Chapter 5

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Problem 1.

1.

$$S_n = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X_n})^2 = \frac{1}{n-1}(\sum_{i=1}^{n} X_i^2 - 2\bar{X_n}\sum_{i=1}^{n} X_i + \sum_{i=1}^{n}\bar{X_n}^2) = \frac{1}{n-1}(\frac{1}{n-1}(X_i^2 - 2n\bar{X_n}\bar{X_n} + n\bar{X_n}^2)$$

$$= \frac{\sum_{i=1}^{n} X_i^2}{n-1} - \frac{n}{n-1}\bar{X_n}^2$$

a)

We know

$$\begin{cases} \mathbb{E}[X_i^2] = \mathbb{V}(X_i) + \mathbb{E}[X_i]^2 = \sigma^2 + \mu^2 \\ \mathbb{E}[\bar{X_n}^2] = \mathbb{V}(\bar{X_n}) + \mathbb{E}[\bar{X_n}]^2 = \frac{\sigma^2}{n} + \mu^2 \end{cases} \tag{1}$$

$$\Rightarrow \mathbb{E}(S_n^2) = \frac{n\sigma^2 + n\mu^2 - n\left(\frac{\sigma^2}{n} + \mu^2\right)}{n-1} = \sigma^2$$

b)

$$S_n^2 = \frac{\sum_{i=1}^{n} X_i^2}{n-1} - \frac{n}{n-1}\bar{X_n}^2 = \frac{n}{n-1}\frac{\sum_{i=1}^{n} X_i^2}{n} - \frac{n}{n-1}\bar{X_n}^2$$

$$\begin{cases} \frac{n}{n-1} \to 1 \\ \frac{\sum_{i=1}^{n} X_i^2}{n} \xrightarrow{\text{P}} \sigma^2 + \mu^2 \text{ (Law of large number)} \\ \bar{X_n}^2 \xrightarrow{\text{P}} \mu^2 \text{ (Law of large number)} \end{cases} \tag{2}$$

$$\Rightarrow S_n^2 \xrightarrow{\text{P}} \sigma^2$$

Problem 2.

$$\mathbb{E}[(\bar{X_n} - b)^2] = \mathbb{E}[(\bar{X_n} - \mathbb{E}[\bar{X_n}] + \mathbb{E}[\bar{X_n}] - b)^2] = \mathbb{V}(X_n) + 2(\mathbb{E}[\bar{X_n}] - b)\mathbb{E}(\bar{X_n} - \mathbb{E}[\bar{X_n}]) + (\mathbb{E}[X_n] - b)^2$$

$$\Rightarrow \mathbb{E}[(\bar{X_n} - b)^2] = \mathbb{V}(X_n) + (\mathbb{E}[X_n] - b)^2$$

Thus if $mathbbE[(\bar{X_n} - b)^2] \to 0$, then $\mathbb{V}(X_n) \to 0$ and $\mathbb{E}[X_n] \to 0$ (because they are non-negative). On the other hands, if $\mathbb{V}(X_n) \to 0$ and $\mathbb{E}[X_n] \to 0$, then $\mathbb{E}[(\bar{X_n} - b)^2] \to 0$

Problem 3.

$$\begin{cases} \mathbb{E}[\bar{X_n}] = \mu \\ \mathbb{V}[\bar{X_n}] = \frac{\sigma^2}{n} \end{cases} \tag{3}$$

$$\Rightarrow \mathbb{E}[\bar{X_n}^2] = \frac{\sigma^2}{n} + \mu^2$$

$$\Rightarrow \mathbb{E}[(\bar{X_n} - \mu)^2] = \mathbb{E}[\bar{X_n}^2] - 2\mathbb{E}[\bar{X_n}]\mu + \mu^2 = \frac{\sigma^2}{n} + \mu^2 - 2\mu^2 + \mu^2 = \frac{\sigma^2}{n} \to 0$$

