Chapter 12

Huy Nguyen, Hoang Nguyen

March 17, 2020

Problem 1. a)

$$X \sim Binomial(n,p) \Rightarrow f(x|p) = \binom{n}{X} p^x (1-p)^{n-x} \propto p^x (1-p)^{n-x}$$

$$\Rightarrow \pi(p|x) = f(x|p)f(p) \propto p^x(1-p)^{n-x}p^{\alpha-1}(1-p)^{\beta-1} = p^{\alpha+x-1}(1-p)^{n-x+\beta-1}$$

$$\Rightarrow p|x \sim Beta(\alpha + x, n-x+\beta)$$

Hence, Bayes estimator:

$$\hat{p} = \mathbb{E}[p|x] = \frac{x+\alpha}{n-x+\beta+x+\alpha} = \frac{x+\alpha}{n+\alpha+\beta}$$

Under least mean square:

$$R(\hat{p},p) = Var(\hat{p}) + (p - \mathbb{E}[\hat{p}])^2 = \frac{Var(x)}{(n+\alpha+\beta)^2} + \big(p - \frac{\mathbb{E}[x]+\alpha}{n+\alpha+\beta}\big)^2 = \frac{np(1-p)}{(n+\alpha+\beta)^2} + \frac{(p(\alpha+\beta)-\alpha)^2}{(n+\alpha+\beta)^2}$$

Hence, Bayes risk :

$$r(f,\hat{p}) = \mathbb{E}_f[R(p,\hat{p})] = \mathbb{E}_f\big[\frac{np(1-p)}{(n+\alpha+\beta)^2}\big] + \mathbb{E}_f\big[\frac{(p(\alpha+\beta)-\alpha)^2}{(n+\alpha+\beta)^2}\big] = \frac{n\alpha\beta}{(\alpha+\beta)^2(n+\alpha+\beta)^2}$$

b)

$$X \sim Poison(\alpha) \Rightarrow f(x|\alpha) \propto \lambda^x e^{-\lambda}$$

$$\Rightarrow \pi(\lambda|x) \propto \lambda^x e^{-\lambda}\lambda^{\alpha-1}e^{-\beta\lambda} = \lambda^{x+\lambda-1}e^{-\lambda(\beta+1)}$$

$$\Rightarrow \lambda|x \sim Gamma(x+\lambda, \beta+1) \Rightarrow \hat{\lambda} = \frac{x+\lambda}{\beta+1}$$

Similarly from above question, we get:

$$R(\hat{\lambda},\lambda) = \frac{\lambda+\alpha}{(\beta+1)^2} + \frac{(\alpha-\lambda\beta)^2}{(\beta+1)^2}$$

$$\Rightarrow r(f,\hat{\lambda}) = \mathbb{E}_f[R(\hat{\lambda},\lambda)] = \frac{\lambda}{\beta(\beta+1)^2}$$

c) Similarly from above questions, we get:

$$\theta|x \sim \mathbb{N}\big(\frac{b^2}{\sigma^2+b^2}x + \frac{\sigma^2}{\sigma^2+b^2}a, (\frac{1}{\sigma^2}+\frac{1}{b^2})^{-1}\big) \Rightarrow \hat{\theta} = \frac{b^2}{\sigma^2+b^2}x + \frac{\sigma^2}{\sigma^2+b^2}a$$

Bayes risk:

$$r(f,\hat{\theta}) = \frac{b^4\sigma^2}{(\sigma^2+b^2)^2} + \frac{\sigma^2 b^2}{(\sigma^2+b^2)^2}$$

