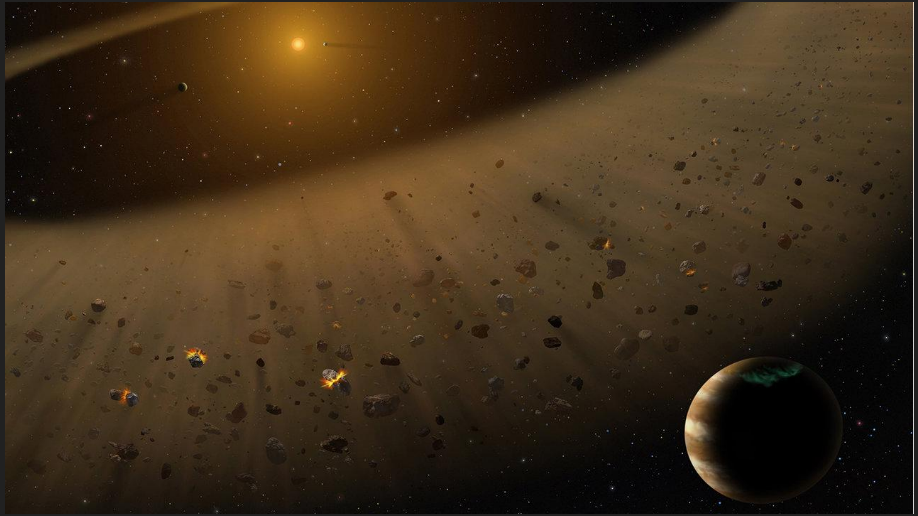
Orbital Dynamics

Everything in the universe with mass experiences the pull of gravity from other objects. This force is pretty accurately approximated with Newton's law of universal gravitation,

$$F_g = G\frac{m_1 m_2}{r^2},$$

where G is the gravitational constant, m_1 and m_2 are the interacting masses, and r is the distance between them. For consistency, m_1 is the more massive object and m_2 is the less massive object.



Credit: NASA/SOFIA/Lynette Cook

Orbital Dynamics

A simple, but useful, type of gravitational system to investigate is the orbit of a planet, comet, asteroid, or something similar around a much more massive star. In these extreme scenarios where one object is more massive than the other, you only need to calculate the motion of the less massive object.

Because F=m*a, where m is the small, orbiting mass and is the same as ‘m_2’, Newton’s law of gravity can be reformulated for acceleration ‘a’,

$$a_2 = G\frac{m_1}{r^2},$$

which is simpler to calculate.

Other useful quantities that can be calculated are the angular momentum,

$$L = rm_2v,$$

and the total orbital energy,

$$E = -G\frac{m_1m_2}{2r},$$

of the orbiting object, where ‘v’ is the orbital velocity.

Zombies!

Everybody has some idea of their plans for a world where zombies are suddenly a real occurrence. It seems pretty far-fetched, but scientists consider this scenario very seriously. Epidemics, and the more devastating pandemics, are one of humankind's looming threats. Understanding how a pathogen can travel from community to community is essential to being able to prevent, stall, or stop a harmful disease.

One useful model is a fixed population model called the SIR model. SIR stands for 'susceptible', 'infected', and 'removed'.

All three terms add to a constant number of people, N, and each term has a differential equation that is coupled to the other terms.

$$\frac{dS}{dt} = -\frac{\beta S I}{N}$$

$$\frac{dI}{dt} = \frac{\beta S I}{N} - \gamma I$$

$$\frac{dR}{dt} = \gamma I$$

Predator-Prey

Complex and quickly changing ecological systems can be difficult to model exactly. However, it is informative to understand general features of local fauna and their interactivity, including that of predator and prey populations.

The Lotka-Volterra equations model predator-prey interactions with two differential equations.

$$\frac{dx}{dt} = \alpha x - \beta xy$$

$$\frac{dy}{dt} = \delta xy - \gamma y$$

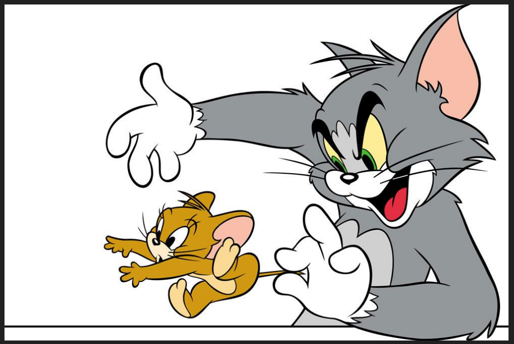
The variables ‘x’ and ‘y’ represent prey and predator population values, respectively. The various parameters (α, □, β, and ɣ). Each term in the equations represent an addition or decrement to the populations over time.

