Module 1

Exercise 1.1 Elementary operators and their properties

An operator S is a transformation of a given signal and is indicated by the notation

$$y[n] = S\{x[n]\}.$$

For instance, the delay operator D is indicated as $D\{x[n]\} = x[n-1]$, and the differentiation operator is indicated as

$$\Delta\{x[n]\} = x[n] - D\{x[n]\} = x[n] - x[n-1].$$

A linear operator is one for which the following holds:

$$\begin{cases} S\{\alpha x[n]\} = \alpha S\{x[n]\} \\ S\{x[n] + y[n]\} = S\{x[n]\} + S\{y[n]\}. \end{cases}$$

(a) Show that the delay operator D is linear.

(b) Show that the differentiation operator Δ is linear.

(c) Show that the squaring operator $S\{x[n]\} = x^2[n]$ is not linear

Solution of Exercise 1.1

(a) $D\{\alpha x[n]\} = \alpha x[n-1] = \alpha D\{x[n]\}$
$D\{x[n] + y[n]\} = x[n-1] + y[n-1] = D\{x[n]\} + D\{y[n]\}$.

(b) Δ is a *linear combination* of the identify operator with the linear operator D, therefore it is also linear.

(c) $S\{\alpha x[n]\} = \alpha^2 x^2[n] = \alpha^2 S\{x[n]\} \neq \alpha S\{x[n]\}$.

Exercise 1.2 Follow-up of Exercise 1.1

In $\mathbb{C}^N$, any linear operator on a vector $\mathbf{x}$ can be expressed as a matrix-vector multiplication for a suitable matrix $\mathbf{A}$. Define the delay operator as the right circular shift of a vector: $D\{\mathbf{x}\} = [x_{N-1} x_0 x_1 \ldots x_{N-2}]^T$.
Assume $N = 4$ for convenience; it is easy to see that

$$D\{\mathbf{x}\} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x} = \mathbf{D}\mathbf{x}.$$

(a) Using the same definition of differentiation operator as in Exercice 1.1, write out the matrix form of the differentiation operator in $\mathbb{C}^4$.

(b) Consider the following matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Which operator do you think it corresponds to?

Solution of Exercise 1.2

(a)

$$\boldsymbol{\Delta} = \mathbf{I} - \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

(b) The matrix realizes an integration operation over a vector in $\mathbb{C}^4$.

Exercise 1.3 Review of complex numbers

(a) Let $s[n] := \frac{1}{2^n} + j\frac{1}{3^n}$. Compute $\sum_{n=1}^{\infty} s[n]$.

(b) Same question with $s[n] := \left(\frac{j}{3}\right)^n$.

(c) Characterize the set of complex numbers satisfying $z^* = z^{-1}$.

(d) Find 3 complex numbers $\{z_0, z_1, z_2\}$ which satisfy $z_i^3 = 1, i = 1, 2, 3$.

(e) What is the following infinite product $\prod_{n=1}^{\infty} e^{j\pi/2^n}$?

Solution of Exercise 1.3

(a) Recall that

$$\sum_{i=0}^{N} z^k = \begin{cases} \frac{1-z^{N+1}}{1-z} & \text{for } z \neq 1 \\ N+1 & \text{for } z = 1\,. \end{cases}$$

Proof for $z \neq 0$ (for $z = 1$ is trivial)

$$s = 1 + z + z^2 + \ldots + z^N\,,$$
$$-zs = -z - z^2 - \ldots - z^N - z^{N+1}\,.$$

Summing the above two equations gives

$$(1-z)s = 1 - z^{N+1} \Rightarrow s = \frac{1-z^{N+1}}{1-z}\,.$$

Similarly

$$\sum_{k=N_1}^{N_2} z^k = z^{N_1}\sum_{k=0}^{N_2-N_1} z^k = \frac{z^{N_1}-z^{N_2+1}}{1-z}\,.$$

We have

$$\begin{aligned}\sum_{n=1}^{N} s[n] &= \sum_{n=1}^{N} 2^{-n} + j\sum_{n=1}^{N} 3^{-n}\\ &= \frac{1}{2}\cdot\frac{1-2^{-N}}{1-2^{-1}} + j\frac{1}{3}\cdot\frac{1-3^{-N}}{1-3^{-1}} = (1-2^{-N}) + j\frac{1}{2}(1-3^{-N})\,.\end{aligned}$$

Now,

$$\lim_{N\to\infty} 2^{-N} = \lim_{N\to\infty} 3^{-N} = 0\,.$$

Therefore,

$$\sum_{n=1}^{\infty} s[n] = 1 + \frac{1}{2}j\,.$$

(b) We can write

$$\sum_{k=1}^{N} s[k] = \frac{j}{3}\cdot\frac{1-(j/3)^N}{1-j/3}\,.$$

Since $|\frac{j}{3}| = \frac{1}{3} < 1$, we have $\lim_{N\to\infty}(j/3)^N = 0$. Therefore,

$$\sum_{k=1}^{\infty} s[k] = \frac{j}{3-j} = \frac{j(3+j)}{10} = -\frac{1}{10} + j\cdot\frac{3}{10}\,.$$

(c) From $z^* = z^{-1}$ with $z \in \mathbb{C}$, we have

$$zz^* = 1, \quad \forall\, z \neq 0.$$

Therefore, $|z|^2 = 1$ and, consequently, $|z| = 1$. It follows that all the z such that $z^* = z^{-1}$ describe the unit circle.

(d) Remark that $e^{j2k\pi} = 1$, for all $k \in \mathbb{Z}$. Therefore, $z_k = e^{j\frac{2k\pi}{3}}$ is such that $z_k^3 = 1$. Now z_k is periodic of period 3, i.e. $z_k = z_{k+3l}$, for all $l \in \mathbb{Z}$. Therefore the (only) three different complex numbers are

$$z_0 = 1, \quad z_1 = e^{j\frac{2\pi}{3}} \quad \text{and} \quad z_2 = e^{j\frac{4\pi}{3}}\,.$$

(e) We have

$$\prod_{n=1}^{N} e^{j\frac{\pi}{2^n}} = e^{j\pi\sum_{n=1}^{N} 2^{-n}} = e^{j\pi\frac{1}{2}\cdot\frac{1-2^{-N}}{1-1/2}}\,.$$

Since $\lim_{N\to\infty} 2^{-N} = 0$,

$$\prod_{n=1}^{\infty} e^{j\frac{\pi}{2^n}} = e^{j\pi} = -1.$$

Exercise 1.4 Review of eigenvalues and eigenvectors

(a) Find the the eigenvalues and eigenvectors of $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

(b) Find the the eigenvalues and eigenvectors of $\mathbf{B} = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$.

(c) Show that $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^T$ where the columns of $\mathbf{V}$ correspond to the eigenvectors of $\mathbf{A}$ and $\mathbf{D}$ is a diagonal matrix whose main diagonal corresponds to the eigenvalues of $\mathbf{A}$.

Solution of Exercise 1.4

(a) The eigenvalues of $\mathbf{A}$ are the roots of the characteristic polynomial.

$$\det\left(\lambda\mathbf{I} - \mathbf{A}\right) = \det\begin{pmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{pmatrix} = \left(\lambda - 1\right)^2 - 4 = \left(\lambda - 3\right)\left(\lambda + 1\right),.$$

Therefore, the eigenvalues are -1 and 3. The eigenvector $\mathbf{x} = (x_1, x_2)^T$ corresponding to the eigenvalue λ is computes as the solution of the equation

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}\,.$$

The eigenvector corresponding to $\lambda = -1$ is given by

$$\mathbf{A}\mathbf{x} = -\mathbf{x}\,.$$

We get $x_2 = -x_1$. Choosing $x_1 = 1$ and $x_2 = -1$, after normalization we obtain

$$\mathbf{x} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\,.$$

Following the same approach, the eigenvector corresponding to $\lambda = 3$ is

$$\mathbf{x} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(b)

$$\det(\lambda \mathbf{I} - \mathbf{B}) = (\lambda - (\alpha + \beta))(\lambda - (\alpha - \beta)) \, .$$

The eigenvalues are $\alpha - \beta$ and $\alpha + \beta$. The eigenvectors are then

$$\mathbf{x} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{for } \lambda = \alpha - \beta$$

$$\mathbf{x} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{for } \lambda = \alpha + \beta.$$

Remark that if $\beta = 0$, there is only an eigenvalue with a corresponding eigenvector

$$\mathbf{x} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{for } \lambda = \alpha.$$

(c)

$$\mathbf{VDV^T} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} = \mathbf{A}$$

.

Module 2

Exercise 2.1 Bases

Let $\{x(k)\}_{k=0,\ldots,N-1}$ be a basis for a subspace S. Prove that any vector $z \in S$ is uniquely represented in this basis. *Hint:* prove by contradiction.

Solution of Exercise 2.1

Suppose by contradiction that the vector $\mathbf{z} \in S$ admits two distinct representations in the basis $\{\mathbf{x}^{(k)}\}_{k=0,\ldots,N-1}$:

$$\begin{aligned}
&\exists \{\alpha_0, \ldots, \alpha_{N-1}\}, \{\beta_0, \ldots, \beta_{N-1}\} \text{ such that:} \\
&(\alpha_0, \ldots, \alpha_{N-1}) \neq (\beta_0, \ldots, \beta_{N-1}) \\
&\mathbf{z} = \Sigma_{k=0}^{N-1} \alpha_k \mathbf{x}^{(k)}, \mathbf{z} = \Sigma_{k=0}^{N-1} \beta_k \mathbf{x}^{(k)}.
\end{aligned}$$

Thus, $\Sigma_{k=0}^{N-1} \alpha_k \mathbf{x}^{(k)} = \Sigma_{k=0}^{N-1} \beta_k \mathbf{x}^{(k)}$ or, equivalently, $\Sigma_{k=0}^{N-1} (\alpha_k - \beta_k) \mathbf{x}^{(k)} = 0$.

Since $\{\mathbf{x}^{(k)}\}_{k=0,\ldots,N-1}$ is a basis, it is a set of independent vectors and, by definition, the above equation admits only the trivial solution $\alpha_k - \beta_k = 0$, $\forall k = 0, \ldots, N-1$. Thus, $\alpha_k = \beta_k$, $\forall k = 0, \ldots, N-1$ which concludes the proof.

Exercise 2.2 Fourier basis

Consider the Fourier basis $\{\mathbf{w}^{(k)}\}_{k=0,\ldots,N-1}$, defined as:

$$\mathbf{w}_n^{(k)} = e^{-j\frac{2\pi}{N}nk}.$$

(a) Prove that it is an orthogonal basis in $\mathbb{C}^N$.

(b) Normalize the vectors in order to get an orthonormal basis.

Solution of Exercise 2.2

(a) Recall one of the most important properties for finite dimensional subspaces: The set of N non-zero orthogonal vectors in an N-dimensional subspace is a basis for the subspace. Therefore, it is sufficient to just prove the orthogonality of the vectors $\{\mathbf{w}^{(k)}\}_{k=0,\ldots,N-1}$. Let us compute the inner product, that is:

$$\begin{aligned}\langle \mathbf{w}^{(k)}, \mathbf{w}^{(h)} \rangle &= \sum_{n=0}^{N-1} \mathbf{w}^{*(k)}[n]\mathbf{w}^{(h)}[n] = \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}nk} e^{-j\frac{2\pi}{N}nh} \\ &= \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}n(h-k)} = \begin{cases} N & \text{if } k = h \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Since the inner product of the vectors is equal to zero, we conclude that

they are orthogonal. However, they do not have a unit norm and therefore are not the orthonormal vectors.

(b) In order to obtain the *orthonormal basis* we normalize the vectors with the factor $1/\sqrt{N}$, having:

$$\begin{aligned}\langle \mathbf{w}^{(k)}_{norm}, \mathbf{w}^{(h)}_{norm} \rangle &= \sum_{n=0}^{N-1} \frac{1}{\sqrt{N}} e^{j\frac{2\pi}{N}nk} \frac{1}{\sqrt{N}} e^{-j\frac{2\pi}{N}nh} \\ &= \frac{1}{N} \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}n(h-k)} = \begin{cases} 1 & \text{if } k = h \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Exercise 2.3 Vector spaces & signals I

(a) Show that the set of all ordered n-tuples $[a_1, a_2, \dots, a_n]$ with the natural definition for the sum:

$$[a_1, a_2, \dots, a_n] + [b_1, b_2, \dots, b_n] = [a_1 + b_1, a_2 + b_2, \dots, a_n + b_n]$$

and the multiplication by a scalar:

$$\alpha[a_1, a_2, \dots, a_n] = [\alpha a_1, \alpha a_2, \dots, \alpha a_n]$$

form a vector space. Give its dimension and find a basis.

(b) Show that the set of signals of the form $y(x) = a\cos(x) + b\sin(x)$ (for arbitrary a, b) with the usual addition and multiplication by a scalar form a vector space. Give its dimension and find a basis.

Solution of Exercise 2.3

(a) It is straight forward to verify that the set of all ordered n-tuples $[a_1, a_2, \dots, a_n]$ with

- $[a_1, a_2, \dots, a_n] + [b_1, b_2, \dots, b_n] = [a_1 + b_1, a_2 + b_2, \dots, a_n + b_n]$

- $\alpha[a_1, a_2, \ldots, a_n] = [\alpha a_1, \alpha a_2, \ldots, \alpha a_n]$

satisfies all the properties of a vector space, which are

(i) Addition is commutative.
(ii) Addition is associative.
(iii) Scalar multiplication is distributive.

- There exists a null vector: $[0, 0, \ldots, 0]$.
- Additive inverse: $[-a_1, -a_2, \ldots, -a_n]$.
- Identity element for scalar multiplication: 1.

The dimension of this vector space is n and a basis is:

$$[1, 0, \ldots, 0], [0, 1, \ldots, 0], ..., [0, 0, \ldots, 1].$$

(b) The set of signals of the form $y(x) = a\cos(x) + b\sin(x)$ (for arbitrary a, b) with the usual addition and multiplication by a scalar form a vector space:

(i) Addition is commutative.
(ii) Addition is associative.
(iii) Scalar multiplication is distributive.

- There exists a null vector: $a, b = 0$.
- Additive inverse: $-y(x) = -a\cos(x) - b\sin(x)$.
- Identity element for scalar multiplication: 1.

The dimension of this vector space is 2 and a possible basis is:

$$y_1(x) = \cos(x), y_2(x) = \sin(x).$$

Exercise 2.4 Vector spaces & signals II

(a) Are the four diagonals of a cube orthogonal?

(b) Express the discrete-time impulse $\delta[n]$ in terms of the discrete-time unit step $u[n]$ and conversely.

(c) Show that any function $f(t)$ can be written as the sum of an odd and an even function, i.e. $f(t) = f_o(t) + f_e(t)$ where $f_o(-t) = -f_o(t)$ and $f_e(-t) = f_e(t)$.

Solution of Exercise 2.4

(a) The eight vertices of the cube can be represented by the following four vectors:

$$v_1 = [0,0,0],\ v_2 = [1,0,0],\ v_3 = [0,1,0],\ v_4 = [1,1,0],$$
$$v_5 = [0,0,1],\ v_6 = [1,0,1],\ v_7 = [0,1,1],\ v_8 = [1,1,1].$$

and the four associated diagonals:

$$\begin{aligned}
d_1 &= \{v_1, v_8\} = v_8 - v_1 = [1,1,1].\\
d_2 &= \{v_2, v_7\} = v_7 - v_2 = [-1,1,1].\\
d_3 &= \{v_3, v_6\} = v_6 - v_3 = [1,-1,1].\\
d_4 &= \{v_4, v_5\} = v_5 - v_4 = [-1,-1,1].
\end{aligned}$$

Two vectors are orthogonal if their inner product is zero. In this case,

$$< \mathbf{d}^i, \mathbf{d}^j >\neq 0 \text{ for all } i, j.$$

Therefore, the four diagonals of a cube are not orthogonal. Remark also that it is not possible to have 4 orthogonal vectors in a space of dimension 3. So this also explains that the 4 diagonals cannot be orthogonal.

