdot t^3 + 86822 \cdot t^4 + O(t^5 + p^5)$
$](-1,3)[$	$t = x + 1$	$\boldsymbol{\alpha}_\phi$	$12887 \cdot 11^1 + 107367 \cdot t^1 + 14912 \cdot t^2 + 241 \cdot 11^1 \cdot t^3 + 104974 \cdot t^4 + O(t^5 + p^5)$
			$11838 \cdot 11^1 + 134209 \cdot t^2 + 63625 \cdot t^3 + 3841 \cdot 11^1 \cdot t^4 + O(t^5 + p^5)$
			$30926 + 161050 \cdot t^1 + 98420 \cdot t^2 + 149121 \cdot t^3 + 12000 \cdot 11^1 \cdot t^4 + O(t^5 + p^5)$
			$99189 + 26842 \cdot t^1 + 5965 \cdot t^2 + 92124 \cdot t^3 + 73272 \cdot t^4 + O(t^5 + p^5)$
		$\boldsymbol{\beta}_\phi$	$132079 + 107378 \cdot t^1 + 59648 \cdot t^2 + 50372 \cdot t^3 + 84538 \cdot t^4 + O(t^5 + p^5)$
			$110837 + 161045 \cdot t^1 + 53685 \cdot t^2 + 43742 \cdot t^3 + 2652 \cdot t^4 + O(t^5 + p^5)$
			$14362 \cdot 11^1 + 161047 \cdot t^1 + 71578 \cdot t^2 + 5965 \cdot t^3 + 116425 \cdot t^4 + O(t^5 + p^5)$
			$5793 \cdot 11^1 + 53681 \cdot t^1 + 119298 \cdot t^2 + 62963 \cdot t^3 + 10678 \cdot t^4 + O(t^5 + p^5)$
		$\boldsymbol{\gamma}_\phi$	$61011 + 127824 \cdot t^1 + 14449 \cdot 11^1 \cdot t^2 + 128648 \cdot t^3 + 111547 \cdot t^4 + O(t^5 + p^5)$

TABLE 1. Approximation to Frobenius structure on Y_{67} in residue discs. N.B. $11^5 = 161051$.