αx --- growth of prey pop. from local resources

βxy --- death of prey from meeting predator

δxy --- growth of predator pop.

ɣy --- death of predator from emigrating or natural death



Free Fall Motion

Free fall motion provides an interesting situation to test the interplay between different forces that an object would experience while not being pulled against a solid object. We present three possible investigation paths for free fall motion:

Skydiving

Bungee Jumping

Each situation assumes that the system takes place primarily within Earth's gravitational control. With this consideration, one may assume that gravitational acceleration, 'g', can be approximated as a constant 9.8 m/s^2.

However, each system will also contain other forces more specific to their situation. These forces comes with more specific initial conditions and formulae.

Free Fall Motion

Skydiving

Skydiving can be treated as a primarily vertical system with the interplay of gravitational acceleration and air resistance. A specific detail is that the human body and parachute experience very different amounts of air resistance, calculated by,

$$F_D = \frac{1}{2}\rho v^2 C_D A$$

where F_D is the drag force, ρ is air density, v is falling speed, C_D is drag coefficient (specific to object), and A is area of object. It will be important to calculate this difference during free fall.

Bungee Jumping

Bungee jumping is also a mostly vertical problem. However, the primary force besides gravity would be the spring force caused by the stretching of the bungee cord. This can be calculated using Hooke's Law,

$$F = -kx$$

where F is the spring force, k is the spring constant, and x is the extra stretched length.

Double Pendulum

The double pendulum is a pendulum with another pendulum attached to its end. This is a simple physical system, yet it contains a strong sensitivity to it's initial conditions. Because of this, it is used as an example for chaotic motion

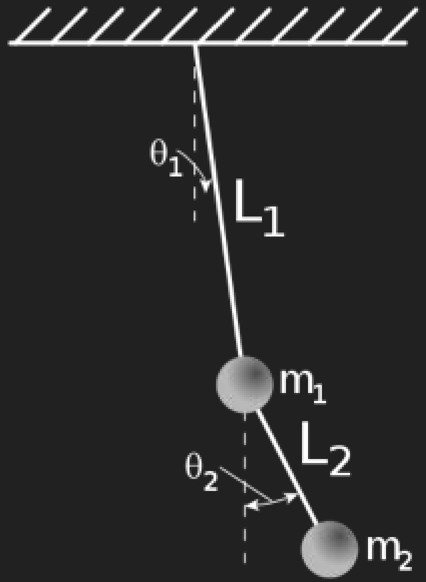
Using the Cartesian coordinate system, if the origin is at the point of suspension of the first pendulum, then the center of mass of the first pendulum is at

$$x_1 = \frac{l}{2}\sin\theta_1$$
$$y_1 = -\frac{l}{2}\cos\theta_1$$

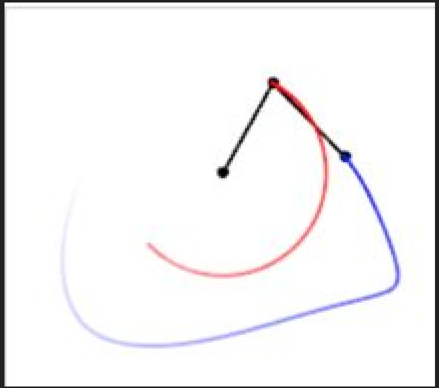
and the center of mass at the second pendulum is at

$$x_2 = l\left(\sin\theta_1 + \tfrac{1}{2}\sin\theta_2\right)$$
$$y_2 = -l\left(\cos\theta_1 + \tfrac{1}{2}\cos\theta_2\right)$$

We can use these to write the Lagrangian of the system!



Double Pendulum



The Lagrangian can be written as:

$$\begin{aligned} L &= \text{kinetic energy} - \text{potential energy} \\ &= \tfrac{1}{2}m\left(v_1^2 + v_2^2\right) + \tfrac{1}{2}I\left({\dot\theta_1}^2 + {\dot\theta_2}^2\right) - mg\left(y_1 + y_2\right) \\ &= \tfrac{1}{2}m\left({\dot x_1}^2 + {\dot y_1}^2 + {\dot x_2}^2 + {\dot y_2}^2\right) + \tfrac{1}{2}I\left({\dot\theta_1}^2 + {\dot\theta_2}^2\right) - mg\left(y_1 + y_2\right) \end{aligned}$$

The first term is the linear kinetic energy of the center of mass, the second term is the rotational kinetic energy, and the third term is the potential energy of the two pendulums due to gravity. One can substitute the position coordinates from the previous slide and then from the Lagrangian one can find the equations for:

$$\dot\theta_1 \quad \dot\theta_2 \quad \dot p_{\theta_1} \quad \dot p_{\theta_2}$$

which are the explicit formulae for the time evolution of the system!