Problem 4. We have:

$$\mathbb{E}[X] = \frac{1}{n}\big(1 - \frac{1}{n^2}\big) + n\frac{1}{n^2} = \frac{1}{n} - \frac{1}{n^3} + \frac{1}{n^2}$$

$$\Rightarrow \mathbb{P}(X_n > t) \leqslant \frac{\mathbb{E}[X]}{t} = \frac{\frac{1}{n} - \frac{1}{n^3} + \frac{1}{n^2}}{t} \to 0$$

Hence, $X_n \xrightarrow{\text{P}} 0$

$$\mathbb{E}[X_n^2] = \frac{1}{n^2}\big(1 - \frac{1}{n^2}\big) + 1 \to 1$$

Hence, X_n does not converge in quadratic mean.

Problem 5. $X_i \sim$ Ber(p) $\iff \frac{1}{n}\sum_{i=1}^n X_i^2 = \frac{1}{n}\sum_{i=1}^n X_i = \bar{X_n} \Rightarrow$

$$\begin{cases} \bar{X_n} \xrightarrow{\text{P}} p \text{ Law of Large Number} \\ \bar{X_n} \xrightarrow{\text{qm}} p \text{ From proof of problem 3} \end{cases} \tag{4}$$

Problem 6. By Central Limit Theorem:

$$\frac{\sqrt{100}(\bar{X_n} - 68)}{2.6} \sim \mathbb{N}(0,1) \Longrightarrow \mathbb{P}(\bar{X_n} > 68) = \mathbb{P}(\mathbb{N}(0,1) > 0) = \frac{1}{2}$$

Problem 7.

$$X_n \sim Poisson(\lambda_n) \Rightarrow \mathbb{E}[X_n] = \mathbb{V}[X_n] = \lambda_n = \frac{1}{n}$$

a.)

$$\mathbb{P}(X_n > t) \leqslant \frac{\mathbb{E}[X_n]}{t} = \frac{1}{nt} \to 0$$

$$\Rightarrow X_n \xrightarrow{\text{P}} 0$$

b.)

$$\mathbb{P}(Y_n > t) = \mathbb{P}(X_n > \frac{t}{n}) = \sum_{i=\frac{t}{n}}^{\infty} \frac{\left(\frac{1}{n}\right)^i \exp(\frac{-1}{n})}{i!} \xrightarrow{n\to\infty} \exp(\frac{-1}{n}) \sum_{i=0}^{\infty} \frac{\left(\frac{1}{n}\right)^i}{i!} =$$

Problem 8. By Central Limit Theorem:

$$Z_n = \frac{\sqrt{100}(\bar{X_n} - 1)}{1} \sim \mathbb{N}(0,1)$$

$$\mathbb{P}(Y = \sum_{i=1}^n X_i < 90) = \mathbb{P}(\bar{X_n} < 0.9) = P(Z_n < -1) = \phi(-1.01)$$

Problem 9.

$$\mathbb{P}(|X_n - X| > t) = \mathbb{P}(X_n \neq X) = \mathbb{P}(X_n = \exp(n)) = \frac{1}{n} \longrightarrow 0$$

$$\longrightarrow X_n \xrightarrow{(P)} X \longrightarrow X_n \rightsquigarrow X$$

$$\mathbb{E}[(X - X_n)^2] = (\exp(n) - 1)^2 \frac{1}{2}\frac{1}{n} + (\exp(n) = 1)^2 \frac{1}{2}\frac{1}{n} = \frac{\exp(2n) + 1}{2n} \to \infty$$

Problem 11.

$$X_n \sim \mathbb{N}(0, \frac{1}{n}) \Rightarrow \sqrt{n}X_n \sim \mathbb{N}(0, 1)$$

$$\mathbb{P}(|X_n - X| > t) = \mathbb{P}(|\sqrt{n}X_n - \sqrt{n}X| > \sqrt{n}t) \leqslant \frac{\mathbb{E}[|\mathbb{N}(0,1) - \sqrt{n}X|]}{\sqrt{n}t} = \frac{\mathbb{E}[\mathbb{N}(-\sqrt{n}X, 1)]}{\sqrt{n}t} = \frac{-\sqrt{n}X}{\sqrt{n}t} = \frac{-X}{t} < 0 \text{ (disprove)}$$