Residue Disc	Uniformizer		Expansion
$]\!(0,1)\![$	$t = x$	$h_p(\mathrm{per}_p(x))$	$1138 \cdot 11^1 + 952 \cdot 11^1 \cdot t^1 + 13548 \cdot t^2 + 11802 \cdot t^3 + O(t^4 + p^4)$
		$\psi_1(\pi(\rho(x)))$	$11331 + 128148 \cdot 11^{-1} \cdot t^1 + 1466547 \cdot 11^{-2} \cdot t^2 + 1432389 \cdot 11^{-2} \cdot t^3 + O(t^4 + p^4)$
		$\psi_1(\pi(\rho(x)))$	$652 \cdot 11^1 + 145206 \cdot 11^{-1} \cdot t^1 + 1484535 \cdot 11^{-2} \cdot t^2 + 1639775 \cdot 11^{-2} \cdot t^3 + O(t^4 + p^4)$
$]\!(-1,3)\![$	$t = x+1$	$h_p(\mathrm{per}_p(x))$	$328 \cdot 11^1 + 2330 \cdot t^1 + 14162 \cdot t^2 + 11768 \cdot t^3 + O(t^4 + p^4)$
		$\psi_1(\pi(\rho(x)))$	$2402 + 140070 \cdot 11^{-1} \cdot t^1 + 891442 \cdot 11^{-2} \cdot t^2 + 1221749 \cdot 11^{-2} \cdot t^3 + O(t^4 + p^4)$
		$\psi_1(\pi(\rho(x)))$	$6043 + 18617 \cdot 11^{-1} \cdot t^1 + 735584 \cdot 11^{-2} \cdot t^2 + 1120107 \cdot 11^{-2} \cdot t^3 + O(t^4 + p^4)$
$]\!(-2,4)\![$	$t = x+2$	$h_p(\mathrm{per}_p(x))$	$964 \cdot 11^1 + 2326 \cdot t^1 + 13326 \cdot t^2 + 13883 \cdot t^3 + O(t^4 + p^4)$
		$\psi_1(\pi(\rho(x)))$	$7070 + 125082 \cdot 11^{-1} \cdot t^1 + 1014117 \cdot 11^{-2} \cdot t^2 + 1295832 \cdot 11^{-2} \cdot t^3 + O(t^4 + p^4)$
		$\psi_1(\pi(\rho(x)))$	$1014 + 123485 \cdot 11^{-1} \cdot t^1 + 1204266 \cdot 11^{-2} \cdot t^2 + 1662212 \cdot 11^{-2} \cdot t^3 + O(t^4 + p^4)$
$]\!(6,5)\![$	$t = x-6$	$h_p(\mathrm{per}_p(x))$	$373 \cdot 11^1 + 12147 \cdot t^1 + 4728 \cdot t^2 + 11464 \cdot t^3 + O(t^4 + p^4)$
		$\psi_1(\pi(\rho(x)))$	$13568 + 140457 \cdot 11^{-1} \cdot t^1 + 720166 \cdot 11^{-2} \cdot t^2 + 520393 \cdot 11^{-2} \cdot t^3 + O(t^4 + p^4)$
		$\psi_1(\pi(\rho(x)))$	$2723 + 23654 \cdot 11^{-1} \cdot t^1 + 1468936 \cdot 11^{-2} \cdot t^2 + 121323 \cdot 11^{-2} \cdot t^3 + O(t^4 + p^4)$
$]\!(1,1)\![$	$t = x-1$	$h_p(\mathrm{per}_p(x))$	$4646 \cdot t^1 + 106 \cdot 11^1 \cdot t^2 + 8667 \cdot t^3 + O(t^4 + p^4)$
		$\psi_1(\pi(\rho(x)))$	$106413 \cdot 11^{-1} \cdot t^1 + 1041741 \cdot 11^{-2} \cdot t^2 + 684782 \cdot 11^{-2} \cdot t^3 + O(t^4 + p^4)$
		$\psi_1(\pi(\rho(x)))$	$78187 \cdot 11^{-1} \cdot t^1 + 418940 \cdot 11^{-2} \cdot t^2 + 1507655 \cdot 11^{-2} \cdot t^3 + O(t^4 + p^4)$

TABLE 2. Approximation to 11-adic height function and bilinear forms on Y_{67} in residue discs. N.B. $11^5 = 161051$.

Residue Disc	Uniformizer	$Q(x)$
$]\!(0,1)\![$	$t = x$	$856 \cdot t^1 + 4218 \cdot 11^{-1} \cdot t^2 + 98595 \cdot 11^{-1} \cdot t^3 + 39132 \cdot 11^{-1} \cdot t^4 + O(t^5 + p^4)$
$]\!(-1,3)\![$	$t = x+1$	$13533 \cdot t^1 + 78652 \cdot 11^{-1} \cdot t^2 + 30075 \cdot 11^{-1} \cdot t^3 + 28231 \cdot 11^{-1} \cdot t^4 + O(t^5 + p^4)$
$]\!(-2,4)\![$	$t = x+2$	$1277 \cdot 11^1 \cdot t^1 + 152509 \cdot 11^{-1} \cdot t^2 + 100231 \cdot 11^{-1} \cdot t^3 + 116176 \cdot 11^{-1} \cdot t^4 + O(t^5 + p^4)$
$]\!(6,5)\![$	$t = x-6$	$190 \cdot 11^1 + 12665 \cdot t^1 + 29894 \cdot 11^{-1} \cdot t^2 + 40650 \cdot 11^{-1} \cdot t^3 + 159219 \cdot 11^{-1} \cdot t^4 + O(t^5 + p^4)$
$]\!(1,1)\![$	$t = x-1$	$1624 \cdot t^1 + 140874 \cdot 11^{-1} \cdot t^2 + 75206 \cdot 11^{-1} \cdot t^3 + 133178 \cdot 11^{-1} \cdot t^4 + O(t^5 + p^4)$

TABLE 3. Power series vanishing on the zeros of each point of $Y_{67}(\mathbf{Q})$ by 11-adic residue disc.

