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A 2-dimensional guillotine cutting stock problem with variable-sized stock for the honeycomb cardboard industry

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ABSTRACT

This paper introduces novel mathematical optimisation models for the 2-Dimensional guillotine Cutting Stock Problem with Variable-Sized Stock that appears in a Spanish company in the honeycomb cardboard industry. This problem mainly differs from the classical cutting stock problems in the stock, which is considered variable-sized, i.e., we have to decide the panel dimensions, width, and length. This approach is helpful in industries where the stock is produced simultaneously with the cutting process. The stock is then cut into smaller rectangular pieces that must meet the customers' requirements, such as the type of item, dimensions, demands, and technical specifications. Furthermore, in the problem tackled in this paper, the cuts are guillotine, performed side to side. The proposed mathematical models are validated using real data from the company, obtaining results that drastically reduce the produced material and leftovers, reducing operation times and economic costs.

KEYWORDS

Cutting Stock Problem; 2-Dimensional cutting; Variable-sized stock; Mixed Integer Linear Optimisation; Cardboard industry.

1. Introduction

The industrial world is increasingly facing complex processes that require expert systems to support decision-making therefore it is necessary to develop useful methods and techniques. This area of research has been extensively studied by Jean-Marie Proth. See Dolgui and Proth (2010), which describes several problems in supply chain management, such as pricing, outsourcing, inventory, manufacturing, etc. On the contrary, Proth and Hillion (1990) present different mathematical tools to solve them such as simulation, linear or dynamic programming, and queueing theory, among others. Finally, Govil and Proth (2002), introduce a general perspective of the supply chain design taking into account the different levels of strategies: strategic, tactical, and operational.

One of the problems found in many industries is the Cutting Stock Problem (CSP), introduced by Kantorovich (1960) and Gilmore and Gomory (1961), is one of the optimisation problems that has been deeply explored in the literature due to its many applications in the industry. The CSP is present in different sectors such as glass (Parreño and Alvarez-Valdes (2021)), stone (Baykasoğlu and Özbel (2021)), wood (Kokten and Sel (2020)), steel (Sierra-Paradinas et al. (2021) and Antonio et al. (1999)), concrete (Signorini, de Araujo, and Melega (2021)), construction (Lemos, Cherri, and de Araujo (2020)), paper (Kallrath et al. (2014)), and textile (Salem et al. (2023)). The cutting process is crucial for companies producing pieces cut from a previous stock. The companies are especially interested in minimising the leftovers and production times, although they can consider other goals, for example in Antonio et al. (1999), reducing computing times as much as possible while achieving a reasonable cost is used by the sales department to respond in 'real time' to the customers' demands.

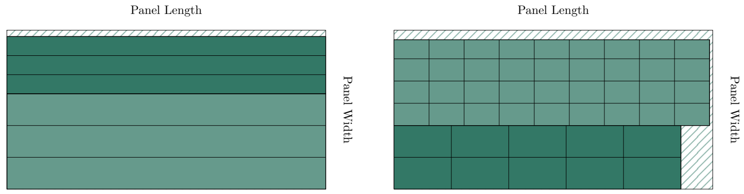
Due to the different industrial processes and their specific restrictions, several versions of the CSP have been developed based on specifications such as dimensions, pattern characteristics, and cutting restrictions, among others. Both exact and non-exact (heuristics, metaheuristics, and/or matheuristics) algorithms have been explored in terms of problem resolution.

The problem studied in this paper is motivated by the collaboration with a medium-sized Spanish company in the honeycomb cardboard sector whose aim for the future medium term is to automatise the operations to gain efficiency and effectiveness. It is part of the 2-Dimensional Cutting Stock Problems (2DCSP). The 2DCSP is concerned with obtaining a set of different rectangular items cut from one or more rectangular panels in stock. In our case, the size of panels is not predefined in advance and is determined by the optimisation model. The cutting process must be defined to produce enough pieces of each item to cover a previously known demand. Additionally, as detailed below, different specifications from the cardboard industry must be considered.

The main contributions of this paper are:

- A 0–1 linear optimisation model is introduced for the Multi-Stock 2-Dimensional CSP. The proposal improves the current operation in the factory.
- A novel mixed 0–1 linear optimisation model for the 2DCSP with variable-sized stock and guillotine cuts, able to decide the dimensions of the panels produced, is also introduced.
- A real problem from a Spanish company in the cardboard industry is tackled. Data from the company are used and extensively analysed, comparing the current operation with the proposals.
- As a result, new, simple and straightforward strategies are proposed for the operation of the company, providing up to a 50% leftover reduction.

The remaining part of this paper is organised as follows: Section 2 reviews the different variants of the Cutting Stock Problem, as well as the specifications to meet in the problem to study; Section 3 describes the problem to deal with and the current operation managed by the company; Section 4 introduces two mathematical optimisation models to solve the problem; Section 5 presents an extensive computational experiment based on real-world data provided by the company; finally, Section 6 concludes and presents the future research lines.



(a) 1DCSP (b) 2DCSP

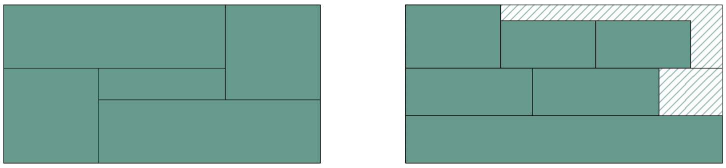
Figure 1. Caption: 1DCSP vs 2DCSP
Figure 1. Alt Text: Two diagrams with a rectangular shape. On the left, caption (a) 1DCSP: Diagram with a rectangular panel divided into different horizontal strips where the top strip in dashed is leftover. On the right, caption (b) 2DCSP: Diagram with a rectangular panel divided into three horizontal strips where the top strip in dashed is leftover. The other two strips are divided in rectangular pieces.

2. The Cutting Stock Problem

The CSP was introduced by Kantorovich (1960) (first published in Russian in 1939), followed by the seminal work presented by Gilmore and Gomory (1965), which extends the works by the same authors for the 1-Dimensional CSP (1DCSP), Gilmore and Gomory (1961) and Gilmore and Gomory (1963). In the latter, a column generation method is presented for tightening the lower bound. However, that method does not obtain good results for the 2-Dimensional CSP (2DCSP).

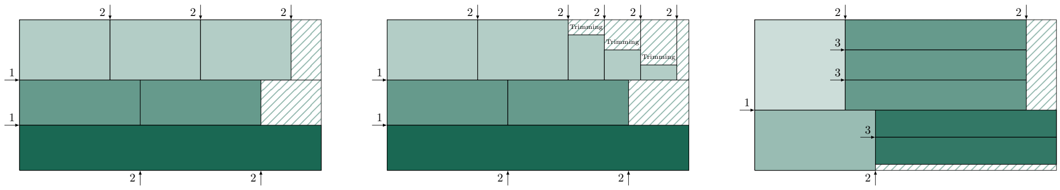
Dyckhoff (1990) presents a consistent and systematic approach for a comprehensive typology integrating the various kinds of problems for the cutting and packing problems. A deeper classification is presented in Dyckhoff and Finke (1992). One of the most important characteristics is the number of dimensions to consider for the cuts. Then, the CSP is classified into 1-Dimensional Cutting Stock Problems (1DCSP), where the stock is cut into strips (performing only lengthwise cuts on the stock), and 2-Dimensional Cutting Stock Problems (2DCSP), where the stock is width and lengthwise cut. Fig. 1 shows the differences between both problems. Fig. 1(a) shows a panel cut lengthwise into different strips whereas Fig. 1(b) shows a panel cut into rectangular pieces. In both cases, the leftovers are represented with the diagonal striped pattern. Some other dimensional problems have been studied in the literature, such as the 1.5-Dimensional Cutting Stock Problem (1.5DCSP). In this variation, the ordered pieces are given by their lengths and widths but fix one dimension and leave the other variable depending on other factors, such as weight (see Sierra-Paradinas et al. (2021)). Haessler and Sweeney (1991) presents a good review of the 1DCSP, 2DCSP and 1.5DCSP and their main differences.

The 2DCSP is related with two classical packing problems: the Bin Packing Problem (BPP) and the Strip Packing Problem (SPP). The 2-Dimensional BPP (2DBPP) is a particular case of the 2DCSP where the demand for each item is equal to one. The SPP considers a strip of fix width and infinite length and consists in cutting all the items from the strip by minimising the used strip length. Lodi, Martello, and Monaci (2002) review the mathematical models, lower bounds, heuristics, exact and approximation algorithms for 2DSPP. Most of the methods proposed are heuristics. Oliveira et al. (2016) review the heuristics methods and classify them according to their type: constructive heuristics, improvement heuristics over sequences, and improvement heuristics over layouts. Lodi, Martello, and Vigo (2004) consider the 2DBPP and SPP, where items must be packed by levels. For further information on other variants of the CSP, together with the evaluation of the different CSPs, see the recent review Bezerra



(a) Non-guillotine cuts (b) Guillotine cuts

Figure 2. Caption: 2DCSP: Non-guillotine vs Guillotine cuts
Figure 2. Alt text: Two diagrams with a rectangular shape. On the left, caption (a) Non-guillotine cuts: Diagram with a rectangular panel divided with non-guillotine cuts forming different pieces. On the right, caption (b) Guillotine cuts: Diagram with a rectangular panel divided into three horizontal strips. Each strip is divided into different rectangular pieces of the same or less width than the strip.



(a) Exact 2-staged cuts (b) Non-exact 2-staged cuts (c) Exact 3-staged cuts

Figure 3. Caption: Exact and non-exact staged cuts
Figure 3. Alt text: Three diagrams with a rectangular shape: caption (a) Exact 2-staged cuts, caption (b) Non-exact 2-staged cuts and caption (c) Exact 3-staged cuts.
Figure 3. Long description: On the left, caption (a) Exact 2-staged cuts: Diagram with a rectangular panel divided in three horizontal strips. Each strip is divided into rectangular pieces of the strip width. The dashed area at the end of the strips is leftover. In the middle, caption (b) Non-exact 2-staged cuts: Diagram with a rectangular panel divided in three horizontal strips. Each strip is divided into rectangular pieces. In the first strip some pieces' widths are smaller than the strip generating some leftover on the top of them, trimming in dashed. On the rigth, caption (c) Exact 3-staged cuts: Diagram with a rectangular panel divided first into two horizontal strips. Each strip, is divided into two rectangular pieces and, third, the two rectangular pieces generated (right hand side of the original strips) are cut into horizontal strips. The dashed area is leftover.

et al. (2019), focused on the 2-Dimensional level strip packing problem.

