Random Weighting in LASSO Regression

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Tun Lee Ng

Email: tng25@wisc.edu

Statistics PhD candidate, University of Wisconsin-Madison

Advisor: Prof. Michael A. Newton

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Motivation

Consider the linear model

$$Y = X\boldsymbol{\beta} + \boldsymbol{\epsilon}.$$

- Bayesian: Gaussian likelihood with Double Exponential Prior under the Bayesian LASSO (Park and Casella, 2008)

$$p(\boldsymbol{\beta}|\boldsymbol{y}) \propto \exp\left\{-\sum_{i=1}^{n}(y_i - \boldsymbol{x}_i'\boldsymbol{\beta})^2 - \lambda_n \sum_{j=1}^{p_n} |\beta_j|\right\}.$$

- Deploy MCMC to draw B posterior samples from $p(\boldsymbol{\beta}|\boldsymbol{y})$:

$$\left\{\boldsymbol{\beta}_{(1)}^{\text{MCMC}}, \cdots, \boldsymbol{\beta}_{(B)}^{\text{MCMC}}\right\}$$

- Other methods? 1 option: Weighted Bootstrap / **Random-weighting**

Random-weighting:

Newton et al. (2020):

- Assign random weights $W_i \overset{iid}{\sim} F_w$ on negative log posterior density $-\log p(\boldsymbol{\beta}|\boldsymbol{y})$ and optimize

$$\arg\min_{\boldsymbol{\beta}} \left\{ \sum_{i=1}^{n} W_i (y_i - \boldsymbol{x}_i' \boldsymbol{\beta})^2 + \lambda_n \sum_{j=1}^{p_n} W_{0,j} |\beta_j| \right\}$$

  to get one solution $\widehat{\boldsymbol{\beta}}^w_{n(1)}$.

- Repeat: draw another set of random weights $W_i \overset{iid}{\sim} F_w$, assign it onto the objective function above, and optimize. Now we have 2 random-weighting (RW) samples $\left\{ \widehat{\boldsymbol{\beta}}^w_{n(1)}, \widehat{\boldsymbol{\beta}}^w_{n(2)} \right\}$.
- Repeat this B times, and obtain B random-weighting samples:

$$\left\{ \widehat{\boldsymbol{\beta}}^w_{n(1)}, \cdots, \widehat{\boldsymbol{\beta}}^w_{n(B)} \right\}.$$

* More on $W_{0,j}$ later.

Random-weighting:

- Assign random weights $W_i \overset{iid}{\sim} F_w$ on negative log posterior density $-\log p(\boldsymbol{\beta}|\boldsymbol{y})$ and optimize with a **2-step procedure** (more on this later)

$$\arg\min_{\boldsymbol{\beta}} \left\{ \sum_{i=1}^{n} W_i (y_i - \boldsymbol{x}_i' \boldsymbol{\beta})^2 + \lambda_n \sum_{j=1}^{p_n} W_{0,j} |\beta_j| \right\}$$

  to get one solution $\widehat{\boldsymbol{\beta}}^w_{n(1)}$.

- Repeat: draw another set of random weights $W_i \overset{iid}{\sim} F_w$, assign it onto the objective function above, and optimize with the **2-step procedure**. Now we have 2 random-weighting (RW) samples $\left\{ \widehat{\boldsymbol{\beta}}^w_{n(1)}, \widehat{\boldsymbol{\beta}}^w_{n(2)} \right\}$.

- Repeat this B times, and obtain B random-weighting samples:

$$\left\{ \widehat{\boldsymbol{\beta}}^w_{n(1)}, \cdots, \widehat{\boldsymbol{\beta}}^w_{n(B)} \right\}.$$

* More on $W_{0,j}$ later.

The **two-step procedure**

- Step 1: Assign random weights, optimize

$$\min_{\boldsymbol{\beta}} \left\{ \sum_{i=1}^{n} W_i (y_i - \boldsymbol{x}_i' \boldsymbol{\beta})^2 + \lambda_n \sum_{j=1}^{p_n} W_{0,j} |\beta_j| \right\}$$

  to select variables. Let $\widehat{S}_n^w \subseteq \{1, \cdots, p_n\}$ be the set of selected variables.

- Step 2: Estimate the selected variables with weighted least squares (same random weights from Step 1).

$$\widehat{\boldsymbol{\beta}}_n^w := \begin{bmatrix} \widehat{\boldsymbol{\beta}}^w_{n,\widehat{S}_n^w} \\ \widehat{\boldsymbol{\beta}}^w_{n,(\widehat{S}_n^w)^c} \end{bmatrix} := \begin{bmatrix} \left(X'_{\widehat{S}_n^w} D_n X_{\widehat{S}_n^w} \right)^{-1} X'_{\widehat{S}_n^w} D_n Y \\ \mathbf{0} \end{bmatrix}, \qquad (1)$$

  where $D_n = diag\,(W_1, \cdots, W_n)$, and $X_{\widehat{S}_n^w}$ is the $n \times |\widehat{S}_n^w|$ submatrix of X.