(b)

$$\delta[n] = u[n] - u[n-1]$$
$$u[n] = \Sigma_{k=0}^{\infty}\delta[n-k]$$

.

(c) By solving the equations,

$$\begin{cases} f(t) = f_o(t) + f_e(t) \\ f(-t) = f_o(-t) + f_e(-t) = -f_o(t) + f_e(t) \end{cases}$$

We have

$$f_o(t) = \frac{f(t) - f(-t)}{2}$$

and

$$f_e(t) = \frac{f(t) + f(-t)}{2}$$

.

Module 3

Exercise 3.1

Derive a simple expression for the DFT of the time-reversed signal

$$\mathbf{x}_r = [x[N-1]\ x[N-2]\ x[1]\ x[0]]^T$$

in terms of the DFT $\mathbf{X}$ of the signal $\mathbf{x}$. Hint: you may find it useful to remark that $W_N^k = W_N^{-(N-k)}$.

Solution of Exercise 3.1

Recall the DFT (analysis) formula

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}.$$

The DFT of the time-reversed signal can be written as

$$X_r[k] = \sum_{n=0}^{N-1} x_r[n] W_N^{nk} = \sum_{n=0}^{N-1} x[N-1-n] W_N^{nk}$$

By replacing W_N^n with $W_N^{-(N-n)}$, we get

$$\begin{aligned}
X_r[k] &= \sum_{n=0}^{N-1} x[N-1-n] W_N^{-(N-n)k} = \sum_{n=0}^{N-1} x[n] W_N^{-(n+1)k} \\
&= W_N^{-k} \sum_{n=0}^{N-1} x[n] W_N^{-nk} = W_N^{-k} \sum_{n=0}^{N-1} x[n] W_N^{n(N-k)} \\
&= W_N^{-k} X[N-k].
\end{aligned}$$

Exercise 3.2 DFT manipulation

Consider a length-N signal $\mathbf{x} = [x[0]\ x[1]\ ...\ x[N-1]]^T$ and the corresponding vector of DFT coefficients $\mathbf{X} = [X[0]\ X[1]\ ...\ X[N-1]]^T$.

Consider now the length-$2N$ signal obtained by interleaving the values of $\mathbf{x}$ with zeros

$$\mathbf{x}_2 = [x[0]\ \ 0\ \ x[1]\ \ 0\ \ x[2]\ \ 0\ \ ...\ \ x[N-1]\ \ 0]^T$$

.
Express $\mathbf{X}_2$ (the $2N$-point DFT of $\mathbf{x}_2$) in terms of $\mathbf{X}$.

Solution of Exercise 3.2

Knowing that

$$W_{2N}^{2nk} = e^{-j\frac{2\pi}{2N}2nk} = e^{-j\frac{2\pi}{N}nk} = \begin{cases} W_N^{nk}, & 0 \leq k < N \\ W_N^{n(k-N)}, & N \leq k < 2N \end{cases}$$

we get

$$X_2[k] = \begin{cases} \sum_{n=0}^{N-1} x[n]W_N^{nk} & = X[k], & 0 \leq k < N \\ \sum_{n=0}^{N-1} x[n]W_N^{n(k-N)} & = X[(k-N)], & N \leq k < 2N \end{cases}$$

Exercise 3.3

Compute the DFT of the length-4 real signal $\mathbf{x} = [a, b, c, d]^T$. For which values of $a, b, c, d \in \mathbb{R}$ is the DFT real?

Solution of Exercise 3.3

The DFT of the length-4 real signal $\mathbf{x} = [a, b, c, d]^T$ is

$$\begin{aligned} X[k] &= a + be^{-j\frac{2\pi}{4}k} + ce^{-j\frac{2\pi}{4}2k} + de^{-j\frac{2\pi}{4}3k} \\ &= a + b(-j)^k + c(-j)^{2k} + d(-j)^{3k} \\ &= \begin{cases} a+b+c+d, & k=0 \\ a-c-j(b-d), & k=1 \\ a-b+c-d, & k=2 \\ a-c+j(b-d), & k=3 \end{cases} \end{aligned}$$

Therefore, the DFT vector $\mathbf{X}$ is real iff $b = d$.

Exercise 3.4 Structure of DFT formulas

The DFT and IDFT formulas are similar, but not identical. Consider a length-N signal $x[n], N = 0, \ldots, N-1$. What is the length-N signal $y[n]$ obtained as

$$y[n] = \text{DFT}\{\text{DFT}\{x[n]\}\}?$$

In other words, what are the effects of applying twice the DFT transform?

Solution of Exercise 3.4

Let $f[k] = \text{DFT}\{x[n]\}$. We have:

$$\begin{aligned} y[n] &= \sum_{k=0}^{N-1} f[k] e^{-j\frac{2\pi}{N}nk} \\ &= \sum_{k=0}^{N-1} \left\{ \sum_{i=0}^{N-1} x[i] e^{-j\frac{2\pi}{N}ik} \right\} e^{-j\frac{2\pi}{N}nk} \\ &= \sum_{i=0}^{N-1} x[i] \sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}(i+n)k}. \end{aligned}$$

Now,

$$\sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}(i+n)k} = \begin{cases} N & \text{for } (i+n) = 0, N, 2N, 3N, \ldots \\ 0 & \text{otherwise} \end{cases} = N\delta[(i+n) \mod N]$$

so that

$$\begin{aligned} y[n] &= \sum_{i=0}^{N-1} x[i] N\delta[(i+n) \mod N] \\ &= \begin{cases} Nx[0] & \text{for } n = 0 \\ Nx[N-n] & \text{otherwise.} \end{cases} \end{aligned}$$

In other words, if $\mathbf{x} = [1\ 2\ 3\ 4\ 5]^T$ then

$$\text{DFT}\{\text{DFT}\{\mathbf{x}\}\} = 5[1\ 5\ 4\ 3\ 2]^T = [5\ 25\ 20\ 15\ 10]^T$$

.

Exercise 3.5 DFT of the Autocorrelation

Consider a sequence $x[n]$ of finite length L. Let $X(k)$ denote the N point DFT of $x[n]$ and define the circular autocorrelation sequence $r_{xx}[m]$ of $x[n]$ as

$$r_{xx}[m] = \sum_{n=0}^{N-1} x[n] x^*[n - m \text{ mod } N].$$

Express $r_{xx}[m]$ in terms of $X(k)$.

Solution of Exercise 3.5

$$
\begin{aligned}
IDFT\left(X(k)X^*(k)\right)[m] &= \frac{1}{N}\sum_{k=0}^{N-1} X(k)X^*(k)e^{j\frac{2\pi}{N}km} \\
&= \frac{1}{N}\sum_{k=0}^{N-1}[\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn}][\sum_{l=0}^{N-1} x^*(l)e^{j\frac{2\pi}{N}lk}]e^{j\frac{2\pi}{N}km} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} x(n)\sum_{l=0}^{N-1} x^*(l)\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(l+m-n)} \\
&= \sum_{n=0}^{N-1} x(n)x^*(n-m \bmod N) \\
&= r_{xx}(m)
\end{aligned}
$$

Exercise 3.6 DFT in matrix form

(a) Express the DFT and inverse DFT (IDFT) formulas (analysis and synthesis) as a matrix - vector multiplication.

(b) Is the DFT matrix Hermitian?

Solution of Exercise 3.6

(a) Recall the DFT (analysis) formula:

$$
X[k] = \sum_{n=0}^{N-1} x[n]W_N^{nk}.
$$

We can define an $N \times N$ square matrix $\mathbf{W}$ by stacking the conjugates of $\{\mathbf{w}^{(k)}\}_k$,

$$
\mathbf{W} = \begin{bmatrix} \mathbf{w}^{*(0)} \\ \mathbf{w}^{*(1)} \\ \mathbf{w}^{*(2)} \\ \dots \\ \mathbf{w}^{*(N-1)} \end{bmatrix}
$$

And get the analysis formula in matrix - vector multiplication form:

$$
\mathbf{X} = \mathbf{W}\mathbf{x}.
$$

Then, knowing that:

$$
\mathbf{W}\mathbf{W}^H = \mathbf{W}^H\mathbf{W} = N\mathbf{I}
$$

we obtain the synthesis formula in matrix-vector multiplication form:

$$\mathbf{x} \quad = \quad \frac{1}{N}\mathbf{W}^H\mathbf{X}.$$

(b) A matrix $\mathbf{A}$ is Hermitian when

$$\mathbf{A} = \mathbf{A}^H$$

Therefore, for $\mathbf{W}$ to be Hermitian, we would need

$$\mathbf{W}_{nk} = \mathbf{W}^*_{kn}$$

for all $n, k \in \{0, 1, \ldots N-1\}$. This translates to having:

$$e^{-j\frac{2\pi}{N}nk} = e^{j\frac{2\pi}{N}kn}.$$

for all $n, k \in \{0, 1, \ldots N-1\}$, which is generally not the case. Consider, for example, the case when $n = k = 1$. We would need to have:

$$e^{-j\frac{2\pi}{N}} = e^{j\frac{2\pi}{N}}$$

which is equivalent to

$$e^{j\frac{4\pi}{N}} = 1.$$

Clearly, this can only happen when $N = 2$.

Exercise 3.7 DFT of a subsampled signal

Consider an N periodic sequence $x[n]$ and its N point DFS $X(k)$. Define $Y(k) = X(2k)$ to be the $N/2$ point sequence that corresponds to the odd terms of $X(k)$. Relate the $N/2$ point inverse DFS of $Y(k)$, $y[n]$ with $x[n]$.

Solution of Exercise 3.7

Given $Y(k) = X(2k)$, $y[n]$, the $\frac{N}{2}$ point inverse of $Y(k)$ can be expressed as

$$\begin{aligned} y[n] &= \frac{1}{N/2}\sum_{l=0}^{\frac{N}{2}-1} Y(l)e^{j\frac{2\pi}{N/2}nl} \\ &= \frac{2}{N}\sum_{l=0}^{\frac{N}{2}-1} X(2l)e^{j\frac{2\pi}{N}2nl} \end{aligned}$$

Now we note that

$$1 + (-1)^n = \begin{cases} 2 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

$y[n]$ can now be rewritten in terms of $X(k)$ as

$$\begin{aligned} y[n] &= \frac{1}{N} \sum_{k=0}^{N-1} (1 + (-1)^k) X(k) e^{j\frac{2\pi}{N}nk} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}nk} + \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}nk} e^{j\pi k} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}nk} + \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}k(n+\frac{N}{2})} \\ &= x[n] + x[n + \frac{N}{2}] \qquad n = 0...(\frac{N}{2} - 1) \end{aligned}$$

Exercise 3.8 DFT & Matlab

(a) Consider the signal $x(n) = \cos(2\pi f_0 n)$. Use the 'fft' function to compute and draw the DFT of the signal in $N = 128$ points, for: $f_0 = 21/128$ and $f_0 = 21/127$.

Explain the differences that we can see in these two signal spectra.

(b) Repeat the process, this time using the 'dftmtrx' function and check that the results are the same. What is the preferred option between the two?

Solution of Exercise 3.8

(a) The spectrum of the signal $x[n]$ is obtained using the Matlab commands given below.

```
N=128;fo1=21/128;fo2=21/127;
n=0:N-1;
x1=cos(2*pi*fo1*n);x2=cos(2*pi*fo2*n);
X1=fft(x1);X2=fft(x2);
subplot(223),stem(n-N/2,fftshift(abs(X1)))
subplot(224),stem(n-N/2,fftshift(abs(X2)))
```

Note that we would expect to see just one sample at the frequency of the signal.

In Figure 3.1, on the left, our prediction is satisfied and the 21st DFT coefficient represents the exact signal frequency. In Figure 3.1 on the

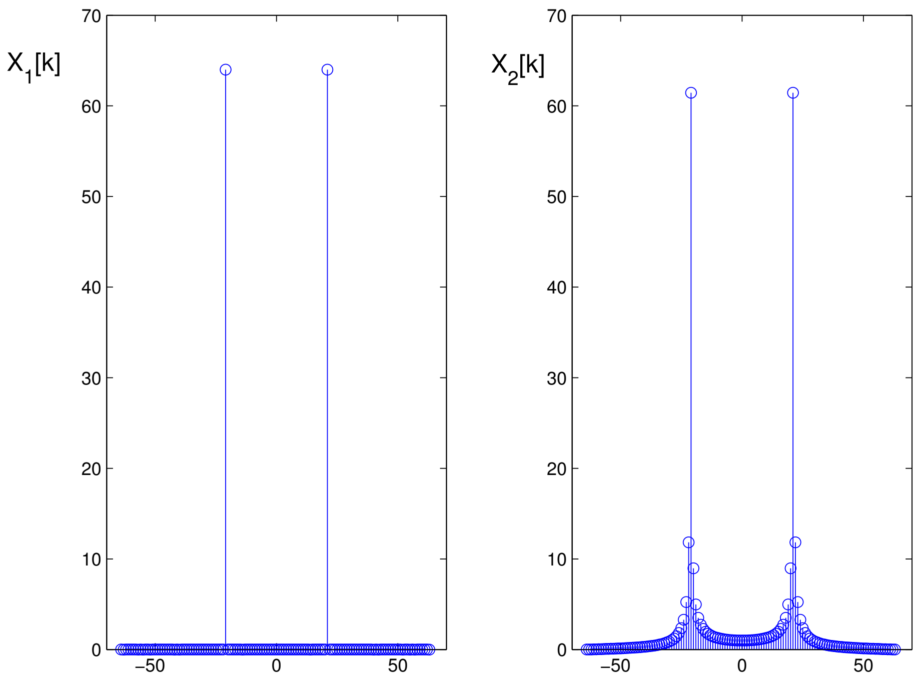


Figure 3.1: Spectrum of signal x_1 and x_2

right, the frequency of the signal $f_0 = 21/127$ does not coincide with any DFT frequency component. The signal energy is spread over all the DFT components. This is called frequency leakage.

(b) We can achieve the same results using the *dftmtx* function instead, as illustrated below.

```
W=dftmtx(N);
X3=W*x1;X4=W*x2;
norm(X1-X3)
norm(X2-X4)
subplot(223),stem(n-N/2,fftshift(abs(X3)))
subplot(224),stem(n-N/2,fftshift(abs(X4)))
```

In practice, however, the discrete Fourier transform is computed more efficiently and uses less memory with an FFT algorithm.

Exercise 3.9 DTFT, DFT & Matlab

Consider the following infinite non-periodic discrete time signal $x[n] = \begin{cases} 0 & n < 0, \\ 1 & 0 \leq n < a, \\ 0 & n \geq a. \end{cases}$

(a) Compute its DTFT $X(e^{jw})$.

(b) We want to visualize the magnitude of $X(e^{jw})$ using Matlab. However, Matlab can not handle continuous sequences as $X(e^{jw})$, thus we need to consider only a finite number of points. Using Matlab, plot 10000 points of one period of$|X(e^{jw})|$ (from 0 to 2π) for $a = 20$.

(c) The DTFT is mostly a theoretical analysis tool, but in many cases, we will compute the DFT. Recall that in Matlab we use the Fast Fourier Transform (FFT), an efficient algorithm to compute the DFT. Generate a finite sequence $x_1[n]$ of length $N = 30$ such that $x_1[n] = x[n]$ for $n = 1, \ldots, N$. Compute its DFT and plot itsmagnitude. Compare it with the plot obtained in (b).