An essential characteristic widespread in the industrial sector is the type of cuts. Often, the cutting equipment can produce only guillotine cuts performed from side to side parallel to the edges producing two new rectangles from a larger one. In the example shown in Fig. 1(a), all the cuts are guillotine. In the example shown in Fig. 1(b), all the cuts are also guillotine but in sequential order. First, lengthwise cuts are performed, generating two types of strips. Each one is then guillotine cut to produce the smaller rectangles. Fig. 2 shows an example of non-guillotine and guillotine cuts.

The 2DCSP is classified as 2-staged (2SCSP) or 3-staged (3SCSP) if two or three sequential cuts are needed, respectively. The first stage consists in performing parallel lengthwise guillotine cuts that produce a set of rectangular strips. Each strip is individually cut in the second stage with the remaining parallel guillotine crosscuts. If there is no need for an additional cut, i.e., all the pieces' widths equal the ordered dimensions, the pattern is called exact 2-staged guillotine. Otherwise, it is called non-exact since the pieces need a third stage, adding a cut to move away some scrap to meet the requested dimensions (see Salem et al. (2023) and Andrade et al. (2014)). Fig. 3 shows three different cases: exact 2-staged (Fig. 3(a)), non-exact 2-staged (Fig. 3(b)), and exact 3-staged (Fig. 3(c)). Observe in Fig. 3(b) that a third cut is needed for the smaller pieces in the first strip to meet the dimensions whereas in the case showed in Fig. 3(c), even being exact, the third cut is needed to obtain the final pieces. See

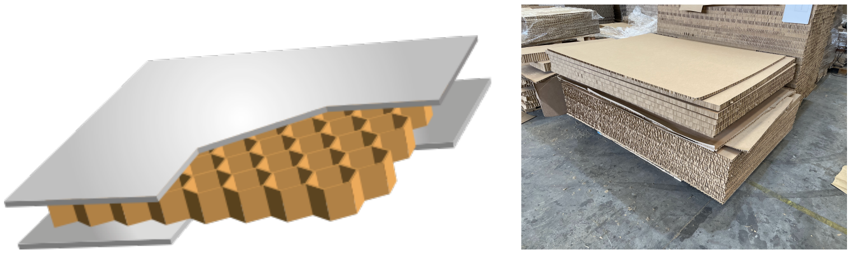
Vanderbeck (2001), Alvarez-Valdes, Parajon, and Tamarit (2002), Yanasse and Morabito (2008) Silva, Alvelos, and de Carvalho (2010), Macedo, Alves, and de Carvalho (2010), among others, for the 2SCSP and 3SCSP. Cintra et al. (2008) study the 2DCSP with guillotine cuts and their variants. They study the SPP with two, three, and four staged patterns. Dolatabadi, Lodi, and Monaci (2012) introduce an exact recursive procedure that constructs the set of guillotine packing associated with a given set of items as part of the 2-Dimensional Knapsack Problem (2DKP). Furini, Malaguti, and Thomopulos (2016) present a framework to model general guillotine constraints in 2DCSP formulated as mixed-integer linear programming, based on the formulation presented in Dyckhoff (1981). Furthermore, as a variant of the 2SCSP, Martin et al. (2022) propose probably for the first time models for 2SCSP and 3SCSP with a limited number of open stacks.

An element to consider in the Cutting problems is the treatment of leftovers. They can be classified as reusable leftovers or scrap. The reusable leftovers are considered when the waste generated can be used again if it meets some specifications (typically minimum dimensions). Then, those reusable leftovers are considered stock in future processes (see do Nascimento, Cherri, and Oliveira (2022)). On the contrary, the scrap is directly discarded.

The stock is also used to classify the CSP. Typically, the CSP considers a stock formed by an infinite number of panels with dimensions $W \times L$. However, it is also common to consider a heterogeneous stock, formed by panels with varied dimensions. This problem is known as the Multi-Stock 2-Dimensional CSP (MS2DCSP). A particular case with heterogeneous stock is the problem with usable leftovers, where there is a high variability of stock dimensions and often only one piece of stock of each size. Furini et al. (2012) introduce a column generation heuristic for the MS2DCSP with 2-Staged guillotine cutting. Lately, Furini and Malaguti (2013) present some mathematical models for MS2DCSP with 2-Staged guillotine cutting. Previously, Pisinger and Sigurd (2005) consider a closely related problem, the 2DBPP with variable bin sizes.

In the literature, the CSP considers that the stock dimensions are known in advance, even if it is heterogeneous. In Mosallaeipour (2017) the problem of supplier material selection and production planning in the carton box production industries is presented. In this work, the panel sizes considered are those offered by the suppliers and the cutting patterns are given by a software. Another problem related to the cardboard sector is presented in Sipahi (2022), where, in the first step, the panel sizes are determined by a simulated annealing algorithm, and in the second step, the products are assigned using an integer linear programming model. Nevertheless, the stock dimensions could be a decision as part of the production process. This variant, known as the 2DCSP with Variable-Sized stock (2D-VSCSP), has been recently presented in Salem et al. (2023). To the best of our knowledge, and as the authors mention in the paper, this is the only work dealing with the 2D-VSCSP. They present two mathematical models. The first is based on the work by Lodi and Monaci (2003), while the second is based on a Bin Packing Problem (BPP). As the authors remark in the paper, the problem can be approximated as a BPP since, in their practical application (textile sector), the demand is not very high, even though the demand for each item is greater than one unit. The problem we are dealing with fits into the 2D-VSCSP.

Note that the 2D-VSCSP could be also treated as a 3-Staged Strip Packing Problem (3S-SPP). The first stage produces the different panels, while the second and third stages produce the strips and the final items, respectively. If more than one width is available in stock, the problem can be considered a Multi-Strip (3S) Packing Problem



(a) Honeycomb structure (b) Produced panels

Figure 4. Caption: Honeycomb cardboard panels
Figure 4. Alt text: Two pictures. On the left, caption (a) Honeycomb structure: Picture of the honeycomb structure, sandwich of two layers of grey paper with the net in the middle. Each cell is hexagonal. On the right, caption (b) Produced panels: Picture of a collection of honeycomb panels piled.

(MSPP).

3. Problem description

The problem considered takes place in the honeycomb cardboard industry. The product's name comes from the honeycomb structure inside the carton panel. The honeycomb cardboard panels are made of paper, with a sandwich structure of two paper layers and a paper net inside (the honeycomb structure). Fig. 4 shows a graphical representation of a honeycomb cardboard panel and some panels produced by the company.

The success of the honeycomb panels is due to their characteristics: green, light, resistant, economical, and easy to manipulate. Green: it is made with recycled and recyclable materials. Light: compared with wood panels, commonly used in transportation, its weight is around 1/6 the wood panels. Resistant: it resists $5kg/cm^2$ to a compression of $50t/m^2$. Economical: compared to wood, or other Extended Polystyrene (EPS) products, the price is significantly lower. Finally, it is easy to manipulate and cut thanks to its low weight.

In its beginnings, the honeycomb panels were used for transportation replacing the EPS. They are very resistant and light, making them perfect for protecting the transported products by absorbing the blows. Recently, this material has gained presence in the advertising and decorating sectors as it is possible to print on it.

The company receives every day different orders that must be served. An order consists of a list of different items (rectangular pieces) given by their length (ℓ_i), width (w_i), and the number of units to serve (d_i). The pieces are obtained by producing carton panels and cutting them into smaller items. Therefore, the company has to decide: (1) how many carton panels to produce, (2) their dimensions, and (3) how to cut them aiming to reduce the leftovers. It is worth pointing out that the pieces cannot be rotated as the material is weaker depending on how the pieces are positioned.

The factory layout encloses two different areas, each devoted to a phase in the production process. The first one contains a machine, called *line*, to produce the honeycomb cardboard panels (see Figs. 5(a) and 5(b)). The second area comprises two cutting machines that cut the panels into smaller pieces (see Fig. 5(c)).

The *line* unrolls the paper rolls used as covers and introduces in between them the honeycomb net, that is glued to the two covers. The paper roll width defines the panel's width, while the length can be decided and adjusted. The cutting machinery

(a) Supply module for the line (b) Line machine (c) Cutting Machine

Figure 5. Caption: Line and cutting machines
Figure 5. Alt text: Three pictures. On the left, caption (a) Supply module for the line: Picture of the machine where the panels are produced. The paper rolls are allocated to feed the machine in four shafts. In the middle, caption (b) Line machine: Picture of the machine producing a continuous panel. On the right, caption (c) Cutting Machine: Picture of a cutting machine. The blades are allocated along a shaft to cut the carton panels.

imposes minimum and maximum lengths to the panels, say $\underline{\ell}$ and $\overline{\ell}$, respectively. A width modification implies paper rolls replacement, whereas a length adjustment only needs to set the new value in the panel control. Both actions force the *line* to stop.

Once the panels are produced, they are moved to the cutting area. The cutting machines have a set of circular blades distributed along a shaft that performs guillotine cuts. The blades can (1) fully cross the panel obtaining separated pieces or (2) partially cross the panel keeping it together. In the latter, the pieces are manually separated by the customer. This option is frequently chosen since packing on pallets is easier. In addition, for operation purposes, many customers prefer to receive pallets with panels containing only one reference, making their manipulation easier. Therefore, each panel can only contain one item reference. The operation produces leftovers but never trimming as all the obtained pieces have the same width as the strip generated. Therefore, the problem leads to the exact 2-Staged 2DCSP.

Currently, the company works with panels of $1200 \times 2400\,\mathrm{mm}^2$, not taking the advantage of the flexibility given by the line to better adjust the panels' dimensions to the orders. As the company serves other products, they also work with different widths (paper rolls) that could be used to produce panels with different widths. In an analogous way, they could work with different lengths that are easily configured in the *line*. The aim of this contribution is to propose mathematical models allowing the company to explore new strategies and even decide on a better panel configuration to serve its demand.

Then, the problem we are tackling fits into the 2DCSP family with variable-sized stock, 2D-VSCSP. As far as we know, 2D-VSCSP was recently introduced in Salem et al. (2023). Models presented in that work only solve cases with low total demand (up to 30 units) while our company manages many different items (up to 90) with high demands (frequently, thousands of pieces per item). These models' size depends on the square of the total number of pieces ordered, making it impossible to apply them to our problem.

Summarising, the main hypotheses considered are:

- Panels are produced in the factory and have a rectangular shape (CSP).
- The panel dimensions (width and length) have to be decided (2D-VSCSP).
- Orders are known before planning the panel production.
- Pieces have a rectangular shape (2D-CSP) and cannot be rotated.
- Only guillotine cuts are allowed.

- Final pieces are supplied in pre-cut panels (they are not separated individually) and each panel includes only one item.
- The objective is to reduce leftovers.

4. Mathematical optimisation models

In this section, two Mixed Integer Linear Optimisation Models to solve the 2D-VSCSP with guillotine constraints are presented. Both models aim to determine the panels' configuration such that the leftovers are minimised. A configuration is given by its length and width.

