$$
\begin{array}{llcccccc}
\min\limits_{x_1,x_2,\ldots,x_n} & z = p_1x_1 + \cdots + p_nx_n & & & & & & \\
\text{subject to} & A_{11}x_1 & + & \cdots & + & A_{1n}x_n & \geq & b_1, \\
& \vdots & & \ddots & & \vdots & & \vdots \\
& A_{m1}x_1 & + & \cdots & + & A_{mn}x_n & \geq & b_m, \\
& & & & x_1, x_2, \ldots, x_n & & \geq & 0.
\end{array}
\tag{1.2}
$$

By grouping the variables $x_1, x_2, \ldots, x_n$ into a vector x and constructing the following matrix and vectors from the problem data,

$$
A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}, \qquad p = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix},
$$

we can restate the standard form compactly as follows:

$$
\begin{array}{ll}
\min\limits_{x} & z = p'x \\
\text{subject to} & Ax \geq b, \quad x \geq 0,
\end{array}
$$

We define linear dependence of the rows of a matrix A formally as follows:

$$
z'A = 0 \qquad \text{for some nonzero } z \in \mathbf{R}^m.
$$

(We see that (2.7) can be expressed in this form by taking $z_i = \lambda_i$, $i \neq k$, $z_k = -1$.) The negation of linear dependence is *linear independence* of the rows of A, which is defined by the implication

$$
z'A = 0 \implies z = 0.
$$

The idea of linear independence extends also to functions, including the linear functions y defined by $y(x) := Ax$ that we have been considering above. The functions $y_i(x)$, $i = 1, 2, \ldots, m$, defined by $y(x) := Ax$ are said to be linearly dependent if

$$
z'y(x) = 0 \qquad \forall x \in \mathbf{R}^n \text{ for some nonzero } z \in \mathbf{R}^m
$$

and linearly independent if

$$
z'y(x) = 0 \qquad \forall x \in \mathbf{R}^n \implies z = 0. \tag{2.8}
$$

The equivalence of the linear independence definitions for matrices and functions is clear when we note that

$$
\begin{array}{ll}
z'Ax = 0 & \forall x \in \mathbf{R}^n \text{ for some nonzero } z \in \mathbf{R}^m \\
\iff z'A = 0 & \text{for some nonzero } z \in \mathbf{R}^m.
\end{array}
$$

Proposition 2.2.1. *If the m linear functions y_i are linearly independent, then any p of them are also linearly independent, where $p \leq m$.*

Proposition 2.2.2. *If the linear functions y defined by $y(x) = Ax$, $A \in \mathbf{R}^{m\times n}$, are linearly independent, then $m \leq n$. Furthermore, in the tableau representation, all m dependent y_i's can be made independent; that is, they can be exchanged with m independent x_j's.*

Theorem 2.2.3 (Steinitz). *For a given matrix $A \in \mathbf{R}^{m\times n}$, the linear functions y, defined by $y(x) := Ax$, are linearly independent if and only if for the corresponding tableau all the y_i's can be exchanged with some m independent x_j's.*

A consequence of this result is that given a matrix A, the number of linearly independent rows in A is the maximum number of components of y that can be exchanged to the top of the tableau for the functions $y(x) := Ax$.

When not all the rows of A are linearly independent, we reach a tableau in which one or more of the y_i's are expressed in terms of other components of y. These relationships show the linear dependencies between the functions $y(x)$ and, therefore, between the rows of the matrix A.

An $n \times n$ matrix is nonsingular if the rows of A are linearly independent; otherwise the matrix is singular.

Theorem 2.3.1. *The system $y = Ax$ with $A \in \mathbf{R}^{n\times n}$ can be inverted to $x = By$ if and only if A is nonsingular. In this case, the matrix B is unique and is called the inverse of A and is denoted by A^{-1}. It satisfies $AA^{-1} = A^{-1}A = I$.*

Theorem 2.4.1. *Let $A \in \mathbf{R}^{n\times n}$. The following are equivalent:*

(a) *A is nonsingular.*

(b) *$Ax = 0 \implies x = 0$, that is, $\ker A := \{x \mid Ax = 0\} = \{0\}$.*

(c) *$Ax = b$ has a unique solution for each $b \in \mathbf{R}^n$.*

(d) *$Ax = b$ has a unique solution for some $b \in \mathbf{R}^n$.*

Suppose we wish to solve $Ax = b$ (or determine that no solution exists), where $A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^m$, with m and n not necessarily equal. The Jordan exchange gives a simple method for solving this problem under no assumption whatsoever on A.

