Chapter 5

Huy Nguyen, Hoang Nguyen

March 1, 2020

Problem 1.

$$bias(\hat{\lambda}) = \mathbb{E}[\hat{\lambda}] - \lambda = \mathbb{E}\big[\frac{\sum_1^n X_i}{n}\big] - \lambda = \frac{n\mathbb{E}X_i}{n} - \lambda = \lambda - \lambda = 0$$

$$se(\hat{\lambda}) = \sqrt{\mathbb{V}(\bar{\lambda})} = \sqrt{\frac{\lambda}{n}}$$

$$MSE = bias(\hat{\lambda}) + se(\hat{\lambda}) = 0 + \frac{\lambda}{n} = \frac{\lambda}{n}$$

Problem 2.

$$F_{\hat{\theta}}(x) = \mathbb{P}(\hat{\theta} < x) = \mathbb{P}(\max X_1, ..., X_n < x) = \mathbb{P}(X_1, X_2, ...X_n < x) = \prod_{i=1}^n \mathbb{P}(X_i < x) = \big(\mathbb{P}(X_i < x)\big)^n = \big(\frac{x}{\theta}\big)^n$$

$$\Rightarrow f_{\hat{\theta}}(x) = \frac{d}{dx}F_{\hat{\theta}}(x) = \frac{d}{dx}\big(\frac{x}{\theta}\big)^n = \frac{nx^{n-1}}{\theta^n}$$

$$\Rightarrow \mathbb{E}[\hat{\theta}] = \int_0^\theta x\frac{nx^{n-1}}{\theta^n}dx = \frac{n}{n+1}\theta$$

$$\Rightarrow bias(\hat{\theta}) = \mathbb{E}[\hat{\theta}] - \theta = \frac{n}{n+1}\theta - \theta = \frac{-\theta}{n+1}$$

$$\mathbb{V}(\hat{\theta}) = \mathbb{E}[\hat{\theta}^2] - (\mathbb{E}[\hat{\theta}])^2 = \int_0^\theta x^2\frac{nx^{n-1}}{\theta^n}dx - \big(\frac{n}{n+1}\theta\big)^2 = \theta^2\frac{n}{(n+2)(n+1)^2}$$

$$\Rightarrow MSE(\hat{\theta}) = \big(\frac{-\theta}{n+1}\big)^2 + \theta^2\frac{n}{(n+2)(n+1)^2} = \frac{2n+2}{(n+2)(n+1)^2}\theta^2$$

Problem 3.

$$bias(\hat{\theta}) = \mathbb{E}[\hat{\theta}] - \theta = \mathbb{E}[\frac{2\sum_{i=1}^n X_i}{n}] - \theta = 2\mathbb{E}[X_i] - \theta = 2\frac{\theta}{2} - \theta = 0$$

$$se(\hat{\theta}) = \sqrt{\frac{\mathbb{V}(\hat{\theta})}{n}} = \sqrt{\frac{\theta}{n}}$$

$$\Rightarrow MSE = \frac{\theta}{n}$$