In the first model, a set of potential configurations (width and length) is predefined by the company, and a subset of them is selected. On the contrary, in the second one, the configurations are proposed by the model: the paper roll selected defines the width, and the length is a variable in $[\underline{\ell}, \overline{\ell}]$.

4.1. *Model 1: Selection Model (SM)*

The SM considers a set of predefined configurations, selects a subset and assigns each item to one of them. Each configuration j is given by its width W_j and length L_j. Upper bounds on the total number of configurations, widths and lengths can be imposed, $\overline{n}^c$, $\overline{n}^w$ and $\overline{n}^\ell$, respectively. This model provides more options than the company's current operation, which only uses the configuration $1200\times2400\,\text{mm}^2$.

Note that if only one configuration is used, there is no place for optimisation since it is easy to calculate the leftover generated. If configuration j is selected the number of panels needed to serve the demand of item i is:

$$np_{ij} = \left\lceil \frac{d_i}{n_{ij}} \right\rceil,$$

where $n_{ij} = \left\lfloor \frac{W_j}{w_i} \right\rfloor \times \left\lfloor \frac{L_j}{\ell_i} \right\rfloor$ is the maximum number of pieces of item i that can be produced with one panel of configuration j.

However, when more than one configuration is selected, it is necessary to decide the assignment of each item to a configuration and, then, there is a combinatorial number of feasible solutions.

4.1.1. Sets

$\mathcal{I} = \{1, \ldots, I\}$, set of items, being I the total number of different items.
$\mathcal{J} = \{1, \ldots, J\}$, set of available configurations, being J the total number of configurations considered.
$\mathcal{W} = \{\hat{w}_1, \hat{w}_2, \ldots, \hat{w}_n\}$, set of available configuration widths.
$\mathcal{L} = \{\hat{\ell}_1, \hat{\ell}_2, \ldots, \hat{\ell}_m\}$, set of available configuration lengths.
$\mathcal{J}^{\hat{w}} = \{j \in \mathcal{J} : W_j = \hat{w}\} \subset \mathcal{J}$, set of configurations of width $\hat{w}$, for $\hat{w} \in \mathcal{W}$.
$\mathcal{J}^{\hat{\ell}} = \{j \in \mathcal{J} : L_j = \hat{\ell}\} \subset \mathcal{J}$, set of configurations of length $\hat{\ell}$, for $\hat{\ell} \in \mathcal{L}$.

4.1.2. Parameters

W_j, width of configuration j, for $j \in \mathcal{J}$.

L_j, length of configuration j, for $j \in \mathcal{J}$.
w_i, width of item i, for $i \in \mathcal{I}$.
ℓ_i, length of item i, for $i \in \mathcal{I}$.
d_i, demand of item i, for $i \in \mathcal{I}$.
s_{ij}, Leftover generated if item i is assigned to configuration j, for $i \in \mathcal{I}$, $j \in \mathcal{J}$. It can be calculated as

$$s_{ij} = np_{ij} \cdot W_j \cdot L_j - d_i \cdot w_i \cdot \ell_i.$$

$\overline{n}^c, \overline{n}^w, \overline{n}^\ell$, maximum number of different configurations, widths and lengths that can be selected.

4.1.3. Decision variables

$x_{ij} = 1$ if item i is assigned to configuration j, 0 otherwise, for $i \in \mathcal{I}, j \in \mathcal{J}$.
$y_j = 1$ if configuration j is selected, 0 otherwise, for $j \in \mathcal{J}$.
$u_{\hat{w}} = 1$ if at least one configuration of $\mathcal{J}^{\hat{w}}$ is selected, 0 otherwise, for $\hat{w} \in \mathcal{W}$.
$v_{\hat{\ell}} = 1$ if at least one configuration of $\mathcal{J}^{\hat{\ell}}$ is selected, 0 otherwise, for $\hat{\ell} \in \mathcal{L}$.

4.1.4. SM mathematical formulation

$$\min \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} s_{ij} x_{ij} \tag{SM.1}$$

subject to

$$\sum_{j \in \mathcal{J}} x_{ij} = 1 \qquad \forall i \in \mathcal{I} \tag{SM.2}$$

$$x_{ij} \leqslant y_j \leqslant \sum_{i \in \mathcal{I}} x_{ij} \qquad \forall i \in \mathcal{I}, j \in \mathcal{J} \tag{SM.3}$$

$$y_j \leqslant u_{\hat{w}} \leqslant \sum_{j \in \mathcal{J}^{\hat{w}}} y_j \qquad \forall \hat{w} \in \mathcal{W}, j \in \mathcal{J}^{\hat{w}} \tag{SM.4}$$

$$y_j \leqslant v_{\hat{\ell}} \leqslant \sum_{j \in \mathcal{J}^{\hat{\ell}}} y_j \qquad \forall \hat{\ell} \in \mathcal{L}, j \in \mathcal{J}^{\hat{\ell}} \tag{SM.5}$$

$$\sum_{j \in \mathcal{J}} y_j \leqslant \overline{n}^c \tag{SM.6}$$

$$\sum_{\hat{w} \in \mathcal{W}} u_{\hat{w}} \leqslant \overline{n}^w \tag{SM.7}$$

$$\sum_{\hat{\ell} \in \mathcal{L}} v_{\hat{\ell}} \leqslant \overline{n}^\ell \tag{SM.8}$$

$$x_{ij}, y_j \in \{0, 1\} \qquad \forall i \in I, j \in \mathcal{J} \tag{SM.9}$$

$$u_{\hat{w}} \in \{0, 1\} \qquad \forall \hat{w} \in \mathcal{W} \tag{SM.10}$$

$$v_{\hat{\ell}} \in \{0, 1\} \qquad \forall \hat{\ell} \in \mathcal{L} \tag{SM.11}$$

The objective function (SM.1) minimises the total leftovers generated in the production process. Constraints (SM.2) ensure that each item is assigned to exactly one configuration. Constraints (SM.3) set that a configuration is used if and only if at least one item is assigned to it. Constraints (SM.4) and (SM.5) control which widths and lengths, respectively, are used at least once. Constraint (SM.6) limits the number of configurations selected. The number of widths and lengths to be used are limited by constraints (SM.7) and (SM.8), respectively. Finally, constraints (SM.9)–(SM.11) define the domain of the decision variables.

4.2. *Model 2: Variable-Sized stock definition Model (VSM)*

Although the previous model is slightly more flexible than the company's operation, it does not take advantage of the *line* machine options to easily fix the panel's length. As it has been mentioned previously, modifying the panel's length only implies stopping the machine for a few seconds to adjust the length parameter in the control panel.

This model is an extension of the SM allowing the length to be defined in the interval $[\underline{\ell}, \overline{\ell}]$ instead of being selected from a finite (and often reduced) set of predefined values. This flexibility allows to better fit the panel's length to each order reducing the leftovers.

Therefore, in this model, the lengths of the selected configurations are included as continuous decision variables. Notice that the higher the number of configurations selected, the lower the amount of material used; however, dealing with many panel configurations is not convenient for the factory operation. To avoid this, the VSM model includes the possibility of limiting the total number of configurations, as SM model does. Configuration j is characterised by the parameter W_j, the width of the roll used to produce it, and the variable L_j, the configuration's length. Remember, that the specifications of the cutting machines impose lower and upper bounds to L_j, given by $\underline{\ell}$ and $\overline{\ell}$.

Let us suppose that item i is assigned to configuration j. As the widths of i and j are known, it is possible to calculate how many rows of i can be placed in configuration j, $r_{ij} = \left\lfloor \frac{W_j}{w_i} \right\rfloor$. Then, the demand of i can be allocated in $c_{ij} = \left\lceil \frac{d_i}{r_{ij}} \right\rceil$ columns. If k panels of configuration j are used to serve the demand of i, the minimum length of each of these panels, say $\underline{\ell}_{ijk}$, must be:

$$\underline{\ell}_{ijk} = \max\left\{\underline{\ell}, \ell_i \left\lceil \frac{c_{ij}}{k} \right\rceil \right\}$$

Consider, for example, a demand of 103 pieces of $500\times\ 250\,\text{mm}^2$ to be served with panels of 1200 mm width. Then,

$$r_{ij} = \left\lfloor \frac{1200}{500} \right\rfloor = 2, \quad c_{ij} = \left\lceil \frac{103}{2} \right\rceil = 52.$$

For $k = 9$, each panel has at least $\left\lceil \frac{52}{9} \right\rceil = 6$ columns, producing a total of 108 pieces (9 panels $\times$ 6 columns $\times$ 2 rows). Note that if each panel includes only 5 columns, 90 pieces are produced, and the demand is not covered. The minimum length of configuration j must be $6 \times 250\,\text{mm} = 1500\,\text{mm}$ (provided that $\underline{\ell} \leqslant 1500\,\text{mm}$).

However, a priori, the number of panels k needed to satisfy the demand of item i if assigned to configuration j cannot be calculated since the length is unknown and is obtained by the model. Only lower and upper bounds can be set on the number of panels of configuration j required to meet the demand for item i:

$$\underline{k}_{ij} = \left\lceil \frac{c_{ij}}{\overline{c}_{ij}} \right\rceil \qquad \overline{k}_{ij} = \left\lceil \frac{c_{ij}}{\underline{c}_{ij}} \right\rceil$$

where $\overline{c}_{ij} = \left\lfloor \frac{\overline{\ell}}{\ell_i} \right\rfloor$ and $\underline{c}_{ij} = \left\lfloor \frac{\max\{\underline{\ell}, \ell_i\}}{\ell_i} \right\rfloor$.

4.2.1. Sets of indices

$\mathcal{I} = \{1, \ldots, I\}$, set of items, being I the total number of different items.
$\mathcal{J} = \{1, \ldots, J\}$, set of available configurations, being J the total number of configurations considered.
$\mathcal{W} = \{\hat{w}_1, \hat{w}_2, \ldots, \hat{w}_n\}$, set of available configuration widths.
$\mathcal{J}^{\hat{w}} = \{j \in \mathcal{J} : W_j = \hat{w}\} \subset \mathcal{J}$, set of configurations of width $\hat{w}$, for $\hat{w} \in \mathcal{W}$.
$\mathcal{K}_{ij} = \{\underline{k}_{ij}, \underline{k}_{ij} + 1, \ldots, \overline{k}_{ij}\}$, set of possible number of panels of configuration j produced to attend the demand of item i, for $i \in \mathcal{I}$, $j \in \mathcal{J}$.

Notice that, if the maximum number of configurations that can be selected is $\overline{n}^c$, then $\mathcal{J}$ includes $\overline{n}^c$ configurations per each element in $\mathcal{W}$.

