# Chapter 6

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**Problem 1.** a)

$$X \sim Binomial(n, p) \Rightarrow f(x|p) = \binom{n}{X} p^x (1-p)^{n-x} \propto p^x (1-p)^{n-x}$$

$$\Rightarrow \pi(p|x) = f(x|p)f(p) \propto p^x (1-p)^{n-x} p^{\alpha-1} (1-p)^{\beta-1} = p^{\alpha+x-1}(1-p)^{n-x+\beta-1}$$

$$\Rightarrow p|x \sim Beta(\alpha + x, n - x + \beta)$$

Hence, Bayes estimator:

$$\hat{p} = \mathbb{E}[p|x] = \frac{x+\alpha}{n-x+\beta+x+\alpha} = \frac{x+\alpha}{n+\alpha+\beta}$$

Under least mean square:

$$R(\hat{p}, p) = Var(\hat{p}) + (p - \mathbb{E}[\hat{p}])^2 = \frac{Var(x)}{(n+\alpha+\beta)^2} + \big(p - \frac{\mathbb{E}[x]+\alpha}{n+\alpha+\beta}\big)^2 = \frac{np(1-p)}{(n+\alpha+\beta)^2} + \frac{(p(\alpha+\beta)-\alpha)^2}{(n+\alpha+\beta)^2}$$

Hence, Bayes risk :

$$r(f, \hat{p}) = \mathbb{E}_f[R(p, \hat{p})] = \mathbb{E}_f\big[\frac{np(1-p)}{(n+\alpha+\beta)^2}\big] + \mathbb{E}_f\big[\frac{(p(\alpha+\beta)-\alpha)^2}{(n+\alpha+\beta)^2}\big] = \frac{n\alpha\beta}{(\alpha+\beta)^2(n+\alpha+\beta)^2}$$

b)

$$X \sim Poison(\alpha) \Rightarrow f(x|\alpha) \propto \lambda^x e^{-\lambda}$$

$$\Rightarrow \pi(\lambda|x) \propto \lambda^x e^{-\lambda} \lambda^{\alpha-1} e^{-\beta\lambda} = \lambda^{x+\lambda-1} e^{-\lambda(\beta+1)}$$

$$\Rightarrow \lambda|x \sim Gamma(x+\lambda, \beta+1) \Rightarrow \hat{\lambda} = \frac{x+\lambda}{\beta+1}$$

Similarly from above question, we get:

$$R(\hat{\lambda}, \lambda) = \frac{\lambda+\alpha}{(\beta+1)^2} + \frac{(\alpha-\lambda\beta)^2}{(\beta+1)^2}$$

$$\Rightarrow r(f, \hat{\lambda}) = \mathbb{E}_f[R(\hat{\lambda}, \lambda)] = \frac{\lambda}{\beta(\beta+1)^2}$$

c) Similarly from above questions, we get:

$$\theta|x \sim \mathbb{N}\big(\frac{b^2}{\sigma^2+b^2}x + \frac{\sigma^2}{\sigma^2+b^2}a, (\frac{1}{\sigma^2} + \frac{1}{b^2})^{-1}\big) \Rightarrow \hat{\theta} = \frac{b^2}{\sigma^2+b^2}x + \frac{\sigma^2}{\sigma^2+b^2}a$$

Bayes risk:

$$r(f, \hat{\theta}) = \frac{b^4\sigma^2}{(\sigma^2+b^2)^2} + \frac{\sigma^2 b^2}{(\sigma^2+b^2)^2}$$

**Problem 2.** a)

$$X_i \sim Uniform(a,b) \Rightarrow$$

$$\begin{cases} \mathbb{E}[X_i] = \frac{a+b}{2} \\ \mathbb{E}[X_i^2] = \frac{a^2+ab+b^2}{3} \end{cases} \tag{1}$$

Let $\hat{\alpha_1} = \mathbb{E}[X_i]$ and $\hat{\alpha_2} = \mathbb{E}[X_i^2] \Rightarrow$

$$\begin{cases} \hat{\alpha_1} = \frac{a+b}{2} \\ \hat{\alpha_2} = \frac{a^2+ab+b^2}{3} \end{cases} \tag{2}$$

Solve the above equations, we obtain:

$$\begin{cases} \hat{\alpha} = \hat{\alpha_1} - \sqrt{3(\alpha_2 - \alpha_1^2)} \\ \hat{\beta} = \hat{\alpha_1} + \sqrt{3(\alpha_2 - \alpha_1^2)} \end{cases} \tag{3}$$

b.)

