Practice problems

Question 1: Direct proofs　　　　　Chapter 2.1　　　　　　10%

Prove the following statements using a **direct proof**. Make sure to write the **final answer** in terms of the appropriate definition.

1. If n is even, then $n^2 - n$ is even.

2. The product of two odd integers is always odd.

Question 2: Division theorem　　　　　Chapter 2.2　　　　　　10%

Fill in the blanks in the style of the **division theorem**.

For all integers a and b (with $b > 0$), there is an integer q and an integer r such that:
1. $a = b \cdot q + r$ and
2. $0 \leq r < b$.

1. $20 = \underline{\quad} \cdot 6 + \underline{\quad}$

2. $-13 = \underline{\quad} \cdot 6 + \underline{\quad}$

3. $(3k^2 + 8) = \underline{\quad} \cdot 3 + \underline{\quad}$

Question 3: Modulus　　　　　Chapter 2.2　　　　　　10%

Solve the following modulus problems

1. 73 mod 6

2. -15 mod 2

Question 4: Proof by induction　　　　　Chapter 2.3　　　　　　10%

Show that the sequence defined by the recursive formula

$$a_k = a_{k-1} + 2 \text{ , where } a_1 = 2, \text{ and for } k \geq 2$$

is equivalently described by the closed formula　　　$a_n = 2n$

Question 5: Proof by induction   Chapter 2.3   10%

Use induction to prove that $\sum_{i=1}^{n}(2i-1) = n^2$ for each $n \geq 1$

Question 6: Deriving a recursive formula from a sum   Chapter 2.4
10%

Consider the sum $\sum_{i=1}^{n}(3i+2)$

Use s_n to denote this sum. Find a recursive description of s_n.

Question 7: Proof by contradiction   Chapter 2.5   10%

Use **proof by contradiction** to prove the following statement:

If $n^2 - 1$ is divisible by 5, then n is not divisible by 5.

Question 8: Numerical representations   Chapter 2.6   10%

Write the following numbers as the sum of multiples of powers of the base. ***Do not simplify!***

1. $(246)_{10} =$

2. $(0100\ 1001)_2 =$

3. $(F00D)_{16} =$

Question 9: Binary $\leftrightarrow$ Hexadecimal   Chapter 2.6   10%

HEX	0	1	2	3	4	5	6	7
BINARY	0000	0001	0010	0011	0100	0101	0110	0111
HEX	8	9	A	B	C	D	E	F
BINARY	1000	1001	1010	1011	1100	1101	1110	1111

Using the table, convert directly between base-2 and base-16.

1. Convert (1111 0000 0000 1101)$_2$ to base-16

2. Convert (CAF3)$_{16}$ to base-2

Question 10: Binary $\leftrightarrow$ Decimal   Chapter 2.6   10%

    Using an algorithm, convert between base-2 and base-10.

1. Convert $(\ 75\ )_{10}$ to base-2

2. Convert $(\ 0101\ 1010\ )_2$ to base-10

Answer key

Solution: Question 1

1. If n is even, then $n^2 - n$ is even.

    1. $n = 2k$

    2. $n^2 - n \Rightarrow (2k)^2 - 2k$

    3. $4k^2 - 2k$

    4. $2(2k^2 - k)$

2. The product of two odd integers is always odd.

    1. Integer 1: $n = 2k + 1$ Integer 2: $m = 2j + 1$.

    2. $n \cdot m \Rightarrow (2k + 1)(2j + 1)$

    3. $4kj + 2k + 2j + 1$

    4. $2(2kj + k + j) + 1$

Solution: Question 2

    1. $20 = 3 \cdot 6 + 2$

    2. $-13 = -3 \cdot 6 + 5$

    3. $(3k^2 + 8) = (k^2 + 2) \cdot 3 + 2$

Solution: Question 3

1. 73 mod 6 = 1

2. -15 mod 2 = 1

Solution: Question 4

1. Check first term:
   Recursive: $a_1 = 2$ Closed: $a_1 = 2(1)$

2. Find a_{m-1} with closed formula:
   $a_{m-1} = 2(m-1) = 2m - 2$

3. Plug a_{m-1} into the recursive formula:
   $a_m = a_{m-1} + 2$
   $a_m = 2m - 2 + 2$
   $a_m = 2m$

4. This matches the closed formula given, so we have proved it.

Solution: Question 5

1. Check first 3 terms:

   (a) $n = 1$:
      Left-hand side: $\sum_{i=1}^{1}(2i - 1) = (2(1) - 1) = 1$
      Right-hand side: $1^2 = 1$ ✓

