Concepts

Sequences

Element: One specific item of a list (or array) at some position. (e.g., For $a_1 = 5$, the index is 1 and the element is 5.)

Index: The position of an item in a sequence. In Discrete Math, these generally begin with a_1, while in programming the index usually starts at 0.

Closed formula: A formula where the element is calculated based on the index (or position) in the sequence. (e.g., $a_n = 2n + 1$)

Recursive formula: A formula where the first element is given and subsequent elements are calculated based on previous elements. (e.g., $a_1 = 1, a_n = a_{n-1} + 2$)

Propositions

A **propositional variable** is a variable that is unambiguously true or false. These variables can also be combined with AND $\wedge$, OR $\vee$, and NOT $\neg$ to build a compound statement. The compound statement will also result in either true or false based on the values of the propositional variables it is made up of.

Predicates

Predicates are generally denoted as $P(x)$, $Q(x)$, etc. where x is some input value. Based on the input, the predicate results in either true or false.

Example: $P(x)$ is the predicate "x is even".
$P(2)$ is true, and $P(3)$ is false.

Further, a **domain** is usually specified with the predicate. Given the domain, a proposition may be **always true**, or **sometimes true**.

Example: $\forall x \in D, P(x)$
For all elements x from the domain D, $P(x)$ is true.

Example: $\exists x \in D, P(x)$
There exists (at least one) element x in the domain D such that $P(x)$ results to true.

In the case where the proposition is always false, we can say that, "for all elements x, $P(x)$ is not true".

Example: $\forall x \in D, \neg P(x)$
For all elements x from the domain D, $P(x)$ is false.

Negations

1. $\neg(p \wedge q) \equiv \neg p \vee \neg q$
2. $\neg(p \vee q) \equiv \neg p \wedge \neg q$
3. $\neg(\forall x \in D, P(x)) \equiv \exists x \in D, \neg P(x)$
4. $\neg(\exists x \in D, P(x)) \equiv \forall x \in D, \neg P(x)$
5. $\neg(p \rightarrow q) \equiv p \wedge \neg q$

Inverse, converse, and contrapositive

Given $p \rightarrow q$, we have:

1. Converse: $q \rightarrow p$
2. Inverse: $\neg p \rightarrow \neg q$
3. Contrapositive: $\neg q \rightarrow \neg p$

Exam 1 Review

1.1 First Examples

No questions from this section.

1.2 Sequences and Summations

Question 1
For sequence "3, 5, 7, 9, 11" find the **closed formula**.

Question 2
For sequence "1, 4, 9, 16, 25" find the **closed formula**.

Question 3
For sequence "2, 4, 6, 8" find the **recursive formula**.

Question 4
For sequence "1, 3, 7, 15, 31" find the **recursive formula**.

Question 5
Evaluate the sum:

$$\sum_{k=1}^{4}(3)$$

Question 6
Evaluate the sum:

$$\sum_{k=1}^{5}(2k)$$

Question 7
Evaluate the sum:

$$\sum_{k=1}^{5}(3k+1)$$

1.3 Propositional Logic

Question 8
Create a truth table for the expression: $(p \vee q) \wedge \neg(p \wedge q)$

Question 9
Create a truth table for the expression: $(p \vee q) \wedge \neg r$

Question 10
Given the following propositional variables:

o. patron has overdue books a. the book is available at this library
m. patron has maximum amount of books checked out

Translate each of the following into symbolic statements:

a. Patron has overdue books and has the maximum amount of books checked out.

b. Patron does not have overdue books, but the book is not available at this library.

c. Patron does not have overdue books and the book is available at the library.

d. Patron does not have overdue books and the book is available at the library, but (and) the patron has the maximum amount of books checked out.

e. Patron either has overdue books, or has the maximum amount of books checked out.

f. The patron doesn't have overdue books, and doesn't have the maximum checked out, and the book is not available at this library.

Question 11
Given three variables p, q, and r, write a compound statement that will meet the following criteria. Also build the truth table for the statement.

a. p and q are true, but not r

b. p and either q or r are true, but not all three variables.

1.4 Predicates

Question 12
Given the following predicate, come up with a domain that matches the criteria. There are many solutions.

a. $P(x)$ is the predicate, "x is divisible by 3".
Quantified predicate: $\forall x \in D, P(x)$
Define domain D.

b. $Q(x)$ is the predicate, "x is divisible by 2".
Quantified predicate: $\forall x \in E, \neg Q(x)$
Define domain E.

c. $R(x)$ is the predicate, "x is positive".
Quantified predicate: $\exists x \in F, R(x)$
Define domain F.

d. $S(x)$ is the predicate, "x is positive".
Quantified predicate: $\exists x \in G, \neg S(x)$
Define domain G.

Question 13
Solve the following.

a. Translate the following statement into a quantified statement using predicate logic...
"For every element x that is a member of the domain D, x is greater than 10."
$D = \{20, 40, 60, 80, 100\}$

b. Write the negation of your statement from (a) and simplify.

c. Which statement is true: (a) or (b)?

1.5 Implications

Question 14
Solve the following.

a. Translate the following statement into a quantified statement using predicate logic...
"for all integers x, if $2 \times x$ is 0, then x is 0."
Make sure to define two predicates: one for the hypothesis and one for the conclusion.

b. Write the negation of your statement from (a) and simplify.

c. Which statement is true: (a) or (b)?