We compute:

$$h_p(\mathrm{per}_p(x_1)) = 7 \cdot 11^1 + 10 \cdot 11^2 + 7 \cdot 11^3 + 4 \cdot 11^4 + O(11^5)$$
$$h_p(\mathrm{per}_p(x_2)) = 9 \cdot 11^1 + 7 \cdot 11^2 + 2 \cdot 11^3 + 1 \cdot 11^4 + O(11^5)$$
$$\psi_1(\pi(\rho(x_1))) = 8 + 4 \cdot 11^1 + 3 \cdot 11^2 + 5 \cdot 11^3 + 9 \cdot 11^4 + O(11^5)$$
$$\psi_2(\pi(\rho(x_1))) = 2 + 4 \cdot 11^1 + 8 \cdot 11^2 + 6 \cdot 11^4 + O(11^5)$$
$$\psi_1(\pi(\rho(x_2))) = 4 + 9 \cdot 11^1 + 8 \cdot 11^2 + 1 \cdot 11^3 + 1 \cdot 11^4 + O(11^5)$$
$$\psi_2(\pi(\rho(x_2))) = 4 + 10 \cdot 11^1 + 5 \cdot 11^2 + 4 \cdot 11^3 + 1 \cdot 11^4 + O(11^5)$$

The top row is listed residue disc by residue disc in Table 2. The quadratic Chabauty function Q is listed residue disc by residue disc in Table 3. Finally, the zeros of Q are listed in Table 4.

Residue Disc	x-Coordinates of Candidate Zeros
$](0, \pm 1)[$	$7 \cdot 11^1 + 3 \cdot 11^3 + 3 \cdot 11^4 + O(11^5)$ 0
$](-1, \pm 3)[$	$10 + 3 \cdot 11^1 + 9 \cdot 11^2 + 10 \cdot 11^3 + 1 \cdot 11^4 + O(11^5)$ -1
$](-2, \pm 4)[$	$9 + 10 \cdot 11^1 + 1 \cdot 11^2 + 8 \cdot 11^3 + O(11^5)$ -2
$](6, \pm 5)[$	$6 + 5 \cdot 11^1 + 8 \cdot 11^2 + 2 \cdot 11^3 + 4 \cdot 11^4 + O(11^5)$ $6 + 7 \cdot 11^1 + 5 \cdot 11^3 + 1 \cdot 11^4 + O(11^5)$
$](1, \pm 1)[$	$1 + 6 \cdot 11^1 + 6 \cdot 11^2 + 8 \cdot 11^3 + 7 \cdot 11^4 + O(11^5)$ 1

TABLE 4. 11-adic points on Y_{67} satisfying the quadratic Chabauty function.

1.2. **Data for the affine patch Y_{67}^*.** In this section, we use coordinates in terms of the affine patch Y_{67}^*, recycling notation as necessary. In these coordinates, the affine patch Y_{67} contains all residue discs except near the new ∞^+ and ∞^-, corresponding to points $(0,1)$ and $(0,-1)$ from the previous section.

We choose differentials

$$\begin{aligned}
\omega_0 &= (-1)\cdot\frac{dx}{y}\\
\omega_1 &= (2-\,x)\cdot\frac{dx}{y}\\
\omega_2 &= \left(6+3\,x+\,x^2+2\,x^3-\,x^4\right)\cdot\frac{dx}{y}\\
\omega_3 &= \left(\frac{19}{10}+\frac{3}{10}\,x+\,x^2-\frac{1}{2}\,x^3\right)\cdot\frac{dx}{y}\\
\omega_4 &= \left(1+2\,x-\,x^2\right)\cdot\frac{dx}{y}
\end{aligned}$$

The basis $(\omega_0,\omega_1,\omega_2,\omega_3)$ for H^1_{dR} induces a K-equivariant splitting of the Hodge filtration. We also choose basepoint $b=(1,1)$.