Problem 2. Let prior probability $f(\theta) = 1$. You can prove your self that:

$$\theta|X \sim \mathbb{N}(\bar{X}, \frac{\sigma^2}{n})$$

The posterior risk:

$$r(\hat{\theta}|X) = \int L(\theta, \hat{\theta}) f(\theta|x) d\theta = \int \frac{(\theta - \hat{\theta})^2}{\sigma^2} \mathbb{N}(\bar{X}, \frac{\sigma^2}{n}) d\theta = \frac{1}{\sigma^2} \Big[\int \theta^2 \mathbb{N}(\bar{X}, \frac{\sigma^2}{n}) d\theta - \int 2\theta\hat{\theta} \mathbb{N}(\bar{X}, \frac{\sigma^2}{n}) d\theta + \int \hat{\theta}^2 \mathbb{N}(\bar{X}, \frac{\sigma^2}{n}) d\theta \Big]$$

$$= \frac{1}{\sigma^2} \big[\mathbb{E}[\theta^2|X] - \hat{\sigma} \mathbb{E}|\mathbb{X}[\theta] + \hat{\theta}^2 \big] = \frac{1}{\sigma^2} \big[Var(\theta|X) + E[\theta|X]^2 - 2\hat{\theta}\mathbb{E}|\mathbb{X}[\theta] + \hat{\theta}^2 \big]$$

$$= \frac{1}{\sigma^2} \Big[\frac{\sigma^2}{n} + \bar{X}^2 - 2\hat{\theta}\bar{X} + \hat{\theta}^2 \Big]$$

Minimize the posterior risk we get the Bayes estimator, take the derivative of the above equation we get $\hat{\theta} = \bar{X}$ is the Bayes estimator $\Rightarrow \bar{\theta}$ is admissible

In addition, with $\theta = \bar{X}$, we get the posterior risk is $\frac{1}{n}$, hence, the Bayes risk is constant $\Rightarrow \hat{\theta}$ is minimax

Problem 3. We have:

$$L(\theta, \hat{\theta}) = I(\theta \neq \hat{\theta}) \Rightarrow R(\theta, \hat{\theta}) = \mathbb{E}[I(\hat{\theta} \neq \theta)] = \sum I(\hat{\theta} \neq \theta) f(\theta|x)$$

Hence, to minimize $R(\hat{\theta}, \theta)$, $\hat{\theta}$ must not be equal θ as much as possible $\Rightarrow \hat{\theta} =$ posterior mode of $f(\theta|x)$

Problem 4. We have:

$$L(\sigma^2, \hat{\sigma}^2) = \frac{\hat{\sigma}^2}{\sigma} - 1 - \log(\frac{\hat{\sigma}^2}{\sigma})$$

$$\Rightarrow R(\sigma^2, \hat{\sigma}^2) = \mathbb{E}[L(\sigma^2, \hat{\sigma}^2)] = \frac{E[bS^2]}{\sigma^2} - 1 - \mathbb{E}[\log(bS^2)] + \log(\sigma^2)$$

Apply Jessen's inequality:

$$R(\sigma^2, \hat{\sigma}^2) \geq \frac{E[bS^2]}{\sigma^2} - 1 - \log(\mathbb{E}[bS^2]) + \log(\sigma^2)$$

In addition, $\mathbb{E}[S^2] = \sigma^2$, hence:

$$R(\sigma^2, \hat{\sigma}^2) \geq b - 1 + \log b$$

Take the derivative of the right-hand side, we get : b=1

Problem 5. We have $L(p, \hat{p}) = \big(1 - \frac{\hat{p}}{p}\big)^2$
Apply Jessen's inequality: $\mathbb{E}_p[\hat{p}^2] \geq \mathbb{E}_p[\tilde{p}]^2$, We get:

$$R(p, \tilde{p}) = \mathbb{E}_p[L(p, \tilde{p})] = 1 - \frac{2\mathbb{E}_p[\tilde{p}]}{p} + \frac{\mathbb{E}_p[\tilde{p}^2]}{p^2} \geq \big(1 - \frac{\mathbb{E}_p[\tilde{p}]}{p}\big)^2$$

In addition $\big(1 - \frac{\mathbb{E}_p[\tilde{p}]}{p}\big)^2 \leq 1$, Hence:

$$inf_{\tilde{p}} sup_p R(p, \tilde{p}) = 1$$

On the other hands:

$$R(p, \hat{p}) = 1 - \frac{2\mathbb{E}_p[\hat{p}]}{p} + \frac{\mathbb{E}_p[\hat{p}^2]}{p^2} = 1$$

$$\Rightarrow sup_p R(p, \hat{p}) = 1 = inf_{\tilde{p}} sup_p R(p, \tilde{p})$$

Therefore, $\hat{p}$ is minimax rule