Model Setup

- $\{\epsilon_i\}$ iid with mean 0, variance σ_ϵ^2, and finite 4^{th} moment. Unlike Bayesian LASSO, no normality assumption needed for our work.
- F_W: $P(W > 0) = 1$ and $\mathbb{E}(W^4) < \infty$ and $\mathbb{E}(W) = 1$.
- Other common linear-model-related regularity assumptions.

3 weighting schemes on the penalty terms are considered (Newton et al., 2020):

$$\min_{\boldsymbol{\beta}} \left\{ \sum_{i=1}^{n} W_i (y_i - \boldsymbol{x}_i' \boldsymbol{\beta})^2 + \lambda_n \sum_{j=1}^{p_n} W_{0,j} |\beta_j| \right\}$$

- `RW1`: $W_{0,j} = 1 \ \forall \ j,$
- `RW2`: $W_{0,j} = W_0 \ \forall \ j,$ where $(W_0, W_i) \overset{iid}{\sim} F_W \ \forall \ i,$
- `RW3`: $(W_{0,j}, W_i) \overset{iid}{\sim} F_W \ \forall \ i, j.$

Theorem 3.4 (Conditional Sparse Normality)

Assume the **strong irrepresentable condition** (Zhao and Yu, 2006)

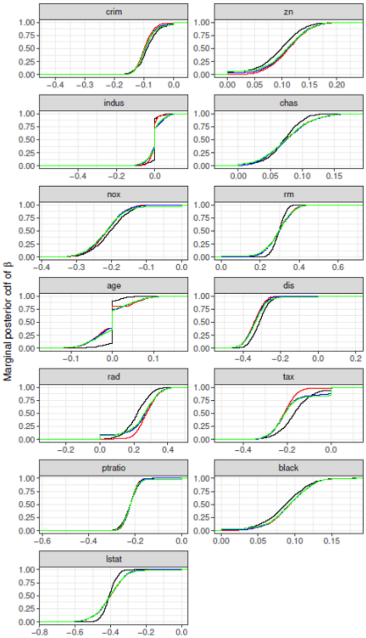
$$\left| C_{n(21)} \left(C_{n(11)} \right)^{-1} \operatorname{sgn} \left(\boldsymbol{\beta}_{0(1)} \right) \right| \leq \mathbf{1} - \boldsymbol{\eta},$$

where inequality holds element-wise for $0 < \eta_j \leq 1 \; \forall \; j = 1, \cdots, p_n - q$. Furthermore, assume proper growth rates of $\lambda_n = \mathcal{O}\left(n^{c_1}\right)$ and $p_n = \mathcal{O}\left(n^{c_2}\right)$, where $0 \leq c_2 < 1/2 < c_1 < 1$, and other regularity assumptions. Let $\widehat{\boldsymbol{\beta}}_n^w$ be the two-step random-weighting samples defined in (1), and let $\widehat{\boldsymbol{\beta}}_n^{LAS+LS}$ be the unweighted two-step LASSO+LS estimator (i.e. a LASSO variable selection step followed by least-squares estimation for the selected variables). Then, for all `RW1`, `RW2` and `RW3`, as $n \to \infty$,

$$P\left(\widehat{S}_n^w = S_0 \middle| \mathcal{F}_n \right) \to 1 \quad a.s.\ P_D,$$

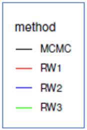
and

$$\sqrt{n}\left(\widehat{\boldsymbol{\beta}}_{n(1)}^w - \widehat{\boldsymbol{\beta}}_{n(1)}^{LAS+LS} \right) \xrightarrow{\text{c.d.}} N_q\left(\mathbf{0}\ ,\ \sigma_W^2 \sigma_\epsilon^2 C_{11}^{-1} \right) \quad a.s.\ P_D.$$



Benchmark Dataset: Boston Housing

Abbreviation	Variable
crim	per capita crime rate by town
zn	proportion of residential land zoned for lots over 25,000 sq.ft.
indus	proportion of non-retail business acres per town
chas	Charles River dummy variable (= 1 if tract bounds river; 0 otherw
nox	nitrogen oxides concentration (parts per 10 million)
rm	average number of rooms per dwelling
age	proportion of owner-occupied units built prior to 1940
dis	weighted mean of distances to five Boston employment centers
rad	index of accessibility to radial highways
tax	full-value property-tax rate per \$10,000
ptratio	pupil-teacher ratio by town
black	proportion of blacks by town
lstat	lower status of the population (percent)



References

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Park, T. and Casella, G. (2008), "The Bayesian Lasso," *Journal of the American Statistical Association*, 103, 681–686.

Zhao, P. and Yu, B. (2006), "On model selection consistency of Lasso," *Journal of Machine Learning Research*, 7, 2541–2563.

Our manuscript for this presentation

Ng, T.L. and Newton, M.A. (2020), "Random-weighting in LASSO Regression," *arXiv:2002.02629*. In revision at *the Electronic Journal of Statistics*.