

Chapter 6

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Problem 1. a)

$$\begin{aligned} X \sim \text{Binomial}(n, p) &\Rightarrow f(x|p) = \binom{n}{x} p^x (1-p)^{n-x} \propto p^x (1-p)^{n-x} \\ \Rightarrow \pi(p|x) = f(x|p)f(p) &\propto p^x (1-p)^{n-x} p^{\alpha-1} (1-p)^{\beta-1} = p^{\alpha+x-1} (1-p)^{n-x+\beta-1} \\ &\Rightarrow p|x \sim \text{Beta}(\alpha+x, n-x+\beta) \end{aligned}$$

Hence, Bayes estimator:

$$\hat{p} = \mathbb{E}[p|x] = \frac{x + \alpha}{n - x + \beta + x + \alpha} = \frac{x + \alpha}{n + \alpha + \beta}$$

Under least mean square:

$$R(\hat{p}, p) = \text{Var}(\hat{p}) + (p - \mathbb{E}[\hat{p}])^2 = \frac{\text{Var}(x)}{(n + \alpha + \beta)^2} + \left(p - \frac{\mathbb{E}[x] + \alpha}{n + \alpha + \beta}\right)^2 = \frac{np(1-p)}{(n + \alpha + \beta)^2} + \frac{(p(\alpha + \beta) - \alpha)^2}{(n + \alpha + \beta)^2}$$

Hence, Bayes risk :

$$r(f, \hat{p}) = \mathbb{E}_f[R(p, \hat{p})] = \mathbb{E}_f\left[\frac{np(1-p)}{(n + \alpha + \beta)^2}\right] + \mathbb{E}_f\left[\frac{(p(\alpha + \beta) - \alpha)^2}{(n + \alpha + \beta)^2}\right] = \frac{n\alpha\beta}{(\alpha + \beta)^2(n + \alpha + \beta)^2}$$

Problem 2. a)

$$X_i \sim \text{Uniform}(a, b) \Rightarrow$$

$$\begin{cases} \mathbb{E}[X_i] = \frac{a+b}{2} \\ \mathbb{E}[X_i^2] = \frac{a^2+ab+b^2}{3} \end{cases} \quad (1)$$

$$\text{Let } \hat{\alpha}_1 = \mathbb{E}[X_i] \text{ and } \hat{\alpha}_2 = \mathbb{E}[X_i^2] \Rightarrow$$

$$\begin{cases} \hat{\alpha}_1 = \frac{a+b}{2} \\ \hat{\alpha}_2 = \frac{a^2+ab+b^2}{3} \end{cases} \quad (2)$$

Solve the above equations, we obtain:

$$\begin{cases} \hat{a} = \hat{\alpha}_1 - \sqrt{3(\hat{\alpha}_2 - \hat{\alpha}_1^2)} \\ \hat{b} = \hat{\alpha}_1 + \sqrt{3(\hat{\alpha}_2 - \hat{\alpha}_1^2)} \end{cases} \quad (3)$$

b.)

The likelihood :

$$\mathbb{L}(a, b) = \prod_{i=1}^n \frac{1}{b-a} \mathbb{I}_{(a,b)} = \frac{1}{(b-a)^n} \mathbb{I}_{(-\infty, x_1)}(a) \mathbb{I}_{(x_n, +\infty)}(b)$$

To maximize the likelihood: $a = X_{(1)}$ and $b = X_{(2)}$

c.)

We have: $\tau = \int x dF(x) = \mathbb{E}[x] = \frac{a+b}{2}$, by the equivariance property:

$$\hat{\tau} = \frac{\hat{\alpha} + \hat{\beta}}{2} = \frac{X_{(1)} + X_{(2)}}{2}$$

d.) Update

Problem 4. $X_i \sim \text{Uniform}(0, \theta) \Rightarrow$ Likelihood of X:

$$\mathcal{L} = \prod_{i=1}^n f(x) = \prod_{i=1}^n \frac{1}{\theta} I(0 \leq X_i \leq \theta) = \frac{1}{\theta^n} I(0 \leq X_1, X_2, \dots, X_n \leq \theta)$$