In this basis, the action of Frobenius on H^1_{dR} is given (up to an 11^6 term) by the matrix

$$F=\begin{pmatrix} 1207756 & 484605 & 529507 & 465850\\ 1120097 & 556292 & 465850 & 337898\\ 1515147 & 1237567 & 563800 & 651474\\ 1237567 & 1643162 & 1286954 & 1215274 \end{pmatrix}+O(11^6)\,.$$

The corresponding Hecke operator is given by

$$T_{11}=F^{\intercal}+11\cdot(F^{\intercal})^{-1}=\begin{pmatrix} -5 & 10 & 0 & 0\\ -2 & 5 & 0 & 0\\ 0 & 0 & -5 & -2\\ 0 & 0 & 10 & 5 \end{pmatrix}$$

and the corresponding p-adic Tate class is encoded by the matrix

$$Z=\begin{pmatrix} 0 & 0 & -20 & 40\\ 0 & 0 & -8 & 20\\ 20 & 8 & 0 & 0\\ -40 & -20 & 0 & 0 \end{pmatrix}.$$

We find that the Hodge filtration of the connection $\mathcal{A}_Z$ is given by

$$\begin{aligned}
\eta_Z &= 0\\
\boldsymbol{\beta}_{\mathrm{Fil}} &= (0,0,0,0)^{\intercal}\\
\gamma_{\mathrm{Fil}} &= -8x+8\,.
\end{aligned}$$

Next we compute the Frobenius structure. Although we cannot write down the global functions compactly, we give approximations to the local expressions for the remaining terms in the Frobenius structure

$$G := \begin{pmatrix} 1 & 0 & 0 \\ \boldsymbol{\alpha}_\phi & F & 0 \\ \gamma_\phi & \boldsymbol{\beta}_\phi^\intercal & 1 \end{pmatrix}$$

in Table 5.

We now choose auxiliary points $x_1 = (-1/2, -7/2)$ and $x_2 = (-1, -3)$.

Remark 1.3. *As in Section 1.1, we set $K_p = \mathbf{Q}(\sqrt{5}) \otimes \mathbf{Q}_p = End_0(J(67)^+) \otimes \mathbf{Q}_p$. We define the projections ψ_1 and ψ_2 from K_p to $\mathbf{Q}_p$ as follows. Let τ satisfy $\tau^2 + 3\tau + 1 = 0$ corresponding to the Hecke operator T_3. Then, we choose ψ_1 and ψ_2 to satisfy*

$$x = \tau \cdot \psi_1(x) + 1 \cdot \psi_2(x)\,.$$

We also set

$$Q(x) = \det \begin{pmatrix} h_p(\mathrm{per}_p(x)) & \psi_1(\pi(\rho(x))) & \psi_2(\pi(\rho(x))) \\ h_p(\mathrm{per}_p(x_1)) & \psi_1(\pi(\rho(x_1))) & \psi_2(\pi(\rho(x_1))) \\ h_p(\mathrm{per}_p(x_2)) & \psi_1(\pi(\rho(x_2))) & \psi_2(\pi(\rho(x_2))) \end{pmatrix}.$$

We compute:

$$\begin{aligned}
h_p(\mathrm{per}_p(x_1)) &= 7 \cdot 11^1 + 6 \cdot 11^2 + 10 \cdot 11^3 + 5 \cdot 11^4 + O(11^5) \\
h_p(\mathrm{per}_p(x_2)) &= 9 \cdot 11^1 + 1 \cdot 11^3 + 8 \cdot 11^4 + O(11^5) \\
\psi_1(\pi(\rho(x_1))) &= 10 + 7 \cdot 11^1 + 10 \cdot 11^2 + 5 \cdot 11^3 + 10 \cdot 11^4 + O(11^5) \\
\psi_2(\pi(\rho(x_1))) &= 1 + 7 \cdot 11^1 + 8 \cdot 11^2 + 9 \cdot 11^3 + 1 \cdot 11^4 + O(11^5) \\
\psi_1(\pi(\rho(x_2))) &= 9 + 2 \cdot 11^1 + 9 \cdot 11^2 + 9 \cdot 11^3 + 2 \cdot 11^4 + O(11^5) \\
\psi_2(\pi(\rho(x_2))) &= 7 + 6 \cdot 11^1 + 4 \cdot 11^3 + 6 \cdot 11^4 + O(11^5)
\end{aligned}$$

