Chapter 6

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Problem 1.

$$X_i \sim Gamma(\alpha, \beta) \Rightarrow \mathbb{E}[X_i] = \frac{\alpha}{\beta} = \bar{X_n}$$

$$\mathbb{E}[X_i^2] = Var(X_i) + (\mathbb{E}[]X_i)^2 = \frac{\alpha}{\beta^2} + \frac{\alpha^2}{\beta^2} = \frac{\alpha^2 + \alpha}{\beta^2}$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n} X_i^2 = \frac{\hat{\alpha}^2 + \hat{\alpha}}{\hat{\beta}^2}$$

$$\Rightarrow \hat{\alpha} = \frac{\bar{X_n}^2}{\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X_n}^2} \text{ and } \hat{\beta} = \frac{\bar{X_n}}{\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X_n}^2}$$

Problem 2. a)

$$X_i \sim Uniform(a, b) \Rightarrow$$

$$\begin{cases} \mathbb{E}[X_i] = \frac{a+b}{2} \\ \mathbb{E}[X_i^2] = \frac{a^2+ab+b^2}{3} \end{cases} \tag{1}$$

Let $\hat{\alpha_1} = \mathbb{E}[X_i]$ and $\hat{\alpha_2} = \mathbb{E}[X_i^2] \Rightarrow$

$$\begin{cases} \hat{\alpha_1} = \frac{a+b}{2} \\ \hat{\alpha_2} = \frac{a^2+ab+b^2}{3} \end{cases} \tag{2}$$

Solve the above equations, we obtain:

$$\begin{cases} \hat{\alpha} = \hat{\alpha_1} - \sqrt{3(\alpha_2 - \alpha_1^2)} \\ \hat{\beta} = \hat{\alpha_1} + \sqrt{3(\alpha_2 - \alpha_1^2)} \end{cases} \tag{3}$$

b.)

The likehood :

$$\mathbb{L}(a, b) = \prod_{i=1}^{n} \frac{1}{b-a}\mathbb{I}_{(a,b)} = \frac{1}{(b-a)^n}\mathbb{I}_{(-\infty, x_1)}(a)\mathbb{I}_{(x_n, +\infty)}(b)$$

To maximize the likelihood: $a = X_{(1)}$ and $b = X_{(2)}$

c.)

We have: $\tau = \int x dF(x) = \mathbb{E}[x] = \frac{a+b}{2}$, by the equivariance property:

$$\hat{\tau} = \frac{\hat{\alpha} + \hat{\beta}}{2} = \frac{X_{(1)} + X_{(2)}}{2}$$

d.) Update

Problem 4. $X_i \sim Uniform(0, \theta) \Rightarrow$ Likelihood of X:

$$\mathcal{L} = \prod_{i=1}^{n} f(x) = \prod_{i=1}^{n} \frac{1}{\theta} I(0 \leq X_i \leq \theta) = \frac{1}{\theta^n} I(0 \leq X_1, X_2, ..., X_n \leq \theta)$$

$\Rightarrow \mathcal{L}$ max when $\theta = \hat{\theta} = X_{(n)}$

We have:

$$\mathbb{P}(|\hat{\theta} - \theta| < \epsilon) = \mathbb{P}(\hat{\theta} < \theta - \epsilon) + \mathbb{P}(\hat{\theta} > \theta + \epsilon) = \mathbb{P}(\hat{\theta} < \theta - \epsilon) = \big(\frac{\theta - \epsilon}{\theta}\big)^n \to 0$$

MLE is consistent