

Volatility in Financial Time Series

MATH 287C - Advanced Time Series Analysis
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Outline

What is Volatility?

Exploratory Data Analysis with NoVaS

Predicting Y_t^2 as a proxy for Volatility

A simple Volatility trading strategy

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Goal - Convince you that WGAN is the “best” GAN

What does it mean to learn a probability distribution?

When learning generative models, we assume the data we have comes from some unknown distribution \mathbb{P}_r .

Want to learn a distribution \mathbb{P}_θ that approximates \mathbb{P}_r , where θ are the parameters of the distribution.

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1. Directly learn the probability density function \mathbb{P}_θ and then optimize through maximum likelihood estimation
2. Learn a function that transforms an existing distribution Z into \mathbb{P}_θ . Here, g_θ is some differentiable function, Z is a common distribution (usually uniform or Gaussian), and $\mathbb{P}_\theta = g_\theta(Z)$

Maximum Likelihood approach is problematic

Recall that for continuous distributions \mathbb{P} and \mathbb{Q} the *KL* divergence is:

$$KL(\mathbb{P}||\mathbb{Q}) = \int_x \mathbb{P}(x) \log \frac{\mathbb{P}(x)}{\mathbb{Q}(x)} dx$$

and given function \mathbb{P}_θ , the MLE objective is

$$\max_{\theta \in \mathbb{R}^d} \frac{1}{m} \sum_{i=1}^m \log \mathbb{P}_\theta(x^{(i)})$$

In the limit (as $m \rightarrow \infty$), samples will appear based on the data distribution \mathbb{P}_r , so

$$\lim_{m \rightarrow \infty} \max_{\theta \in \mathbb{R}^d} \frac{1}{m} \sum_{i=1}^m \log \mathbb{P}_\theta(x^{(i)}) = \max_{\theta \in \mathbb{R}^d} \int_x \mathbb{P}_r(x) \log \mathbb{P}_\theta(x) dx$$

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- ▶ Typical remedy is to add a noise term to the model distribution to ensure distribution is defined everywhere
- ▶ This unfortunately introduces some error, and empirically people have needed to add a lot of random noise to make models train

Transforming an existing distribution

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GANs offer much more flexibility but their training is unstable.

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How do we define d ?

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χ - compact metric set (such as the space of images $[0, 1]^d$)

Σ - set of all Borel subsets of χ

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Total Variation (TV) distance

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Kullback-Leibler (KL) divergence

$$KL(\mathbb{P}_r || \mathbb{P}_\theta) = \int_{\chi} \mathbb{P}_r(x) \log \frac{\mathbb{P}_r(x)}{\mathbb{P}_\theta(x)} d\mu(x)$$

where both \mathbb{P}_r and \mathbb{P}_θ are assumed to be absolutely continuous with respect to a same measure μ defined on χ .

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Jensen-Shannon (JS) Divergence

$$JS(\mathbb{P}_r, \mathbb{P}_\theta) = KL(\mathbb{P}_r || \mathbb{P}_m) + KL(\mathbb{P}_\theta || \mathbb{P}_m)$$

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Earth-Mover (EM) distance or Wasserstein-1

$$W(\mathbb{P}_r, \mathbb{P}_\theta) = \inf_{\gamma \in \Pi((\mathbb{P}_r, \mathbb{P}_\theta))} \mathbb{E}_{(x,y) \sim \gamma} [||x - y||]$$

where $\Pi(\mathbb{P}_r, \mathbb{P}_\theta)$ denotes the set of all join distributions $\gamma(x, y)$ whose marginals are respectively \mathbb{P}_r and \mathbb{P}_θ

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Transport Plan

Each $\gamma \in \Pi$ is a transport plan and to execute the plan, for all x, y move $\gamma(x, y)$ mass from x to y .