The top row of the matrix used to determine the quadratic Chabauty function is listed residue disc by residue disc in Table 6. We summarize the quadratic Chabauty function Q in Table 7. In particular, from the valuation of the coefficients and the theory of Newton polygons, we can see that there are 2 zeros (with multiplicity) on each residue disc and that all of the zeros are simple. We collect the zeros in Table 8

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Residue Disc	Uniformizer		Expansion
$](0,1)[$	$t=x$	$\boldsymbol{\alpha}_\phi$	$290\cdot 11^1+161050\cdot t^1+80526\cdot t^2+107367\cdot t^3+O(t^5+p^5)$
			$3039\cdot 11^1+2\cdot t^1+80524\cdot t^2+1\cdot t^3+120788\cdot t^4+O(t^5+p^5)$
			$132455+6\cdot t^1+80524\cdot t^2+53685\cdot t^3+1\cdot t^4+O(t^5+p^5)$
			$140915+16107\cdot t^1+128840\cdot t^2+75158\cdot t^3+48315\cdot t^4+O(t^5+p^5)$
		$\boldsymbol{\beta}_\phi$	$152892+4\cdot 11^1\cdot t^1+2\cdot t^2+161043\cdot t^3+32\cdot t^4+O(t^5+p^5)$
			$46663+10\cdot t^1+4\cdot t^2+53677\cdot t^3+14\cdot t^4+O(t^5+p^5)$
			$11326\cdot 11^1+4\cdot t^1+2\cdot t^2+107366\cdot t^3+2\cdot t^4+O(t^5+p^5)$
			$5800\cdot 11^1+161041\cdot t^2+107374\cdot t^3+161046\cdot t^4+O(t^5+p^5)$
		$\boldsymbol{\gamma}_\phi$	$155658+38164\cdot t^1+123213\cdot t^2+84977\cdot t^3+76737\cdot t^4+O(t^5+p^5)$
$](2,4)[$	$t=x-2$	$\boldsymbol{\alpha}_\phi$	$1554\cdot 11^1+142249\cdot t^1+137898\cdot t^2+5405\cdot 11^1\cdot t^3+2385\cdot 11^1\cdot t^4+O(t^5+p^5)$
			$8964\cdot 11^1+151650\cdot t^2+91932\cdot t^3+7714\cdot 11^1\cdot t^4+O(t^5+p^5)$
			$1495\cdot 11^1+139781\cdot t^1+38945\cdot t^2+91701\cdot t^3+73228\cdot t^4+O(t^5+p^5)$
			$42949+47005\cdot t^1+74111\cdot t^2+64901\cdot t^3+125634\cdot t^4+O(t^5+p^5)$
		$\boldsymbol{\beta}_\phi$	$12802\cdot 11^1+10015\cdot 11^1\cdot t^1+69174\cdot t^2+43235\cdot t^3+143373\cdot t^4+O(t^5+p^5)$
			$111597+17097\cdot t^1+117748\cdot t^2+79792\cdot t^3+5756\cdot t^4+O(t^5+p^5)$
			$11851\cdot 11^1+53938\cdot t^1+55115\cdot t^2+8056\cdot t^3+7716\cdot 11^1\cdot t^4+O(t^5+p^5)$
			$13809\cdot 11^1+53175\cdot t^1+13217\cdot t^2+29514\cdot t^3+783\cdot 11^1\cdot t^4+O(t^5+p^5)$
		$\boldsymbol{\gamma}_\phi$	$82888+154335\cdot t^1+43370\cdot t^2+84574\cdot t^3+118477\cdot t^4+O(t^5+p^5)$
$](-\frac{1}{2},\frac{7}{8})[$	$t=x+\frac{1}{2}$	$\boldsymbol{\alpha}_\phi$	$11201\cdot 11^1+92028\cdot t^1+84985\cdot t^2+32035\cdot t^3+134100\cdot t^4+O(t^5+p^5)$
			$1106\cdot 11^1+92032\cdot t^1+75128\cdot t^2+3411\cdot t^3+51141\cdot t^4+O(t^5+p^5)$