4.2.2. Additional parameters

$\underline{\ell}_{ijk}$, minimum length of panels of configuration j if item i is served using k panels, for $i \in \mathcal{I}$, $j \in \mathcal{J}$, $k \in \mathcal{K}_{ij}$.
$\underline{\ell}, \overline{\ell}$, lower and upper bounds for the panel length, respectively.

4.2.3. Decision variables

$x_{ij} = 1$ if item i is assigned to configuration j, 0 otherwise, for $i \in \mathcal{I}, j \in \mathcal{J}$.
$y_j = 1$ if configuration j is selected, 0 otherwise, for $j \in \mathcal{J}$.
L_j, length of configuration j, for $j \in \mathcal{J}$.
$u_{\hat{w}} = 1$ if at least one configuration of $\mathcal{J}^{\hat{w}}$ is selected, 0 otherwise, for $\hat{w} \in \mathcal{W}$.
$z_{ijk} = 1$ if a total of k panels of configuration j are produced to serve item i demand, 0 otherwise, for $i \in \mathcal{I}, j \in \mathcal{J}, k \in \mathcal{K}_{ij}$.
δ_{ijk}, auxiliary continuous variable needed to linearised the product $L_j z_{ijk}$, for $i \in \mathcal{I}$, $j \in J$, $k \in \mathcal{K}_{ij}$. Notice that $\delta_{ijk} = L_j$ if $z_{ijk} = 1$, and 0 otherwise.

4.2.4. VSM mathematical formulation

$$\min \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}_{ij}} W_j k \delta_{ijk} \tag{VSM.1}$$

subject to

$$\sum_{j \in \mathcal{J}} x_{ij} = 1 \qquad \forall i \in \mathcal{I} \tag{VSM.2}$$

$$x_{ij} \leqslant y_j \leqslant \sum_{i \in \mathcal{I}} x_{ij} \qquad \forall i \in \mathcal{I}, j \in \mathcal{J} \tag{VSM.3}$$

$$y_j \leqslant u_{\hat{w}} \leqslant \sum_{j \in \mathcal{J}^{\hat{w}}} y_j \qquad \forall \hat{w} \in \mathcal{W}, j \in \mathcal{J}^{\hat{w}} \tag{VSM.4}$$

$$\sum_{k \in \mathcal{K}_{ij}} z_{ijk} = x_{ij} \qquad \forall i \in \mathcal{I}, j \in \mathcal{J} \tag{VSM.5}$$

$$\sum_{k \in \mathcal{K}_{ij}} \underline{\ell}_{ijk} z_{ijk} \leqslant L_j \qquad \forall i \in \mathcal{I}, j \in \mathcal{J} \tag{VSM.6}$$

$$\delta_{ijk} \leqslant L_j \qquad \forall i \in \mathcal{I}, j \in \mathcal{J}, k \in \mathcal{K}_{ij} \tag{VSM.7}$$

$$\delta_{ijk} \leqslant \overline{\ell} z_{ijk} \qquad \forall i \in \mathcal{I}, j \in \mathcal{J}, k \in \mathcal{K}_{ij} \tag{VSM.8}$$

$$L_j - \delta_{ijk} \leqslant \overline{\ell}(1 - z_{ijk}) \qquad \forall i \in \mathcal{I}, j \in \mathcal{J}, k \in \mathcal{K}_{ij} \tag{VSM.9}$$

$$\underline{\ell} y_j \leqslant L_j \leqslant \overline{\ell} y_j \qquad \forall j \in \mathcal{J} \tag{VSM.10}$$

$$\sum_{j \in \mathcal{J}} y_j \leqslant \overline{n}^c \tag{VSM.11}$$

$$\sum_{\hat{w} \in \mathcal{W}} u_{\hat{w}} \leqslant \overline{n}^w \tag{VSM.12}$$

$$y_{j+1} \leqslant y_j \qquad \forall \hat{w} \in \mathcal{W}, j, j+1 \in \mathcal{J}^{\hat{w}} \tag{VSM.13}$$

$$x_{ij}, y_j \in \{0, 1\} \qquad \forall i \in \mathcal{I}, j \in \mathcal{J} \tag{VSM.14}$$

$$u_{\hat{w}} \in \{0, 1\} \qquad \forall \hat{w} \in \mathcal{W} \tag{VSM.15}$$

$$z_{ijk} \in \{0, 1\} \qquad \forall i \in \mathcal{I}, j \in \mathcal{J}, k \in \mathcal{K}_{ij} \tag{VSM.16}$$

$$L_j \geqslant 0 \qquad \forall j \in \mathcal{J} \tag{VSM.17}$$

$$\delta_{ijk} \geqslant 0 \qquad \forall i \in \mathcal{I}, j \in \mathcal{J}, k \in \mathcal{K}_{ij} \tag{VSM.18}$$

The objective function (VSM.1) minimises the total area used. Note that as the items area is constant, minimising the total area is equivalent to minimising the leftovers generated. Constraints (VSM.2) ensure that each item is assigned to one configuration. Analogously to SM, constraints (VSM.3)–(VSM.4) control which configurations and widths are selected. Constraints (VSM.5) select the number of panels k to be produced when item i is assigned to configuration j. Constraints (VSM.6) impose the minimum length for these panels. It is worth mentioning that demand coverage is implicit in the calculation of $\underline{\ell}_{ijk}$. Constraints (VSM.7)-(VSM.9) assure all panels of configuration j to have the same length. Constraints (VSM.10) bound the length of each configuration. Constraints (VSM.11) and (VSM.12) limit the number of configurations and widths selected. Constraints (VSM.13) impose an order in the configurations with the same width; these constraints are added to eliminate symmetries in the solution. Finally, constraints (VSM.14)–(VSM.18) define the variables domain.

5. Computational experiment

In this section, the proposed models are validated using a set of representative real orders received in the company during 2022. In the following, the study cases are described, the different experiments are presented and, finally, based on the results, some strategies are proposed to improve the company's operation.

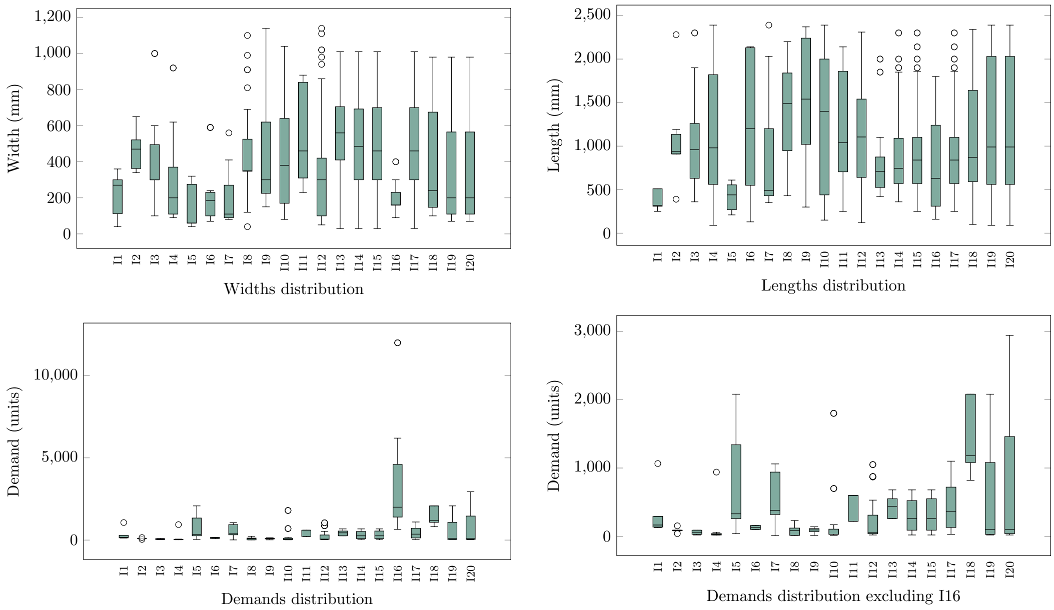


Figure 6. Caption: Summary of instances specifications
Figure 6. Alt Text: Four boxplots to represent the variability of widths, lengths and demands for the twenty instances evaluated in the computational experiment. The last set of boxplots excludes the instance I16 to better observe the distribution.

5.1. *Study cases*

The testbed includes twenty instances, corresponding to real orders provided by the company[1]. Their main characteristics are reported in Table 1 whose headings are as follows: number of items (#) and total area demanded (m^2); for the current company's operation (only $1200\times2400\,\text{mm}^2$ panels are used), total area used (m^2) and percentage of this area considered leftover (Left.(%)); and for the items' widths, lengths and demands, the minimum (Min), mean, median, maximum (Max), and standard deviation (St. Dev.).

Additionally, Fig. 6 shows the boxplots of the widths, lengths, and demands distributions for the twenty instances, demonstrating the variability of these parameters even for the items of the same instance. Notice that in terms of demand, instance I16 differs from the others; in order to better illustrate the demand distribution, boxplots excluding this instance are also shown.

The factory imposes some specifications on the configurations. The company currently manages four different paper roll widths: 1200 mm, 1400 mm, 1550 mm, and 1600 mm, being 1200 mm the most one used. Therefore, only configurations with these widths will be considered. Regarding the lengths, the cutting machines impose minimum and maximum limits on the panels' length, $\underline{\ell} = 700\,\text{mm}$ and $\overline{\ell} = 3100\,\text{mm}$, respectively.

Finally, as it can be observed in Fig. 5(a), the *line* can allocate up to four paper rolls, so that a maximum of two configurations widths can be managed without roll changes. As paper roll changes are a time-consuming operation, the company avoids using more than two widths for the same order.

[1]Due to confidentiality issues, the data are not available. However, input data and results for six instances have been published in Terán-Viadero, Alonso-Ayuso, and Martín-Campo (2023).

