4.2 The Composition Operation

4.2.1 Finding $f \circ g$, given f and g

If $f : A \to B$ and $g : B \to C$, then we can build a new function called $(f \circ g)(x)$ that has the domain A and the codomain C, and that follows the rule $(f \circ g)(x) = f(g(x))$. We read $f \circ g$ as "f of g", or the composition of f with g.

To find $f \circ g$, you plug $f(x)$ into $g(x)$ and simplify.

Example: $f(x) = 2x + 1$ and $g(x) = x^2 - 1$. What is $(f \circ g)(x)$?

1. Plug $f(x)$ into $g(x)$: $f(g(x))$
   $= f(x^2 - 1)$
   $= 2(x^2 - 1) + 1$

2. Simplify: $= 2x^2 - 2x + 1$
   $= 2x^2 - 1$

Question 1

Solve the following:

a. $f(x) = 2x - 1$ and $g(x) = 3x$, what is $(f \circ g)(x)$?

b. $f(x) = 2x - 1$ and $g(x) = 3x$, what is $(g \circ f)(x)$?

c. $f(x) = x^2$ and $g(x) = x + 1$, what is $(f \circ g)(x)$?

d. $f(x) = x^2$ and $g(x) = x + 1$, what is $(g \circ f)(x)$?

4.2.2 Finding g based on f and $f \circ g$

If we have f and $f \circ g$, but need to find the function g, we can do this with substitution...

Example: $f(x) = 2x + 1$ and $(f \circ g)(x) = 2x^2 - 1$. What is $g(x)$?

1. Use a to symbolize $g(x)$. $a = g(x)$.
2. Rewrite $f(g(x))$ $f(g(x)) = f(a)$
3. Find $f(a)$ via the $f(x)$ function. $f(a) = 2a + 1$
4. Set $f(a) = 2a + 1$ equal to $f(g(x)) = 2x^2 - 1$. $2a + 1 = 2x^2 - 1$
5. Solve for a to find $g(x)$.
$$2a + 1 = 2x^2 - 1$$
$$2a = 2x^2 - 1 - 1$$
$$2a = 2x^2 - 2$$
$$a = x^2 - 1$$

Therefore, $g(x) = x^2 - 1$.

Question 2

Solve the following:

a. $f(x) = 2x - 1$ and $(f \circ g)(x) = 6x - 1$, what is $g(x)$?

b. $f(x) = x^2$ and $(f \circ g)(x) = x^2 + 2x + 1$, what is $g(x)$?

c. $f(x) = 3x - 2$ and $(f \circ g)(x) = 12x + 7$, what is $g(x)$?

4.2.3 Finding f based on g and $f \circ g$

If we have g and $f \circ g$, but need to find the function f, we can use substitution in another way...

Example: $g(x) = x^2 - 1$ and $(f \circ g)(x) = 2x^2 - 1$. What is $f(x)$?

1. Beginning with $f(g(x))$,
   Set a to the LHS of $g(x)$. $\quad a = x^2 - 1.$

2. Solve for x: $\quad x^2 = a + 1$
   $x = \sqrt{a+1}$

3. Rewrite $f(g(x))$: $\quad f(g(x)) = f(x^2 - 1)$

4. Plug in $x = \sqrt{a+1}$ $\quad f(a) = 2(\sqrt{a+1})^2 - 1$

5. Simplify: $\quad f(a) = 2(a+1) - 1$
   $f(a) = 2a + 2 - 1$
   $f(a) = 2a + 1$

So, $f(x) = 2x + 1$.

Question 3

Solve the following:

a. $g(x) = 3x$ and $(f \circ g)(x) = 6x - 1$. What is $f(x)$?

b. $g(x) = x + 1$ and $(f \circ g)(x) = x^2 + 2x + 1$. What is $f(x)$?

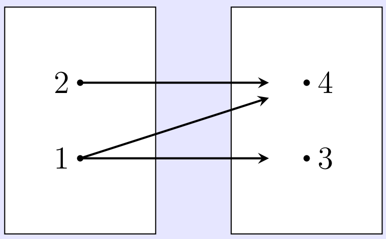
c. $g(x) = 2x - 1$ and $(f \circ g)(x) = 6x - 1$. What is $f(x)$?