			$84240+34516\cdot t^1+52830\cdot t^2+104304\cdot t^3+9726\cdot t^4+O(t^5+p^5)$
			$47535+1046\cdot 11^1\cdot t^1+57379\cdot t^2+18321\cdot t^3+124246\cdot t^4+O(t^5+p^5)$
		$\boldsymbol{\beta}_\phi$	$24874+69029\cdot t^1+49848\cdot t^2+64832\cdot t^3+5110\cdot 11^1\cdot t^4+O(t^5+p^5)$
			$79084+46008\cdot t^1+80315\cdot t^2+145910\cdot t^3+8653\cdot t^4+O(t^5+p^5)$
			$13544\cdot 11^1+115041\cdot t^2+137267\cdot t^3+129892\cdot t^4+O(t^5+p^5)$
			$8273\cdot 11^1+46026\cdot t^1+70430\cdot t^2+61212\cdot t^3+9621\cdot 11^1\cdot t^4+O(t^5+p^5)$
		$\boldsymbol{\gamma}_\phi$	$54484+77987\cdot t^1+90744\cdot t^2+48622\cdot t^3+20959\cdot t^4+O(t^5+p^5)$
$](-1,3)[$	$t=x+1$	$\boldsymbol{\alpha}_\phi$	$9973\cdot 11^1+107367\cdot t^1+65613\cdot t^2+123936\cdot t^3+58562\cdot t^4+O(t^5+p^5)$
			$10911\cdot 11^1+1\cdot t^1+98421\cdot t^2+155087\cdot t^3+78317\cdot t^4+O(t^5+p^5)$
			$136018+53684\cdot t^1+68598\cdot t^2+110683\cdot t^3+85093\cdot t^4+O(t^5+p^5)$
			$150562+102000\cdot t^1+145543\cdot t^2+101138\cdot t^3+35230\cdot t^4+O(t^5+p^5)$
		$\boldsymbol{\beta}_\phi$	$160374+53649\cdot t^1+59668\cdot t^2+100752\cdot t^3+131609\cdot t^4+O(t^5+p^5)$
			$43278+161033\cdot t^1+53689\cdot t^2+151112\cdot t^3+137195\cdot t^4+O(t^5+p^5)$
			$3587\cdot 11^1+107366\cdot t^1+155086\cdot t^2+145808\cdot t^3+134836\cdot t^4+O(t^5+p^5)$
			$8843\cdot 11^1+107374\cdot t^1+7592\cdot 11^1\cdot t^2+6630\cdot t^3+43596\cdot t^4+O(t^5+p^5)$
		$\boldsymbol{\gamma}_\phi$	$146140+1146\cdot 11^1\cdot t^1+24257\cdot t^2+123778\cdot t^3+69096\cdot t^4+O(t^5+p^5)$
$](1,1)[$	$t=x-1$	$\boldsymbol{\alpha}_\phi$	$161050\cdot t^1+161049\cdot t^2+14640\cdot 11^1\cdot t^3+3654\cdot 11^1\cdot t^4+O(t^5+p^5)$
			$1\cdot t^1+80527\cdot t^2+107377\cdot t^3+666\cdot 11^2\cdot t^4+O(t^5+p^5)$
			$1\cdot 11^1\cdot t^1+80551\cdot t^2+107498\cdot t^3+81340\cdot t^4+O(t^5+p^5)$
			$48318\cdot t^1+32216\cdot t^2+64451\cdot t^3+64612\cdot t^4+O(t^5+p^5)$
		$\boldsymbol{\beta}_\phi$	$80423+112\cdot t^1+278\cdot t^2+55073\cdot t^3+8626\cdot t^4+O(t^5+p^5)$
			$33762+34\cdot t^1+8\cdot 11^1\cdot t^2+54117\cdot t^3+244\cdot 11^1\cdot t^4+O(t^5+p^5)$
			$12156\cdot 11^1+12\cdot t^1+28\cdot t^2+107510\cdot t^3+81\cdot 11^1\cdot t^4+O(t^5+p^5)$
			$6625\cdot 11^1+161031\cdot t^1+161001\cdot t^2+53437\cdot t^3+14501\cdot 11^1\cdot t^4+O(t^5+p^5)$
		$\boldsymbol{\gamma}_\phi$	$53678\cdot t^3+107341\cdot t^4+O(t^5+p^5)$

TABLE 5. Approximation to Frobenius structure on Y_{67} in residue discs. N.B. $11^5=161051$.