Question 15
Given the following statement, find the contrapositive, converse, and inverse.
$G(x)$ is x glitters $A(x)$ is x is gold.
$G(x) \rightarrow A(x)$:   "If x glitters, then x is gold"

Question 16
Given the following predicates:

$S(x)$ is x studies hard, $G(x)$ is x gets a good grade,
$P(x)$ is x passes the class.

Use the logical operators $\wedge$, $\vee$, $\neg$, and the quantifiers $\forall$ and $\exists$, and the implication $\rightarrow$ where appropriate. Assume the domain is P, the domain of all people.

a. For all students x, if x studies hard, then x passes the class.

b. There exists some student x such that x didn't study hard and x passed the class.

c. There exists some student x such that x studied hard and x didn't pass the class.

d. For all students x, if x studies hard and x gets a good grade, then x passes the class.

e. For all students x, if x doesn't pass the class, then x didn't get a good grade.

Answer Key

1. $a_n = 2n + 1$
2. $a_n = n^2$
3. $a_1 = 2; a_n = a_{n-1} + 2$
4. $a_1 = 1; a_n = 2 \cdot a_{n-1} + 1$
5. $\sum_{k=1}^{4}(3) = 3 + 3 + 3 + 3 = 4(3) = 12$
6. $\sum_{k=1}^{5}(2k) = (2 \cdot 1) + (2 \cdot 2) + (2 \cdot 3) + (2 \cdot 4) + (2 \cdot 5)$
   $= 2 + 4 + 6 + 8 + 10$
   $= 30$
7. $\sum_{k=1}^{5}(3k+1) = (3 \cdot 1 + 1) + (3 \cdot 2 + 1) + (3 \cdot 3 + 1) + (3 \cdot 4 + 1) + (3 \cdot 5 + 1)$
   $= 4 + 7 + 10 + 13 + 16$
   $= 50$
8.

p	q	$(p \vee q)$	$(p \wedge q)$	$\neg(p \wedge q)$	$(p \vee q) \wedge \neg(p \wedge q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

9.

p	q	r	$(p \vee q)$	$\neg r$	$(p \vee q) \wedge \neg r$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	F	F	F
F	F	F	F	T	F

10. a. $o \wedge m$

b. $\neg o \wedge \neg a$

c. $\neg o \wedge a$

d. $\neg o \wedge a \wedge m$

e. $o \vee m$

f. $\neg o \wedge \neg m \wedge \neg a$
or $\neg(o \vee m \vee a)$

11. a. $p \wedge q \wedge \neg r$

p	q	r	$p \wedge q \wedge \neg r$	
T	T	T	F	
T	T	F	T	p and q are true, but not r
T	F	T	F	
T	F	F	F	
F	T	T	F	
F	T	F	F	
F	F	T	F	
F	F	F	F	

b. $p \wedge (q \vee r) \wedge \neg(p \wedge q \wedge r)$

p	q	r	$(q \vee r)$	$\neg(p \wedge q \wedge r)$	$p \wedge (q \vee r)$ $\wedge \neg(p \wedge q \wedge r)$	
T	T	T	T	F	F	
T	T	F	T	T	T	p and q or r are true, but not all 3.
T	F	T	T	T	T	p and q or r are true, but not all 3.
T	F	F	F	T	F	
F	T	T	T	T	F	
F	T	F	T	T	F	
F	F	T	T	T	F	
F	F	F	F	T	F	

12. a. Multiple solutions; **ALL** elements of D should be divisible by 3.
Example: $D = \{3, 6, 9, 12\}$

b. Multiple solutions; **NO** elements of E should be divisible by 2.
Example: $E = \{5, 7, 9, 11\}$

c. Multiple solutions; **some** element(s) of F should be positive.
Example: $F = \{-2, -1, 0, 1, 2\}$

d. Multiple solutions; **some** element(s) of G should not be positive.
Example: $G = \{-5, 5, 10, 15\}$

13. a. $\forall x \in D, P(x)$ where $P(x)$ is $x > 10$.

b. $\neg(\forall x \in D, P(x)) = \exists x \in D, \neg P(x)$;
There is some x in D such that $x \leq 10$.

c. (a) is true.

14. a. $\forall x \in \mathbb{Z}, P(x) \rightarrow Q(x)$, where $P(x)$ is $2x = 0$ and $Q(x)$ is $x = 0$.

b. $\neg(\forall x \in \mathbb{Z}, P(x) \rightarrow Q(x)) = \exists x \in \mathbb{Z}, P(x) \wedge \neg Q(x)$

c. (a) is true.

15. Original: $G(x) \rightarrow A(x)$
"If x glitters, then x is gold"
Inverse: $\neg G(x) \rightarrow \neg A(x)$
"If x doesn't glitter, then x isn't gold"
Converse: $A(x) \rightarrow G(x)$
"If x is gold, then x glitters"
Contrapositive: $\neg A(x) \rightarrow \neg G(x)$
"If x doesn't glitter, then x isn't gold"

16. a. $\forall x \in P, S(x) \rightarrow P(x)$

b. $\exists x \in P, \neg S(x) \wedge P(x)$

c. $\exists x \in P, S(x) \wedge \neg P(x)$

d. $\forall x \in P, (S(x) \wedge G(x)) \rightarrow P(x)$

e. $\forall x \in P, \neg P(x) \rightarrow \neg G(x)$