

Problem Set 1

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Problem 1.

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We have:

$$\begin{aligned}\ell_n &= \sum_{i=1}^n (\log \theta + \theta \log \tau - (\theta + 1) \log X_i) \mathbf{1}(X_i \geq \tau) \\ \Rightarrow \frac{\partial \ell_n}{\partial \theta} &= \sum_{i=1}^n \left(\frac{1}{\theta} + \log \tau - \log X_i \right) \mathbf{1}(X_i \geq \tau) = 0 \\ \Leftrightarrow \theta &= \frac{\sum_{i=1}^n (\mathbf{1}(X_i \geq \tau))}{\sum_{i=1}^n (\log X_i - \log \tau) \mathbf{1}(X_i \geq \tau)}\end{aligned}$$

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$$\begin{aligned}\ell_n &= \sum_{i=1}^n \left(\frac{1}{2} \log \theta + (\sqrt{\theta} - 1) \log X_i \right) \mathbf{1}(0 \leq X_i \leq 1) \\ \Rightarrow \frac{\partial \ell_n}{\partial \theta} &= \left(\frac{1}{2\theta} + \frac{\log X_i}{2\sqrt{\theta}} \right) \mathbf{1}(0 \leq X_i \leq 1) = 0 \\ \Leftrightarrow \theta &= \left(\frac{\sum_{i=1}^n 1}{\sum_{i=1}^n \log X_i} \right)^2\end{aligned}$$

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$$\begin{aligned}\ell_n &= \sum_{i=1}^n \left(\log X_i - 2 \log \theta - \frac{X_i^2}{2\theta^2} \right) \mathbf{1}(X_i \geq 0) \\ \Leftrightarrow \frac{\partial \ell_n}{\partial \theta} &= \sum_{i=1}^n \left(-\frac{2}{\theta} + \frac{X_i^2}{\theta^3} \right) \mathbf{1}(X_i \geq 0) = 0 \\ \Leftrightarrow \theta &= \frac{\sum_{i=1}^n (X_i) \mathbf{1}(X_i \geq 0)}{\sum_{i=1}^n \sqrt{2} \mathbf{1}(X_i \geq 0)}\end{aligned}$$

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$$\begin{aligned}\ell_n &= \sum_{i=1}^n (\log \theta + \log \tau + (\tau - 1) \log X_i - \theta X_i^\tau) \mathbf{1}(X_i \geq 0) \\ \Leftrightarrow \frac{\partial \ell_n}{\partial \theta} &= \sum_{i=1}^n \left(\frac{1}{\theta} - X_i^\tau \right) \mathbf{1}(X_i \geq 0) \\ \Leftrightarrow \theta &= \frac{\sum_{i=1}^n \mathbf{1}(X_i \geq 0)}{\sum_{i=1}^n (X_i^\tau) \mathbf{1}(X_i \geq 0)}\end{aligned}$$

Problem 2. We have :

$$\begin{aligned}
\mathcal{L}(\mu, \sigma) &= \prod_{i=1}^n \left(\frac{1}{\sigma} \exp\left(-\frac{1}{2\sigma^2}(X_i - \mu)^2\right) \right) \\
\ell_n = \log \mathcal{L}(\mu, \sigma) &= \sum_{i=1}^n \left(-\log \sigma - \frac{1}{2\sigma^2}(X_i - \mu)^2 \right) \\
&= \sum_{i=1}^n \left(-\log \sigma - \frac{1}{2\sigma^2}(X_i - \bar{X} + \bar{X} - \mu)^2 \right) \\
&= \sum_{i=1}^n \left(-\log \sigma - \frac{1}{2\sigma^2} \left((X_i - \bar{X})^2 + 2(X_i - \bar{X})(\bar{X} - \mu) + (\bar{X} - \mu)^2 \right) \right) \\
&= -n \log \sigma - \frac{1}{2\sigma^2} \left(\sum_{i=1}^n (X_i - \bar{X})^2 + 2(\bar{X} - \mu) \sum_{i=1}^n (X_i - \bar{X}) + n(\bar{X} - \mu)^2 \right)
\end{aligned}$$

Because $\sum_{i=1}^n (X_i - \bar{X}) = 0$, we get :

$$\begin{aligned}
\ell_n &= -n \log \sigma - \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{2\sigma^2} - \frac{n(\bar{X} - \mu)}{2\sigma^2} \\
\Rightarrow \begin{cases} \frac{\partial \ell_n}{\partial \mu} = \frac{n(\bar{X} - \mu)}{\sigma^2} = 0 \\ \frac{\partial \ell_n}{\partial \sigma} = \frac{-n}{\sigma} + \frac{1}{\sigma^3} \left(\sum_{i=1}^n (X_i - \bar{X})^2 + n(\bar{X} - \mu)^2 \right) = 0 \end{cases} \\
&\Leftrightarrow \begin{cases} \mu = \bar{X} \\ \sigma^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n} \end{cases}
\end{aligned}$$

Prove this this is consitent:

To do this, we need to prove bias $\rightarrow 0$ and $\mathbf{V}(\mu, \sigma^2) \rightarrow 0$ as $n \rightarrow \infty$, which means that prove $\mathbb{E}[\bar{X}] = \mu$ and $\mathbb{E}\left[\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n}\right] = 0$

$$\begin{aligned}
\mathbb{E}[\bar{X}] &= \mathbb{E}\left[\frac{\sum_{i=1}^n X_i}{n}\right] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[X_i] = \frac{n\mu}{n} = \mu \\
\mathbb{E}\left[\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n}\right] &= \mathbb{E}\left[\frac{\sum_{i=1}^n (X_i - \mu + \mu - \bar{X})^2}{n}\right] = \mathbb{E}\left[\frac{\sum_{i=1}^n (X_i - \mu)^2 + 2(\mu - \bar{X})n(\bar{X} - \mu) + n(\mu - \bar{X})^2}{n}\right] \\
&= \mathbb{E}\left[\frac{\sum_{i=1}^n (X_i - \mu)^2 - n(\mu - \bar{X})^2}{n}\right] = \frac{\sigma^2 - n \frac{\sigma^2}{n}}{n} = 0
\end{aligned}$$