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This shows why the marginals of $\gamma \in \Pi$ must be \mathbb{P}_r and \mathbb{P}_θ . For scoring, the effort spent is

$$\int_x \int_y \gamma(x, y) \|x - y\| dy dx = \mathbb{E}_{(x, y) \sim \gamma} [\|x - y\|]$$

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$$\int_x \int_y \gamma(x, y) \|x - y\| dy dx = \mathbb{E}_{(x, y) \sim \gamma} [\|x - y\|]$$

Computing the infimum of this over all valid γ gives the earth mover distance

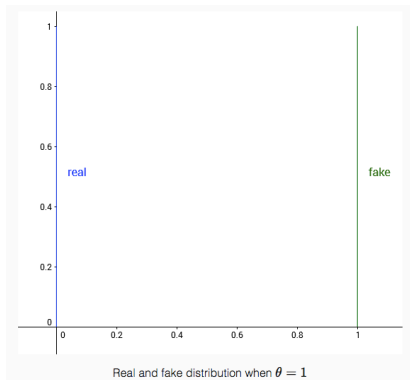
Learning Parallel Lines Example

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Let $Z \sim U[0, 1]$ and let \mathbb{P}_0 be the distribution of $(0, Z) \in \mathbb{R}^2$, uniform on a straight vertical line passing through the origin. Now let $g_\theta(z) = (\theta, z)$ with θ a single real parameter.

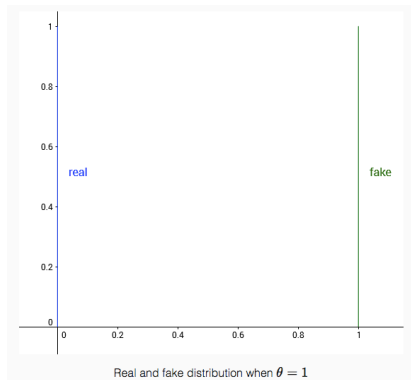
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We'd like our optimization algorithm to learn to move θ to 0. As $\theta \rightarrow 0$, the distance $d(\mathbb{P}_0, \mathbb{P}_\theta)$ should decrease.

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For many common distance functions, this doesn't happen.

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$$\delta(\mathbb{P}_0, \mathbb{P}_\theta) = \begin{cases} 1 & \text{if } \theta \neq 0 \\ 0 & \text{if } \theta = 0 \end{cases}$$

$$JS(\mathbb{P}_0, \mathbb{P}_\theta) = \begin{cases} \log 2 & \text{if } \theta \neq 0 \\ 0 & \text{if } \theta = 0 \end{cases}$$

$$KL(\mathbb{P}_\theta, \mathbb{P}_0) = KL(\mathbb{P}_0, \mathbb{P}_\theta) = \begin{cases} +\infty & \text{if } \theta \neq 0 \\ 0 & \text{if } \theta = 0 \end{cases}$$

$$W(\mathbb{P}_0, \mathbb{P}_\theta) = |\theta|$$

Theoretical Justification

Theorem (1)

Let \mathbb{P}_r be a fixed distribution over χ . Let Z be a random variable over another space \mathcal{Z} . Let $g : \mathcal{Z} \times \mathbb{R}^d \rightarrow \chi$ be a function, that will be denote $g_\theta(z)$ with z the first coordinate and θ the second. Let \mathbb{P}_θ denote the distribution of $g_\theta()$. Then,

1. If g is continuous in θ , so is $W(\mathbb{P}_r, \mathbb{P}_\theta)$.
2. If g is locally Lipschitz and satisfies regularity assumption 1, then $W(\mathbb{P}_r, \mathbb{P}_\theta)$ is continuous everywhere, and differentiable almost everywhere
3. Statements 1-2 are false for the Jensen-Shannon divergence $JS(\mathbb{P}_r, \mathbb{P}_\theta)$ and all the KLs.

Theoretical Justification

Theorem (2)

Let \mathbb{P} be a distribution on a compact space χ and $(\mathbb{P}_n)_{n \in \mathbb{N}}$ be a sequence of distributions on χ . Then, considering all limits as $n \rightarrow \infty$,

1. The following statements are equivalent
 - ▶ $\delta(\mathbb{P}_n, \mathbb{P}) \rightarrow 0$
 - ▶ $JS(\mathbb{P}_n, \mathbb{P}) \rightarrow 0$
2. The following statements are equivalent
 - ▶ $W(\mathbb{P}_n, \mathbb{P}) \rightarrow 0$
 - ▶ $\mathbb{P}_n \xrightarrow{D} \mathbb{P}$ where \xrightarrow{D} represents convergence in distribution for random variables
3. $KL(\mathbb{P}_n || \mathbb{P}) \rightarrow 0$ or $KL(\mathbb{P} || \mathbb{P}_n) \rightarrow 0$ imply the statements in (1).
4. The statements in (1) imply the statements in (2).

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The *generator* network G maps a source of noise to the input space.

The *discriminator* network D receives either a generated sample or a true data sample and must distinguish between the two.