   (b) $n = 2$:
      Left-hand side: $\sum_{i=1}^{2}(2i - 1) = (2(1) - 1) + (2(2) - 1)$
      $= 1 + 3 = 4$
      Right-hand side: $2^2 = 4$ ✓

   (c) $n = 3$:
      Left-hand side: $\sum_{i=1}^{3}(2i - 1) = (2(1) - 1) + (2(2) - 1) + (2(3) - 1)$
      $= 1 + 3 + 5 = 9$
      Right-hand side: $3^2 = 9$ ✓

2. Rewrite the sum:
   $\sum_{i=1}^{m}(2i - 1) = \sum_{i=1}^{m-1}(2i - 1) + (2m - 1)$

3. Find $\sum_{i=1}^{m-1}(2i - 1)$ using the proposition:
   $\sum_{i=1}^{n}(2i - 1) = n^2$
   $\sum_{i=1}^{m-1}(2i - 1) = (m - 1)^2$
   $\sum_{i=1}^{m-1}(2i - 1) = m^2 - 2m + 1$

4. Plug $\sum_{i=1}^{m-1}(2i - 1)$ into the sum from (2):
   $\sum_{i=1}^{m}(2i - 1) = \sum_{i=1}^{m-1}(2i - 1) + (2m - 1)$
   $\sum_{i=1}^{m}(2i - 1) = m^2 - 2m + 1 + 2m - 1 \sum_{i=1}^{m}(2i - 1) = m^2$

5. This matches the original proposition, so we have proved it.

Solution: Question 6

1. Rewrite the sum:
   $\sum_{i=1}^{m}(3i+2) = \sum_{i=1}^{m-1}(3i+2) + 3m + 2$
   $s_m = s_{m-1} + 3m + 2$

2. Find s_1:
   $s_1 = \sum_{i=1}^{1}(3i+2) = 3(1) + 2 = 5$

3. Write out together:
   $s_m = s_{m-1} + 3m + 2$ $s_1 = 5$

Solution: Question 7

1. Hypothesis: $n^2 - 1$ is divisible by 5.
   Conclusion: n is not divisible by 5.

2. Negation:
   $n^2 - 1$ is divisible by 5 and n IS divisible by 5.

3. Symbolically:
   $n^2 - 1 = 5k$ $n = 5j$

4. Equation:
   $(5j)^2 - 1 = 5k$

5. Simplify until contradiction:
   $25j^2 - 5k = 1$ $\rightarrow 5(5j^2 - k) = 1$ $\rightarrow 5j^2 - k = \frac{1}{5}$

Solution: Question 8

1. $(246)_{10} = 2 \cdot 10^2 + 4 \cdot 10^1 + 6 \cdot 10^0$

2. $(0100\ 1001)_2 = 1 \cdot 2^6 + 1 \cdot 2^3 + 1 \cdot 2^0$

3. $(F00D)_{16} = 15 \cdot 16^3 + 13 \cdot 16^0$

Solution: Question 9

1. Convert (`1111 0000 0000 1101`)$_2$ to base-16
   = F 0 0 D

2. Convert (`CAF3`)$_{16}$ to base-2
   = 1100 1010 1111 0011

Solution: Question 10

1. Convert (`75`)$_{10}$ to base-2

   (a) 75 / 2 = 37 r 1
   (b) 37 / 2 = 18 r 1
   (c) 18 / 2 = 9 r 0
   (d) 9 / 2 = 4 r 1
   (e) 4 / 2 = 2 r 0
   (f) 2 / 2 = 1 r 0
   (g) 1 / 2 = 0 r 1

   = 0100 1011

2. Convert (`0101 1010`)$_2$ to base-10

   (a) $= 1 \cdot 2^6 + 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^1 = 90$