Problem Set 1

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Problem 1.

1

We have :

$$\ell_n = \sum_{i=1}^{n} (\log\theta + \theta\log\tau - (\theta+1)\log X_i)\mathbf{1}(X_i \geqslant \tau)$$

$$\Rightarrow \frac{\partial \ell_n}{\partial\theta} = \sum_{i=1}^{n} (\frac{1}{\theta} + \log\tau - \log X_i)\mathbf{1}(X_i \geqslant \tau) = 0$$

$$\Leftrightarrow \theta = \frac{\sum_{i=1}^{n}(\mathbf{1}(X_i \geqslant \tau))}{\sum_{i=1}^{n}(\log X_i - \log\tau)\mathbf{1}(X_i \geqslant \tau)}$$

3

$$\ell_n = \sum_{i=1}^{n} (\frac{1}{2}\log\theta + (\sqrt{\theta}-1)\log X_i)\mathbf{1}(0 \leqslant X_i \leqslant 1)$$

$$\Rightarrow \frac{\partial \ell_n}{\partial\theta} = (\frac{1}{2\theta} + \frac{\log X_i}{2\sqrt{\theta}})\mathbf{1}(0 \leqslant X_i \leqslant 1) = 0$$

$$\Leftrightarrow \theta = \Big(\frac{\sum_{i=1}^{n} 1}{\sum_{i=1}^{n}\log X_i}\Big)^2$$

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$$\ell_n = \sum_{i=1}^{n} (\log X_i - 2\log\theta - \frac{{X_i}^2}{2\theta^2})\mathbf{1}(X_i \geqslant 0)$$

$$\Leftrightarrow \frac{\partial \ell_n}{\partial\theta} = \sum_{i=1}^{n} (-\frac{2}{\theta} + \frac{{X_i}^2}{\theta^3})\mathbf{1}(X_i \geqslant 0) = 0$$

$$\Leftrightarrow \theta = \frac{\sum_{i=1}^{n}(X_i)\mathbf{1}(X_i \geqslant 0)}{\sum_{i=1}^{n}\sqrt{2}\mathbf{1}(X_i \geqslant 0)}$$

5

$$\ell_n = \sum_{i=1}^{n} (\log\theta + \log\tau + (\tau-1)\log X_i - \theta {X_i}^{\tau})\mathbf{1}(X_i \geqslant 0)$$

$$\Leftrightarrow \frac{\partial \ell_n}{\partial\theta} = \sum_{i=1}^{n} (\frac{1}{\theta} - {X_i}^{\tau})\mathbf{1}(X_i \geqslant 0)$$

$$\Leftrightarrow \theta = \frac{\sum_{i=1}^{n} 1\mathbf{1}(X_i \geqslant 0)}{\sum_{i=1}^{n}({X_i}^{\tau})\mathbf{1}(X_i \geqslant 0)}$$

Problem 2. We have :

$$\mathcal{L}(\mu,\sigma) = \prod_{i=1}^{n} \Big(\frac{1}{\sigma}\exp(-\frac{1}{2\sigma^2}(X_i-\mu)^2)\Big)$$

$$\ell_n = \log \mathcal{L}(\mu,\sigma) = \sum_{i=1}^{n} \Big(-\log\sigma - \frac{1}{2\sigma^2}(X_i-\mu)^2\Big)$$

$$= \sum_{i=1}^{n} \Big(-\log\sigma - \frac{1}{2\sigma^2}(X_i-\bar{X}+\bar{X}-\mu)^2\Big)$$

$$= \sum_{i=1}^{n} \Big(-\log\sigma - \frac{1}{2\sigma^2}\Big((X_i-\bar{X})^2 + 2(X_i-\bar{X})(\bar{X}-\mu) + (\bar{X}-\mu)^2\Big)\Big)$$

$$-n\log\sigma - \frac{1}{2\sigma^2}\Big(\sum_{i=1}^{n}(X_i-\bar{X})^2 + 2(\bar{X}-\mu)\sum_{i=1}^{n}(X_i-\bar{X}) + n(\bar{X}-\mu)^2\Big)$$

Because $\sum_{i=1}^{n}(X_i-\bar{X}=0)$, we get :

$$\ell_n = -n\log\sigma - \frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{2\sigma^2} - \frac{n(\bar{X}-\mu)}{2\sigma^2}$$

$$\Rightarrow \begin{cases} \frac{\partial \ell_n}{\partial \mu} = \frac{n(\bar{X}-\mu)}{\sigma^2} = 0 \\ \frac{\partial \ell_n}{\partial \sigma} = \frac{-n}{\sigma} + \frac{1}{\sigma^3}\Big(\sum_{i=n}^{n}(X_i-\bar{X})^2 + n(\bar{X}-\mu)^2\Big) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \mu = \bar{X} \\ \sigma^2 = \frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{n} \end{cases}$$

Prove this this is consitent:

To do this, we need to prove bias $\to 0$ and $\mathbf{V}(\mu,\sigma^2) \to 0$ as n $\to \infty$, which means that prove $\mathbb{E}[\bar{X}] = \mu$ and $\mathbb{E}[\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{n}] = 0$

$$\mathbb{E}[\bar{X}] = \mathbb{E}[\frac{\sum_{i=1}^{n} X_i}{n}] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i] = \frac{n\mu}{n} = \mu$$

$$\mathbb{E}[\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{n}] = \mathbb{E}[\frac{\sum_{i=1}^{n}(X_i-\mu+\mu-\bar{X})^2}{n}] = \mathbb{E}[\frac{\sum_{i=1}^{n}(X_i-\mu)^2 + 2(\mu-\bar{X})n(\bar{X}-\mu) + n(\mu-\bar{X})^2}{n}]$$

$$= \mathbb{E}\Big[\frac{\sum_{i=1}^{n}(X_i-\mu)^2 - n(\mu-\bar{X})^2}{n}\Big] = \frac{\sigma^2 - n\frac{\sigma^2}{n}}{n} = 0$$