4.4 Properties of Relations

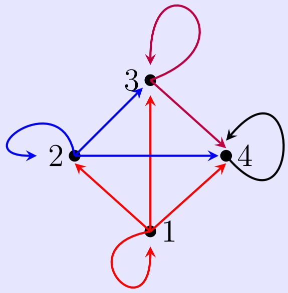
4.4.1 Relations

Relations: A Relation is a way to relate two sets of data together. The two sets are the Domain and Codomain, and there is a Rule that associates them together.

Example: $R : \{1, 2, 3\} \to \{1, 2, 3\}$;

The relation R has a domain of $\{1, 2, 3\}$ and a codomain of $\{1, 2, 3\}$.

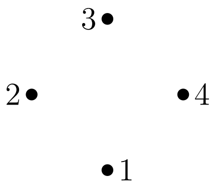
The rule is: $(x, y) \in R$ if $x \le y$; so there is a relation (an arrow) from x to y if x is less than or equal to y. 1 points to 1, 2, 3, and 4, 2 points to 2, 3, and 4, and 3 points to 3 and 4, and 4 only points to itself.



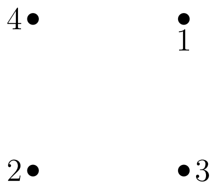
Question 1

Draw the arrows for the following relations:

R



S



Set A is $A = \{1, 2, 3, 4\}$.
Relation R: $R : A \to A$, with the rule: $\{(1, 2), (2, 3), (3, 4)\}$
Relation S: $S : A \to A$, with the rule: $\{(1, 1), (2, 3), (2, 4), (3, 4)\}$

Properties of Binary Relations
Let R be a binary relation on set A.

Reflexive: R is said to be reflexive if $(a, a) \in R$ for all $a \in A$. In terms of the arrow diagram, this means that **every node has a loop**.

Irreflexive: A relation R on set A is irreflexive if, for all $a \in A$, $(a, a) \notin R$. On the arrow diagram, this means **there are no loops.**

Symmetric: R is called symmetric if for all $a, b \in A$, if $a \neq b$ and $(a, b) \in R$, then $(b, a) \in R$. In terms of the arrow diagram, this means that **every arrow goes in both directions**.

Antisymmetric: R is called antisymmetric if for all $a, b \in A$, if $a \neq b$ and $(a, b) \in R$, then $(b, a) \notin R$. In terms of the arrow diagram, this means that **arrows only go in one direction**.

Transitive: R is transitive if, whenever $(a, b) \in R$ and $(b, c) \in R$, it must also be the case that $(a, c) \in R$. In terms of the arrow diagram, this means that **whenever you can follow two arrows to get from node a to node c, you can also get there along a single arrow**. [a]

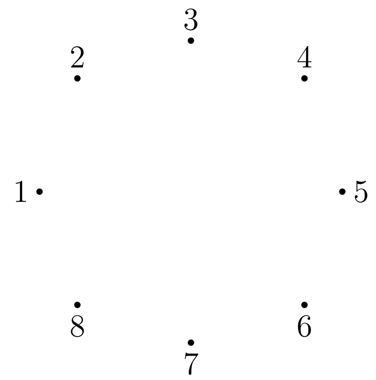
Note that relations can be Reflexive, Irreflexive, or Neither, as well as Symmetric, Antisymmetric, or Neither.

[a] Discrete Mathematics, Ensley and Crawley

Question 2

Complete the arrow diagram for the relation on $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Also identify the properties of each. Give reasons why the relation is transitive, intransitive, neither, symmetric, antisymmetric, neither, and/or reflexive.

$R_1 = \{$ (1,1), (1,2), (1,4), (1,8), (2,2), (2,4), (2,8), (3,3), (3,6), (4,4), (4,8) (5,5), (6,6), (7,7), (8,8) $\}$



☐ Reflexive? ☐ Irreflexive? ☐ Neither? Why?

☐ Symmetric? ☐ Antisymmetric? ☐ Neither? Why?