$\Rightarrow \mathcal{L}$ max when $\theta = \hat{\theta} = X_{(n)}$

We have:

$$\mathbb{P}(|\hat{\theta} - \theta| < \epsilon) = \mathbb{P}(\hat{\theta} < \theta - \epsilon) + \mathbb{P}(\hat{\theta} > \theta + \epsilon) = \mathbb{P}(\hat{\theta} < \theta - \epsilon) = \left(\frac{\theta - \epsilon}{\theta}\right)^n \rightarrow 0$$

MLE is consistent

Problem 5. We have $X_i \sim \text{Poisson}(\lambda)$

$$\Rightarrow E[X_i] = \lambda$$

Method of moment:

$$\hat{\lambda} = \frac{1}{n} \sum_{i=1}^n X_i$$

Log-likelihood of Poisson:

$$\mathcal{L} = \log(\lambda) \sum_{i=1}^n X_i - n\lambda - \sum_{i=1}^n \log(X_i!)$$

$$\frac{d\mathcal{L}}{d\lambda} = 0 \Leftrightarrow \frac{\sum_{i=1}^n X_i}{\hat{\lambda}} - n = 0$$

$$\Leftrightarrow \hat{\lambda} = \frac{\sum_{i=1}^n X_i}{n}$$

Fisher Information:

$$I(\lambda) = -\mathbb{E}\left[\frac{d^2\mathcal{L}}{d\lambda^2}\right] = -\mathbb{E}\left[-\frac{\sum_{i=1}^n X_i}{\lambda^2}\right] = \frac{1}{\lambda}$$

Problem 6. a)

MLE of θ : $\hat{\theta} = \bar{X}_n$

We have

$$\psi = \mathbb{P}(Y_1 = 1) = \mathbb{P}(X_1 > 0) = \phi(\theta)$$

By the equivariance property of MLE :

$$\hat{\psi} = \phi(\hat{\theta})$$

b)

We have Standard deviation of $\hat{\theta}$: $se(\hat{\theta}) = \sqrt{\frac{\phi^2}{n}} = \sqrt{\frac{1}{n}}$

Apply Delta Method we get :

$$\hat{se}(\hat{\psi}) = |\sigma'(\hat{\theta})| \hat{se}(\hat{\theta}) = |\phi'(\hat{\theta})| \sqrt{\frac{1}{n}}$$

We get the Confident Interval: $C_n = \hat{\psi} \pm |\sigma'(\hat{\theta})| \sqrt{\frac{1}{n}} c$

From Central Limit Theorem, we get:

$$\sqrt{n}(\hat{\theta} - \theta) \rightarrow \mathbb{N}(0, 1)$$

Apply Delta Method with $\psi = \phi(\theta)$

$$\sqrt{n}(\phi(\hat{\theta}) - \phi(\theta)) \rightarrow \mathbb{N}(0, |\phi'(\theta)|)$$

In addition : $\phi'(\theta) = \frac{1}{\sqrt{2\pi}}$

Hence:

$$\sqrt{n}(\hat{\psi} - \psi) \rightarrow \mathbb{N}(0, \frac{1}{\sqrt{2\pi}})$$

On the other hand, $Y_i \sim \text{Bernulli}(\psi)$ from Central Limit Theorem, we get:

$$\sqrt{n}(\tilde{\psi} - \psi) \rightarrow \mathbb{N}(0, \psi(1 - \psi)) = \mathbb{N}(0, \phi(\theta)(1 - \phi(\theta))) = \mathbb{N}(0, \frac{1}{4}) \text{ because } (\phi(\theta) = \frac{1}{2})$$

Hence, the asymptotic relative efficiency of $\tilde{\psi}$ and $\hat{\psi} : \sqrt{\frac{2}{\pi}}$

d)

Updated

Problem 7. a)

$X_1 \sim \text{Binomial}(n_1, p_1)$, $X_2 \sim \text{Binomial}(n_2, p_2) \Rightarrow$ MLE of p_1 and $p_2 : \hat{p}_1 = \frac{X_1}{n_1}$, $\hat{p}_2 = \frac{X_2}{n_2}$

By the equivariance property of MLE, MLE of $\psi : \hat{\psi} = \hat{p}_1 - \hat{p}_2 = \frac{X_1}{n_1} - \frac{X_2}{n_2}$

b)