(d) Repeat now for different values of $N = 50, 100, 1000$. What can you conclude?

Solution of Exercise 3.9

(a) First, we note that $x[n]$ can also be expressed as: $x[n] = u[n] - u[n-a]$. Using the DTFT shift property,

$$X(e^{jw}) = \frac{1}{1-e^{-jw}} + \frac{1}{2}\tilde{\delta}(w) - \frac{e^{-jaw}}{1-e^{-jw}} - \frac{e^{-jaw}}{2}\tilde{\delta}(w).$$

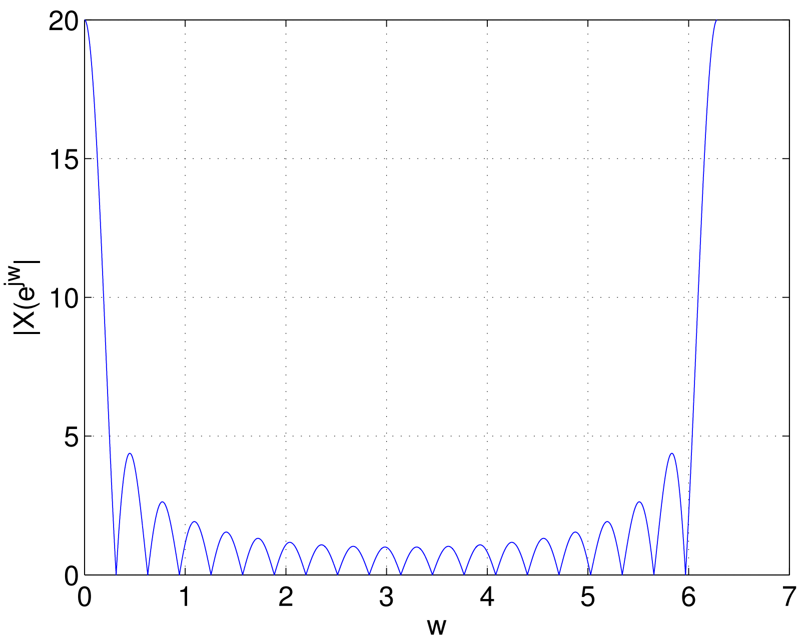


Figure 3.2: DTFT of $x[n]$.

Note that $e^{-jaw}\tilde{\delta}(w) = \tilde{\delta}(w)$. Therefore,

$$X(e^{jw}) = \frac{1 - e^{-jaw}}{1 - e^{-jw}}.$$

(b) We visualize the magnitude of $X(e^{jw})$ using Matlab with the following code:

```
n=1:10000;
w=(n.*2*pi/max(n));
X=((1-exp(-j.*w.*20))./(1-exp(-j.*w)));
plot(w,abs(X));
```

The result is represented in Figure .

(c) To plot the DFT for $N = 30$, we use the following Matlab code:

```
N=30;
x1=[ones(1,20),zeros(1,N-20)];
X1=fft(x1);
plot(abs(X1));
```

The result is depicted in Figure 3.3

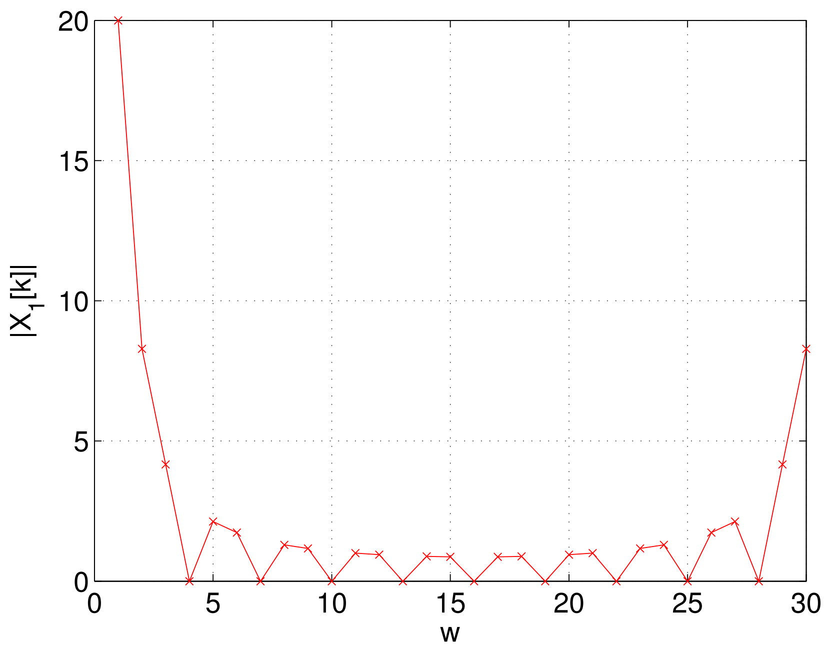


Figure 3.3: DFT of $x[n]$ for $N = 30$.

(d) The DFT of $x[n]$ for various values of N is represented in Figure 3.4.

As we increase N, the DFT becomes closer and closer to the DTFT of $x[n]$. We know that the DFT and the DFS are formally identical, and as N grows, the DFS converges to the DTFT. We can use Matlab to approximate the DTFT of any signal.

Exercise 3.10 Plancherel-Parseval Equality

Let $x[n]$ and $y[n]$ be two complex valued sequences and $X(e^{jw})$ and $Y(e^{jw})$ their corresponding DTFTs.

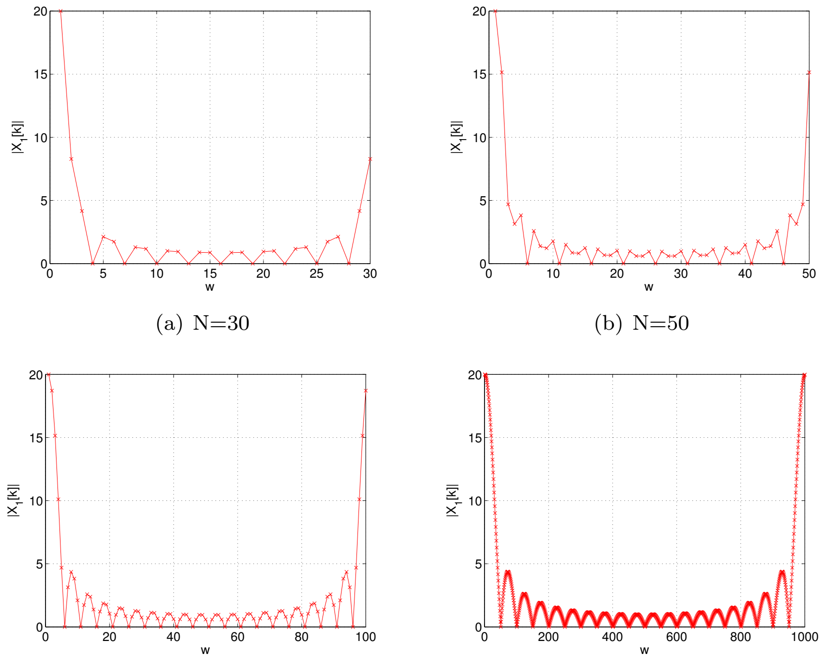
(a) Show that

$$\langle x[n], y[n] \rangle = \frac{1}{2\pi} \langle X(e^{jw}), Y(e^{jw}) \rangle,$$

where we use the inner products for $l_2(\mathbb{Z})$ and $L_2([-\pi, \pi])$ respectively.

(b) What is the physical meaning of the above formula when $x[n] = y[n]$?

Solution of Exercise 3.10



(a) N=30

(b) N=50

(c) N=100

(d) N=1000

Figure 3.4: DFT of $x[n]$ for various values of N

(a) The inner product in $l_2(\mathbb{Z})$ is defined as

$$\langle x[n], y[n] \rangle = \Sigma_n x^*[n] y[n],$$

and in $L_2([-\pi, \pi])$ as

$$\langle X(e^{jw}), Y(e^{jw}) \rangle = \int_{-\pi}^{\pi} X^*(e^{jw}) Y(e^{jw}) dw.$$

Thus,

$$\begin{aligned}
\frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(e^{jw}) Y(e^{jw}) dw &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\Sigma_n x[n] e^{-jwn} \right)^* \Sigma_m y[m] e^{-jwm} dw \\
&\overset{(1)}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} \Sigma_n x^*[n] e^{jwn} \Sigma_m y[m] e^{-jwm} dw \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \Sigma_n \Sigma_m x^*[n] y[m] e^{jw(n-m)} dw \\
&\overset{(2)}{=} \frac{1}{2\pi} \Sigma_n \Sigma_m x^*[n] y[m] \int_{-\pi}^{\pi} e^{jw(n-m)} dw \\
&\overset{(3)}{=} \Sigma_n x^*[n] y[n],
\end{aligned}$$

where (1) follows from the properties of the complex conjugate, (2) from swapping the integral and the sums and (3) from the fact that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jw(n-m)}dw = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}.$$

(b) $\langle x[n], x[n]\rangle$ corresponds to the energy of the signal in the time domain; $\langle X(e^{jw}), X(e^{jw})\rangle$ corresponds to the energy of the signal in the frequency domain. The Plancherel-Parseval equality illustrates an energy conservation property from the time domain to the frequency domain. This property is known as the *Parseval's theorem*.

Exercise 3.11 DTFT properties

Derive the time-reverse and time-shift properties of the DTFT.

Solution of Exercise 3.11

Define the DTFT transform as

$$x[n] \leftrightarrow DTFT X(e^{j\omega}).$$

We have:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} x[-n]e^{-j\omega n} &= \sum_{m=-\infty}^{\infty} x[m]e^{j\omega m} \\
&= \sum_{m=-\infty}^{\infty} x[m]e^{-j(-\omega)m} \\
&= X(e^{-j\omega})
\end{aligned}$$

with the change of variable $m = -n$. Hence, we obtain that the DTFT of the time-reversed sequence $x[-n]$ is:

$$x[-n] \leftrightarrow DTFT X(e^{-j\omega})$$

Similarly:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} x[n-n_0]e^{-j\omega n} &= \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega(m+n_0)} \\
&= e^{-j\omega n_0}\sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m} \\
&= e^{-j\omega n_0}X(e^{j\omega})
\end{aligned}$$

with the change of variable $m = n - n_0$. We, therefore, obtain the DTFT of the time-shifted sequence $x[n - n_0]$:

$$x[n - n_0] \leftrightarrow DTFT e^{-j\omega n_0} X(e^{j\omega})$$

Exercise 3.12 DTFT and DFS

Consider the Discrete Time Fourier Transform (DTFT) $X(e^{j\omega})$ of a sequence $x(n) \in \ell^2(Z)$, of finite support, L.

(a) Show that by observing N uniform samples of $X(e^{j\omega})$, one obtains the DFS $X(k)$ of the periodically extended version of the same signal, for all values of N.

(b) Show that by observing $N \geq L$ uniform samples of $X(e^{j\omega})$, one obtains the DFS of the periodically extended version of the same signal, $X(k)$.

(c) Show that the DFT of $x[n]$ is one period of an uniformly sampled version of $X(e^{j\omega})$ at an appropriate sampling rate.

Solution of Exercise 3.12

(a) Using the definition of DTFT, $X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-i\omega n}$.

Since $X(\omega)$ is periodic in 2π, the discretization with N samples is obtained with $\omega = 2\pi k/N$

$$\begin{aligned} X\left(\frac{2\pi}{N}k\right) &= \sum_{n=-\infty}^{\infty} x(n)e^{-i2\pi kn/N} \quad k = 0, 1..N-1 \\ &= \sum_{r=-\infty}^{\infty} \sum_{n=rN}^{(r+1)N-1} x(n)e^{-i2\pi kn/N} \\ &= \sum_{n=0}^{N-1} \sum_{r=-\infty}^{\infty} x(n+rN)e^{-i2\pi kn/N} \end{aligned}$$

(b) Let,

$$x_p(n) = \sum_{r=-\infty}^{\infty} x(n+rN)$$

be the periodic extension of the finite length sequence of interest, $x(n)$. Then,

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} x_p(n)e^{-i2\pi kn/N}$$

is the DFS of the periodic sequence. Applying the inverse transform, we get the signal $x_p(n)$. The sequence of interest, $x(n)$ corresponds to one period of the periodic signal, $x_p(n)$ when $N \geq L$

$$x(n) = \begin{cases} x_p(n) & \text{if } 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

When $N < L$,

$$x_p(n) = \sum_{r=-\infty}^{\infty} x(n + rN)$$

$x_p(n)$ is no longer the exact periodic repition of $x(n)$. It is now the sum of *N-shifted* versions of the signal $x(n)$ and since $N < L$, each N-tap signal, is corrupted by a shifted version of itself.

(c) Consider the previous result for $N > L$,

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} x_p(n)e^{-i2\pi kn/N},$$

where we note that $x_p(n) = x(n)$, for $n \in [0, N-1]$. Then, (**??**) can be rewritten as

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} x(n)e^{-i2\pi kn/N}.$$

The right hand side of the above equation is the DFT sum, while the left hand side is one period of the uniformly sampled version of the DTFT, sampled such that $N > L$.

Exercise 3.13 Sinusoidal Modulation

Given an ideal low pass filter

$$H_{lp}(e^{j\omega}) = \begin{cases} 1 & |\omega| \leq \omega_b/2, \\ 0 & \text{elsewhere.} \end{cases} \quad ,$$

relate the impulse response of a bandpass filter with center frequency ω_0 and passband ω_b:

$$H_{bp}(e^{j\omega}) = \begin{cases} 1 & \omega_0 - \omega_b/2 \leq \omega \leq \omega_0 + \omega_b/2, \\ 1 & -\omega_0 - \omega_b/2 \leq \omega \leq -\omega_0 + \omega_b/2, \\ 0 & \text{elsewhere.} \end{cases}$$

with the lowpass filter.

Solution of Exercise 3.13

Consider a lowpass filter $h_{lp}[n]$ with bandwidth ω_b. If we consider the sequence

$$h[n] = 2\cos(\omega_0 n) h_{lp}[n]$$

the modulation theorem tells us that its Fourier transform is

$$H(e^{j\omega}) = H_{lp}(e^{j(\omega-\omega_0)}) + H_{lp}(e^{j(\omega+\omega_0)}) = H_{bp}(e^{j\omega})$$

Therefore the impulse response of the bandpass filter is

$$h_{bp}[n] = 2\cos(\omega_0 n) h_{lp}[n] = 2\cos(\omega_0 n)\, \frac{\omega_b}{2\pi}\, \mathrm{sinc}\left(\frac{\omega_b}{2\pi} n\right)$$

Module 4

Exercise 4.1 LTI systems

Consider the transformation $\mathcal{H}\{x[n]\} = nx[n]$. Does $\mathcal{H}$ define an LTI system?

Solution of Exercise 4.1

The system is not time-invariant. To see this consider the following signals

$$\begin{aligned} x[n] &= \delta[n] \\ y[n] &= \delta[n-1] \end{aligned}$$

We have $\mathcal{H}\{x[n]\} = w[n] = 0$ and, clearly, it is $y[n] = x[n-1]$ However,

$$\mathcal{H}\{y[n]\} = \delta[n-1] \neq w[n-1] = 0.$$

Exercise 4.2 Convolution

Let $x[n]$ be a discrete-time sequence defined as

$$x[n] = \begin{cases} M-n & 0 \leq n \leq M, \\ M+n & -M \leq n \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

for some odd integer M.

(a) Show that $x[n]$ can be expressed as the convolution of two discrete-time sequences $x_1[n]$ and $x_2[n]$.

(b) Using the results found in (a), compute the DTFT of $x[n]$.