The likehood :

$$\mathbb{L}(a,b) = \prod_{i=1}^{n} \frac{1}{b-a} \mathbb{I}_{(a,b)} = \frac{1}{(b-a)^n} \mathbb{I}_{(-\infty,x_1)}(a) \mathbb{I}_{(x_n,+\infty)}(b)$$

To maximize the likelihood: $a = X_{(1)}$ and $b = X_{(2)}$
c. )

We have: $\tau = \int x dF(x) = \mathbb{E}[x] = \frac{a+b}{2}$ , by the equivariance property:

$$\hat{\tau} = \frac{\hat{\alpha} + \hat{\beta}}{2} = \frac{X_{(1)} + X_{(2)}}{2}$$

d. ) Update

**Problem 4.** $X_i \sim Uniform(0,\theta) \Rightarrow$ Likelihood of X:

$$\mathcal{L} = \prod_{i=1}^{n} f(x) = \prod_{i=1}^{n} \frac{1}{\theta} I(0 \le X_i \le \theta) = \frac{1}{\theta^n} I(0 \le X_1, X_2, ..., X_n \le \theta)$$

$\Rightarrow \mathcal{L}$ max when $\theta = \hat{\theta} = X_{(n)}$
We have:

$$\mathbb{P}(|\hat{\theta} - \theta| < \epsilon) = \mathbb{P}(\hat{\theta} < \theta - \epsilon) + \mathbb{P}(\hat{\theta} > \theta + \epsilon) = \mathbb{P}(\hat{\theta} < \theta - \epsilon) = \big(\frac{\theta - \epsilon}{\theta}\big)^n \to 0$$

MLE is consistent

**Problem 5.** We have $Xi \sim Poissin(\lambda)$

$$\Rightarrow E[X_i] = \lambda$$

Method of moment:

$$\hat{\lambda} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

Log-likelihood of Poisson:

$$\mathcal{L} = \log(\lambda)\sum_{i=1}^{n} X_i - n\lambda - \sum_{i=1}^{n}\log(X_i!)$$

$$\frac{d\mathcal{L}}{d\lambda} = 0 \Leftrightarrow \frac{\sum_{i=1}^{n} X_i}{\hat{\lambda}} - n = 0$$

$$\Leftrightarrow \hat{\lambda} = \frac{\sum_{i=1}^{n} X_i}{n}$$

Fisher Information:

$$I(\lambda) = -\mathbb{E}[\frac{d^2\mathcal{L}}{d\lambda^2}] = -\mathbb{E}[-\frac{\sum_{i=1}^{n} X_i}{\lambda^2}] = \frac{1}{\lambda}$$

**Problem 6.** a)
MLE of $\theta$ : $\hat{\theta} = \bar{X_n}$

We have

$$\psi = \mathbb{P}(Y_1 = 1) = \mathbb{P}(X_1 > 0) = \phi(\theta)$$

By the equivariance property of MLE :

$$\hat{\psi} = \phi(\hat{\theta})$$

b)

We have Standard deviation of $\hat{\theta}$: $\hat{se}(\hat{\theta}) = \sqrt{\frac{\phi^2}{n}} = \sqrt{\frac{1}{n}}$
Apply Delta Method we get :

$$\hat{se}(\hat{\psi}) = |\sigma'(\hat{\theta})|\hat{se}(\hat{\theta}) = |\phi'(\hat{\theta})|\sqrt{\frac{1}{n}}$$

We get the Confident Interval: $C_n = \hat{\psi} \pm |\sigma'(\hat{\theta})|\sqrt{\frac{1}{n}}$ c)

From Central Limit Theorem, we get:

$$\sqrt{n}(\hat{\theta} - \theta) \to \mathbb{N}(0, 1)$$

Apply Delta Method with $\psi = \phi(\theta)$

$$\sqrt{n}\big(\phi(\hat{\theta}) - \phi(\theta)\big) \to \mathbb{N}(0, |\phi'(\theta)|)$$

In addition : $\phi'(\theta) = \frac{1}{\sqrt{2\pi}}$
Hence:

$$\sqrt{n}(\hat{\psi} - \psi) \to \mathbb{N}(0, \frac{1}{\sqrt{2\pi}})$$

On the other hand, $Y_i \sim Bernulli(\psi)$ from Central Limit Theorem, we get:

$$\sqrt{n}(\tilde{\psi} - \psi) \to \mathbb{N}\big(0, \psi(1-\psi)\big) = \mathbb{N}\big(0, \phi(\theta)(1-\phi(\theta))\big) = \mathbb{N}(0, \frac{1}{4}) \text{ because } (\phi(\theta) = \frac{1}{2})$$

Hence, the asymptotic relative efficiency of $\tilde{\psi}$ and $\hat{\psi}$ : $\sqrt{\frac{2}{\pi}}$
d)