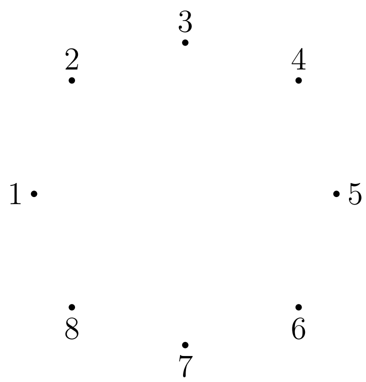
☐ Transitive? Why?

Question 3

Complete the arrow diagram for the relation on $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Also identify the properties of each. Give reasons why the relation is transitive, intransitive, neither, symmetric, antisymmetric, neither, and/or reflexive.

FIXME: KEY SHOULD HAVE LOOPS

$R_2 = \{$ (1,1), (1,3), (1,5), (1,7) (2,2), (2,4), (2,8),
(3,3), (3,5), (3,7), (4,2), (4,4), (4,8), (5,3), (5,7),
(6,6), (6,8), (8,2), (8,4), (8,8) $\}$



☐ Reflexive? ☐ Irreflexive? ☐ Neither? Why?

☐ Symmetric? ☐ Antisymmetric? ☐ Neither? Why?

☐ Transitive? Why?

Recap

- **Reflexive:** $(a, a) \in R$ for all $a \in A$
- **Irreflexive:** $(a, a) \notin R$ for all $a \in A$
- **Antisymmetric:** for all $a, b \in A$, if $a \neq b$ and $(a, b) \in R$, then $(b, a) \notin R$
- **Transitive:** if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$

Question 4

Given the relation, $R_1 = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a + b \text{ is even}\}$.

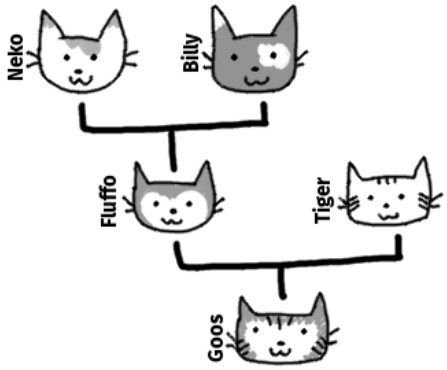
a. This relation is **reflexive**. Find an example to illustrate why.

a and b are both in the set of integers. We are checking to see if the result of (a, a) is always in the relation R_1... so, if you plug in (a, a) into the relation, is the output still "is even"?

b. This relation is **symmetric**. Find an example to illustrate why.

Find some (a, b) and (b, a) that are both in the relation. If you can, it's symmetric.

Question 5



Let C be the set of all cats who have ever lived. For each of the following relations on the set C, decide if the given is reflexive irreflexive, transitive, or antisymmetric. Some of these can satisfy more than one property. Give explanations on how you decided each of these.

a. $R_1 = \{(a, b) \in C \times C : a \text{ is a child of } b\}$

- Reflexive - Is $(a, a) \in C$ for all a valid?
  (For some cat a, (a, a) means "a is a child of a". Is this valid?)

- Irreflexive - Is $(a, a) \notin C$ for all a valid?

- Transitive - Is there some $(a, b) \in C$ and $(b, c) \in C$? [1]
  (For three cats a, b, and c, if a is a chlid of b, and b is a child of c, can a be a child of c?)

- Antisymmetric - Is $(a, b) \in C$ and $(b, a) \notin C$ valid?
  For some cat a and b, can both of the following be true? "a is a child of b, and b is a child of a")

[1] Just assume a cat isn't going to mate with its child. : |

b. $R_2 = \{(a, b) \in C \times C : a$ is a descendant of $b\}$

- Reflexive - Is $(a, a) \in C$ for all a valid?

- Irreflexive - Is $(a, a) \notin C$ for all a valid?

- Transitive - Is there some $(a, b) \in C$ and $(b, c) \in C$?

- Antisymmetric - Is $(a, b) \in C$ and $(b, a) \notin C$ valid?