t-SNE

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January 2022

1 EX1:

$$p_{ij} = \frac{p_{j|i} + p_{i|j}}{2n}$$

$$q_{ji} = q_{ij} = \frac{(1 + \|y_i - y_j\|^2)^{-1}}{\sum_{k \neq l}(1 + \|y_k - y_l\|^2)^{-1}} = \frac{E_{ij}^{-1}}{\sum_{k,l \neq k} E_{kt}^{-1}} = \frac{E_{ij}^{-1}}{Z}$$

The loss function:
(Since E$_{ij} = E_{ji}$)

$$C = \sum_{k,l \neq k} p_{lk} \log \frac{p_{lk}}{q_{lk}} = \sum_{k,l \neq k} p_{lk} \log p_{lk} - p_{lk} \log q_{lk} = \sum_{k,l \neq k} p_{lk} \log p_{lk} - p_{lk} \log E_{kl}^{-1} + p_{lk} \log Z$$

$$\frac{\partial C}{\partial y_i} = \sum_{k,l \neq k} -p_{lk} \partial \log E_{kl}^{-1} + \sum_{k,l \neq k} p_{lk} \partial \log Z$$

We start with the first term, noting that the derivative is non-zero when j, k = i or l = i, that p$_{ji} = p_{ij} and E_{ji} = E_{ij}$

$$\sum_{k,l \neq k} -p_{lk} \partial \log E_{kl}^{-1} = -2 \sum_{j \neq i} p_{ji} \partial \log E_{ij}^{-1}$$

Since $\partial E_{ij}^{-1} = E_{ij}^{-2}(-2(y_i - y_j)) we have :$

$$-2 \sum_{j \neq i} p_{ji} \frac{E_{ij}^{-2}}{E_{ij}^{-1}} (-2(y_i - y_j)) = 4 \sum_{j \neq i} p_{ji} E_{ij}^{-1} (y_i - y_j)$$

(1)

Since $\sum_{k,l \neq k} p_{kl} = 1 and that Z does not depend on k or l$

$$\sum_{k,l \neq k} p_{lk} \partial \log Z = \frac{1}{Z} \sum_{k',l' \neq k'} \partial E_{kl}^{-1} = 2 \sum_{j \neq i} \frac{E_{ji}^{-2}}{Z} (-2(y_j - y_i)) = -4 \sum_{j \neq i} q_{ij} E_{ji}^{-1} (y_j - y_i)$$

(2)
(1) and (2) we have:

$$\frac{\partial C}{\partial y_i} = 4\sum_{j\neq i}(p_{ji} - q_{ji})E_{ji}^{-1}(y_i - y_j)$$

$$\frac{\partial C}{\partial y_i} = 4\sum_{j\neq i}(p_{ji} - q_{ji})(1 + \|y_i - y_j\|^2)^{-1}(y_i - y_j)$$