	Items		Current operation		Widths					Lengths					Demands				
	#	m^2	m^2	Left. (%)	Min	Mean	Median	Max	St. Dev.	Min	Mean	Median	Max	St. Dev.	Min	Mean	Median	Max	St. Dev.
I1	10	168	225	25.2	40	227	270	360	118	250	382	320	510	107	126	282	166	1065	270
I2	6	333	461	27.7	340	466	470	650	110	390	1108	940	2280	576	40	91	90	154	33
I3	19	412	613	32.9	100	406	300	1000	290	360	1085	960	2300	580	20	60	60	90	30
I4	37	567	861	34.1	90	284	200	920	233	90	1165	980	2390	722	20	56	30	940	148
I5	15	803	1204	33.3	40	137	60	320	115	210	415	440	610	146	40	810	330	2080	777
I6	18	856	1020	16.1	70	239	185	590	194	130	1312	1200	2140	746	100	130	130	160	30
I7	15	1367	1659	17.6	80	195	110	560	154	350	869	490	2390	719	10	505	380	1060	386
I8	24	1405	2454	42.8	40	455	350	1100	290	430	1406	1490	2200	529	14	95	84	232	85
I9	23	1446	2022	28.5	150	451	300	1140	301	300	1514	1540	2370	709	14	88	94	140	37
I10	29	3645	5630	35.3	80	433	380	1040	281	150	1254	1400	2390	694	15	291	30	1800	551
I11	19	4037	6261	35.5	230	515	460	880	242	250	1137	1040	2140	614	220	428	600	600	182
I12	80	5645	9521	40.7	50	352	300	1140	289	120	1139	1105	2310	629	20	230	60	1050	299
I13	27	6365	10555	39.7	30	527	560	1010	274	420	857	710	2000	479	260	444	440	680	147
I14	46	6777	11169	39.3	30	477	485	1010	287	360	951	745	2300	535	20	286	260	680	221
I15	65	10814	17430	38.0	30	488	460	1010	275	250	1006	840	2300	565	20	327	260	680	220
I16	27	14020	19267	27.2	90	201	160	400	78	160	801	630	1800	524	650	3628	2000	12000	3407
I17	65	14445	23394	38.3	30	488	460	1010	275	250	1006	840	2300	565	30	432	360	1100	321
I18	32	23536	36746	35.9	100	394	240	980	303	100	1102	870	2340	736	820	1403	1180	2080	505
I19	87	24959	38627	35.4	70	315	200	980	262	90	1172	990	2390	736	20	567	100	2080	715
I20	87	34239	52618	34.9	70	315	200	980	262	90	1172	990	2390	736	20	778	100	2940	994

Table 1. Instances summarised information

5.2. *Experiments design*

Different experiments have been performed with both models, SM and VSM, varying the maximum number of configurations and widths that can be used. Four scenarios have been explored, whose main characteristics are reported in Table 2:

- In scenario 1, model SM is validated. Width is set to 1200 mm for all configurations, while four different lengths can be selected: 1200 mm, 1800 mm, 2400 mm, and 3100 mm. That is, $\mathcal{W} = \{1200\}$ and $\mathcal{L} = \{1200, 1800, 2400, 3100\}$, i.e., four configurations are available. These values have been chosen based on the standards that the company normally manages in the factory.
- Scenarios 2, 3, and 4 have been used to validate model VSM. Notice that configurations in this model are given only by their widths whereas lengths are part of the solution. In Scenario 2 only configurations of 1200 mm width are available ($\mathcal{W} = \{1200\}$) while in Scenarios 3 and 4, all widths in stock are available ($\mathcal{W} = \{1200, 1400, 1550, 1600\}$). The difference between Scenario 3 and Scenario 4 is the maximum number of different widths that can be selected, $\overline{n}^w = 1$ and $\overline{n}^w = 2$, respectively. Remember that using more than two different widths is not an option for the company.

These four scenarios have been tested varying $\overline{n}^c$, the maximum number of configurations that can be selected, from 1 to 4 (for scenario 4, $\overline{n}^c = 1$ has been discarded since $\overline{n}^c \geqslant \overline{n}^w$). In total, combining the four scenarios with the different values of $\overline{n}^c$, 15 experiments have been designed for the 20 instances.

As a reference, two additional scenarios have been explored:

- The company's current operation (Scenario 0). Only the configuration $1200 \times 2400\,\text{mm}^2$ is available.
- An extreme scenario (Scenario 5) with $\mathcal{W} = \{1200, 1400, 1550, 1600\}$, $\overline{n}^w = 4$ and $\overline{n}^c = 8$. From an operation point of view, this situation is not suitable, but it allows us to know a bound for better improvement.

5.3. *Computational results*

The models have been implemented using the algebraic modelling language GAMS v.41.5.0 (see Rosenthal (2007)) and solved with Gurobi v.9.5.2 optimiser (see Gurobi Optimization, LLC (2022)). Default Gurobi settings are used, except for the number of threads used (set to all minus 1) and the time limit (set to 1800s). The study was performed on a personal computer with an Intel(R) Core(TM) i7-1280P processor, 2.00GHz, 16GB of RAM, on Windows 11 OS.

Table 3 reports, for each instance, the model dimensions in Scenarios 1, 2 and 3 with $\overline{n}^c = 1$. Notice that Scenarios 3 and 4 have the same dimensions. The headings are as follows: number of constraints (Cons.), number of variables (Vars.), number of binary

	Model	Widths	Lengths	$\overline{n}^c$
Scenario 1	SM	1200mm.	discrete set	$1, \ldots, 4$
Scenario 2	VSM	1200mm.	continuous	$1, \ldots, 4$
Scenario 3	VSM	$\overline{n}^w = 1$	continuous	$1, \ldots, 4$
Scenario 4	VSM	$\overline{n}^w = 2$	continuous	$2, \ldots, 4$

Table 2. Experiments design

	SM - $\lvert\mathcal{W}\rvert = 1$ - $\overline{n}^c = 1$				VSM - $\lvert\mathcal{W}\rvert = 1$ - $\overline{n}^c = 1$				VSM - $\lvert\mathcal{W}\rvert = 4$ - $\overline{n}^c = 1$			
	Cons.	Vars.	0–1 Vars.	NonZero	Cons.	Vars.	0–1 Vars.	NonZero	Cons.	Vars.	0–1 Vars.	NonZero
I1	56	53	52	217	783	507	259	2529	2655	1729	890	8633
I2	34	53	52	145	410	269	140	1319	1277	845	448	4133
I3	91	89	88	357	756	474	247	2373	2715	1727	907	8659
I4	167	185	184	685	1407	884	464	4417	4485	2847	1515	14235
I5	81	65	64	309	3641	2402	1208	12017	11396	7525	3794	37665
I6	80	77	76	305	521	317	167	1595	1907	1185	630	5977
I7	73	81	80	293	1985	1302	660	6501	7151	4711	2395	23531
I8	103	109	108	405	1847	1193	609	5973	6320	4107	2107	20579
I9	97	101	100	377	982	617	320	3096	3310	2101	1100	10561
I10	126	137	136	502	7417	4900	2466	24505	24838	16437	8286	82217
I11	89	89	88	349	4251	2804	1412	14023	14433	9539	4813	47719
I12	355	349	348	1429	8875	5789	2937	28993	30826	20199	10273	101199
I13	133	113	112	515	13427	8906	4466	44549	45035	29895	15003	149563
I14	218	197	196	863	14178	9377	4712	46915	47727	31609	15902	158177
I15	301	281	280	1203	18424	12178	6123	60931	62137	41135	20707	205851
I16	133	113	112	515	24287	16146	8086	80749	83591	55599	27855	278083
I17	301	281	280	1203	26062	17270	8669	86391	88135	58467	29373	292511
I18	148	133	132	576	20716	13757	6894	68809	71047	47213	23672	236173
I19	383	389	388	1552	22634	14953	7524	74808	77393	51223	25805	256299
I20	383	389	388	1552	30524	20213	10154	101108	104213	69103	34745	345699

Table 3. Models dimensions

Sce.	$\overline{n}^c$ 1	2	3	4
1	2	21	26	26
2	7	32	39	42
3	11	33	37	43
4	–	49	55	62

Table 5. Leftover reduction %

Sce.	$\overline{n}^c$ 1	2	3	4
1	99	93	91	91
2	98	89	86	85
3	96	88	87	85
4	–	83	81	78

Table 6. Area improvement %

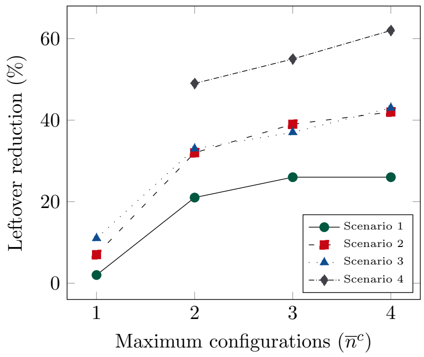


Figure 7. Caption: Leftover lines for each scenario
Figure 7. Alt Text: Lines chart for each scenario showing the percentage of leftover reduction for $\overline{n}^c = 1, 2, 3, 4$

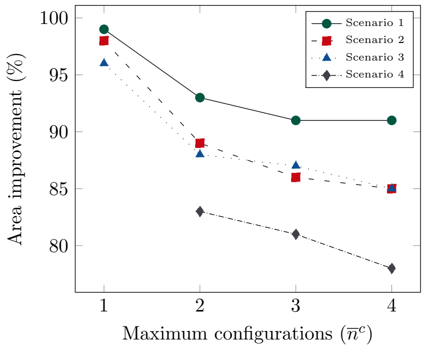


Figure 8. Caption: Area lines for each scenario
Figure 8. Alt Text: Lines chart for each scenario showing the percentage of area improvement for $\overline{n}^c = 1, 2, 3, 4$

variables (0–1 Vars.), and non-zero elements (NonZero). It is worth pointing out that for $\overline{n}^c > 1$, the number of variables and constraints is approximately proportional to the value of $\overline{n}^c$.

To evaluate the results, as the total area of the demanded items is known, a key element is the percentage of material produced that is considered leftover. These percentages are reported in Table 4 for each scenario and each value of $\overline{n}^c$. For each instance, a colour scale is used, where the worst percentage is dark, and the best percentage is light.

It can be observed that the company operation always reports the worst percentage compared to the scenarios tested. However, for Scenario 1 (model SM) and $\overline{n}^c = 1$, in most of the instances, the selected configuration is the one used by the company. As expected, for each scenario, the higher the value of $\overline{n}^c$, the better the results obtained. Observe that the largest improvement is from $\overline{n}^c = 1$ to $\overline{n}^c = 2$. In some specific instances, in particular instances I5 and I14 (scenario 3) and I5, I10, and I13 (scenario 4), it can be observed that the percentage of leftovers does not improve when the value of $\bar{n}^c$ is increased from 3 to 4. This is because, due to the increased complexity of the model, the optimiser does not achieve the optimal solution and only provides a (good) feasible solution within 1800 seconds.

Finally, the last row of this table provides the average percentage that is considered leftover. Currently, the company discards on average 36% of the material, while with the extreme Scenario 5, the leftover is reduced to a third (13%).

To better illustrate these results, Tables 5 and 6 report the average percentage of leftover reduction and material used improvement, respectively, using SM and VSM against the factory's current operation.