Solution of Exercise 4.2

(a) $x[n]$ can be written as the convolution of $x_1[n]$ and $x_2[n]$ defined as

$$\begin{aligned} x_1[n] = x_2[n] &= \begin{cases} 1 & -(M-1)/2 \leq n \leq (M-1)/2 \\ 0 & \text{otherwise.} \end{cases} \\ &= u[n+(M-1)/2] - u[n-(M+1)/2]. \end{aligned}$$

Then,

$$\begin{aligned}
x_1[n] * x_2[n] &= \Sigma_k x_1[k] x_2[n-k] \\
&\overset{(1)}{=} \Sigma_k x_1[k] x_1[k-n] \\
&\overset{(2)}{=} x[n]
\end{aligned}$$

(1) follows from the fact that $x_1[n] = x_2[n]$ and the symmetry of $x_1[n]$. (2) follows by noticing that the sum corresponds to the size of the overlapping area between $x_1[k]$ and its n-shifted version $x_1[k-n]$.

(b)

$$\begin{aligned}
X_1(e^{j\omega}) &\overset{(1)}{=} \left(\frac{1}{1-e^{-j\omega}} + \frac{1}{2}\tilde{\delta}(\omega)\right)\left(e^{j\omega(M-1)/2} - e^{-j\omega(M+1)/2}\right) \\
&\overset{(2)}{=} \frac{e^{j\omega(M-1)/2} - e^{-j\omega(M+1)/2}}{1-e^{-j\omega}} = \frac{e^{-j\omega/2}(e^{j\omega M/2} - e^{-j\omega M/2})}{e^{-j\omega/2}(e^{j\omega/2} - e^{-j\omega/2})} \\
&= \frac{\sin(\omega M/2)}{\sin(\omega/2)}
\end{aligned}$$

(1) follows from the DTFT of $u[n]$ (2) follows from $e^{jw(M-1)/2}\tilde{\delta}(w) = e^{-jw(M+1)/2}\tilde{\delta}(w) = \tilde{\delta}(w)$. Now, using the convolution theorem, we can write

$$\begin{aligned}
X(e^{jw}) &= X_1(e^{jw})X_2(e^{jw}) \\
&= X_1(e^{jw})X_1(e^{jw}) \\
&= \left(\frac{sin(\omega M/2)}{sin(\omega/2)}\right)^2.
\end{aligned}$$

Exercise 4.3 Impulse response

(a) The impulse response of an LTI system is shown in Figure 4.1. Determine and carefully sketch the response of this system to the input $x[n] = u[n-4]$.

(b) Calculate the impulse response of the system of Figure 4.2 given the impulse responses of the separate blocks

- $h_1[n] = 3(-1)^n(\frac{1}{4})^n u[n-2]$
- $h_2[n] = h_3[n] = u[n+2]$
- $h_4[n] = \delta[n-1]$

(c) Determine the latter system BIBO stability and causality.

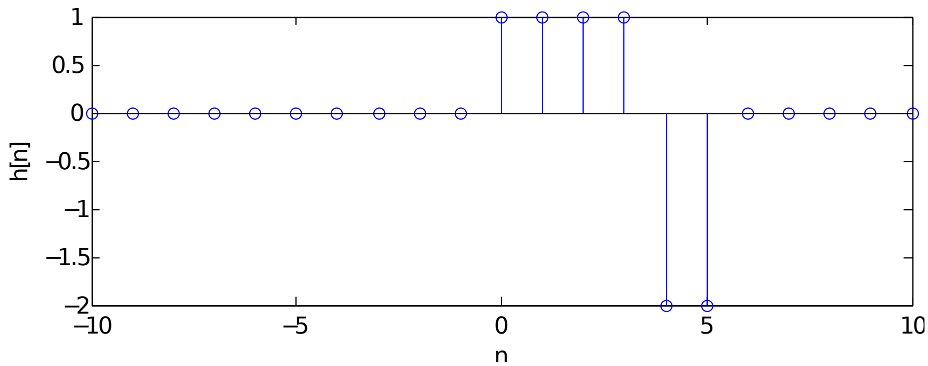


Figure 4.1: Impulse response

Solution of Exercise 4.3

(a) $x[n] * h[n]$ is illustrated in Figure 4.3.

(b) We have

$$\begin{aligned} h[n] &= h_1[n] * (h_2[n] - h_3[n] * h_4[n]) \\ &= h_1[n] * (u[n+2] - u[n+2] * \delta[n-1]) \\ &= h_1[n] * (u[n+2] - u[n+1]) \\ &= h_1[n] * \delta[n+2] \\ &= h_1[n+2] \\ &= 3(-1)^n (\frac{1}{4})^{n+2} u[n]. \end{aligned}$$

(c) A discrete system is BIBO stable if the impulse response is absolutely summable. We have

$$\sum_{n=-\infty}^{\infty} |h[n]| = 3\frac{1}{1-1/4} - 3\left(1+\frac{1}{4}\right) = \frac{1}{4},$$

which means the system is BIBO stable. The system is causal because $h[n] = 0$ for $n < 0$.

Exercise 4.4 System properties

Let $x[n]$ be a signal. Consider the following systems with output $y[n]$. Determine if such systems are: linear, time-invariant, stable (BIBO) and causal or anti-causal. Characterize the systems by their impulse response. Hint: Since the system is causal and satisfies initial-rest conditions, we can recursively find the response to any input as, for instance, $\delta[n]$.

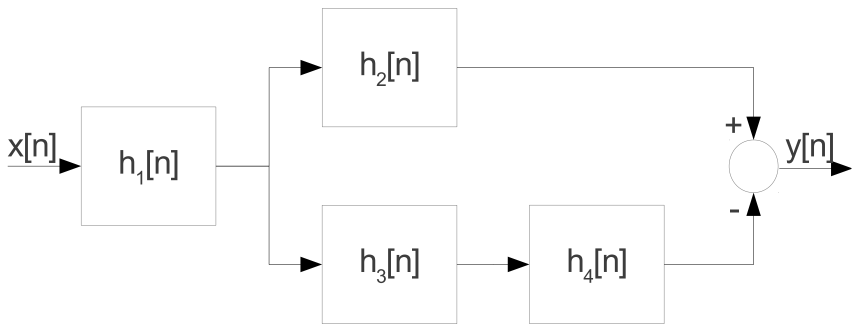


Figure 4.2: Impulse response block schema

(a) $y[n] = x[-n]$,

(b) $y[n] = e^{-j\omega n}x[n]$,

(c) $y[n] = \sum_{k=n-n_0}^{n+n_0} x[k]$,

(d) $y[n] = ny[n-1] + x[n]$, such that if $x[n] = 0$ for $n < n_0$, then $y[n] = 0$ for $n < n_0$.

Solution of Exercise 4.4

(a)
- $\mathcal{H}$ is linear:

$$\mathcal{H}\{ax_1[n] + bx_2[n]\} = ax_1[-n] + bx_2[-n] = a\mathcal{H}\{x_1[n]\} + b\mathcal{H}\{x_2[n]\}.$$

- $\mathcal{H}$ is NOT time invariant:

$$\mathcal{H}\{x[n-n_0]\} = x[-n-n_0] \neq y[n-n_0].$$

- $\mathcal{H}$ is BIBO stable:

$$|x[n]| \leq M \Rightarrow |\mathcal{H}\{x[n]\}| \leq M.$$

- $\mathcal{H}$ is not causal.
- $\mathcal{H}$ is not LTI, therefore $h[n]$ does not characterize the system.

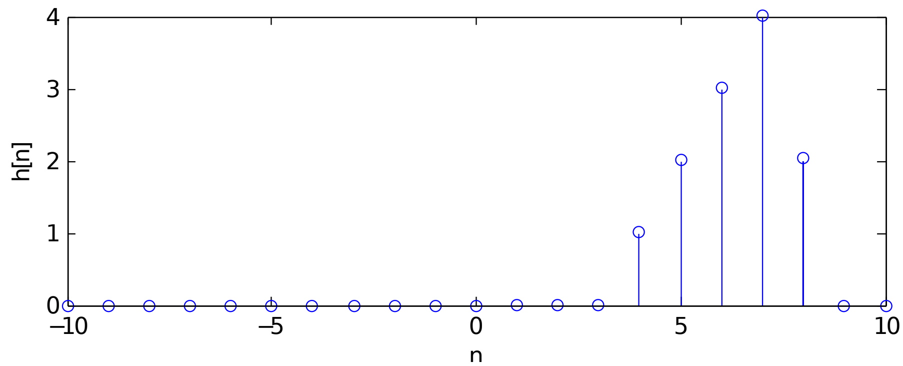


Figure 4.3: The system solution to the input signal $x[n]$

(b)
- $\mathcal{H}$ is linear:

$$\mathcal{H}\{ax_1[n]+bx_2[n]\} = e^{-j\omega n}(ax_1[n]+bx_2[n]) = a\mathcal{H}\{x_1[n]\}+b\mathcal{H}\{x_2[n]\}.$$

- $\mathcal{H}$ is Not time invariant (only for $\omega = 0$):

$$\mathcal{H}\{x[n-n_0]\} = e^{-j\omega n}x[n-n_0] = e^{j\omega n_0}y[n-n_0].$$

- $\mathcal{H}$ is BIBO stable:

$$|x[n]| \leq M \Rightarrow |\mathcal{H}\{x[n]\}| = |x[n]| \leq M.$$

- $\mathcal{H}$ is causal.
- $\mathcal{H}$ is not LTI, therefore $h[n]$ does not characterize the system.

(c)
- $\mathcal{H}$ is linear:

$$\mathcal{H}\{ax_1[n]+bx_2[n]\} = \sum_{k=n-n_0}^{n+n_0}(ax_1[k]+bx_2[k]) = a\mathcal{H}\{x_1[n]\}+b\mathcal{H}\{x_2[n]\}.$$

- $\mathcal{H}$ is time invariant:

$$\mathcal{H}\{x[n-n_0]\} = \sum_{k=n-n_0}^{n+n_0}x[k-n_0] = \sum_{k=n-2n_0}^{n}x[k] = y[n-n_0].$$

(d)
- $\mathcal{H}$ is BIBO stable:

$$|x[n]| \leq M \Rightarrow \mathcal{H}\{x[n]\} \leq |2n_0+1|M.$$

- $\mathcal{H}$ is not causal.

- The system impulse response is

$$h[n] = \begin{cases} 1 & \text{if } |n| \leq |n_0|, \\ 0 & \text{otherwise.} \end{cases}$$

(e)
- Note that all inputs $x[n]$ can be expressed as a linear combination of delayed impulses:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

Therefore, to show that $\mathcal{H}$ is linear or time invariant, we can restrict the input to delayed impulses.

- $\mathcal{H}$ is linear: For $x[n] = \delta[n]$, we obtain $y[n]$ by recursion:

$$h[n] = y[n] = n!u[n].$$

Also, if $x[n] = a\delta[n] + b\delta[n]$:

$$y[n] = (a+b)n!u[n].$$

- $\mathcal{H}$ is time invariant: it is easy to check that $\mathcal{H}\{\delta[n-1]\} = h[n-1]$.
- The system is non stable.
- $\mathcal{H}$ is causal.
- $\mathcal{H}$ is LTI, therefore $h[n]$ characterizes the system.

Exercise 4.5 Zero phase filtering

Let $\mathcal{R}$ be the time reversal operator for sequences.

$$\mathcal{R}\{x[n]\} = x[-n]$$

Let $\mathcal{H}$ be a linear time invariant system. What can you say about the following system?

$$y[n] = \mathcal{R}\{\mathcal{H}\{\mathcal{R}\{\mathcal{H}\{x[n]\}\}\}\}$$

Solution of Exercise 4.5

Let $X(e^{j\omega})$ be the DTFT of the sequence $x(n)$ and $H(e^{j\omega})$ be the frequency response of the LTI operator $\mathcal{H}$. By the time reversal property of DTFT, the DTFT of $x(-n)$ is $X(e^{-j\omega})$. Since $x(n)$ is real, the DTFT of $x(-n)$ is $X^*(e^{j\omega})$

$$\begin{aligned}
\mathcal{H}\{x(n)\} &= H(e^{j\omega})X(e^{j\omega}) \\
\mathcal{R}\{\mathcal{H}\{x(n)\}\} &= H^*(e^{j\omega})X^*(e^{j\omega}) \\
\mathcal{H}\{\mathcal{R}\{\mathcal{H}\{x(n)\}\}\} &= H(e^{j\omega})H^*(e^{j\omega})X^*(e^{j\omega}) \\
\mathcal{R}\{\mathcal{H}\{\mathcal{R}\{\mathcal{H}\{x(n)\}\}\}\} &= H^*(e^{j\omega})H(e^{j\omega})X(e^{j\omega}) \\
&= |H(e^{j\omega})|^2 X(e^{j\omega})
\end{aligned}$$

The above system is a linear, time invariant system with zero phase delay.

Exercise 4.6 Modulation theorem

Given an ideal low pass filter $H_{lp}(\omega)$ with cut-off frequency ω_c and a sinusoid generator at frequency $\frac{\pi}{2}$, design an ideal high-pass filter $H_{hp}(\omega)$ with cut-off frequency $\pi - \omega_c$. You are required to write H_{hp} with respect to H_{lp} and a sinusoid.

Solution of Exercise 4.6

By using the modulation theorem we can see that a high-pass filter with cut-off frequency at $\pi - \omega_c$ can be written as

$$h_{hp}(n) = cos(\pi n)h_{lp}(n)$$

Since we are only given a sinusoid generator of frequency $\frac{\pi}{2}$,

$$h_{hp}(n) = (2(cos(\frac{\pi}{2}n))^2 - 1)h_{lp}(n)$$

Exercise 4.7 Convolution theorem

Let

$$w_R[n] = \begin{cases} 1 & 0 \le n \le M, \\ 0 & \text{elsewhere.} \end{cases}$$

and

$$w_T[n] = \begin{cases} n & 0 \le n \le M/2, \\ M - n & M/2 < n \le M, \\ 0 & \text{elsewhere.} \end{cases} .$$

Express the Fourier transform W_T of $w_T[n]$ with respect to the Fourier transform W_R of $w_R[n]$.

Solution of Exercise 4.7

Let $\hat{w}_T[n]$ defined as

$$\hat{w}_T[n] = \begin{cases} n & 0 \le n \le M, \\ 2M - n & M < n \le 2M, \\ 0 & \text{elsewhere.} \end{cases}$$

We observe that

$$\hat{w}_T[n-1] = w_R[n] \star w_R[n]$$

, which becomes in the Fourier domain

$$\hat{W}_T(e^{j\omega}) = e^{j\omega} W_R^2(e^{j\omega})$$

using the convolution property. We also have that $w_T[n] = \frac{1}{2}\hat{w}_T[2n]$, which becomes in the Fourier domain, (see for example the proof in Chapter 11 of the book, the formula for downsampling by 2)

$$W_T(e^{j\omega}) = \frac{1}{4}\hat{W}_T(e^{j\omega/2}) + \frac{1}{4}\hat{W}_T(e^{j\omega/2-\pi}).$$

Combining these two relations, we obtain

$$W_T(e^{j\omega}) = \frac{1}{4}e^{j\omega/2}\left(W_R^2(e^{j\omega/2}) + W_R^2(e^{j\omega/2-\pi})\right).$$

Exercise 4.8 Infinite impulse response

Let $H(\omega)$ denote the system response of an ideal low-pass filter. How should the input sequence $x[n]$ be, so that the output sequence is finite length and non zero?

Solution of Exercise 4.8

The impulse response of an ideal low pass filter is,

$$h(n) = \frac{\omega_b}{2\pi}\,\text{sinc}\left(\frac{\omega_b}{2\pi}n\right)$$

which is of infinite length. Since any output sequence of the ideal low-pass filter can be written as the sum of scaled and shifted versions of this signal, any output sequence is of infinite length, as long as it is non-zero.