1. Write the system in the following tableau form:

$$
\begin{array}{cc|c|c|}
 & & x & 1 \\ \cline{3-4}
y & = & A & -b \\ \cline{3-4}
\end{array}
$$

Our aim is to seek x and y related by this tableau *such that* $y = 0$.

2. Pivot as many of the y_i's to the top of the tableau, say y_{I_1}, until no more can be pivoted, in which case we are blocked by a tableau as follows (with row and column reordering):

$$
\begin{array}{cc|cc|c|}
 & & y_{\mathrm{I}_1} & x_{\mathrm{J}_2} & 1 \\ \cline{3-5}
x_{\mathrm{J}_1} & = & B_{\mathrm{I}_1\mathrm{J}_1} & B_{\mathrm{I}_1\mathrm{J}_2} & d_{\mathrm{I}_1} \\
y_{\mathrm{I}_2} & = & B_{\mathrm{I}_2\mathrm{J}_1} & 0 & d_{\mathrm{I}_2} \\ \cline{3-5}
\end{array}
$$

We now ask the question: Is it possible to find x and y related by this tableau such that $y = 0$?

3. The system is solvable if and only if $d_{\mathrm{I}_2} = 0$, since *we require* $y_{\mathrm{I}_1} = 0$ *and* $y_{\mathrm{I}_2} = 0$. When $d_{\mathrm{I}_2} = 0$, we obtain by writing out the relationships in the tableau explicitly that

$$
\begin{aligned}
y_{\mathrm{I}_1} &= 0, \\
y_{\mathrm{I}_2} &= B_{\mathrm{I}_2\mathrm{J}_1} y_{\mathrm{I}_1} = 0, \\
x_{\mathrm{J}_2} & \quad \text{is arbitrary}, \\
x_{\mathrm{J}_1} &= B_{\mathrm{I}_1\mathrm{J}_2} x_{\mathrm{J}_2} + d_{\mathrm{I}_1}.
\end{aligned}
$$

Note that m could be less than or greater than n.

All linear programs can be reduced to the following *standard form*:

$$
\begin{array}{ll}
\min\limits_{x} & z = p'x \\
\text{subject to} & Ax \geq b, \quad x \geq 0,
\end{array}
\tag{3.1}
$$

where $p \in \mathbf{R}^n$, $b \in \mathbf{R}^m$, and $A \in \mathbf{R}^{m\times n}$. To create the initial tableau for the simplex method, we rewrite the problem in the following *canonical form*:

$$
\begin{array}{ll}
\min\limits_{x_{\mathrm{B}}, x_{\mathrm{N}}} & z = p'x_{\mathrm{N}} + 0'x_{\mathrm{B}} \\
\text{subject to} & x_{\mathrm{B}} = Ax_{\mathrm{N}} - b, \quad x_{\mathrm{B}}, x_{\mathrm{N}} \geq 0,
\end{array}
\tag{3.2}
$$

where the index sets N and B are defined initially as $\mathrm{N} = \{1, 2, \ldots, n\}$ and $\mathrm{B} = \{n + 1, \ldots, n + m\}$. The variables $x_{n+1}, \ldots, x_{n+m}$ are introduced to represent the slack in the inequalities $Ax \geq b$ (the difference between left- and right-hand sides of these inequalities) and are called *slack variables*. We shall represent this canonical linear program by the following tableau:

$$
\begin{array}{cc|ccc|c|}
 & & x_1 & \cdots & x_n & 1 \\ \cline{3-6}
x_{n+1} & = & A_{11} & \cdots & A_{1n} & -b_1 \\
\vdots & & \vdots & \ddots & \vdots & \vdots \\
x_{n+m} & = & A_{m1} & \cdots & A_{mn} & -b_m \\ \cline{3-6}
z & = & p_1 & \cdots & p_n & 0 \\ \cline{3-6}
\end{array}
\tag{3.3}
$$

1. *Pricing* (selection of pivot column s): The pivot column is a column s with a negative element in the bottom row. These elements are called *reduced costs*.