As mentioned, with Scenario 5, it is possible to reduce the leftover to only 13%

	Sce. 0 company 1.2×2.4	Scenario 1 SM - $\mathcal{W}^1$				Scenario 2 VSM - $\mathcal{W}^1$				Scenario 3 VSM - $\mathcal{W}^2$ - $\overline{n}^w = 1$				Scenario 4 VSM - $\mathcal{W}^2$ - $\overline{n}^w = 2$			Scenario 5 VSM - $\mathcal{W}^2$
		$\overline{n}^c = 1$	$\overline{n}^c = 2$	$\overline{n}^c = 3$	$\overline{n}^c = 4$	$\overline{n}^c = 1$	$\overline{n}^c = 2$	$\overline{n}^c = 3$	$\overline{n}^c = 4$	$\overline{n}^c = 1$	$\overline{n}^c = 2$	$\overline{n}^c = 3$	$\overline{n}^c = 4$	$\overline{n}^c = 2$	$\overline{n}^c = 3$	$\overline{n}^c = 4$	$\overline{n}^w = 4$-$\overline{n}^c = 8$
I1	25.2	19.4	18.0	17.6	17.6	15.8	13.7	12.8	12.6	14.3	13.7	12.8	12.6	10.3	9.1	8.7	6.3
I2	27.7	27.7	24.1	24.1	24.1	27.1	21.7	19.9	18.9	18.2	12.3	9.9	8.6	8.9	6.3	4.7	4.1
I3	32.9	31.7	23.1	21.4	21.3	21.7	16.8	14.2	12.9	21.7	16.8	14.2	12.9	16.8	13.5	11.0	7.4
I4	34.1	34.1	27.5	25.6	25.3	29.0	23.3	21.9	21.1	29.0	23.3	21.9	21.2	19.8	15.5	14.5	10.5
I5	33.3	19.2	18.7	18.6	18.6	19.2	18.1	17.5	17.3	7.1	4.1	3.7	5.5	4.6	2.8	3.2	1.4
I6	16.1	16.1	14.7	14.3	14.1	13.9	6.7	3.8	2.9	13.9	6.7	3.8	2.9	6.7	3.8	2.9	1.8
I7	17.6	17.6	15.7	15.3	15.2	17.2	8.5	5.4	4.1	17.2	8.5	5.4	4.1	8.5	5.8	4.3	3.0
I8	42.8	42.8	37.9	35.6	35.6	38.8	28.9	26.5	25.7	38.8	28.9	26.7	25.4	25.6	23.5	21.4	17.0
I9	28.5	28.5	19.5	19.3	19.3	23.1	18.7	16.1	14.5	23.1	18.7	16.1	14.5	15.1	10.9	9.5	6.3
I10	35.3	35.3	34.3	34.1	34.1	35.3	27.3	26.3	26.1	24.0	15.2	13.8	13.6	13.0	8.0	11.2	4.7
I11	35.5	35.5	33.6	33.6	33.6	29.1	23.0	22.0	21.5	29.1	23.0	22.0	21.4	20.3	19.0	18.1	16.8
I12	40.7	40.7	33.5	31.9	31.9	34.8	28.2	26.5	24.8	34.8	28.2	26.5	24.8	24.5	21.2	19.6	15.9
I13	39.7	39.7	37.6	37.2	37.2	36.8	29.7	29.1	28.3	32.2	26.9	23.9	23.9	22.1	19.0	23.6	11.8
I14	39.3	39.3	36.9	36.4	36.4	37.6	30.1	28.8	27.6	33.8	27.9	24.9	26.1	23.1	19.3	18.2	13.6
I15	38.0	38.0	36.0	35.4	35.4	36.0	28.4	26.9	25.9	36.0	28.4	27.1	26.3	22.8	20.6	17.4	15.6
I16	27.2	22.4	14.5	11.2	11.2	20.9	10.0	6.7	4.5	21.2	10.0	6.7	4.4	9.7	5.3	3.0	2.0
I17	38.3	38.3	36.2	35.7	35.7	36.4	28.8	27.1	27.7	35.9	28.8	31.1	27.0	22.8	20.6	19.7	16.1
I18	35.9	35.9	29.8	28.2	28.2	35.9	29.8	27.6	26.3	34.9	29.4	27.3	26.3	25.2	22.8	19.2	15.3
I19	35.4	35.4	29.5	27.7	27.7	35.4	29.6	26.9	26.2	34.8	32.0	29.2	26.4	24.5	22.6	19.5	13.7
I20	34.9	34.9	29.2	27.5	27.5	34.9	29.2	27.1	26.2	34.2	28.9	29.5	27.3	23.7	23.2	18.9	13.1
av.	35.5	35.1	30.4	29.1	29.1	34	27.2	25.2	24.3	32.8	27	25.8	24	22.1	19.9	17.4	12.9

$\mathcal{W}^1 = \{1200\}$
$\mathcal{W}^2 = \{1200, 1400, 1550, 1600\}$

Table 4. Percentage of leftover

(73% reduction with respect to the company's operation) resulting in a total material used of 178,827 m^2 instead of 241,735 m^2 to serve the 20 orders. However, this strategy is not an option for the company as it supposes to work with eight different paper rolls (two per width). In these tables, it can be observed that some strategies using not more than four paper rolls improve its current operation significantly. From the analysis of these results, it is possible to identify simple strategies that can even halve the leftover amount.

If only one configuration can be used, the best strategy would be to consider Scenario 3 with $\overline{n}^c = 1$. For each instance, one width is selected from the four available in the stock, and only one length is fixed (defined by the VSM model) for all the produced panels. The leftover reduction is 11% on average (for instance I5, the reduction is from 33% to 7%), obtaining in global total material saving of 9,674 m^2.

Nevertheless, considering the use of two configurations can lead to an important progress. If $\overline{n}^c$ is set to 2, Scenario 2 (only 1200 mm width is available) and Scenario 3 (four widths are available and just one can be selected), obtain a big reduction in the percentage of leftover, 32% and 33%, respectively. It could be observed that both scenarios gave very similar improvements, therefore keeping the use of 1200 mm width (Scenario 2) is more convenient for operation purposes. With this operation, the company will have to adjust the machines only to fix the panels' length to the two best lengths defined by the model. In this case, the 32% in leftover reduction turns into 27,586 m^2 of material savings. Notice that this saving is equivalent to $2\times 23 =$ 46 km of paper rolls with 1200 mm width.

Finally, the best results are obtained when the company considers the option of selecting two different widths per order (Scenario 4). Different to the previous operation presented, in this case, the *line* needs to stop to change the paper rolls, which is not too time-consuming as four rolls can be allocated. If $\overline{n}^c = 2$, the percentage of leftovers is halved with respect to the amount generated with the current operative (49%). With this operation, the total amount of material used is reduced by 42,244 m^2. Notice that when $\overline{n}^c = 4$, the leftover reduction is 62%; however, manipulating four different configurations makes the production process more complicated.

Summarising, it can be observed that there are three strategies that can improve drastically the leftover reduction without implying significant changes in the company's current operation:

- Strategy 1 (Scenario 3 - $\overline{n}^c = 1$): 11% leftover reduction / 96% area.
- Strategy 2 (Scenario 2 - $\overline{n}^c = 2$): 32% leftover reduction / 89% area.
- Strategy 3 (Scenario 4 - $\overline{n}^c = 2$): 49% leftover reduction / 83% area.

Finally, Table 7 reports some statistics on the computational performance of the selected strategies. The following information is provided: the value for the incumbent solution obtained ($\overline{z}_{IP}$), the optimality gap (in %), and the computing time in seconds (Time). For comparison purposes, the last column reports the total area of the company's current operation.

Notice, that the total area of the demanded items is a lower bound for the VSM optimal solution (z_{IP}) and, therefore, it can be considered to compute the optimality gap. In Table 7, the optimality gap reported is the minimum between the one obtained by Gurobi (GAP_G) and the one calculated with the total demanded area (a) as lower bound: $GAP(\%) = \min\{GAP_G, 100\frac{\overline{z}_{IP}-a}{\overline{z}_{IP}}\}$. Results using a as a lower bound of z_{IP} in the VSM model are not reported since they are worse in terms of computing time and/or incumbent solution.

	Strategy 1			Strategy 2			Strategy 3			Company
	$\overline{z}_{IP}$	Gap	Time	$\overline{z}_{IP}$	Gap	Time	$\overline{z}_{IP}$	Gap	Time	z_{IP}
I1	196	0.0	7.6	195	0.0	35.9	187	10.3	1800*	225
I2	407	0.0	0.8	425	0.0	2.1	366	0.0	1.9	461
I3	526	0.0	1.5	495	0.0	4.8	495	0.0	44.7	613
I4	799	0.0	2.1	740	0.0	13.3	708	0.0	124.9	861
I5	864	7.1	1800*	980	18.1	1800*	841	4.6	1800*	1204
I6	994	0.0	0.9	917	0.0	1.2	917	0.0	15.0	1020
I7	1652	0.0	30.7	1494	0.0	96.5	1494	8.5	1800*	1659
I8	2297	0.0	1.9	1975	0.0	56.1	1889	5.9	1800*	2454
I9	1881	0.0	1.1	1779	0.0	6.5	1703	0.0	25.8	2022
I10	4795	0.0	8.5	5015	27.3	1800*	4190	13.0	1800*	5630
I11	5691	0.0	64.0	5245	0.0	733.7	5063	10.2	1800*	6261
I12	8663	0.0	781.8	7861	4.4	1800*	7473	4.4	1800*	9521
I13	9392	0.0	23.2	9059	13.4	1800*	8173	22.1	1800*	10555
I14	10242	0.0	6.2	9697	13.0	1800*	8818	23.1	1800*	11169
I15	16902	36.0	1800*	15111	12.2	1800*	14011	22.8	1800*	17430
I16	17800	21.2	1800*	15569	10.0	1800*	15523	9.7	1800*	19267
I17	22520	35.9	1800*	20281	21.2	1800*	18709	22.8	1800*	23394
I18	36138	26.9	1800*	33537	27.7	1800*	31480	25.2	1800*	36746
I19	38266	31.9	1800*	35441	27.6	1800*	33039	24.5	1800*	38627
I20	52036	31.7	1800*	48333	29.2	1800*	44864	23.7	1800*	52618

* time limit exceeded

Table 7. Model results for the three best operation strategies

It can be observed in Table 7 that the areas do not increase from Strategy 1 to Strategy 3. Notice that in general, the more complex the strategy is, the higher the computing times. Furthermore, there is a relation between the size of the instance and the solution times/GAPS. Certainly, there are other factors that increase the difficulty of finding the optimal solution in a reasonable time, such as in instance I5, which corresponds to an order with small items (narrow and short) and a large variability in the quantity demanded for each of them (see Fig. 6). Regardless, although some of the GAPs are high, the solution obtained clearly improves the current operation of the company.

6. Conclusions

The results reveal that using expert systems based on mathematical optimisation to support decision-making processes provides significant advantages compared to the operational processes designed based on the operator's experience, even if they have a deep knowledge of the problem.