Exercise 4.9 Pole-zero plot and stability

Consider a causal LTI system with the following transfer function

$$H(z) = \frac{3 + 4.5z^{-1}}{1 + 1.5z^{-1}} - \frac{2}{1 - 0.5z^{-1}}$$

Sketch the pole-zero plot of the transfer function and specify its region of convergence. Is the system stable?

Solution of Exercise 4.9

We have to factorize the filter expression to make explicit the numerator and denominator roots

$$H(z) = \frac{3 + 4.5z^{-1}}{1 + 1.5z^{-1}} - \frac{2}{1 - 0.5z^{-1}} = \frac{(z - 1.5)(z + 1.5)}{(z - 0.5)(z + 1.5)}$$

In this way we see that the zeros of this system are in $z_{01} = 1.5$ and $z_{02} = -1.5$ and the poles in $z_{p1} = 0.5$ and $z_{p2} = -1.5$. Now we can draw the pole-zero plot represented in Figure 4.4

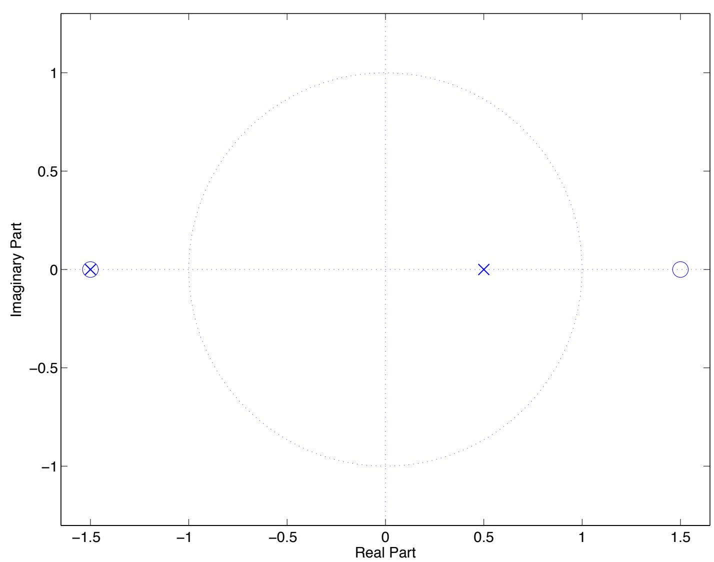


Figure 4.4: Pole-zero plot

In theory, the zero $z_{01} = -1.5$ cancels out the pole $z_{p2} = -1.5$. As our system is causal, the ROC extends outward from the remaining outermost pole $z_{p1} = 0.5$. As the unit circle is included in this ROC, the system is stable. However, in practice, the stabilization of a system with this technique is a risky enterprise since the exact cancellation of the pole outside the unit circle will be extremely sensitive to numerical precision issues. If the coefficients of the filter (or the internal accumulators) are subject to truncation or rounding, the implicit position of the zero may drift ever so slightly from the implicit position of the pole and the system will no longer be stable.

Exercise 4.10 Stability

Consider a causal discrete system represented by the following difference equation

$$y[n] - 3.25y[n-1] + 0.75y[n-2] = x[n-1] + 3x[n-2].$$

(a) Compute the transfer function and check the stability of this system both analytically and graphically.

(b) If the input signal is $x[n] = \delta[n] - 3\delta[n-1]$, compute the z-transform of the output signal and discuss the stability.

(c) Take an arbitrary input signal that does not cancel the unstable pole of the transfer function. Compute the z-transform of the output signal and discuss the stability.

Solution of Exercise 4.10

(a) The transfer function of the system is given by

$$Y(z)(1 - 3.25z^{-1} + 0.75z^{-2}) = X(z)(z^{-1} + 3z^{-2}),$$

and we obtain

$$\begin{aligned} H(z) = \frac{Y(z)}{X(z)} &= \frac{z^{-1} + 3z^{-2}}{1 - 3.25z^{-1} + 0.75z^{-2}} = \\ &= \frac{z^{-1}(1 + 3z^{-1})}{(1 - 0.25z^{-1})(1 - 3z^{-1})} = \frac{z+3}{(z-0.25)(z-3)}. \end{aligned}$$

Since the system is causal, the convergence region is $|z| > 3$. We can see that there is the pole $z = 3$ that is out of the unit circle and therefore the system is unstable. See Figure 4.5.

(b) Z-transform of the output signal is

$$\begin{aligned} Y(z) &= H(z)X(z) = \frac{z^{-1}(1 + 3z^{-1})}{(1 - 0.25z^{-1})(1 - 3z^{-1})}(1 - 3z^{-1}) \\ &= \frac{z^{-1} + 3z^{-2}}{1 - 0.25z^{-1}}. \end{aligned}$$

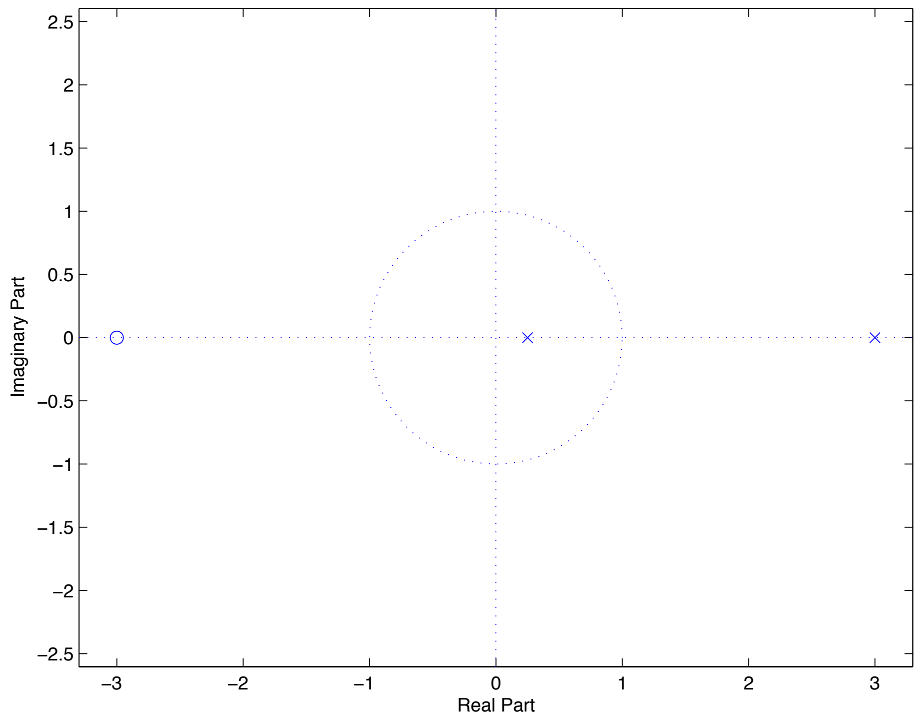


Figure 4.5: Pole-zero plot

From $Y(z)$ we can see that the unstable pole $z = 3$ is canceled and only the pole $z = 0.25$ of $Y(z)$ is left. Since the system is causal, even from the unstable system we can get the stable output if the unstable pole is canceled by the input signal.

(c) See for example the input and output signals depicted in Figure 4.6.

Note that the unstable pole is not canceled and the output signal is therefore $Y(z)$ is unstable function.

Exercise 4.11 Properties of the z-transform

Let $x[n]$ be a discrete-time sequence and $X(z)$ its corresponding z-transform with appropriate ROC.

(a) Prove that the following relation holds:

$$nx[n] \overset{Z}{\longleftrightarrow} -z\frac{d}{dz}X(z).$$

(b) Show that

$$(n+1)\alpha^n u[n] \overset{Z}{\longleftrightarrow} \frac{1}{(1-\alpha z^{-1})^2}, \qquad |z| > |\alpha|.$$

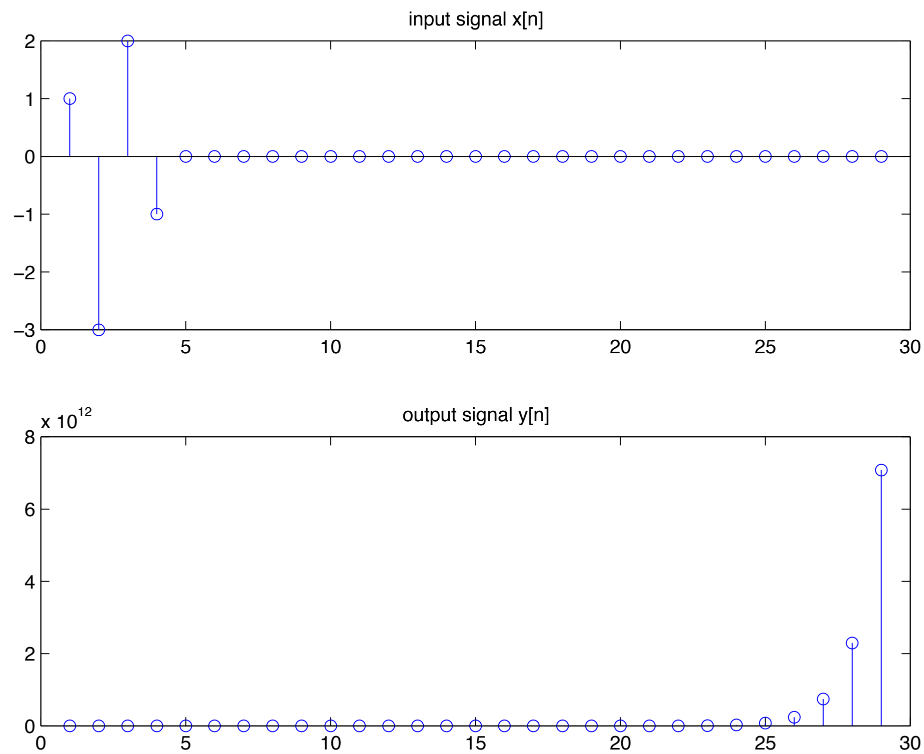


Figure 4.6: Input and output signals

(c) Suppose that the above expression corresponds to the impulse response of an LTI system. What can you say about the causality of such a system? About its stability?

(d) Let $\alpha = 0.8$, what is the spectral behavior of the corresponding filter? What if $\alpha = -0.8$?

Solution of Exercise 4.11

(a) Let $X(z) = \Sigma_n x[n]z^{-n}$. We have that

$$\begin{aligned}\frac{d}{dz}X(z) &= \frac{d}{dz}\left(\Sigma_n x[n]z^{-n}\right)\\ &= \Sigma_n(-n)x[n]z^{-n-1}\\ &= -z^{-1}\Sigma_n nx[n]z^{-n}\end{aligned}$$

and the relation follows directly.

(b) We have that

$$\alpha^n u[n] \stackrel{Z}{\longleftrightarrow} \frac{1}{1-\alpha z^{-1}}.$$

Using the previous result, we find

$$n\alpha^n u[n] \stackrel{Z}{\longleftrightarrow} -z\frac{d}{dz}\left(\frac{1}{1-\alpha z^{-1}}\right) = \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}.$$

Thus,

$$(n+1)\alpha^{n+1}u[n+1] \stackrel{Z}{\longleftrightarrow} z\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$$

and

$$(n+1)\alpha^n u[n+1] \stackrel{Z}{\longleftrightarrow} \frac{1}{(1-\alpha z^{-1})^2}.$$

The relation follows by noticing that

$$(n+1)\alpha^n u[n+1] = (n+1)\alpha^n u[n]$$

since when $n=-1$ both sides are equal to zero.

(c) The system is causal since the ROC corresponds to the outside of a circle of radius α (or equivalently since the impulse response is zero when $n<0$). The system is stable when the unit circle lies inside the ROC, i.e. when $|\alpha| \leq 1$.

(d) When $\alpha = 0.8$, the angular frequency of the pole is $\omega = 0$. Thus the filter is lowpass. When $\alpha = -0.8$, $\omega = \pi$ and the filter is highpass.

Exercise 4.12 Interleaving sequences

Consider two two-sided sequences $h[n]$ and $g[n]$ and consider a third sequence $x[n]$ which is built by interleaving the values of $h[n]$ and $g[n]$:

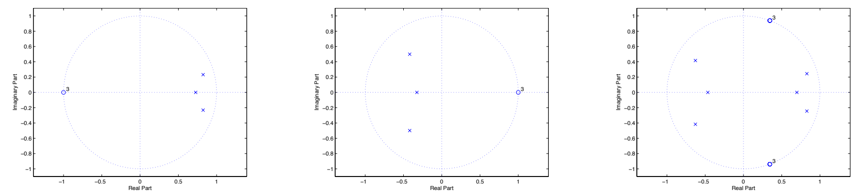
$$x[n] = \ldots, h[-3], g[-3], h[-2], g[-2], h[-1], g[-1], h[0], g[0], h[1], g[1], h[2], g[2], h[3], g[3], \ldots$$

with $x[0] = h[0]$.

(a) Express the z-transform of $x[n]$ in terms of the z-transforms of $h[n]$ and $g[n]$.

(b) Assume that the ROC of $H(z)$ is $0.64 < |z| < 4$ and that the ROC of $G(z)$ is $0.25 < |z| < 9$. What is the ROC of $X(z)$?

Solution of Exercise 4.12



(a) Diagram 1 (b) Diagram 2 (c) Diagram 3

Figure 4.7: Zeros and Poles Diagrams

(a) We have that:

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} h[n]z^{-2n} + g[n]z^{-(2n+1)} \\ &= H(z^2) + z^{-1}G(z^2) \end{aligned}$$

(b) The ROC is determined by the zeros of the transform. Since the sequence is two sided, the ROC is a ring bounded by two poles z_L and z_R such that $|z_L| < |z_R|$ and no other pole has magnitude between $|z_L|$ and $|z_R|$. Consider $H(z)$; if z_0 is a pole of $H(z)$, $H(z^2)$ will have two poles at $\pm z_0^{1/2}$; however, the square root preserves the monotonicity of the magnitude and therefore no new poles will appear between the circles $|z| = \sqrt{|z_L|}$ and $|z| = \sqrt{|z_R|}$. Therefore the ROC for $H(z^2)$ is the ring $\sqrt{|z_L|} < |z| < \sqrt{|z_R|}$. The ROC of the sum $H(z^2) + z^{-1}G(z^2)$ is the intersection of the ROCs, and so

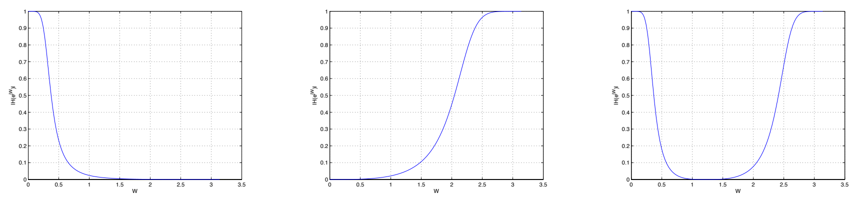
$$\text{ROC } = 0.8 < |z| < 2.$$

Exercise 4.13 Transfer function, zeros and poles

Figure 4.7 shows the zeros and poles of three different filters with the unit circle for reference. Each zero is represented with a 'o' and each pole with a 'x' on the plot. Multiple zeros and poles are indicated by the multiplicity number shown to the upper right of the zero or pole.Sketch the magnitude of each frequency response and determine the type of filter.

Solution of Exercise 4.13

To obtain the frequency response of a filter, we analyze the z-transform on the unit circle, that is, in $z = e^{j\omega}$. Figure 4.8 shows the exact magnitude of each frequency response.



