4. Predicates

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About

In this lecture, we are working with predicates. Predicates behave similarly to our propositions from the last lecture, but this time they receive input. This input affects whether the predicate will be true or false.

We are also going to work with the quantifiers, "For All" and "There Exists" along with our predicates, to show that some predicate "P(x)" is *always true* or *sometimes true* for any given input passed in as *x*.

Topics

1. Predicates
2. Quantifiers
3. Counter-examples
4. Negations
5. Common numerical sets
6. Predicates with two variables

1. Predicates

Predicates are similar to propositions, except that predicates take an input, and the result will be a proposition that is either true or false based on that input.

If we make a statement like,
"x is positive", then whether the statement is true or not depends on what *x* is: **True** for 10, **false** for -5.

Notes

Predicate P(x):
A statement that takes in some input *x*, and results in either a **true** or **false** proposition.

1. Predicates

While propositions are written with lowercase letters (usually *p* and *q*),

Predicates are written with uppercase letters and their input variable, like P(x) and Q(x).

Example:

P(x) is the predicate, “*x* is a cool dude.”

Notes

<u>Predicate P(x):</u>
A statement that takes in some input *x*, and results in either a **true** or **false** proposition.

1. Predicates

In programming terms, think of propositions like our **boolean variables**, and predicates like defining a **function** that will return either **true** or **false**.

Proposition

```
bool pizzaIsLate = false;
```

Predicate

```
bool PizzaIsLate( int minutesElapsed )
{
    return ( minutesElapsed > 45 );
}
```

Predicate returns a proposition

```
bool pizzaIsLate = PizzaIsLate( 60 );
```

Notes

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Predicate returns a proposition

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bool pizzaIsLate = PizzaIsLate( 60 );
```

```
if ( pizzaIsLate )
{
    Output( "WHERE" );
    Output( "IS" );
    Output( "MY" );
    Output( "PIZZA?!" );
}
```

Notes

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if ( pizzaIsLate )
{
    Output( "WHERE" );
    Output( "IS" );
    Output( "MY" );
    Output( "PIZZA?!" );
}
```

```
WHERE
IS
MY
PIZZA?!
```

Notes

Predicate P(x):
A statement that takes in some input *x*, and results in either a **true** or **false** proposition.

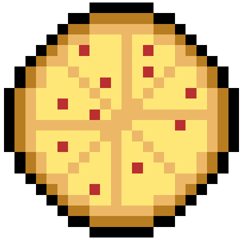
1. Predicates

We could translate this from programming terms:

```
bool PizzaIsLate( int minutesElapsed )
{
    return ( minutesElapsed > 45 );
}
```

To Discrete Math terms:

P(x) is the predicate, "*x* is greater than 45".



Notes

Predicate P(x):
A statement that takes in some input *x*, and results in either a **true** or **false** proposition.

1. Predicates

We can also use our logic operators ∧, ∨, and ¬ with predicates.

P(x) is the predicate, "x is even"
Q(x) is the predicate, "x ends in 0"

$P(x) \wedge Q(x)$: x *is even* and x *ends in 0*

$P(x) \vee Q(x)$: x *is even* or x *ends in 0*

$\neg P(x)$: x *is not even*

Are these statements **true** or **false**?
We don't know until we specify *x*!

Notes

Predicate P(x):
A statement that takes in some input *x*, and results in either a **true** or **false** proposition.

Logic Operators:
∧ And
∨ Or
¬ Not

1. Predicates

Practice 1:

Given the following predicate:
P(x) is the predicate, “x is prime”
Decide whether each of the following results to ***true*** *or* ***false***.

x is 3

x is 9

Notes

Predicate P(x):
A statement that takes in some input *x*, and results in either a **true** or **false** proposition.

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1. Predicates

Practice 1:

Given the following predicate:
P(x) is the predicate, “x is prime”
Decide whether each of the following results to ***true*** *or* ***false***.

x is 3

P(3) is true

x is 9

P(9) is false

Notes

Predicate P(x):
A statement that takes in some input *x*, and results in either a **true** or **false** proposition.

Logic Operators:
∧ And
∨ Or
¬ Not

Sometimes, no matter what you plug into a predicate, it will *always* come out to the same result
(either always **true** or always **false**)

If this is the case, we can use a symbol that means *"for all x"*, to specify that, *"For all input values of x"*, the predicate will evaluate to the same result.

Notes

On the other hand, if a predicate may be **true** for some values of x, and **false** for other values of x, we can use a symbol to say, *"There exists some input value x"* such that the the predicate will give some result.

Notes

2. Quantifiers

“For all” is denoted by **∀**
And
“There exists” is denoted by **∃**

So $\forall x$ would be read as, “For all x…”,

and $\exists x$ would be read as, “There exists some x…”

Notes

For all: $\forall$

There exists: $\exists$

2. Quantifiers

Further, when we're making a quantified statement, we also need to specify what the **domain** is.

"In mathematics, and more specifically in naive set theory, the domain of definition (or simply the domain) of a function is the set of "input" or argument values for which the function is defined."
(From Wikipedia https://en.wikipedia.org/wiki/Domain_of_a_function)

In other words, the **domain** is a **set** of all possible input values we can use.

Notes

For all: $\forall$

There exists: $\exists$

Domain:
Set of input values

$x \in D$
x is an element of D

2. Quantifiers

So, putting this all together, let's say we have a set **D** with a set of numbers…

$$D = \{ 2, 4, 6, 8, 10 \}$$

When we choose some *x* value to plug into our predicate, *x* will be selected from the domain **D**.