We have presented two novel linear optimisation models for a cutting stock problem in the honeycomb cardboard industry suggested by a Spanish company. This problem belongs to the family of 2-Dimensional Cutting Stock Problems with Variable-Sized stock (2D-VSCSP) recently introduced by Salem et al. (2023). The models have been validated using real data from the company. The results improve the current operation in the factory by reducing leftovers and panel production. Furthermore, three straightforward strategies have been proposed to the company that slightly modify its current operation. Concerning the number of configurations, the obtained results have shown that it is necessary to consider two configurations at most for a notable leftover reduction (until 49%). Taking into account the *line* specifications, the impact of the proposed strategies is not relevant in terms of time consumption and operation.

Finally, we propose different research lines: (1) tight the VSM model with new valid cuts, trying to reduce the GAP and the computing times; (2) develop (meta)heuristics to obtain good feasible solutions in a short time, avoiding the use of commercial optimisers that are expensive for the company; (3) as the company has two cutting machines, consider job sequencing to minimise the production times.

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Disclosure of interest

The authors report there are no competing interests to declare.

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Data availability statement

Due to the nature of the research, due to commercial supporting not all data is available. We refer the readers to Terán-Viadero, Alonso-Ayuso, and Martín-Campo (2023) where for six instances, input data and results obtained are reported.

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Appendix A. Illustrative small-scale case study

This section presents a small-case study for illustrative purposes. The order includes the following items:

Items	$w_i(mm)$	$l_i(mm)$	$d_i(mm)$
i_1	60	500	340
i_2	265	600	72
i_3	320	700	24

For the sake of clarity, we assume that only 1200 mm wide rolls are available and that only one configuration can be defined. In addition, the company imposes a lower and upper limit of $\underline{\ell} = 1800$ mm and $\overline{\ell} = 3100$ mm for the panels' length.

Then, using the notation defined above, we have: $\mathcal{I} = \{i_1, i_2, i_3\}$, $\mathcal{J} = \{j_1\}$, $W = \{1200\}$, $\mathcal{J}^{1200} = \{j_1\}$, $\overline{n}^c = 1$, $\underline{\ell} = 1800$ and $\overline{\ell} = 3100$.

The first step is to calculate r_{ij}, the number of rows of each item i that fits in configuration j:

$$r_{i_1j_1} = \left\lfloor \frac{W_{j_1}}{w_{i_1}} \right\rfloor = \left\lfloor \frac{1200}{60} \right\rfloor = 20, \quad r_{i_2j_1} = \left\lfloor \frac{1200}{265} \right\rfloor = 4, \quad r_{i_3j_1} = \left\lfloor \frac{1200}{320} \right\rfloor = 3$$

Then, it is possible to calculate c_{ij}, the number of columns needed to meet the demand of item i using configuration j:

$$c_{i_1j_1} = \left\lceil \frac{d_{i_1}}{r_{i_1j_1}} \right\rceil = \left\lceil \frac{340}{20} \right\rceil = 17, \quad c_{i_2j_1} = \left\lceil \frac{72}{4} \right\rceil = 18, \quad c_{i_3j_1} = \left\lceil \frac{24}{3} \right\rceil = 8$$

The next step is to calculate a lower and an upper bound for the number of panels needed to produce this number of columns for each item. These bounds depend on $\underline{\ell} = 1800$ mm and $\overline{\ell} = 3100$ mm, the minimum and maximum lengths allowed for the configuration, respectively:

$$\underline{k}_{i_1j_1} = \left\lceil \frac{c_{i_1j_1}}{\lfloor \frac{\overline{\ell}}{\ell_{i_1}} \rfloor} \right\rceil = \left\lceil \frac{17}{6} \right\rceil = 3, \qquad \underline{k}_{i_2j_1} = \left\lceil \frac{18}{5} \right\rceil = 4, \quad \underline{k}_{i_3j_1} = \left\lceil \frac{8}{4} \right\rceil = 2,$$

$$\overline{k}_{i_1j_1} = \left\lceil \frac{c_{i_1j_1}}{\lfloor \frac{\max\{\ell_{i_1}, \underline{\ell}\}}{\ell_{i_1}} \rfloor} \right\rceil = \left\lceil \frac{17}{3} \right\rceil = 6, \quad \overline{k}_{i_2j_1} = \left\lceil \frac{18}{3} \right\rceil = 6, \quad \overline{k}_{i_3j_1} = \left\lceil \frac{8}{2} \right\rceil = 4.$$

Therefore, $K_{i_1j_1} = \{3, 4, 5, 6\}$, $K_{i_2j_1} = \{4, 5, 6\}$ and $K_{i_3j_1} = \{2, 3, 4\}$.

Finally, the values of $\underline{\ell}_{ijk}$ are calculated. For item i_1:

$$\begin{aligned}
\underline{\ell}_{i_1 j_1 3} &= \max\Big\{\underline{\ell}, \ell_{i_1}\Big\lceil\frac{c_{i_1 j_1}}{3}\Big\rceil\Big\} = \max\Big\{1800, 500\Big\lceil\frac{17}{3}\Big\rceil\Big\} = 3000,\\
\underline{\ell}_{i_1 j_1 4} &= \max\Big\{\underline{\ell}, \ell_{i_1}\Big\lceil\frac{c_{i_1 j_1}}{4}\Big\rceil\Big\} = \max\Big\{1800, 500\Big\lceil\frac{17}{4}\Big\rceil\Big\} = 2500,\\
\underline{\ell}_{i_1 j_1 5} &= \max\Big\{\underline{\ell}, \ell_{i_1}\Big\lceil\frac{c_{i_1 j_1}}{5}\Big\rceil\Big\} = \max\Big\{1800, 500\Big\lceil\frac{17}{5}\Big\rceil\Big\} = 2000,\\
\underline{\ell}_{i_1 j_1 6} &= \max\Big\{\underline{\ell}, \ell_{i_1}\Big\lceil\frac{c_{i_1 j_1}}{6}\Big\rceil\Big\} = \max\Big\{1800, 500\Big\lceil\frac{17}{6}\Big\rceil\Big\} = 1800.
\end{aligned}$$

Analogously:

- Item i_2: $\underline{\ell}_{i_2 j_1 4} = 3000$, $\underline{\ell}_{i_2 j_1 5} = 2400$ and $\underline{\ell}_{i_2 j_1 6} = 1800$.
- Item i_3: $\underline{\ell}_{i_3 j_1 2} = 2800$, $\underline{\ell}_{i_3 j_1 3} = 2100$ and $\underline{\ell}_{i_3 j_1 4} = 1800$.

The objective function for the VSM model results:

$$\min 1200\big(3\delta_{i_1 j_1 3} + 4\delta_{i_1 j_1 4} + 5\delta_{i_1 j_1 5} + 6\delta_{i_1 j_1 6} + 4\delta_{i_2 j_1 4} + 5\delta_{i_2 j_1 5} + 6\delta_{i_2 j_1 6} + 2\delta_{i_3 j_1 2} + 3\delta_{i_3 j_1 3} + 4\delta_{i_3 j_1 4}\big)$$

Constraints (VSM.2), all items must be assigned to a configuration:

$$x_{i_1 j_1} = 1, \qquad x_{i_2 j_1} = 1, \qquad x_{i_3 j_1} = 1.$$

Constraints (VSM.3), a configuration is selected if and only if at least one item is assigned to it:

$$x_{i_1 j_1} \leqslant y_{j_1} \leqslant x_{i_1 j_1} + x_{i_2 j_1} + x_{i_3 j_1}, \qquad x_{i_2 j_1} \leqslant y_{j_1} \leqslant x_{i_1 j_1} + x_{i_2 j_1} + x_{i_3 j_1}, \qquad x_{i_3 j_1} \leqslant y_{j_1} \leqslant x_{i_1 j_1} + x_{i_2 j_1} + x_{i_3 j_1}.$$

Constraints (VSM.4) assure that a width is used if and only if a configuration of that width is selected.

$$y_{j_1} \leqslant u_{1200} \leqslant y_{j_1}.$$

Constraints (VSM.5), a number of panels to be produced must be selected for each item assigned to a configuration:

$$\begin{aligned}
z_{i_1 j_1 3} + z_{i_1 j_1 4} + z_{i_1 j_1 5} + z_{i_1 j_1 6} &= x_{i_1 j_1},\\
z_{i_2 j_1 4} + z_{i_2 j_1 5} + z_{i_2 j_1 6} &= x_{i_2 j_1},\\
z_{i_3 j_1 2} + z_{i_3 j_1 3} + z_{i_3 j_1 4} &= x_{i_3 j_1}.
\end{aligned}$$

Constraints (VSM.6), the minimum configurations' length is imposed by the $\underline{\ell}_{ijk}$ calculated:

$$\begin{aligned}
3000 z_{i_1 j_1 3} + 2500 z_{i_1 j_1 4} + 2000 z_{i_1 j_1 5} + 1800 z_{i_1 j_1 6} &\leqslant L_{j_1},\\
3000 z_{i_2 j_1 4} + 2400 z_{i_2 j_1 5} + 1800 z_{i_2 j_1 6} &\leqslant L_{j_1},\\
2800 z_{i_3 j_1 2} + 2100 z_{i_3 j_1 3} + 1800 z_{i_3 j_1 4} &\leqslant L_{j_1}.
\end{aligned}$$

Constraints (VSM.7)–(VSM.9) assure all panels of configuration j have the same length. Constraints (VSM.7):

$$\begin{array}{lll} \delta_{i_1j_13} \leqslant L_{j_1}, & \delta_{i_2j_14} \leqslant L_{j_1}, & \delta_{i_1j_12} \leqslant L_{j_1}, \\ \delta_{i_1j_14} \leqslant L_{j_1}, & \delta_{i_2j_15} \leqslant L_{j_1}, & \delta_{i_1j_13} \leqslant L_{j_1}, \\ \delta_{i_1j_15} \leqslant L_{j_1}, & \delta_{i_2j_16} \leqslant L_{j_1}, & \delta_{i_1j_14} \leqslant L_{j_1}, \\ \delta_{i_1j_16} \leqslant L_{j_1}. & & \end{array}$$

Constraints (VSM.8):

$$\begin{array}{lll} \delta_{i_1j_13} \leqslant 3100z_{i_1j_13}, & \delta_{i_2j_14} \leqslant 3100z_{i_2j_14}, & \delta_{i_1j_12} \leqslant 3100z_{i_1j_12}, \\ \delta_{i_1j_14} \leqslant 3100z_{i_1j_14}, & \delta_{i_2j_15} \leqslant 3100z_{i_2j_15}, & \delta_{i_1j_13} \leqslant 3100z_{i_1j_13}, \\ \delta_{i_1j_15} \leqslant 3100z_{i_1j_15}, & \delta_{i_2j_16} \leqslant 3100z_{i_2j_16}, & \delta_{i_1j_14} \leqslant 3100z_{i_1j_14}, \\ \delta_{i_1j_16} \leqslant 3100z_{i_1j_16}. & & \end{array}$$

Constraints (VSM.9):

$$\begin{array}{lll} L_{j_1} - \delta_{i_1j_13} \leqslant 3100(1 - z_{i_1j_13}), & L_{j_1} - \delta_{i_2j_14} \leqslant 3100(1 - z_{i_2j_14}), & L_{j_1} - \delta_{i_1j_12} \leqslant 3100(1 - z_{i_1j_12}), \\ L_{j_1} - \delta_{i_1j_14} \leqslant 3100(1 - z_{i_1j_14}), & L_{j_1} - \delta_{i_2j_15} \leqslant 3100(1 - z_{i_2j_15}), & L_{j_1} - \delta_{i_1j_13} \leqslant 3100(1 - z_{i_1j_13}), \\ L_{j_1} - \delta_{i_1j_15} \leqslant 3100(1 - z_{i_1j_15}), & L_{j_1} - \delta_{i_2j_16} \leqslant 3100(1 - z_{i_2j_16}), & L_{j_1} - \delta_{i_1j_14} \leqslant 3100(1 - z_{i_1j_14}), \\ L_{j_1} - \delta_{i_1j_16} \leqslant 3100(1 - z_{i_1j_16}). & & \end{array}$$

Constraints (VSM.10) bound the length of each configuration:

$$1800y_{j_1} \leqslant L_{j_1} \leqslant 3100y_{j_1}.$$

Constraints (VSM.11) limit the number of configurations selected:

$$y_{j_1} \leqslant 1.$$

Constraints (VSM.12) limit the number of widths selected:

$$u_{1200} \leqslant 1.$$

Constraints (VSM.13) ensure that there are no symmetry constraints, in this case as there is only configuration there are no symmetries.