2. *Ratio Test* (selection of pivot row r): The pivot row is a row r such that

$$
-h_r/H_{rs} = \min_i \{-h_i/H_{is} \mid H_{is} < 0\}.
$$

3.2 Vertices

The concept of a *vertex* of the feasible region plays an important role in the geometric interpretation of the simplex method. Given the feasible set for (3.1) defined by

$$
\mathrm{S} := \{x \in \mathbf{R}^n \mid Ax \geq b, x \geq 0\}, \tag{3.7}
$$

a point $x \in \mathrm{S}$ is a *vertex* of S if it is not possible to define a line segment lying entirely in S that contains x in its interior. Each vertex can be represented as the intersection of the n hyperplanes defined by $x_i = 0$, for all $i \in \mathrm{N}$, where N is the set of nonbasic variables for some feasible tableau. This representation is apparent from Figure 3.1, where for instance Vertex 2 is at the intersection of the lines defined by $x_1 = 0$ and $x_5 = 0$, where x_1 and x_5 are the two nonbasic variables in the tableau corresponding to this vertex.

Definition 3.2.1. For the feasible region S of (3.1) defined by (3.7), let $x_{n+i} := A_{i\cdot}x - b_i$, $i = 1, 2, \ldots, m$. A *vertex* of S is any point in $(x_1, x_2, \ldots, x_n)' \in \mathrm{S}$ that satisfies

$$
x_{\mathrm{N}} = 0,
$$

where N is any subset of $\{1, 2, \ldots, n + m\}$ containing n elements such that the linear functions defined by x_j, $j \in \mathrm{N}$, are linearly independent.

Theorem 3.2.2. *Suppose that $\bar{x}$ is a vertex of* S *with corresponding index set* N. *Then if we define*

$$
\mathcal{A} := [A \quad -I], \qquad \mathrm{B} := \{1, 2, \ldots, n + m\} \backslash \mathrm{N},
$$

then $\bar{x}$ satisfies the relationships

$$
\mathcal{A}_{\cdot\mathrm{B}} x_{\mathrm{B}} + \mathcal{A}_{\cdot\mathrm{N}} x_{\mathrm{N}} = b, \qquad x_{\mathrm{B}} \geq 0, \qquad x_{\mathrm{N}} = 0, \tag{3.8}
$$

where $\mathcal{A}_{\cdot\mathrm{B}}$ is invertible. Moreover, $\bar{x}$ can be represented by a tableau of the form

$$
\begin{array}{cc|c|c|}
 & & x_{\mathrm{N}} & 1 \\ \cline{3-4}
x_{\mathrm{B}} & = & H & h \\ \cline{3-4}
z & = & c' & \alpha \\ \cline{3-4}
\end{array}
\tag{3.9}
$$

with $h \geq 0$.

total of 2^l active/inactive combinations. The situation hardly improves if we make use of the fact that a solution occurs at one of the vertices of the feasible region, defined as a point at which at least n of the constraints are active. A problem in $\mathbf{R}^n$ with a total of l inequality constraints and bounds (and no equality constraints) may have as many as

$$
\binom{l}{n} = \frac{l!}{(l-n)!n!}
$$

Algorithm 3.1 (Simplex Method).

1. *Construct an initial tableau. If the problem is in standard form (3.1), this process amounts to simply adding slack variables.*

2. *If the tableau is not feasible, apply a Phase I procedure to generate a feasible tableau, if one exists (see Section 3.4). For now we shall assume the origin $x_{\mathrm{N}} = 0$ is feasible.*

3. *Use the pricing rule to determine the pivot column s. If none exists,* **stop**; (a)*: tableau is optimal.*

4. *Use the ratio test to determine the pivot row r. If none exists,* **stop**; (b)*: tableau is unbounded.*

5. *Exchange $x_{\mathrm{B}(r)}$ and $x_{\mathrm{N}(s)}$ using a Jordan exchange on H_{rs}.*

6. *Go to Step 3.*

$$
\begin{array}{lll}
\min\limits_{x_0, x} & z_0 = x_0 & \\
\text{subject to} & x_{n+i} = A_{i\cdot}x - b_i + x_0 & \text{if } b_i > 0, \\
& x_{n+i} = A_{i\cdot}x - b_i & \text{if } b_i \leq 0, \\
& x_0, x \geq 0. &
\end{array}
$$