Problem 12. if $X_n \rightsquigarrow X$, then :

$$F_n(k) \to F(k) \Leftrightarrow$$

$$\begin{cases} \mathbb{P}(X_n = k) - \mathbb{P}(X_n = k-1) \to \mathbb{P}(X = k) - \mathbb{P}(X = k-1) \\ \mathbb{P}(X_n = k-2) - \mathbb{P}(X_n = k-2) \to \mathbb{P}(X = k-1) - \mathbb{P}(X = k-2) \\ \ldots \\ \ldots \\ \ldots \\ \mathbb{P}(X_n = 0) = \mathbb{P}(X = 0) = 0 \end{cases} \tag{5}$$

Add those equations above we get:

$$\Rightarrow \mathbb{P}(X_n = k) \to \mathbb{P}(X = k)$$

if $\mathbb{P}(X_n = k) \to \mathbb{P}(X = k)$, then:

$$\begin{cases} \mathbb{P}(X_n = k) \to \mathbb{P}(X = k) \\ \mathbb{P}(X_n = k-1) \to \mathbb{P}(X = k-1) \\ \ldots \\ \ldots \\ \ldots \end{cases} \tag{6}$$

Add those equations above we get:

$$F_n(k) \to F(k)$$

Problem 13.

$$F_n(t) = \mathbb{P}(X_n \leqslant t) = 1 - \mathbb{P}(X_n > t) = 1 - \mathbb{P}(\min(Z_1, Z_2, \ldots)) > \frac{t}{n}) = 1 - \prod_1^n \mathbb{P}(Z_i > \frac{t}{n}) = 1 - \big(1 - F_n(\frac{t}{n})\big)^n$$

Taylor series:

$$F_n(\frac{t}{n}) \leqslant F_n(0) + \frac{F'(0)}{1}\frac{t}{n} = 0 + f(0)\frac{t}{n} \to \frac{\lambda t}{n}$$

$$\Rightarrow F_n(t) \leqslant 1 - \big(1 - \frac{\lambda t}{n}\big)^n \to 1 - \exp(-\lambda t)$$

Problem 14.

$$X_i \sim \mathbb{U}(0,1) \Rightarrow \mu = \frac{1}{2} \text{ and } \sigma^2 = \frac{1}{12}$$

Delta method with $g(x) = x^2$:

$$\bar{X_n}^2 \sim \mathbb{N}(\mu^2, (2\mu)^2\frac{\sigma^2}{n}) = \mathbb{N}(\frac{1}{2}, \frac{1}{12n})$$

Problem 15. Let $\bar{X} = \begin{pmatrix}\bar{X}_1 & \bar{X}_2\end{pmatrix}^T$. By Delta Method with $g(x,y) = \frac{x}{y}$:

$$\sqrt{n}\big(g(\bar(X)) - g(\mu) \sim \mathbb{N}(0, \nabla_\mu^T \sum \nabla_\mu)\big)$$

where

$$\sum = \begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{pmatrix} \text{ and } \nabla_\mu = \nabla g(\mu_1, \mu_2) \begin{pmatrix}1/\mu_2\\ -\mu_1/\mu_2^2\end{pmatrix}$$

Explicitly the multiplication of $\nabla_\mu^T \sum \nabla_\mu$, we get:

$$\sqrt{n}(\frac{X_1}{X_2} - \frac{\mu_1}{\mu_2}) \sim \mathbb{N}(0, \frac{\sigma_{11}}{\mu_2^2} - \frac{2\mu_1\sigma_{12}}{\mu_2^3} + \frac{\mu_1^2\sigma_{22}}{\mu_2^4})$$

Problem 16.

$$\begin{cases} X_n = X = \mathbb{U}(0,1) \\ Y_n = -X_n \\ Y = \mathbb{U}(-1,0) \end{cases} \tag{7}$$