Residue Disc	Uniformizer		Expansion
$]（0,1)[$	$t = x$	$h_p(\mathrm{per}_p(x))$	$28 \cdot 11^2 + 3754 \cdot t^1 + 13292 \cdot t^2 + 82 \cdot 11^1 \cdot t^3 + O(t^4 + p^4)$
		$\psi_1(\pi(\rho(x)))$	$3687 + 153768 \cdot 11^{-1} \cdot t^1 + 1663713 \cdot 11^{-2} \cdot t^2 + 992380 \cdot 11^{-2} \cdot t^3 + O(t^4 + p^4)$
		$\psi_1(\pi(\rho(x)))$	$1337 + 47754 \cdot 11^{-1} \cdot t^1 + 1587215 \cdot 11^{-2} \cdot t^2 + 168700 \cdot 11^{-2} \cdot t^3 + O(t^4 + p^4)$
$](2,4)[$	$t = x - 2$	$h_p(\mathrm{per}_p(x))$	$1323 \cdot 11^1 + 5722 \cdot t^1 + 860 \cdot 11^1 \cdot t^2 + 13048 \cdot t^3 + O(t^4 + p^4)$
		$\psi_1(\pi(\rho(x)))$	$5581 + 51954 \cdot 11^{-1} \cdot t^1 + 1731418 \cdot 11^{-2} \cdot t^2 + 287741 \cdot 11^{-2} \cdot t^3 + O(t^4 + p^4)$
		$\psi_1(\pi(\rho(x)))$	$11020 + 14220 \cdot t^1 + 1756516 \cdot 11^{-2} \cdot t^2 + 79924 \cdot 11^{-2} \cdot t^3 + O(t^4 + p^4)$
$](-\frac{1}{2},\frac{7}{8})[$	$t = x + \frac{1}{2}$	$h_p(\mathrm{per}_p(x))$	$1283 \cdot 11^1 + 629 \cdot t^1 + 8367 \cdot t^2 + 9954 \cdot t^3 + O(t^4 + p^4)$
		$\psi_1(\pi(\rho(x)))$	$7952 + 85314 \cdot 11^{-1} \cdot t^1 + 1510522 \cdot 11^{-2} \cdot t^2 + 1673282 \cdot 11^{-2} \cdot t^3 + O(t^4 + p^4)$
		$\psi_1(\pi(\rho(x)))$	$13025 + 98563 \cdot 11^{-1} \cdot t^1 + 593529 \cdot 11^{-2} \cdot t^2 + 334533 \cdot 11^{-2} \cdot t^3 + O(t^4 + p^4)$
$](-1,3)[$	$t = x + 1$	$h_p(\mathrm{per}_p(x))$	$130 \cdot 11^1 + 7801 \cdot t^1 + 104 \cdot 11^2 \cdot t^2 + 3710 \cdot t^3 + O(t^4 + p^4)$
		$\psi_1(\pi(\rho(x)))$	$13099 + 101439 \cdot 11^{-1} \cdot t^1 + 1011025 \cdot 11^{-2} \cdot t^2 + 73020 \cdot 11^{-2} \cdot t^3 + O(t^4 + p^4)$
		$\psi_1(\pi(\rho(x)))$	$5397 + 279 \cdot 11^{-1} \cdot t^1 + 589404 \cdot 11^{-2} \cdot t^2 + 718060 \cdot 11^{-2} \cdot t^3 + O(t^4 + p^4)$
$](1,1)[$	$t = x - 1$	$h_p(\mathrm{per}_p(x))$	$2746 \cdot t^1 + 7794 \cdot t^2 + 12047 \cdot t^3 + O(t^4 + p^4)$
		$\psi_1(\pi(\rho(x)))$	$91448 \cdot 11^{-1} \cdot t^1 + 1564841 \cdot 11^{-2} \cdot t^2 + 1701796 \cdot 11^{-2} \cdot t^3 + O(t^4 + p^4)$
		$\psi_1(\pi(\rho(x)))$	$153768 \cdot 11^{-1} \cdot t^1 + 160195 \cdot 11^{-2} \cdot t^2 + 142931 \cdot 11^{-1} \cdot t^3 + O(t^4 + p^4)$

