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We have dataset,

$$X = \{x_1, x_2, x_3, \dots, x_n \in R^h\}$$

SNE converts euclidean distances to similarities

$$p_{j|i} = \frac{\exp(-\|x_i - x_j\|^2 / 2\sigma_i^2)}{\sum_{k \neq i} \exp(-\|x_i - x_k\|^2 / 2\sigma_i^2)},$$

$$q_{j|i} = \frac{\exp(-\|y_i - y_j\|^2)}{\sum_{k \neq i} \exp(-\|y_i - y_k\|^2)} = \frac{w_{ij}}{\sum_{k \neq i} w_{ij}} = \frac{w_{ij}}{Z_i}$$

Kullback-Leiber Divergence (KL) compares two distributions

$$C = \sum_i KL(P_i \| Q_i) = \sum_i \sum_j p_{j|i} \log \frac{p_{j|i}}{q_{j|i}}$$

Cost fuction

$$L = \sum_i \sum_j p_{j|i} \log \frac{p_{j|i}}{q_{j|i}}$$

$$= \cancel{\sum_i \sum_j p_{j|i} \log \frac{\exp \|y_j - y_i\|^2}{\exp \|x_j - x_i\|^2}}$$

$$= \sum_{i,j} p_{j|i} \log p_{j|i} - \log q_{j|i}$$

$$= \sum_{i,j} p_{j|i} \log p_{j|i} - p_{j|i} \log w_{ij} + p_{j|i} \log Z_i$$

$$\frac{\partial L}{\partial y_i} = \sum_{i,j} - p_{j|i} \partial \log w_{ij} + \sum_{i,j} p_{j|i} \partial \log Z_i$$

$$* \sum_{i,j} - p_{j|i} \partial \log w_{ij} = \sum_{j \neq i} - p_{j|i} \partial \log w_{ij} - p_{i|j} \partial \log w_{ji}$$

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Because $\partial w_{ij} = w_{ij}(-2(y_i - y_j))$

$$\sum_{j \neq i} -p_{j|i} \frac{w_{ij}}{w_{ij}} (-2(y_i - y_j)) - p_{i|j} \frac{w_{ji}}{w_{ji}} (2(y_j - y_i))$$

$$= 2 \sum_{j \neq i} (p_{j|i} + p_{i|j})(y_i - y_j)$$

$$* \sum_{j,i} p_{j|i}\, \partial \log Z_i = \sum_i \partial \log Z_j$$

$$\sum_j \frac{1}{Z_j} \sum_{j,i} \partial w_{ji}$$

$$= \sum_{j \neq i} \frac{w_{ji}}{Z_j} (2(y_j - y_i)) + \sum_{j \neq i} \frac{w_{ij}}{Z_i} (-2 \cdot (y_i - y_j))$$

$$= 2 \sum_{j \neq i} (-q_{j|i} - q_{i|j})(y_i - y_j)$$

$$\Rightarrow \frac{\partial L}{\partial y_i} = 2 \sum_{j \neq i} (p_{j|i} - q_{j|i} + p_{i|j} - q_{i|j})(y_i - y_j)$$

t-SNE

$$q_{ji} = q_{ij} = \frac{(1 + \|y_i - y_j\|^2)^{-1}}{\sum_{k,l} (1 + \|y_k - y_l\|^2)^{-1}} = \frac{w_{ij}^{-1}}{\sum_{k,l} w_{kl}^{-1}} = \frac{w_{ij}^{-1}}{Z}$$

The loss function

$$L = \sum_{k,l} p_{lk} \log \frac{p_{lk}}{q_{lk}} = \sum_{k,l} p_{lk} \log p_{lk} - p_{lk} \log q_{lk}$$

$$= \sum_{k,l} p_{lk} \log p_{lk} - p_{lk} \log w_{kl}^{-1} + p_{lk} \log Z$$

$$\frac{\partial L}{\partial y_i} = \sum_{k,l} -p_{lk}\, \partial \log w_{kl}^{-1} + \sum_{k,l} p_{lk}\, \partial \log Z$$

$$* \sum_{k,l} -p_{lk}\, \partial \log w_{kl}^{-1} = -2 \sum_{j \neq i} p_{ji}\, \partial \log w_{ij}^{-1}$$

Because $\partial w_{ij}^{-1} = w_{ji}^{-2}(-2(y_i - y_j))$

$$-2\sum_{j\neq i} p_{ji} \frac{w_{ij}^{-2}}{w_{ij}^{-1}}(-2(y_i - y_j)) = 4\sum_{j\neq i} p_{ji}\, w_{ij}^{-1}(y_i - y_j)$$

* $\sum_{k,l} p_{kl}\, \partial \log Z = \frac{1}{Z}\sum_{k,l} \partial w_{kl}^{-1}$

$$= 2\sum_{j\neq i} \frac{w_{ji}^{-2}}{Z}(-2(y_j - y_i))$$

$$= -4\sum_{j\neq i} q_{ij}\, w_{ji}^{-1}(y_i - y_j)$$

In conclusion

$$\frac{\partial L}{\partial y_i} = 4\sum_{j\neq i}(p_{ji} - q_{ji})\, w_{ji}^{-1}(y_i - y_j)$$

$$= 4\sum_{j\neq i}(p_{ji} - q_{ji})(1 + \|y_i - y_j\|^2)^{-1}(y_i - y_j)$$