Chapter 6

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March 12, 2020

Problem 1.

$$X_i \sim Gamma(\alpha, \beta) \Rightarrow \mathbb{E}[X_i] = \frac{\alpha}{\beta} = \bar{X_n}$$

$$\mathbb{E}[X_i^2] = Var(X_i) + (\mathbb{E}[]X_i)^2 = \frac{\alpha}{\beta^2} + \frac{\alpha^2}{\beta^2} = \frac{\alpha^2 + \alpha}{\beta^2}$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n} X_i^2 = \frac{\hat{\alpha}^2 + \hat{\alpha}}{\hat{\beta}^2}$$

$$\Rightarrow \hat{\alpha} = \frac{\bar{X_n}^2}{\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X_n}^2} \text{ and } \hat{\beta} = \frac{\bar{X_n}}{\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X_n}^2}$$

Problem 2. a)

$$X_i \sim Uniform(a, b) \Rightarrow$$

$$\begin{cases} \mathbb{E}[X_i] = \frac{a+b}{2} \\ \mathbb{E}[X_i^2] = \frac{a^2+ab+b^2}{3} \end{cases} \tag{1}$$

Let $\hat{\alpha_1} = \mathbb{E}[X_i]$ and $\hat{\alpha_2} = \mathbb{E}[X_i^2] \Rightarrow$

$$\begin{cases} \hat{\alpha_1} = \frac{a+b}{2} \\ \hat{\alpha_2} = \frac{a^2+ab+b^2}{3} \end{cases} \tag{2}$$

Solve the above equations, we obtain:

$$\begin{cases} \hat{\alpha} = \hat{\alpha_1} - \sqrt{3(\alpha_2 - \alpha_1^2)} \\ \hat{\beta} = \hat{\alpha_1} + \sqrt{3(\alpha_2 - \alpha_1^2)} \end{cases} \tag{3}$$

b.)

The likehood :

$$\mathbb{L}(a, b) = \prod_{i=1}^{n} \frac{1}{b-a}\mathbb{I}_{(a,b)} = \frac{1}{(b-a)^n}\mathbb{I}_{(-\infty, x_1)}(a)\mathbb{I}_{(x_n, +\infty)}(b)$$

To maximize the likelihood: $a = X_{(1)}$ and $b = X_{(2)}$

c.)

We have: $\tau = \int xdF(x) = \mathbb{E}[x] = \frac{a+b}{2}$, by the equivariance property:

$$\hat{\tau} = \frac{\hat{\alpha} + \hat{\beta}}{2} = \frac{X_{(1)} + X_{(2)}}{2}$$

d.) Update

Problem 4. $X_i \sim Uniform(0, \theta) \Rightarrow$ Likelihood of X:

$$\mathcal{L} = \prod_{i=1}^{n} f(x) = \prod_{i=1}^{n} \frac{1}{\theta} I(0 \leq X_i \leq \theta) = \frac{1}{\theta^n} I(0 \leq X_1, X_2, ..., X_n \leq \theta)$$

$\Rightarrow \mathcal{L}$ max when $\theta = \hat{\theta} = X_{(n)}$

We have:

$$\mathbb{P}(|\hat{\theta} - \theta| < \epsilon) = \mathbb{P}(\hat{\theta} < \theta - \epsilon) + \mathbb{P}(\hat{\theta} > \theta + \epsilon) = \mathbb{P}(\hat{\theta} < \theta - \epsilon) = (\frac{\theta - \epsilon}{\theta})^n \to 0$$

MLE is consistent

Problem 5. We have $Xi \sim Poissin(\lambda)$

$$\Rightarrow E[X_i] = \lambda$$

Method of moment:

$$\hat{\lambda} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

Log-likelihood of Poisson:

$$\mathcal{L} = \log(\lambda) \sum_{i=1}^{n} X_i - n\lambda - \sum_{i=1}^{n} \log(X_i!)$$

$$\frac{d\mathcal{L}}{d\lambda} = 0 \Leftrightarrow \frac{\sum_{i=1}^{n} X_i}{\hat{\lambda}} - n = 0$$

$$\Leftrightarrow \hat{\lambda} = \frac{\sum_{i=1}^{n} X_i}{n}$$

Fisher Information:

$$I(\lambda) = -\mathbb{E}[\frac{d^2\mathcal{L}}{d\lambda^2}] = -\mathbb{E}[-\frac{\sum_{i=1}^{n} X_i}{\lambda^2}] = \frac{1}{\lambda}$$

Problem 6. a)

MLE of θ : $\hat{\theta} = \bar{X_n}$

We have

$$\psi = \mathbb{P}(Y_1 = 1) = \mathbb{P}(X_1 > 0) = \phi(\theta)$$

By the equivariance property of MLE :

$$\hat{\psi} = \phi(\hat{\theta})$$

b)