- We can obtain a feasible point for (3.15) by setting $x_0 = \max\left(\max_{1\leq i\leq m} b_i, 0\right)$ and $x_{\mathrm{N}} = 0$ for $\mathrm{N} = \{1, \ldots, n\}$. The dependent variables x_{B}, where $\mathrm{B} = \{n + 1, n + 2, \ldots, n + m\}$, then take the following initial values:

$$
\begin{aligned}
b_i > 0 &\implies x_{n+i} = A_{i\cdot}x - b_i + x_0 = -b_i + \max_{1\leq j\leq m,\, b_j>0} b_j \geq -b_i + b_i = 0, \\
b_i \leq 0 &\implies x_{n+i} = A_{i\cdot}x - b_i = -b_i \geq 0,
\end{aligned}
$$

so that $x_{\mathrm{B}} \geq 0$, and these components are also feasible.

- If there exists a point $\bar{x}$ that is feasible for the original problem, then the point $(x_0, x) = (0, \bar{x})$ is feasible for the Phase I problem. (It is easy to check this fact by verifying that $x_{n+i} \geq 0$ for $i = 1, 2, \ldots, m$.)
- If (x_0, x) is a solution of the Phase I problem and x_0 is *strictly positive*, then the original problem must be infeasible. This fact follows immediately from the observations above: If the original problem were feasible, it would be possible to find a feasible point for the Phase I problem with objective zero.

We can set up this starting point by forming the initial tableau for (3.15) in the usual way and performing a "special pivot." We select the x_0 column as the pivot column and choose the pivot row to be a row with the most negative entry in the last column of the tableau.

After the special pivot, the tableau contains only nonnegative entries in its last column, and the simplex method can proceed, using the usual rules for pivot column and row selection. Since the objective of (3.15) is bounded below (by zero), it can terminate only at an optimal tableau. Two possibilities then arise.

- The optimal objective z_0 is *strictly positive*. In this case, we conclude that the original problem (3.1) is *infeasible*, and so we terminate without going to Phase II.
- The optimal objective z_0 is *zero*. In this case, x_0 must also be zero, and the remaining components of x are a feasible initial point for the original problem. We can construct a feasible table for the initial problem from the optimal tableau for the Phase I problem as follows. First, if x_0 is still a dependent variable in the tableau (that is, one of the row labels), perform a Jordan exchange to make it an independent variable. (Since $x_0 = 0$, this pivot will be a degenerate pivot, and the values of the other variables will not change.) Next, delete the column labeled by x_0 and the row labeled by z_0 from the tableau. The tableau that remains is feasible for the original problem (3.1), and we can proceed with Phase II, as described in Section 3.3.

Algorithm 3.2 (Phase I).

1. *If $b \leq 0$, then $x_{\mathrm{B}} = -b$, $x_{\mathrm{N}} = 0$ is a feasible point corresponding to the initial tableau and no Phase I is required. Skip to Phase II.*

2. *If $b \not\leq 0$, introduce the artificial variable x_0 (and objective function $z_0 = x_0$) and set up the Phase I problem (3.15) and the corresponding tableau.*

3. *Perform the "special pivot" of the x_0 column with a row corresponding to the most negative entry of the last column to obtain a feasible tableau for Phase I.*

4. *Apply standard simplex pivot rules until an optimal tableau for the Phase I problem is attained. If the optimal value (for z_0) is positive,* **stop***: The original problem has no feasible point. Otherwise, perform an extra pivot (if needed) to move x_0 to the top of the tableau.*

5. *Strike out the column corresponding to x_0 and the row corresponding to z_0 and proceed to Phase II.*

Definition 3.5.1. A feasible tableau is *degenerate* if the last column contains any zero elements. If the elements in the last column are all strictly positive, the tableau is *nondegenerate*. A linear program is said to be *nondegenerate* if all feasible tableaus for that linear program are nondegenerate.

Geometrically, a tableau is nondegenerate if the vertex it defines is at the intersection of exactly n hyperplanes of the form $x_j = 0$; namely, those hyperplanes defined by $j \in \mathrm{N}$. Vertices that lie at the intersection of more than n hyperplanes correspond to degenerate