We have data $X = (X_1, X_2)$

The log-likelihood of data:

$$\begin{aligned} \mathcal{L} = \log(f(X_1, X_2)) &= \log \left(\prod_{i=1}^2 \binom{n_i}{X_i} p_i^{X_i} (1 - p_i)^{n_i - X_i} \right) = \log \left(\binom{n_1}{X_1} \right) + X_1 \log(p_1) + (n_1 - X_1) \log(1 - p_1) \\ &\quad + \log \left(\binom{n_2}{X_2} \right) + X_2 \log(p_2) + (n_2 - X_2) \log(1 - p_2) \end{aligned}$$

Hence:

$$H_{11} = \frac{\partial^2 \mathcal{L}(X_1, X_2)}{\partial X_1^2} = \frac{-X_1}{p_1^2} - \frac{n_1 - X_1}{(1 - p_1)^2} \Rightarrow \mathbb{E}[H_{11}] = \frac{-n_1}{p_1} - \frac{n_1}{1 - p_1} = \frac{-n_1}{p_1(1 - p_1)}$$

$$H_{12} = H_{21} = \frac{\partial^2 \mathcal{L}(X_1, X_2)}{\partial X_1 \partial X_2} = 0 \Rightarrow \mathbb{E}[H_{11}] = \mathbb{E}[H_{21}] = 0$$

$$H_{22} = \frac{\partial^2 \mathcal{L}(X_1, X_2)}{\partial X_2^2} = \frac{-X_2}{p_2^2} - \frac{n_2 - X_2}{(1 - p_2)^2} \Rightarrow \mathbb{E}[H_{22}] = \frac{-n_2}{p_2} - \frac{n_2}{1 - p_2} = \frac{-n_2}{p_2(1 - p_2)}$$

Hence:

$$I_n(p_1, p_2) = -\mathbb{E}[H] = \begin{bmatrix} \frac{-n_1}{p_1(1-p_1)} & 0 \\ 0 & \frac{-n_2}{p_2(1-p_2)} \end{bmatrix}$$

c)

$$J_n = I_n^{-1}(p_1, p_2) = \begin{bmatrix} \frac{p_1(1-p_1)}{n_1} & 0 \\ 0 & \frac{p_2(1-p_2)}{n_2} \end{bmatrix}$$

On the other hand, we have function g : $g(p_1, p_2) = p_1 - p_2 \Rightarrow \nabla g = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\Rightarrow \hat{se}(\hat{\psi}) = \sqrt{\nabla g^T J_n \nabla g} = \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$$

d)

$$C_n = (\hat{\psi} - z_{\alpha/2} \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}, \hat{\psi} + z_{\alpha/2} \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}})$$

Problem 8. $X_i \sim \mathbb{N}(0, \mu)$, log-likelihood is :

$$\mathcal{L}(\mu, \sigma) = \log \left(\prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}(x - \mu)^2\right) \right) = \sum_{i=1}^n \log\left(\frac{1}{2\pi}\right) + \sum_{i=1}^n \log(\sigma) - \frac{1}{2\sigma^2} \sum_{i=1}^n (x - \mu)^2$$

Hence, :

$$\frac{\partial^2 \mathcal{L}(\mu, \sigma)}{\partial \mu^2} = -\frac{n}{\sigma^2} \Rightarrow H_{11} = -\mathbb{E}\left[\frac{-n}{\sigma^2}\right] = \frac{n}{\sigma^2}$$

$$\frac{\partial^2 \mathcal{L}}{\partial \mu \partial \sigma} = \frac{\partial^2 \mathcal{L}}{\partial \sigma \partial \mu} = 0 \Rightarrow H_{12} = H_{21} = 0$$

$$\frac{\partial^2 \mathcal{L}(\mu, \sigma)}{\partial \sigma^2} = \frac{n}{\sigma^2} - \frac{3}{\sigma^2} \sum_{i=1}^n (x - \mu)^2 \Rightarrow H_{22} = -\mathbb{E}\left[\frac{n}{\sigma^2} - \frac{3}{\sigma^2} \sum_{i=1}^n (x - \mu)^2\right] = \frac{2n}{\sigma^2}$$

$$\Rightarrow I_n(\mu, \sigma) = \begin{bmatrix} \frac{n}{\sigma^2} & 0 \\ 0 & \frac{2n}{\sigma^2} \end{bmatrix}$$