Updated

**Problem 7.** a)

$X_1 \sim Binomial(n_1, p_1)$, $X_2 \sim Binomial(n_2, p_2) \Rightarrow$ MLE of $p_1$ and $p_2$ : $\hat{p_1} = \frac{X_1}{n_1}$ , $\hat{p_2} = \frac{X_2}{n_2}$
By the equivariance property of MLE, MLE of $\psi$ : $\hat{\psi} = \hat{p_1} - \hat{p_2} = \frac{X_1}{n_1} - \frac{X_2}{n_2}$

b)

We have data $X = (X_1, X_2)$
The log- likelihood of data:

$$\mathcal{L} = \log(f(X_1, X_2)) = \log\big(\prod_{i=1}^{2} \binom{n_i}{X_i} p_i^{X_i}(1-p_i)^{n_i - X_i}\big) = \log\big(\binom{n_1}{X_1}\big) + X_1 \log(p_1) + (n_1 - X_1)\log(1-p_1)$$

$$+ \log\big(\binom{n_2}{X_2}\big) + X_2 \log(p_2) + (n_2 - X_2)\log(1-p_2)$$

Hence:

$$H_{11} = \frac{\partial^2 \mathcal{L}(X_1, X_2)}{\partial X_1^2} = \frac{-X_1}{p_1^2} - \frac{n_1 - X_1}{(1-p_1)^2} \Rightarrow \mathbb{E}[H_1 1] = \frac{-n_1}{p_1} - \frac{n_1}{1-p_1} = \frac{-n_1}{p_1(1-p_1)}$$

$$H_{12} = H_{21} = \frac{\partial^2 \mathcal{L}(X_1, X_2)}{\partial X_1 \partial X_2} = 0 \Rightarrow \mathbb{E}[H_{11}] = \mathbb{E}[H_{21}] = 0$$

$$H_{22} = \frac{\partial^2 \mathcal{L}(X_1, X_2)}{\partial X_2^2} = \frac{-X_2}{p_2^2} - \frac{n_2 - X_2}{(1-p_2)^2} \Rightarrow \mathbb{E}[H_2 2] = \frac{-n_2}{p_2} - \frac{n_2}{1-p_2} = \frac{-n_2}{p_2(1-p_2)}$$

Hence:

$$I_n(p_1, p_2) = -\mathbb{E}[H] = \begin{bmatrix} \frac{-n_1}{p_1(1-p_1)} & 0 \\ 0 & \frac{-n_2}{p_2(1-p_2)} \end{bmatrix}$$

c)

$$J_n = I_n^{-1}(p_1, p_2) = \begin{bmatrix} \frac{p_1(1-p_1)}{n_1} & 0 \\ 0 & \frac{p_2(1-p_2)}{n_2} \end{bmatrix}$$

On the other hand, we have function g: $g(p_1, p_2) = p_1 - p_2 \Rightarrow \nabla g = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\Rightarrow \hat{se}(\hat{\psi}) = \sqrt{\nabla g^T J_n \nabla g} = \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$$

d)

$$C_n = \big(\hat{\psi} - z_{\alpha/2}\sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}, \hat{\psi} + z_{\alpha/2}\sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}\big)$$

**Problem 8.** $X_i \sim \mathbb{N}(0, \mu)$, log- likelihood is :

$$\mathcal{L}(\mu, \sigma) = \log\big(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp(\frac{-1}{2\sigma^2}(x-\mu)^2)\big) = \sum_{i=1}^{n} \log(\frac{1}{2\pi}) + \sum_{i=1}^{n} \log(\sigma) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x-\mu)^2$$

Hence, :

$$\frac{\partial^2 \mathcal{L}(\mu, \sigma)}{\partial \mu^2} = -\frac{n}{\sigma^2} \Rightarrow H_{11} = -\mathbb{E}[\frac{-n}{\sigma^2}] = \frac{n}{\sigma^2}$$

$$\frac{\partial^2 \mathcal{L}}{\partial \mu \partial \sigma} = \frac{\partial^2 \mathcal{L}}{\partial \sigma \partial \mu} = 0 \Rightarrow H_{12} = H_{21} = 0$$

$$\frac{\partial^2 \mathcal{L}(\mu,\sigma)}{\partial \sigma^2} = \frac{n}{\sigma^2} - \frac{3}{\sigma^2}\sum_{i=1}^{n}(x-\mu)^2 \Rightarrow H_{22} = -\mathbb{E}[\frac{n}{\sigma^2} - \frac{3}{\sigma^2}\sum_{i=1}^{n}(x-\mu)^2] = \frac{2n}{\sigma^2}$$

$$\Rightarrow I_n(\mu,\sigma) = \begin{bmatrix} \frac{n}{\sigma^2} & 0 \\ 0 & \frac{2n}{\sigma^2} \end{bmatrix}$$