(a) Diagram 1 (b) Diagram 2 (c) Diagram 3

Figure 4.8: Frequency response

The first filter is a low-pass filter. Note that there are three poles located in low frequency (near $\omega = 0$), while there is a zero located in high frequency ($\omega = \pi$). The second filter is just the opposite. The zero is located in low frequency, while the influence of the three poles is maximum in high frequency ($\omega = \pi$). Therefore, it is a high-pass filter. In the third system, there are poles which affect low and high frequency and two zeros close to $w = \pi/2$. Therefore, this system is a stop-band filter.

Module 5

Exercise 5.1 LTI systems

Consider a discrete-time sequence $x[n]$ with DTFT $X(e^{j\omega})$ and the continuous-time interpolated signal,

$$x_0(t) = \sum_{n=-\infty}^{\infty} x[n]\mathrm{rect}\,(t-n)$$

i.e. the signal interpolated with a zero-centered zero-order hold and $T_s = 1$ sec.

(a) Express $X_0(j\Omega)$ (the spectrum of $x_0(t)$) in terms of $X(e^{j\omega})$.

(b) Compare $X_0(j\Omega)$ to $X(j\Omega)$. We can look at $X(j\Omega)$ as the Fourier transform of the signal obtained from the sinc interpolation of $x[n]$ (always with $T_s = 1$):

$$x(t) = \sum_{n\in\mathbb{Z}} x[n]\mathrm{sinc}(t-n)$$

Comment on the result: you should point out two major problems.

(c) The signal $x(t)$ can be obtained from the zero-order hold interpolation $x_0(t)$ as $x(t) = x_0(t) * g(t)$ for some filter $g(t)$. Sketch the frequency response of $g(t)$

(d) Propose two solutions (one in the continuous-time domain, and another in the discrete-time domain) to eliminate or attenuate the distortion due to the zero-order hold. Discuss the advantages and disadvantages of each.

Solution of Exercise 5.1

(a)

$$\begin{aligned}
X_0(j\Omega) &= \int_{-\infty}^{\infty} x_0(t)e^{-j\Omega t}dt \\
&= \int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} x[n]\mathrm{rect}\,(t-n)e^{-j\Omega t}dt \\
&= \sum_{n=-\infty}^{\infty} x[n]\int_{-\infty}^{\infty}\mathrm{rect}\,(t-n)e^{-j\Omega t}dt \\
&= \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}\int_{-1/2}^{1/2} e^{-j\Omega\tau}d\tau = \frac{\sin(\Omega/2)}{\Omega/2}\,X(e^{j\Omega}) \\
&= \mathrm{sinc}(\Omega/2\pi)\,X(e^{j\Omega}).
\end{aligned}$$

(b) Take for instance a discrete-time signal with a triangular spectrum represented in Figure (b).

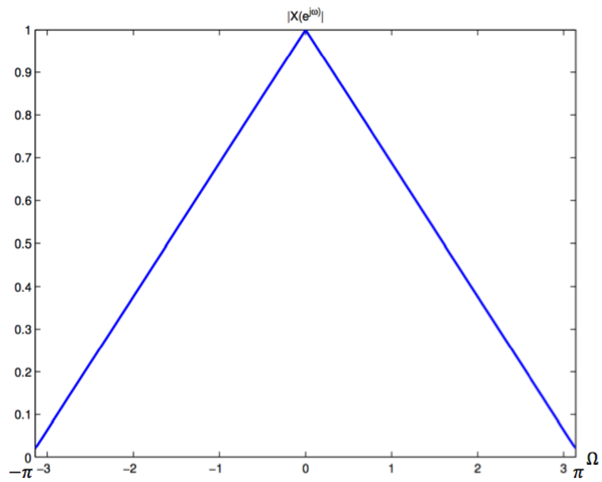


Figure 5.1: Triangular spectrum

We know that the sinc interpolation will give us a continuous-time signal which is strictly bandlimited to the $[-\Omega_N, \Omega_N]$ interval (with $\Omega_N = \pi/T_s = \pi$) and whose shape is exactly triangular, like the one represented in Figure 5.2.

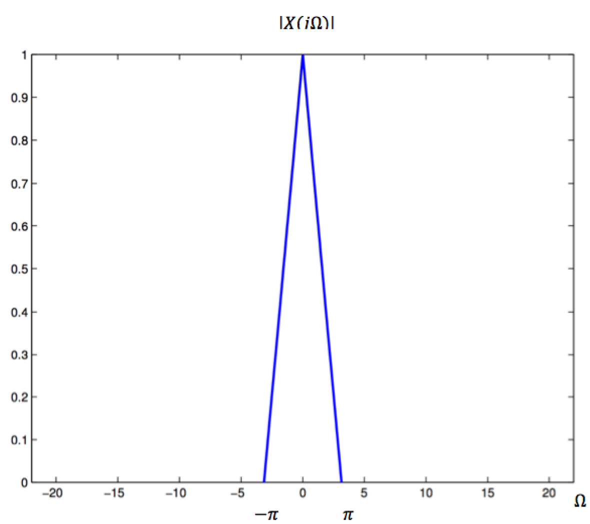


Figure 5.2: Continuous-time spectrum with sinc interpolation

Conversely, the spectrum of the continuous-time signal interpolated by the zero-order hold looks like the on in Figure 5.3

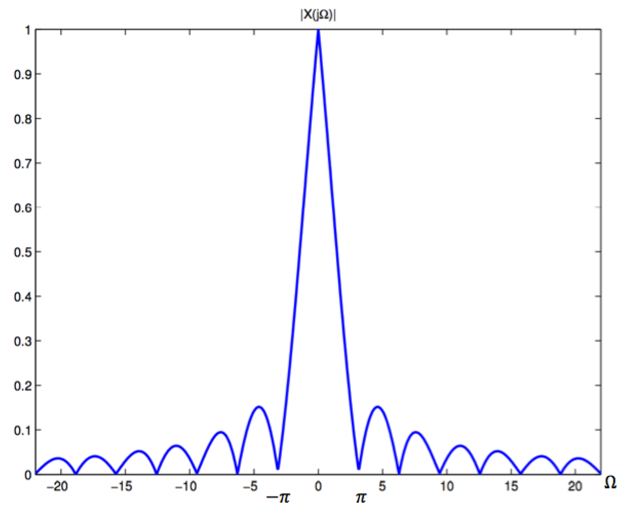


Figure 5.3: Continuous-time spectrum with zero-order hold interpolation

There are two main problems in the zero-order hold interpolation as compared to the sinc interpolation:

- The zero-order hold interpolation is NOT bandlimited: the 2π-periodic replicas of the digital spectrum leak through in the continuous-time signal as high frequency components. This is due to the sidelobes of the interpolation function in the frequency domain (rect in time $\leftrightarrow$ sinc in frequency) and it represents an undesirable high-frequency content which is typical of all local interpolation schemes.
- There is a distortion in the main portion of the spectrum (that between $-\Omega_N$ and Ω_N, with $\Omega_N = \pi$) due to the non-flat frequency response of the interpolation function. It can be seen in the zoom in version of the main portion of the spectrum in the next slide.

(c) See Figure 5.4.

(d) Observe that $X(j\Omega)$ can be expressed as

$$X(j\Omega) = \begin{cases} X(e^{j\Omega}) & \text{if } \Omega \in [-\pi, \pi] \\ 0 & \text{otherwise,} \end{cases}$$

where $X(e^{j\Omega})$ is the DTFT of the sequence $x[n]$ evaluated at $\omega = \Omega$. So

$$X(j\Omega) = X(e^{j\Omega})\mathrm{rect}\left(\frac{\Omega}{2\pi}\right) = X_0(j\Omega)\mathrm{sinc}^{-1}\left(\frac{\Omega}{2\pi}\right)\mathrm{rect}\left(\frac{\Omega}{2\pi}\right).$$

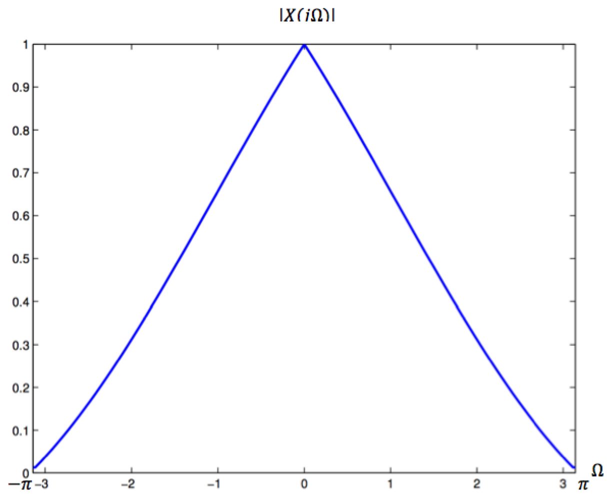


Figure 5.4: Frequency response of $g(t)$

Hence

$$G(j\Omega) = \text{sinc}^{-1}\left(\frac{\Omega}{2\pi}\right)\text{rect}\left(\frac{\Omega}{2\pi}\right),$$

(e)

- A first solution is to compensate for the distortion introduced by $G(j\Omega)$ in the discrete-time domain. This is equivalent to pre-filtering $x[n]$ with a discrete-time filter of magnitude $1/G(e^{j\Omega})$. The advantages of this method is that digital filters such as this one are very easy to design and that the filtering can be done in the discrete-time domain. The disadvantage is that this approach does not eliminate or attenuate the high frequency leakage outside of the baseband.
- Alternatively, one can cascade the interpolator with an analog lowpass filter to eliminate the leakage. The disadvantage is that it is hard to design an analog lowpass which can also compensate for the in-band distortion introduced by $G(j\Omega)$; such a filter will also introduce unavoidable phase distortion (no analog filter has linear phase).

Exercise 5.2 A bizarre interpolator

Consider the local interpolation scheme of the previous exercise but assume that the characteristic of the interpolator is the following:

$$I(t) = \begin{cases} 1 - 2|t| & \text{for} \quad |t| \leq 1/2 \\ 0 & \text{otherwise} \end{cases}$$

This is a triangular characteristic with the same support as the zero-order hold. If we pick an interpolation interval T_s and interpolate a given discrete-time signal $x[n]$ with $I(t)$, we obtain a continuous-time signal:

$$x(t) = \sum_n x[n] I(\frac{t - nT_s}{T_s})$$

Assume that the spectrum of x[n] between $-\pi$ and π is

$$X(e^{j\omega}) = \begin{cases} 1 & \text{for} \quad |\omega| \leq 2\pi/3 \\ 0 & \quad \text{otherwise} \end{cases}$$

(with the obvious 2π-periodicity over the entire frequency axis).

(a) Compute and sketch the Fourier transform $I(j\Omega)$ of the interpolating function $I(t)$ – Recall that the triangular function can be expresses as the convolution of rect(2t) with itself.

(b) Sketch the Fourier transform $X(j\Omega)$ of the interpolated signal $x(t)$; in particular, clearly mark the Nyquist frequency $\Omega_N = \pi/T_s$.

(c) The use of I(t) instead of a sinc interpolator introduces two types of errors: briefly describe them.

(d) To eliminate the error in the baseband $[-\Omega_N, \Omega_N]$ we can pre-filter the signal $x[n]$ before interpolating with $I(t)$. Write the frequency response of the discrete-time filter $H(e^{j\omega})$.

Solution of Exercise 5.2

(a) From last exercise, we know that the Fourier Transform of rect(t) is

$$\frac{\sin(\Omega/2)}{\Omega/2} = \text{sinc}(\Omega/2\pi)$$

In our case,

$$I(t) = 2 \ \text{rect}(2t) * \text{rect}(2t)$$

so

$$I(j\Omega) = \left(2 \cdot \frac{1}{2} \frac{\sin(\Omega/4)}{\Omega/4}\right)^2 = \frac{1}{2}\text{sinc}^2(\Omega/4\pi)$$

which is represented in Figure 5.5

(b)

$$x(t) = \sum_n x[n] I(\frac{t - nT_s}{T_s})$$

Then,

$$\begin{aligned} X(j\Omega) = \quad & \textstyle\sum_n x[n] \text{FT}\{I(\frac{t-nT_s}{T_s})\} = \\ & \textstyle\sum_n x[n] e^{-j\Omega n} \frac{T_s}{4} \text{sinc}^2(\frac{\Omega T_s}{4\pi}) = \\ & \tfrac{T_s}{4} X(e^{j\Omega}) \text{sinc}^2(\tfrac{\Omega T_s}{4\pi}) \end{aligned}$$

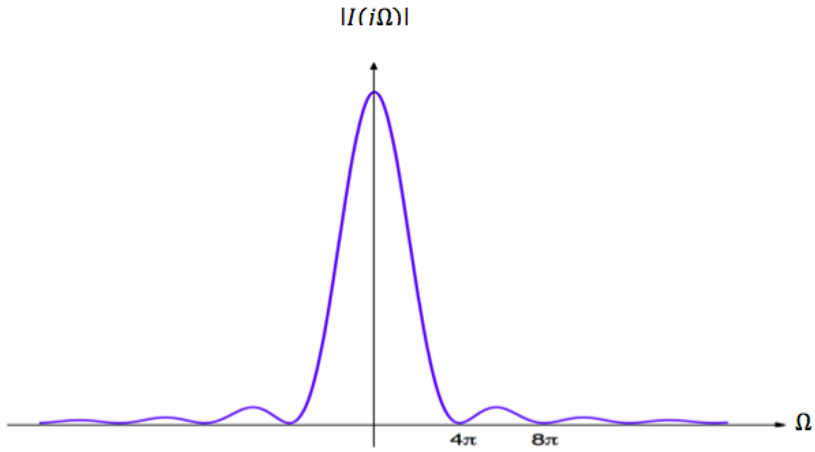


Figure 5.5: Spectrum of the interpolating function

So the Fourier transform of the interpolated signal is composed of the products of two parts (recall that, as usual, $\Omega_N = \pi/T_S$):

- The $2\Omega_N$-periodic spectrum $X(e^(j\pi\Omega/\Omega_N))$
- The Fourier transform of the interpolating function.

The result is represented in Figure 5.6

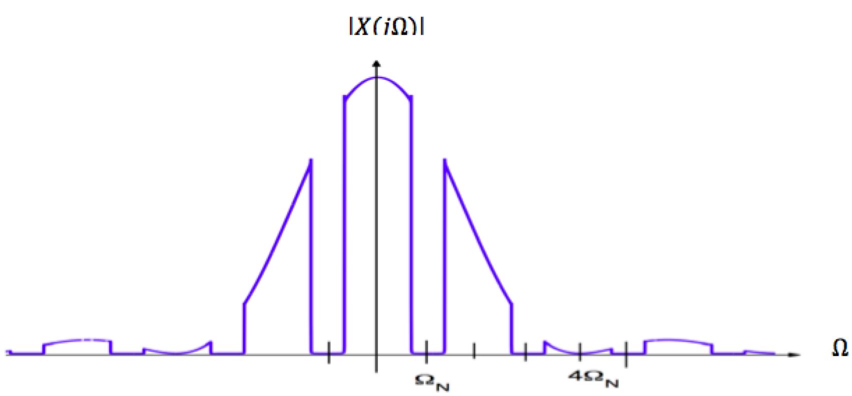


Figure 5.6: Spectrum of the interpolated signal

(c) There are two types of error, in-band and out-of-band:

- **In − band**: The spectrum between $[-\Omega_N, \Omega_N]$ (the baseband) is distorted by the non-flat response of the interpolating function over the baseband.
- **Out − of − band**:The periodic copies of $X(e^(j\pi\Omega/\Omega_N))$ outside of $[-\Omega_N, \Omega_N]$ are not eliminated by the interpolation filter, since it is not an ideal lowpass.