We have Standard deviation of $\hat{\theta}$: $\hat{se}(\hat{\theta}) = \sqrt{\frac{\phi^2}{n}} = \sqrt{\frac{1}{n}}$

Apply Delta Method we get :

$$\hat{se}(\hat{\psi}) = |\sigma'(\hat{\theta})|\hat{se}(\hat{\theta}) = |\phi'(\hat{\theta})|\sqrt{\frac{1}{n}}$$

We get the Confident Interval: $C_n = \hat{\psi} \pm |\sigma'(\hat{\theta})|\sqrt{\frac{1}{n}}$ c)

From Central Limit Theorem, we get:

$$\sqrt{n}(\hat{\theta} - \theta) \to \mathbb{N}(0, 1)$$

Apply Delta Method with $\psi = \phi(\theta)$

$$\sqrt{n}\big(\phi(\hat{\theta}) - \phi(\theta)\big) \to \mathbb{N}(0, |\phi'(\theta)|)$$

In addition : $\phi'(\theta) = \frac{1}{\sqrt{2\pi}}$
Hence:

$$\sqrt{n}(\hat{\psi} - \psi) \to \mathbb{N}(0, \frac{1}{\sqrt{2\pi}})$$

On the other hand, $Y_i \sim Bernulli(\psi)$ from Central Limit Theorem, we get:

$$\sqrt{n}(\tilde{\psi} - \psi) \to \mathbb{N}\big(0, \psi(1-\psi)\big) = \mathbb{N}\big(0, \phi(\theta)(1-\phi(\theta))\big) = \mathbb{N}(0, \frac{1}{4}) \text{ because } (\phi(\theta) = \frac{1}{2})$$

Hence, the asymptotic relative efficiency of $\tilde{\psi}$ and $\hat{\psi}$: $\sqrt{\frac{2}{\pi}}$

d)

Updated

Problem 7. a)

$X_1 \sim Binomial(n_1, p_1)$, $X_2 \sim Binomial(n_2, p_2) \Rightarrow$ MLE of p_1 and p_2 : $\hat{p_1} = \frac{X_1}{n_1}$, $\hat{p_2} = \frac{X_2}{n_2}$
By the equivariance property of MLE, MLE of ψ : $\hat{\psi} = \hat{p_1} - \hat{p_2} = \frac{X_1}{n_1} - \frac{X_2}{n_2}$

b)

We have data $X = (X_1, X_2)$
The log- likelihood of data:

$$\mathcal{L} = \log(f(X_1, X_2)) = \log\big(\prod_{i=1}^{2} \binom{n_i}{X_i} p_i^{X_i}(1-p_i)^{n_i - X_i}\big) = \log\big(\binom{n_1}{X_1}\big) + X_1\log(p_1) + (n_1 - X_1)\log(1-p_1)$$

$$+\log\big(\binom{n_2}{X_2}\big) + X_2\log(p_2) + (n_2 - X_2)\log(1-p_2)$$

Hence:

$$H_{11} = \frac{\partial^2 \mathcal{L}(X_1, X_2)}{\partial X_1^2} = \frac{-X_1}{p_1^2} - \frac{n_1 - X_1}{(1-p_1)^2} \Rightarrow \mathbb{E}[H_1 1] = \frac{-n_1}{p_1} - \frac{n_1}{1-p_1} = \frac{-n_1}{p_1(1-p_1)}$$

$$H_{12} = H_{21} = \frac{\partial^2 \mathcal{L}(X_1, X_2)}{\partial X_1 \partial X_2} = 0 \Rightarrow \mathbb{E}[H_{11}] = \mathbb{E}[H_{21}] = 0$$

$$H_{22} = \frac{\partial^2 \mathcal{L}(X_1, X_2)}{\partial X_2^2} = \frac{-X_2}{p_2^2} - \frac{n_2 - X_2}{(1-p_2)^2} \Rightarrow \mathbb{E}[H_2 2] = \frac{-n_2}{p_2} - \frac{n_2}{1-p_2} = \frac{-n_2}{p_2(1-p_2)}$$

Hence:

$$I_n(p_1, p_2) = -\mathbb{E}[H] = \begin{bmatrix} \frac{-n_1}{p_1(1-p_1)} & 0 \\ 0 & \frac{-n_2}{p_2(1-p_2)} \end{bmatrix}$$

c)

$$J_n = I_n^{-1}(p_1, p_2) = \begin{bmatrix} \frac{p_1(1-p_1)}{n_1} & 0 \\ 0 & \frac{p_2(1-p_2)}{n_2} \end{bmatrix}$$

On the other hand, we have function g: $g(p_1, p_2) = p_1 - p_2 \Rightarrow \nabla g = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\Rightarrow \hat{se}(\hat{\psi}) = \sqrt{\nabla g^T J_n \nabla g} = \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$$

d)

$$C_n = \left(\hat{\psi} - z_{\alpha/2}\sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}, \hat{\psi} + z_{\alpha/2}\sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}\right)$$