The generator is trained to fool the discriminator.

Generative Adversarial Networks

Formally we can express the game between the generator G and the discriminator D with the minimax objective:

$$\min_G \max_D \mathbb{E}_{x \sim \mathbb{P}_r} [\log(D(x))] + \mathbb{E}_{\tilde{x} \sim \mathbb{P}_g} [\log(1 - D(\tilde{x}))]$$

where \mathbb{P}_r is the data distribution and \mathbb{P}_g is the model distribution implicitly defined by $\tilde{x} = G(z)$, $z \sim p(z)$

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- ▶ If the discriminator is trained to optimality before each generator parameter update, minimizing the value function amounts to minimizing the Jensen-Shannon divergence between the data and model distributions on x
- ▶ This is expensive and often leads to vanishing gradients as the discriminator saturates
- ▶ In practice, this requirement is relaxed, and the generator and discriminator are update simultaneously

Kantorovich-Rubinstein Duality

Unfortunately, computing the Wasserstein distance exactly is intractable.

$$W(\mathbb{P}_r, \mathbb{P}_\theta) = \inf_{\gamma \in \Pi((\mathbb{P}_r, \mathbb{P}_\theta))} \mathbb{E}_{(x,y) \sim \gamma} [\|x - y\|]$$

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However, a result from the Kantorovich-Rubinstein Duality (Villani 2008) shows W is equivalent to

$$W(\mathbb{P}_r, \mathbb{P}_\theta) = \sup_{\|f\|_L \leq 1} \mathbb{E}_{x \sim \mathbb{P}_r} [f(x)] - \mathbb{E}_{x \sim \mathbb{P}_\theta} [f(x)]$$

where the supremum is taken over all 1-Lipschitz functions

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Calculating this is still intractable, but now it's easier to approximate.

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Suppose we have a parametrized function family $\{f_w\}_{w \in W}$, where w are the weights and W is the set of all possible weights

Furthermore suppose these functions are all K -Lipschitz for some K .

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Note that if we replace the supremum over 1-Lipschitz functions with the supremum over K -Lipschitz functions, then the supremum is $K \cdot W(\mathbb{P}_r, \mathbb{P}_\theta)$ instead.

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In practice this won't be true!

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We can then backpropagate through $W(\mathbb{P}_r, g_\theta(Z))$ to get the gradient for θ .

$$\begin{aligned}\nabla_\theta W(\mathbb{P}_r, \mathbb{P}_\theta) &= \nabla_\theta (\mathbb{E}_{x \sim \mathbb{P}_r} [f_w(x)] - \mathbb{E}_{z \sim Z} [f_w(g_\theta(z))]) \\ &= -\mathbb{E}_{z \sim Z} [\nabla_\theta f_w(g_\theta(z))]\end{aligned}$$

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Important Detail

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To guarantee this is true, the authors use weight clamping. The weights w are constrained to lie within $[-c, c]$, by clipping w after every update to w .

Wasserstein GAN Algorithm

Algorithm 1 WGAN, our proposed algorithm. All experiments in the paper used the default values $\alpha = 0.00005$, $c = 0.01$, $m = 64$, $n_{\text{critic}} = 5$.

Require: : α , the learning rate. c , the clipping parameter. m , the batch size. n_{critic} , the number of iterations of the critic per generator iteration.

Require: : w_0 , initial critic parameters. θ_0 , initial generator's parameters.

```
1: while  $\theta$  has not converged do
2:   for  $t = 0, \dots, n_{\text{critic}}$  do
3:     Sample  $\{x^{(i)}\}_{i=1}^m \sim \mathbb{P}_r$  a batch from the real data.
4:     Sample  $\{z^{(i)}\}_{i=1}^m \sim p(z)$  a batch of prior samples.
5:      $g_w \leftarrow \nabla_w [\frac{1}{m} \sum_{i=1}^m f_w(x^{(i)}) - \frac{1}{m} \sum_{i=1}^m f_w(g_\theta(z^{(i)}))]$ 
6:      $w \leftarrow w + \alpha \cdot \text{RMSPProp}(w, g_w)$ 
7:      $w \leftarrow \text{clip}(w, -c, c)$ 
8:   end for
9:   Sample  $\{z^{(i)}\}_{i=1}^m \sim p(z)$  a batch of prior samples.
10:   $g_\theta \leftarrow -\nabla_\theta \frac{1}{m} \sum_{i=1}^m f_w(g_\theta(z^{(i)}))$ 
11:   $\theta \leftarrow \theta - \alpha \cdot \text{RMSPProp}(\theta, g_\theta)$ 
12: end while
```