4.2.4 More arrow diagrams

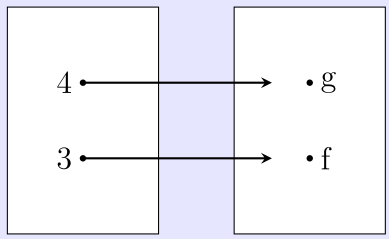
We can also use arrow diagrams to visually represent functions and compositions of functions.

Example:
$f(x)$: Domain $A = \{1, 2\}$, Codomain $B = \{3, 4\}$,
Rule { (1,3), (1,4), (2,4) },
$g(x)$: Domain $B = \{3, 4\}$, Codomain $D = \{f, g\}$,
Rule { (3,f), (4,g) }.

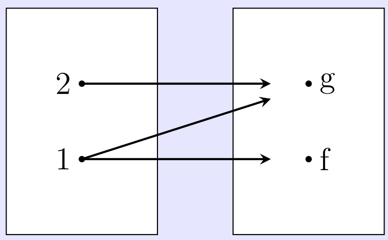
Draw the diagrams for $f(x)$, $g(x)$, and $f(g(x))$.



$f : A \to B$



$g : B \to C$

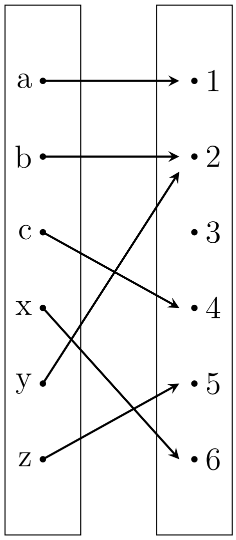


$f \circ g : A \to C$

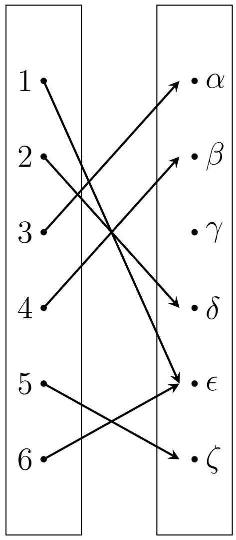
Question 4

Finish the following diagrams:

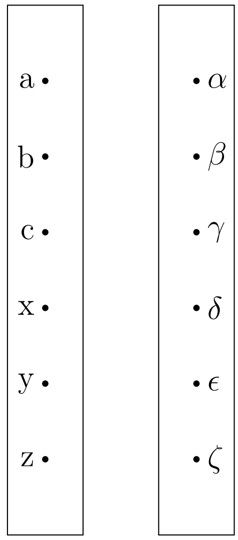
a.



$f : A \to B$

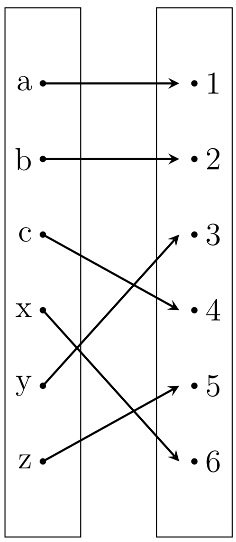


$g : B \to C$

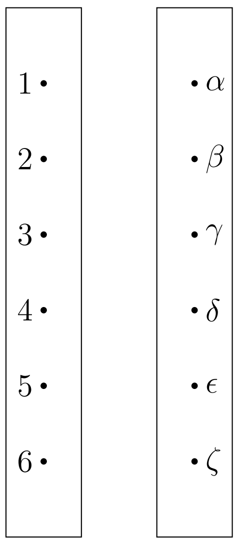


$(g \circ f) : A \to C$

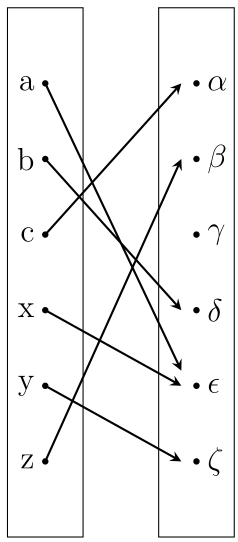
b.



$f : A \to B$



$g : B \to C$



$(g \circ f) : A \to C$