TABLE 6. Approximation to 11-adic height function and bilinear forms on Y_{67}^* in residue discs. N.B. $11^5 = 161051$.

Residue Disc	Uniformizer	$Q(x)$
$](0,1)[$	$t = x$	$9373 \cdot t^1 + 158211 \cdot 11^{-1} \cdot t^2 + 79270 \cdot 11^{-1} \cdot t^3 + 119233 \cdot 11^{-1} \cdot t^4 + O(t^5 + p^4)$
$](2,4)[$	$t = x - 2$	$116 \cdot 11^2 + 3361 \cdot t^1 + 82938 \cdot 11^{-1} \cdot t^2 + 12274 \cdot t^3 + 4116 \cdot 11^{-1} \cdot t^4 + O(t^5 + p^4)$
$](-\frac{1}{2},\frac{7}{8})[$	$t = x + \frac{1}{2}$	$1167 \cdot 11^1 \cdot t^1 + 11926 \cdot 11^{-1} \cdot t^2 + 12279 \cdot 11^{-1} \cdot t^3 + 21971 \cdot 11^{-1} \cdot t^4 + O(t^5 + p^4)$
$](-1,3)[$	$t = x + 1$	$9018 \cdot t^1 + 126538 \cdot 11^{-1} \cdot t^2 + 120441 \cdot 11^{-1} \cdot t^3 + 94103 \cdot 11^{-1} \cdot t^4 + O(t^5 + p^4)$
$](1,1)[$	$t = x - 1$	$11043 \cdot t^1 + 111120 \cdot 11^{-1} \cdot t^2 + 21539 \cdot 11^{-1} \cdot t^3 + 8768 \cdot t^4 + O(t^5 + p^4)$

TABLE 7. Power series vanishing on the zeros of each point of $Y_{67}^*(\mathbf{Q})$ by 11-adic residue disc.

Residue Disc	x-Coordinates of Candidate Zeros
$](0,\pm 1)[$	$0 + 6 \cdot 11^1 + 4 \cdot 11^2 + 10 \cdot 11^3 + O(11^5)$
	0
$](2,\pm 4)[$	$2 + 10 \cdot 11^2 + 10 \cdot 11^3 + 10 \cdot 11^4 + O(11^5)$
	$2 + 3 \cdot 11^1 + 7 \cdot 11^2 + 2 \cdot 11^3 + 1 \cdot 11^4 + O(11^5)$
$](-\frac{1}{2},\pm\frac{7}{8})[$	$5 + 5 \cdot 11^1 + 10 \cdot 11^2 + 8 \cdot 11^3 + 4 \cdot 11^4 + O(11^5)$
	$-\frac{1}{2}$
$](-1,\pm 3)[$	$10 + 6 \cdot 11^1 + 7 \cdot 11^2 + 5 \cdot 11^3 + 5 \cdot 11^4 + O(11^5)$
	-1
$](1,\pm 1)[$	$1 + 5 \cdot 11^1 + 7 \cdot 11^2 + 4 \cdot 11^3 + O(11^5)$
	1

TABLE 8. 11-adic points on Y_{67}^* satisfying the quadratic Chabauty function.

References

[BBB+19] Jennifer S. Balakrishnan, Alex J. Best, Francesca Bianchi, Brian Lawrence, J. Steffen Müller, Nicholas Triantafillou, and Jan Vonk. Two recent p-adic approaches towards the (effective) Mordell conjecture. https://arxiv.org/abs/1910.12755, 2019. ↑2.