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$$\frac{1}{m} \sum_{i=1}^m \log D(x^{(i)}) + \frac{1}{m} \sum_{i=1}^m \log(1 - D(g_{\theta}(z^{(i)})))$$

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- ▶ Although GANs are formulated as a min max problem, in practice we never train D to convergence
- ▶ Consequently, we're updating G against an objective that kind of aims towards the JS divergence, but doesn't go all the way
- ▶ In contrast, because the Wasserstein distance is differentiable nearly everywhere, we can (and should) train f_w to convergence before each generator update, to get as accurate an estimate of $W(\mathbb{P}_r, \mathbb{P}_{\theta})$ as possible

Properties of optimal WGAN critic

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To understand why weight clipping is problematic in WGAN critic we need to understand what are the properties of the optimal WGAN critic?

Properties of optimal WGAN critic

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If the optimal critic under the Kantorovich-Rubinstein dual D^* is differentiable, and x is a point from our generator distribution \mathbb{P}_θ , then there is a point y sampled from the true distribution \mathbb{P}_r such that the gradient of D^* at all points $x_t = (1 - t)x + ty$ lie on a straight line between x and y .

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In other words, $\nabla D^*(x_t) = \frac{y - x_t}{\|y - x_t\|}$.

This implies that the optimal WGAN critic has gradients with norm 1 almost everywhere under \mathbb{P}_r and \mathbb{P}_θ

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This implies we should directly constrain the gradient norm of our critic function with respect to its input.

Enforcing a soft version of this we get:

$$L = \underbrace{\mathbb{E}_{\tilde{\mathbf{x}} \sim \mathbb{P}_g} [D(\tilde{\mathbf{x}})] - \mathbb{E}_{\mathbf{x} \sim \mathbb{P}_r} [D(\mathbf{x})]}_{\text{Original critic loss}} + \lambda \underbrace{\mathbb{E}_{\hat{\mathbf{x}} \sim \mathbb{P}_{\hat{\mathbf{x}}}} [(\|\nabla_{\hat{\mathbf{x}}} D(\hat{\mathbf{x}})\|_2 - 1)^2]}_{\text{Our gradient penalty}}$$

WGAN with gradient penalty

Algorithm 1 WGAN with gradient penalty. We use default values of $\lambda = 10$, $n_{\text{critic}} = 5$, $\alpha = 0.0001$, $\beta_1 = 0.5$, $\beta_2 = 0.9$.

Require: The gradient penalty coefficient λ , the number of critic iterations per generator iteration n_{critic} , the batch size m , Adam hyperparameters α, β_1, β_2 .

Require: initial critic parameters w_0 , initial generator parameters θ_0 .

```
1: while  $\theta$  has not converged do
2:   for  $t = 1, \dots, n_{\text{critic}}$  do
3:     for  $i = 1, \dots, m$  do
4:       Sample real data  $\mathbf{x} \sim \mathbb{P}_r$ , latent variable  $\mathbf{z} \sim p(\mathbf{z})$ , a random number  $\epsilon \sim U[0, 1]$ .
5:        $\tilde{\mathbf{x}} \leftarrow G_\theta(\mathbf{z})$ 
6:        $\hat{\mathbf{x}} \leftarrow \epsilon \mathbf{x} + (1 - \epsilon) \tilde{\mathbf{x}}$ 
7:        $L^{(i)} \leftarrow D_w(\tilde{\mathbf{x}}) - D_w(\mathbf{x}) + \lambda(\|\nabla_{\hat{\mathbf{x}}} D_w(\hat{\mathbf{x}})\|_2 - 1)^2$ 
8:     end for
9:      $w \leftarrow \text{Adam}(\nabla_w \frac{1}{m} \sum_{i=1}^m L^{(i)}, w, \alpha, \beta_1, \beta_2)$ 
10:   end for
11:   Sample a batch of latent variables  $\{\mathbf{z}^{(i)}\}_{i=1}^m \sim p(\mathbf{z})$ .
12:    $\theta \leftarrow \text{Adam}(\nabla_\theta \frac{1}{m} \sum_{i=1}^m -D_w(G_\theta(\mathbf{z})), \theta, \alpha, \beta_1, \beta_2)$ 
13: end while
```