(d) We need to undo the linear distortion introduced by the nonflat response of the interpolation filter in the baseband. The idea is to have a modified spectrum $H(e^{j\omega})X(e^{j\omega})$ so that, in the $[-\Omega_N, \Omega_N]$, we have

$$X(j\Omega) = X(e^{j\Omega T_s}).$$

If we use $H(e^{j\omega})X(e^{j\omega})$ in the interpolation formula, we have

$$X(j\Omega) = \frac{T_s}{4} H(e^{j\Omega T_s})X(e^{j\Omega T_s}) \operatorname{sinc}^2\left(\frac{\Omega T_s}{4\pi}\right)$$

so that

$$H(e^{j\Omega T_s}) = [\frac{T_s}{4}\operatorname{sinc}^2\left(\frac{\Omega T_s}{4\pi}\right)]^{-1}.$$

Therefore, the frequency response of the digital filter will be

$$H(e^{j\omega}) = \frac{4}{T_s}\operatorname{sinc}^{-2}\left(\frac{\omega}{4\pi}\right), \qquad -\pi \le \omega \le \pi$$

prolonged by 2π-periodicity over the entire frequency axis.

Exercise 5.3 Sampling and interpolation for bandlimited vectors

In the lecture we saw how to sample and interpolate π-bandlimited functions. In this problem we will do the same but, instead, with band limited vectors. A vector $\mathbf{x} \in \mathbb{C}^M$ is called bandlimited when there exists $k_0 \in \{0, 1, \ldots, M-1\}$ such that its DFT coefficient sequence X satisfies $X_k = 0$ for all k with $|k - \frac{M}{2}| > \frac{k_0-1}{2}$. The smallest such k_0 is called the bandwidth of $\mathbf{x}$. A vector in $\mathbb{C}^M$ that is not bandlimited is called full band. The set of vectors in $\mathbb{C}^M$ with bandwidth at most k_0 is a subspace. For $\mathbf{x}$ in such a bandlimited subspace, find a linear mapping $\Phi : \mathbb{C}^{k_0} \to \mathbb{C}^M$ (i.e., a basis) so that the system described by $\Phi\Phi^*$ achieves perfect recovery $\hat{\mathbf{x}} = \mathbf{x}$ (i.e. $\Phi\Phi^*$ equals the identity matrix).

Solution of Exercise 5.3

Call our bandlimited subspace S. If we find a orthogonal basis Φ such that $S = \mathcal{R}(\Phi)$, sampling followed by interpolation $\Phi\Phi^*$ will lead to perfect recovery. Since all the $X_k = 0$ for $|k - \frac{M}{2}| < \frac{(K_0-1)}{2}$, we build our Φ^* by taking the DFT matrix W and removing the rows whose indices satisfy $|k - \frac{M}{2}| < \frac{(K_0-1)}{2}$. We call this matrix $\widehat{W}$,

$$\Phi^* = \widehat{W}.$$

For the interpolation we use the IDFT matrix with the columns pruned according to the indices satisfying the aforementioned relationship,

$$\Phi \;=\; \widehat{W}^{-1} \;=\; \widehat{W}^*.$$

Note now that we can also multiply Φ^* by any unitary matrix to get a plausible sampling operator. That is $U\Phi^*$ and ΦU^* are still ideally matched since $UU^* = U^*U = I$.

Exercise 5.4 Another view of Sampling

One of the standard ways of describing the sampling operation relies on the concept of "modulation by a pulse train". Choose a sampling interval T_s and define a continuous-time pulse train $p(t)$ as:

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s).$$

The Fourier Transform of the pulse train is

$$P(j\Omega) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\frac{2\pi}{T_s})$$

This is tricky to show, so just take the result as is. The "sampled" signal is simply the modulation of an arbitrary-continuous time signal $x(t)$ by the pulse train:

$$x_s(t) = p(t)\, x(t).$$

Derive the Fourier transform of $x_s(t)$ and show that if $x(t)$ is bandlimited to π/T_s then we can reconstruct $x(t)$ from $x_s(t)$.

Solution of Exercise 5.4

By using the modulation theorem, we have

$$\begin{aligned} X_s(j\Omega) &= X(j\Omega)P(j\Omega) \\ &= \int_{\mathbb{R}} X(j\tilde{\Omega})P(j(\Omega-\tilde{\Omega}))d\tilde{\Omega} = \frac{2\pi}{T_s}\int_{\mathbb{R}} X(j\tilde{\Omega}) \sum_{k\in\mathbb{Z}} \delta\left(\Omega - \tilde{\Omega} - k\frac{2\pi}{T_s}\right) d\tilde{\Omega} \end{aligned}$$

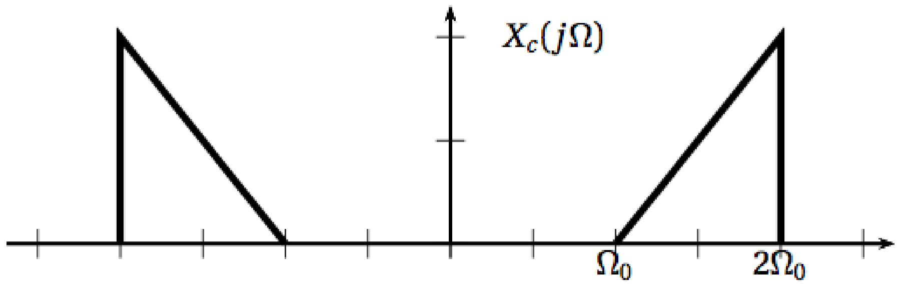


Figure 5.7: Spectrum of $x_c(t)$

$$
\begin{aligned}
X_s(j\Omega) &= X(j\Omega) * P(j\Omega) \\
&= \int_{\mathbb{R}} X(j\tilde{\Omega}) P(j(\Omega - \tilde{\Omega})) d\tilde{\Omega} = \frac{2\pi}{T_s} \int_{\mathbb{R}} X(j\tilde{\Omega}) \sum_{k \in \mathbb{Z}} \delta \left(\Omega - \tilde{\Omega} - k\frac{2\pi}{T_s} \right) d\tilde{\Omega} \\
&= \frac{2\pi}{T_s} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} X(j\tilde{\Omega}) \delta \left(\Omega - \tilde{\Omega} - k\frac{2\pi}{T_s} \right) d\tilde{\Omega} = \frac{2\pi}{T_s} \sum_{k \in \mathbb{Z}} X \left(j \left(\Omega - k\frac{2\pi}{T_s} \right) \right).
\end{aligned}
$$

In other words, the spectrum of the delta-modulated signal is just the periodic repetition (with period $(2\pi/T_s)$) of the original spectrum. If the latter is bandlimited to (π/T_s) there will be no overlap and therefore $x(t)$ can be obtained simply by lowpass filtering $x_s(t)$ (in the continuous-time domain).

Exercise 5.5 Aliasing can be good!

Consider a signal $x_c(t)$ with spectrum represented in Figure 5.7.

(a) What is the bandwidth of the signal? What is the minimum sampling period in order to satisfy the sampling theorem?

(b) Take a sampling period $T_s = \frac{\pi}{\Omega_0}$; clearly, with this sampling period, there will be aliasing. Plot the DTFT of the discrete-time signal $x_a[n] = x_c(nT_s)$.

(c) Suggest a block diagram to reconstruct $x_c(t)$ from $x_a[n]$.

(d) Therefore, we can exploit aliasing to reduce the sampling frequency necessary to sample a bandpass signal. What is the minimum sampling frequency to be able to reconstruct with the above strategy a real signal whose frequency support on the positive axis is $[\Omega_0, \Omega_1]$?

Solution of Exercise 5.5

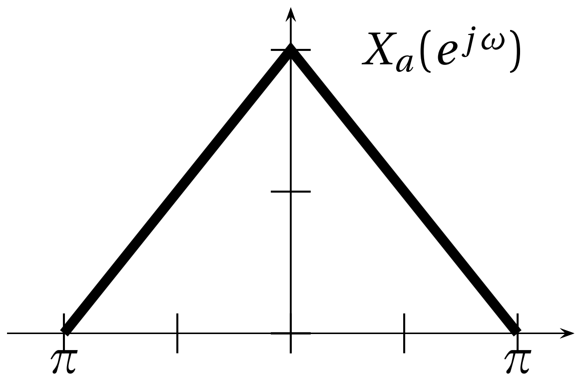


Figure 5.8: DTFT of $x_a[n]$

(a) The highest nonzero frequency is $2\Omega_0$ and therefore $x_c(t)$ is $2\Omega_0$-bandlimited for a total bandwidth of $4\Omega_0$. The maximum sampling period (i.e. the inverse of the *minimum* sampling frequency) which satisfies the sampling theorem is $T_s = \pi/(2\Omega_0)$. Note however that the total support over which the (positive) spectrum is nonzero is the interval $[\Omega_0, 2\Omega_0]$ so that one could say that the total *effective* positive bandwidth of the signal is just Ω_0.

(b) The digital spectrum will be the rescaled version of the periodized continuous-time spectrum

$$\tilde{X}_c(j\Omega) = \sum_{k=-\infty}^{\infty} X_c(j(\Omega - 2k\Omega_0)).$$

The general term $X_c(j\Omega - j2k\Omega_0)$ is nonzero for $\Omega_0 \leq |\Omega - 2k\Omega_0| \leq 2\Omega_0 \quad$ for $k \in \mathbb{Z}$. or equivalently

$$\begin{aligned} (2k+1)\Omega_0 &\leq \Omega \leq & (2k+2)\Omega_0 \\ (2k-2)\Omega_0 &\leq \Omega \leq & (2k-1)\Omega_0 \end{aligned}$$

Non-overlapping intervals, therefore, no disruptive superpositions of the copies of the spectrum! The digital spectrum is

$$X(e^{j\omega}) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_c(j\frac{\omega}{T_s} - j\frac{2\pi k}{T_s})$$

which is represented in Figure 5.8 (with 2π-periodicity, of course)

(c) Here's a possible scheme (verify that it works):

- Sinc-interpolate $x_a[n]$ with period T_s to obtain $x_b(t)$
- Multiply $x_b(t)$ by $\cos(2\Omega_0 t)$ in the continuous time domain to obtain $x_p(t)$ (i.e. modulate by a carrier at frequency (Ω_0/π) Hz).
- Bandpass filter $x_p(t)$ with an ideal bandpass filter with (positive) passband equal to $[\Omega_0, 2\Omega_0]$ to obtain $x_c(t)$.

(d) The effective *positive* bandwidth of such a signal is $\Omega_\Delta = (\Omega_1 - \Omega_0)$. The sampling frequency must be at least equal to the effective total bandwidth, so a first condition on the maximum allowable sampling period: $T_{\max} < \pi/\Omega_\Delta$.

- Case 1: assume Ω_1 is a multiple of the bandwidth, i.e. $\Omega_1 = M\Omega_\Delta$ for some integer M (in the previous case, it was $M = 2$).
  In this case, the argument we made in the previous point can be easily generalized: if we pick $T_s = \pi/\Omega_\Delta$ and sample we have that

  $$\tilde{X}_c(j\Omega) = \sum_{k=-\infty}^{\infty} X_c(j(\Omega - 2k\Omega_\Delta)).$$

  The general term $X_c(j\Omega - j2k\Omega_\Delta)$ is nonzero only for

  $$\Omega_0 \leq |\Omega - 2k\Omega_\Delta| \leq \Omega_1 \quad \text{for } k \in \mathbb{Z}.$$

  Since $\Omega_0 = \Omega_1 - \Omega_\Delta = (M-1)\Omega_\Delta$, this translates to

  $$\begin{aligned} (2k + M - 1)\Omega_\Delta &\leq \Omega \leq (2k + M)\Omega_\Delta \\ (2k - M)\Omega_\Delta &\leq \Omega \leq (2k - M + 1)\Omega_\Delta \end{aligned}$$

  again non-overlapping intervals!
- Case 2: Ω_1 is *not* a multiple of the bandwidth.The easiest thing to do is to change the lower frequency Ω_0 to a new frequency Ω_0' so that the new bandwidth $\Omega_1 - \Omega_0'$ divides Ω_1 exactly. In other words we set a new lower frequency Ω_0' so that it will be $\Omega_1 = M(\Omega_1 - \Omega_0')$ for some integer M; it is easy to see that

  $$M = \left\lfloor \frac{\Omega_1}{\Omega_1 - \Omega_0} \right\rfloor.$$

  since this is the maximum number of copies of the Ω_Δ-wide spectrum which fit *with no overlap* in the $[0, \Omega_0]$ interval. If $\Omega_\Delta > \Omega_0$ we cannot hope to reduce the sampling frequency and we have to use normal sampling. This artificial change of frequency will leave a small empty "gap" in the new bandwidth $[\Omega_0', \Omega_1]$, but that's no problem. Now we use the previous result and sample with $T_s = \pi/(\Omega_1 - \Omega_0')$ with no overlap. Since $(\Omega_1 - \Omega_0') = \Omega_1/M$, we have that, in conclusion, the maximum sampling period is

  $$T_{\max} = \frac{\pi}{\Omega_1} \left\lfloor \frac{\Omega_1}{\Omega_1 - \Omega_0} \right\rfloor$$

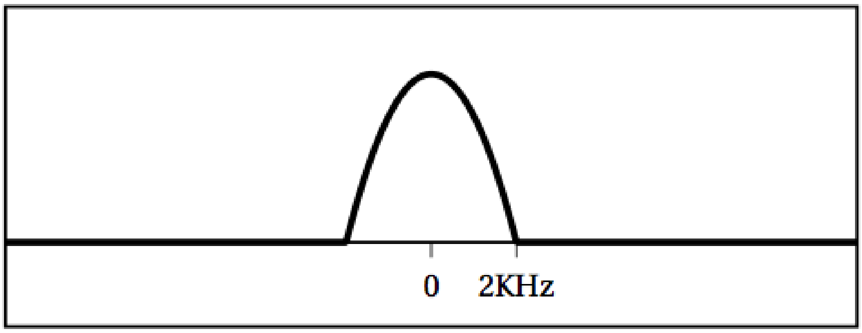


Figure 5.9: Spectrum of $x(t)$

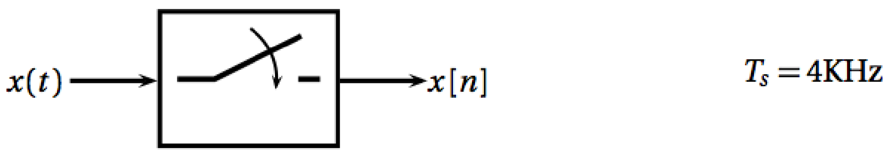


Figure 5.10: System (a)

i.e. we obtain a sampling frequency reduction factor of $\lfloor \Omega_1/(\Omega_1 - \Omega_0) \rfloor$.

Exercise 5.6

Consider a bandlimited continuous-time signal $x(t)$ with a spectrum $X(j\Omega)$ as sketched in Figure 5.9
Sketch the DTFT of the output signal for each of the following systems, where the frequency of the raw sampler is indicated on each line.

