9 PROJECTS

9.1 Project 1 - Relaxation Schemes

Problem Set #1 and Computing Project #1
Due Date - TBA

Nothing is to be done on paper. Both the problem set and your project report need to be completely typed. Then create two pdf files, one for the problem set and one for the project report. Each pdf file should then be uploaded to Canvas. I recommend that you prepare both in Latex, but you can use any word processor that you wish, as long as both are completely typed (including math symbols).

1. Do problems 1-9 in the notes. Make certain to type your solutions.

2. Consider the discretization of the boundary value problem

$$-\left(\epsilon u_{xx} + u_{yy}\right) = 0\,, \quad 0 \le x \le 1\,,\ \ 0 \le y \le 1\,, \tag{9.1}$$

   with $u = 0$ on all boundaries.

   Discretize as described in the notes (see equation (1.44) and the text around this equation). Consider three sets of starting data,

   (a) $u_{j,k} = (-1)^{j+k}$
   (b) $u_{j,k} = \sin(8\pi jh)\sin(8\pi kh)$
   (c) $u_{j,k} = \sin(\pi jh)\sin(\pi kh)$

   Employ the following relaxation methods: point Jacobi, weighted Jacobi, Gauss-Seidel, red-black Gauss-Seidel, SOR, SSOR and Kaczmarz relaxation (make sure you review this scheme in 7.4) with $N = 16, 32, 64, 128$ and $\epsilon = 1, .1, .01, .001$. In each case, the exact solution is $u = 0$, so you can monitor convergence simply by examining max $\mid u_{jk} \mid$ over the grid. You should then have a convergence check in your program to stop the iterations when

$$\max \mid u_{jk} \mid < tol\,,$$

   where you can take $tol = 10^{-7}$.

   For the case of weighted Jacobi, start out with $\omega = 2/3$ and then experiment to find the optimal ω. For SOR and SSOR start with $\omega = 1.8$ and vary ω to find the optimal value of the relaxation parameter. Note, the optimal ω will depend on both ϵ and N.

   For each series of computations tabulate (and/or illustrate graphically) the number of iterations required to satisfy your convergence criteria as a function of N and ϵ. Also illustrate the sensitivity of the convergence results to ω for weighted

Jacobi, SOR and SSOR. As part of your convergence studies do runs where you set the relaxation parameter $\omega = 1$ so that SSOR is simply a symmetrized version of Gauss-Seidel and make certain that you report on this method as well.

Note, that for small ϵ the solutions to (9.1) can exhibit boundary layer behavior. You need not focus on this behavior since your solution will be simply $u = 0$, but you should be aware that it will in fact happen for nontrivial boundary data and it is a very common effect for elliptic equations that occur in practice. This project will give an indication as to how some of these point relaxation schemes deal with boundary layer behavior, i.e., the role of ϵ in (9.1). and you should include a discussion of this in your report.

If you are using a programming language rather than Matlab, make certain your program is in double precision.

Write a report giving a concise description and analysis of your results. **Your report must be typed.** All figures and tables must have a number and a descriptive caption and must be referred to in the text. Attach a fully commented program to your report. Make sure that your program is clearly documented so that it can be followed by someone who has not written it. Also, your program should be modularized so that each relaxation scheme corresponds to one subprogram ($m-$file in Matlab). Please be aware that your report **must** include a "bottom line" indicating which method is preferred as ϵ decreases ((9.1) becomes more singular) and as N increases (you use finer and finer grids). In assessing the efficiency of the different schemes that you are testing, make certain that you account for the fact that the computational resources required for each sweep of SSOR are double that of one sweep of SOR and other relaxation schemes. Thus, for example, SSOR has to converge in half or less the iterations required for SOR in order to be competitive. You can assume that the computational cost of one sweep of the other relaxation schemes are the same so you can just compare the number of iterations required for convergence.

As part of your report, clearly indicate for the three sets of starting data, whether they represent smooth modes or high frequency modes on the grid (the first and third should be clear while for the second set, smoothness relative to the grid should depend on N).

Note, that you will generate a lot of data from these computations. Be aware that it can be difficult to summarize all of this data in a small number of tables and graphs, but this is something that many of you may have to do for your research in the future, so this can be an important learning exercise for you over and above the math and programming.