Constraints (VSM.14) define the variables domain:

$$\begin{aligned} x_{i_1j_1}, x_{i_2j_1}, x_{i_3j_1} &\in \{0,1\}, \\ y_{j_1} &\in \{0,1\}, \\ u_{1200} &\in \{0,1\}, \\ z_{i_1j_13}, z_{i_1j_14}, z_{i_1j_15}, z_{i_1j_16}, z_{i_2j_14}, z_{i_2j_15}, z_{i_2j_16}, z_{i_3j_12}, z_{i_3j_13}, z_{i_3j_14} &\in \{0,1\}, \\ L_{j1} &\geqslant 0, \\ \delta_{i_1j_13}, \delta_{i_1j_14}, \delta_{i_1j_15}, \delta_{i_1j_16}, \delta_{i_2j_14}, \delta_{i_2j_15}, \delta_{i_2j_16}, \delta_{i_3j_12}, \delta_{i_3j_13}, \delta_{i_3j_14} &\geqslant 0. \end{aligned}$$

The optimal solution for this model is to use 1200 mm×3000 mm panels. In particular, three, four, and two panels for items i_1, i_2, and i_3 respectively. A total of 32.40

m^2 of raw material. The items' demand is 27.02 m^2, therefore this solution produces 5.37 m^2 of leftover, which corresponds to 17% of the total material used. The current company's operation, which uses standard 1200 mm$\times$2400 mm panels, produces 37.44 m^2 of raw material, which is equivalent to 28% of leftover.

If $\overline{n}^c = 2$ is considered, i.e., two different configurations of 1200 mm width can be used, the model is modified as follows. The terms related to configurations j_2 are added to the objective function:

$$\begin{aligned}
\min 1200\big(&3\delta_{i_1j_13} + 4\delta_{i_1j_14} + 5\delta_{i_1j_15} + 6\delta_{i_1j_16} + \\
&3\delta_{i_1j_23} + 4\delta_{i_1j_24} + 5\delta_{i_1j_25} + 6\delta_{i_1j_26} + \\
&4\delta_{i_2j_14} + 5\delta_{i_2j_15} + 6\delta_{i_2j_16} + \\
&4\delta_{i_2j_24} + 5\delta_{i_2j_25} + 6\delta_{i_2j_26} + \\
&2\delta_{i_3j_12} + 3\delta_{i_3j_13} + 4\delta_{i_3j_14} + \\
&2\delta_{i_3j_22} + 3\delta_{i_3j_23} + 4\delta_{i_3j_24}\big)
\end{aligned}$$

The (VSM.2) constraints, for each item terms related to j_2 are added.

$$x_{i_1j_1} + x_{i_1j_2} = 1, \qquad x_{i_2j_1} + x_{i_2j_2} = 1, \qquad x_{i_3j_1} + x_{i_3j_2} = 1.$$

For the constraints (VSM.3)–(VSM.10), the following inequalities related to j_2 are appended.

Constraints (VSM.3):

$$x_{i_1j_2} \leqslant y_{j_2} \leqslant x_{i_1j_2} + x_{i_2j_2} + x_{i_3j_2}, \qquad x_{i_2j_2} \leqslant y_{j_2} \leqslant x_{i_1j_2} + x_{i_2j_2} + x_{i_3j_2}, \qquad x_{i_3j_2} \leqslant y_{j_2} \leqslant x_{i_1j_2} + x_{i_2j_2} + x_{i_3j_2}.$$

Constraints (VSM.4) are modified as:

$$y_{j_1} \leqslant u_{1200}, \qquad y_{j_2} \leqslant u_{1200}, \qquad u_{1200} \leqslant y_{j_1} + y_{j_2}.$$

Constraints (VSM.5):

$$\begin{aligned}
z_{i_1j_23} + z_{i_1j_24} + z_{i_1j_25} + z_{i_1j_26} &= x_{i_1j_2}, \\
z_{i_2j_24} + z_{i_2j_25} + z_{i_2j_26} &= x_{i_2j_2}, \\
z_{i_3j_22} + z_{i_3j_23} + z_{i_3j_24} &= x_{i_3j_2}.
\end{aligned}$$

Constraints (VSM.6):

$$\begin{aligned}
3000z_{i_1j_23} + 2500z_{i_1j_24} + 2000z_{i_1j_25} + 1800z_{i_1j_26} &\leqslant L_{j_2}, \\
3000z_{i_2j_24} + 2400z_{i_2j_25} + 1800z_{i_2j_26} &\leqslant L_{j_2}, \\
2800z_{i_3j_22} + 2100z_{i_3j_23} + 1800z_{i_3j_24} &\leqslant L_{j_2}.
\end{aligned}$$

Constraints (VSM.7):

$$\begin{array}{lll}
\delta_{i_1j_23} \leqslant L_{j_2}, & \delta_{i_2j_24} \leqslant L_{j_2}, & \delta_{i_1j_22} \leqslant L_{j_2}, \\
\delta_{i_1j_24} \leqslant L_{j_2}, & \delta_{i_2j_25} \leqslant L_{j_2}, & \delta_{i_1j_23} \leqslant L_{j_2}, \\
\delta_{i_1j_25} \leqslant L_{j_2}, & \delta_{i_2j_26} \leqslant L_{j_2}, & \delta_{i_1j_24} \leqslant L_{j_2}, \\
\delta_{i_1j_26} \leqslant L_{j_2}. & &
\end{array}$$

Constraints (VSM.8):

$$\begin{array}{lll}
\delta_{i_1j_23} \leqslant 3100z_{i_1j_23}, & \delta_{i_2j_24} \leqslant 3100z_{i_2j_24}, & \delta_{i_1j_22} \leqslant 3100z_{i_1j_22}, \\
\delta_{i_1j_24} \leqslant 3100z_{i_1j_24}, & \delta_{i_2j_25} \leqslant 3100z_{i_2j_25}, & \delta_{i_1j_23} \leqslant 3100z_{i_1j_23}, \\
\delta_{i_1j_25} \leqslant 3100z_{i_1j_25}, & \delta_{i_2j_26} \leqslant 3100z_{i_2j_26}, & \delta_{i_1j_24} \leqslant 3100z_{i_1j_24}, \\
\delta_{i_1j_26} \leqslant 3100z_{i_1j_26}. & &
\end{array}$$

Constraints (VSM.9):

$$\begin{array}{lll}
L_{j_2} - \delta_{i_1j_23} \leqslant 3100(1 - z_{i_1j_23}), & L_{j_2} - \delta_{i_2j_24} \leqslant 3100(1 - z_{i_2j_24}), & L_{j_2} - \delta_{i_1j_22} \leqslant 3100(1 - z_{i_1j_22}), \\
L_{j_2} - \delta_{i_1j_24} \leqslant 3100(1 - z_{i_1j_24}), & L_{j_2} - \delta_{i_2j_25} \leqslant 3100(1 - z_{i_2j_25}), & L_{j_2} - \delta_{i_1j_23} \leqslant 3100(1 - z_{i_1j_23}), \\
L_{j_2} - \delta_{i_1j_25} \leqslant 3100(1 - z_{i_1j_25}), & L_{j_2} - \delta_{i_2j_26} \leqslant 3100(1 - z_{i_2j_26}), & L_{j_1} - \delta_{i_1j_24} \leqslant 3100(1 - z_{i_1j_24}), \\
L_{j_2} - \delta_{i_1j_26} \leqslant 3100(1 - z_{i_1j_26}). & &
\end{array}$$

Constraints (VSM.10):

$$1800y_{j_2} \leqslant L_{j_2} \leqslant 3100y_{j_2}.$$

For constraints(VSM.11) terms related to j_2 are added:

$$y_{j_1} + y_{j_2} \leqslant 2.$$

Constraints (VSM.13) the symmetry constraint is added:

$$y_{j_2} \leqslant y_{j_1}.$$

The variables domain related to j_2 are appended. Constraints (VSM.14):

$$\begin{aligned}
x_{i_1j_2}, x_{i_2j_2}, x_{i_3j_2} &\in \{0, 1\}, \\
y_{j_2} &\in \{0, 1\}, \\
z_{i_1j_23}, z_{i_1j_24}, z_{i_1j_25}, z_{i_1j_26}, z_{i_2j_24}, z_{i_2j_25}, z_{i_2j_26}, z_{i_3j_22}, z_{i_3j_23}, z_{i_3j_24} &\in \{0, 1\}, \\
L_{j_2} &\geqslant 0, \\
\delta_{i_1j_23}, \delta_{i_1j_24}, \delta_{i_1j_25}, \delta_{i_1j_26}, \delta_{i_2j_24}, \delta_{i_2j_25}, \delta_{i_2j_26}, \delta_{i_3j_22}, \delta_{i_3j_23}, \delta_{i_3j_24} &\geqslant 0.
\end{aligned}$$

The optimal solution in this case is to keep using configuration 1200 mm×3000 mm to produce items i_1 and i_3, and introduce configuration 1200 mm×1800 mm to produce item i_2. This solution needs $30.96m^2$ of raw material, reducing the amount of leftover generated to 13%. This means halving the amount of leftovers compared to the companys' current operation, and it reduces to almost a quarter compared to the solution obtained above using a single configuration.