(a) The system represented in Figure 5.10

(b) The system represented in Figure 5.11

Solution of Exercise 5.6

(a) The solution is represented in Figure 5.12

(b) The solution is represented in Figure 5.13

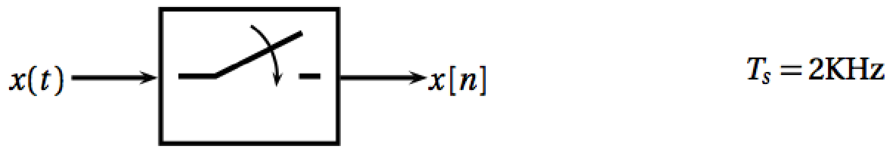


Figure 5.11: System (b)

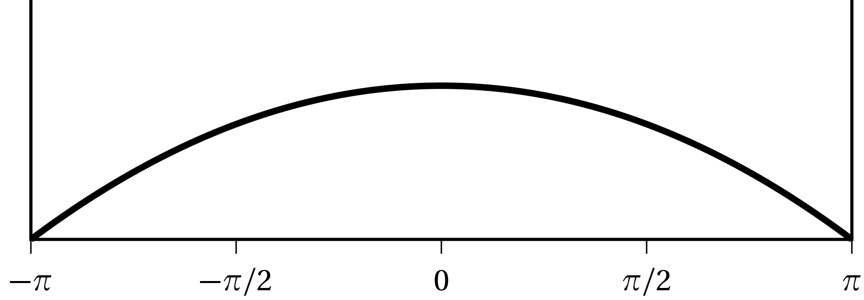


Figure 5.12: DTFT for system (a)

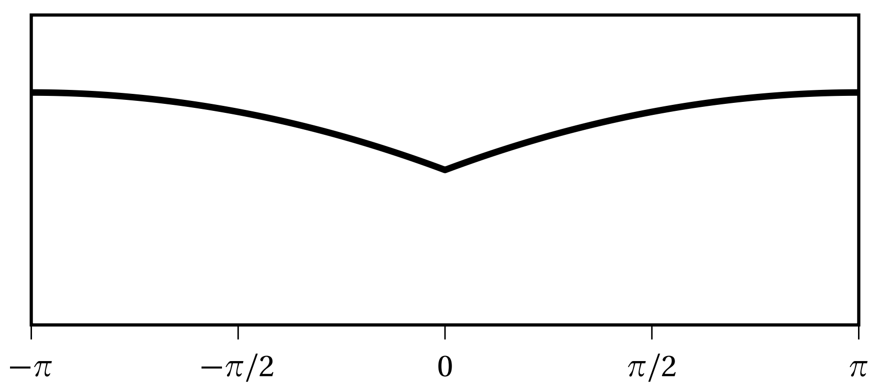


Figure 5.13: DTFT for system (b)

Exercise 5.7

Consider a stationary i.i.d. random process $x[n]$ whose samples are uniformly distributed over the $[-1, 1]$ interval. Consider a quantizer $\mathcal{Q}\{\cdot\}$ with the following characteristic:

$$\mathcal{Q}\{x\} = \begin{cases} -1 & \text{if } -1 \leq x < -0.5 \\ 0 & \text{if } -0.5 \leq x \leq 0.5 \\ 1 & \text{if } 0.5 < x \leq 1 \end{cases}$$

The quantized process $y[n] = \mathcal{Q}\{x[n]\}$ is still i.i.d.; compute its error energy.

Solution of Exercise 5.7

In this quantizer, $K = 4$, $B = 1$, $A = -1$ and the input is uniformly distributed. Thus $P_e = \frac{4}{2} \int_0^{0.5} \tau^2 \, \mathrm{d}\tau = \frac{\Delta^2}{3} = (0.25)/3$

Exercise 5.8

Consider a stationary i.i.d. random process $x[n]$ whose samples are uniformly distributed over the $[-1, 2]$ interval.The process is uniformly quantized with a 1-bit quantizer $\mathcal{Q}\{\cdot\}$ with the following characteristic:

$$\mathcal{Q}\{x\} = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

Compute the signal to noise ratio at the output of the quantizer

Solution of Exercise 5.8

In this quantizer, $K = 3$, $B = -1$, $A = 2$. Thus $P_e = \frac{3}{3} \int_0^1 \tau^2 \, \mathrm{d}\tau = \frac{\Delta^2}{3} = (1)/3$. $P_x = \frac{(B-A)^2}{12} = 0.25$. Thus $SNR = \frac{P_x}{P_e} = \frac{9}{4}$

Module 6

Exercise 6.1

Let $x[n]$ be the discrete-time version of an audio signal, originally bandlimited to $20KHz$ and sampled at $40KHz$; assume that we can model $x[n]$ as an i.i.d. process with variance σ_x^2 . The signal is converted to continuous time, sent over a noisy analog channel and resampled at the receiving end using the following scheme, where both the ideal interpolator and sampler work at a frequency $F_s = 40KHz$. The channel is represented in Figure 6.1

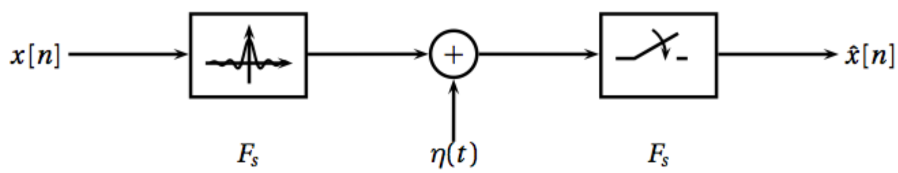


Figure 6.1: Channel

The channel introduces zero-mean, additive white Gaussian noise. At the receiving end, after the sampler, assume that the effect of the noise introduced by the channel can be modeled as a zero-mean white Gaussian stochastic signal $\eta[n]$ with power spectral density $P_\eta(e^{j\omega}) = \sigma_0^2$.

(a) What is the signal to noise ratio (SNR) of $\hat{x}[n]$, i.e. the ratio of the power of the "good" signal and the power of the noise?

(b) The SNR obtained with the transmission scheme above is too low for our purposes. Unfortunately the power constraint of the channel prevents us from simply amplifying the audio signal (in other words: the total power $\int_{-\pi}^{\pi} P_x(e^{j\omega})$ cannot be greater than $2\pi\sigma_x^2$). In order to improve the quality of the received signal, we modify the transmission scheme as shown in Figure6.2, by adding pre-processing and post-processing digital blocks at the transmitting and receiving ends, while F_s is still equal to $40000Hz$. Design the processing blocks A and B so that the signal to noise ratio of $\hat{x}[n]$ is at least twice that of the simple scheme above. You should use upsamplers, downsamplers and lowpass filters only.

(c) Using your new scheme, how long does it take to transmit a 3-minute song signal?

Solution of Exercise 6.1

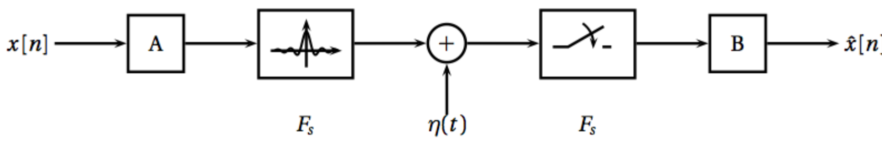


Figure 6.2: Modified channel

(a) The power of the good signal is simply

$$\int_{-\pi}^{\pi} P_x(e^{j\omega} d\omega) = 2\pi\sigma_x^2$$

The power of the noise is

$$\int_{-\pi}^{\pi} P_\eta(e^{j\omega} d\omega) = 2\pi\sigma_0^2$$

so that the SNR is simply σ_x^2/σ_0^2. Just as in oversampling, the idea is to send the signal more slowly as to occupy less bandwidth. If the signal has a smaller bandwidth, we can increase its amplitude without exceeding the power constraint, which will allow us to have a better SNR over the band of interest.

(b) Consider the following preprocessing chain depicted in Figure 6.3, where the lowpass filter has a cutoff frequency $\frac{\pi}{3}$

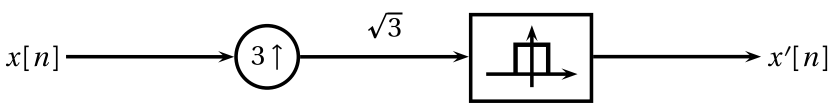


Figure 6.3: Preprocessing chain

The power of $x'[n]$ is

$$\int_{-\pi}^{\pi} P_{x'}(e^{j\omega} d\omega) = \int_{-\pi/3}^{\pi/3} 3\sigma_x^2 = 2\pi\sigma_x^2$$

so that the power constraint is fulfilled. The signal and the noise at the receiver after the sampler have PSDs represented in Figure 6.4.

At the receiver we can filter out the out-of-band noise with the following scheme depicted in Figure 6.5, where, once again, the lowpass has cutoff frequency $\pi/3$.

After the filter the psd is represented in Figure 6.6 and after the down-

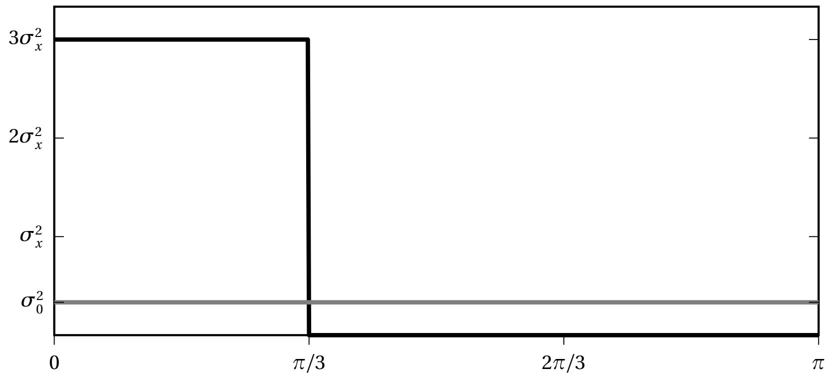


Figure 6.4: PSD of the signal (black) and noise (gray) after sampler

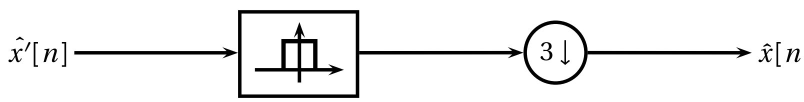


Figure 6.5: Postprocessing scheme

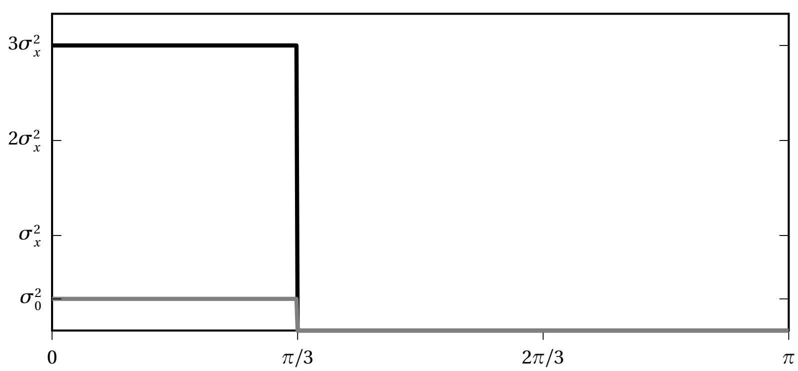


Figure 6.6: PSD of the signal (black) and noise (gray) after filtering

sampler in Figure 6.7 and the signal-to-noise ratio becomes

$$\mathrm{SNR}_2 = 2\pi\sigma_x^2/2\pi(\sigma_0^2/3) = 3\mathrm{SNR}_1$$

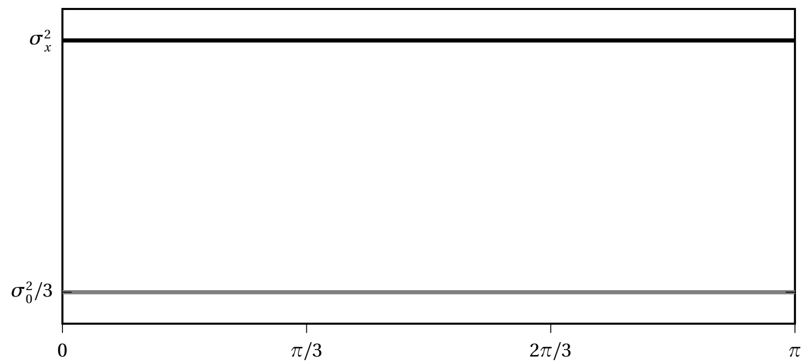


Figure 6.7: PSD of the signal (black) and noise (gray) after downsampler

(c) 9 minutes

Exercise 6.2 The shape of a constellation

One of the reasons for designing non-regular constellations, or constellations on lattices, different than the upright square grid, is that the energy of the transmitted signal is directly proportional to the parameter σ_α^2, i.e., if we assume that the mapper includes a multiplicative factor G_0, $a[n] = G_0\alpha[n], \alpha[n] \in \mathcal{A}$ and

$$\sigma_\alpha^2 = E \mid a[n] \mid^2 = G_0^2 \sum_{\alpha \in \mathcal{A}} \mid \alpha \mid^2 p_a(\alpha) = G_0^2 \sigma_\alpha^2,$$

where $p_a(\alpha)$ is the probability assigned by the mapper to symbol $\alpha \in \mathcal{A}$. By arranging the same number of alphabet symbols in a different manner, we can sometimes reduce σ_α^2 and therefore use a larger amplification gain while keeping the total output power constant, which in turn lowers the probability of error. Consider this two 8-point constellations represented in Figure 6.8 in which the outer points in the irregular constellation (on the left) are at a distance of $1 + \sqrt{3}$. In both constellations consider a minimum distance of 1, considering circular decision boundaries centered upon the constellation points. Compute their intrinsic power σ_α^2 for uniform symbol distributions. What do you notice?

Solution of Exercise 6.2

We know that $\sigma_\alpha^2 = \sum_{\alpha \in \mathcal{A}} |\alpha|^2 p_a(\alpha)$. Then, we have to compute $|\alpha|$ to solve the exercise.

- Constellation (a):

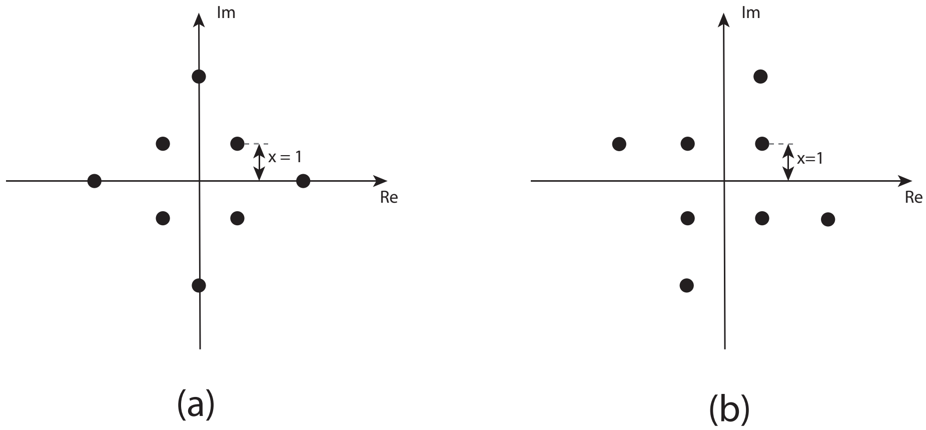


Figure 6.8: Constellations

- $|\alpha_1| = \sqrt{x^2 + x^2} = \sqrt{2}x$
- $|\alpha_2| = 1 + \sqrt{3}$

- Constellation (b):
  - $|\alpha_1| = \sqrt{x^2 + x^2} = \sqrt{2}x$
  - $|\alpha_2| = \sqrt{x^2 + (3x)^2} = \sqrt{10}x$

Now, we just have to compute σ_α^2

Constellation(a):

- $\sigma_\alpha^2 = \sum_{\alpha \in \mathcal{A}} |\alpha|^2 p_a(\alpha) = \frac{4(\sqrt{2}x)^2 + 4(1+\sqrt{3})^2}{8} = 4.73$

Constellation(b):

- $\sigma_\alpha^2 = \sum_{\alpha \in \mathcal{A}} |\alpha|^2 p_a(\alpha) = \frac{4(\sqrt{2}x)^2 + 4(\sqrt{10}x)^2}{8} = 6$

In other words, the irregular constellation(a) offers more than a 1 dB gain over the regular one (b). This gain can be translated into a reliability gain by increasing G_0 while the transmitted signal remains within the power constraint.