Equivalence between the digital error model and the noisy quantum channel description

The digital error model simplifies continuous noise by discretizing it into Pauli errors (I, X, Y, Z) with probabilities. A noisy quantum channel is described using Kraus operators (E_i) acting on a density matrix ρ. Both models are equivalent because any noise channel can be expressed as a combination of Pauli operators. Thus, for low error rates, Pauli errors accurately capture general noise processes.

Knill-Laflamme conditions for correctable errors

Errors $\{E_a\}$ are correctable on a code space $\mathcal{C}$ if:

$$\langle\psi_i|E_a^\dagger E_b|\psi_j\rangle = \alpha_{ab}\delta_{ij},$$

where $|\psi_i\rangle$ are codewords, and α_{ab} is independent of i, j. This ensures errors act consistently on codewords, enabling a recovery operation.

Shor's code basics

Shor's code protects one logical qubit using 9 physical qubits:

1. Encode against Z errors using repetition:

$$\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|000\rangle + \beta|111\rangle.$$

2. Encode against X errors by applying Hadamard gates, resulting in:

$$|0_L\rangle = \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle)^{\otimes 3}, \quad |1_L\rangle = \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle)^{\otimes 3}.$$

Encoding the X and H gates on Shor's code

Logical X is applied transversally as X on all 9 qubits. Logical H is applied as H on each qubit. These operations preserve the logical structure by acting consistently across blocks.

Circuits for detecting and correcting X and Z errors

- X-errors: Detected using Hadamard gates and parity checks among blocks of 3 qubits.
- Z-errors: Detected using CNOT gates to compare phases within each block.
- Linear combinations of Pauli errors: Decomposed into individual errors, which are correctable if they satisfy Knill-Laflamme conditions.

Theoretical fidelity

Shor's code succeeds if:

- No errors occur.
- 1-qubit errors (bit-flip or phase-flip) are corrected.
- 2-qubit errors are corrected if they occur in specific configurations (e.g., different blocks for bit-flips).
- 3-qubit errors are corrected if in distinct or same blocks, depending on the type.

The success probability is:

$$P_{\text{success}} = P_{\text{bit}} \cdot P_{\text{phase}},$$

where:

$$P_{\text{bit}} = (1-p_{\text{bit}})^9 + 9p_{\text{bit}}(1-p_{\text{bit}})^8 + \frac{9 \cdot 6}{2} p_{\text{bit}}^2 (1-p_{\text{bit}})^7 + \frac{9 \cdot 6 \cdot 3}{6} p_{\text{bit}}^3 (1-p_{\text{bit}})^6,$$

$$P_{\text{phase}} = (1-p_{\text{phase}})^9 + 9p_{\text{phase}}(1-p_{\text{phase}})^8 + \frac{9 \cdot 2}{2} p_{\text{phase}}^2 (1-p_{\text{phase}})^7 + \frac{9 \cdot 2 \cdot 1}{6} p_{\text{phase}}^3 (1-p_{\text{phase}})^6.$$

Without correction, the success probability is:

$$P_{\text{no correction}} = (1 - p_{\text{bit}}) \cdot (1 - p_{\text{phase}}).$$

Non-degenerate vs. degenerate error correction codes

- **Non-degenerate codes:** Each correctable error produces a unique syndrome, allowing unambiguous recovery.
- **Degenerate codes:** Some errors produce the same syndrome but are correctable because they act identically on the code space.

Extending Shor's code to larger distance d

Shor's code can be extended by concatenation or larger block structures. For instance, using 5- or 7-qubit codes instead of 3-qubit repetition increases distance d and improves error tolerance.

Simulated noise and break-even

Using QASM simulator, Shor's code achieves break-even if physical error rates are below a threshold. At low noise levels, logical error rates drop below physical error rates, demonstrating the effectiveness of error correction.