Notes

For all: ∀

There exists: ∃

Domain:
Set of input values

$x \in D$
x is an element of D

2. Quantifiers

$D = \{ 2, 4, 6, 8, 10 \}$

In this case, we can see that all elements of the domain D are even numbers, so we could say…

E(x) is the predicate, "x is even"

$\forall$	x	$\in$	$D,$	$E(x)$
For all	elements x	in the set	D,	The predicate E(x) is true.

Notes

For all: $\forall$

There exists: $\exists$

Domain:
Set of input values

$x \in D$
x is an element of D

2. Quantifiers

$$D = \{ 2, 4, 6, 8, 10 \}$$

E(x) is the predicate, “x is even”

$$\forall\, x \in D, E(x)$$

So for our entire **quantified statement**, we need:
1) The quantifier (∀ or ∃)
2) Defining the input variable (x) as a member of the domain D (or some other domain)
3) The statement that will be true for all input values of *x*.

Notes

For all: ∀

There exists: ∃

Domain:
Set of input values

$x \in D$
x is an element of D

2. Quantifiers

Let's say we changed this around so that not all elements of D were even:

$$D = \{ 1, 2, 4, 6, 8, 10 \}$$

E(x) is the predicate, "x is even"

In this case, **$\forall x \in D, E(x)$** is **not** valid here.

We could say that,
there exists an x in D, such that x is not even.
Or, symbolically, **$\exists x \in D, \neg E(x)$**

Notes

For all: $\forall$

There exists: $\exists$

Domain:
Set of input values

$x \in D$
x is an element of D

D = { 1, 2, 4, 6, 8, 10 }
E(x) is the predicate, "x is even"

So what can we say about this domain and this predicate?

- **∃x∈D, E(x)** There exists some element *x* in *D* such that *x* is even.
- **∃x∈D, ¬E(x)** There exists some element *x* in *D* such that *x* is not even.

Notes

For all: ∀

There exists: ∃

Domain:
Set of input values

x ∈ D
x is an element of D

2. Quantifiers

D = { 1, 2, 4, 6, 8, 10 }
E(x) is the predicate, "x is even"

And the following statements would be false:

- **$\forall x \in D, E(x)$** For all elements *x* in *D*, *x* is even.
- **$\forall x \in D, \neg E(x)$** For all elements *x* in *D*, *x* is not even.

Notes

For all: ∀

There exists: ∃

Domain:
Set of input values

$x \in D$
x is an element of D

2. Quantifiers

So in order to state $\forall x \in D, P(x)$
it must be true for ***all input variables.***

And in order to state $\exists x \in D, P(x)$
it must be true for ***at least one input variable.***

Notes

For all: $\forall$

There exists: $\exists$

Domain:
Set of input values

$x \in D$
x is an element of D

2. Quantifiers

Practice 2:

Translate the following into English, and specify whether the quantified statement is true or false.

$\forall x \in D, P(x)$,

$P(x)$ is the predicate, "x is negative".

$D = \{ -5, -10, -15, -20 \}$

Notes

For all: $\forall$

There exists: $\exists$

Domain:
Set of input values

$x \in D$
x is an element of D

Practice 2: Translate the following into English, and specify whether the quantified statement is true or false.

$\forall x \in D, P(x)$,

P(x) is the predicate, “x is negative”.

D = { -5, -10, -15, -20 }

For all elements x in D, x is negative.

For this domain D, this quantified statement is true.

Notes

For all: ∀

There exists: ∃

Domain:
Set of input values

$x \in D$
x is an element of D

Practice 3: Translate the following into English.

$\exists d \in D, G(d),$

G(d) is the predicate, "*d* is a good boy"

D is the set of all dogs.

Notes

For all: $\forall$

There exists: $\exists$

Domain:
Set of input values

$x \in D$
x is an element of D

Practice 3: Translate the following into English.

$\exists d \in D, G(d)$,

G(d) is the predicate, “*d* is a good boy”

D is the set of all dogs.

There exists some dog in the set of all dogs, such that dog *d* is a good boy.

(One may argue that $\forall d \in D, G(d)$, however)

Notes

For all: $\forall$

There exists: $\exists$

Domain:
Set of input values

$x \in D$
x is an element of D

Practice 3: Translate the following into a quantified statement. You must define the domain and the predicate.

All numbers between 1 and 10 are positive.

Hints:
- Is this ∀ or ∃?
- What is the domain?
- Define the predicate.

Notes

For all: ∀

There exists: ∃

Domain:
Set of input values

$x \in D$
x is an element of D

Practice 3:

Translate the following into a quantified statement. You must define the domain and the predicate.

All numbers between 1 and 10 are positive.

$\forall x \in D, P(x)$

Where D = { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 }
And P(x) is the predicate, “x > 0”.

Notes

For all: ∀

There exists: ∃

Domain:
Set of input values

$x \in D$
x is an element of D

Practice 4: Translate the following into a quantified statement. You must define the predicate.

Given the domain D = { 1, 1, 2, 3, 5, 8 }, some numbers are even.

Hints:

- For your predicate, just write **"is even"** in English. We have not yet covered how to specify that an integer is even mathematically; that's for another lesson.

Notes

For all: ∀

There exists: ∃

Domain:
Set of input values

$x \in D$
x is an element of D

Practice 4: Translate the following into a quantified statement. You must define the predicate.

Given the domain D = { 1, 1, 2, 3, 5, 8 }, some numbers are even.

$\exists x \in D, E(x)$
Where E(x) is the predicate, "x is even".*

*** A common error starting out is for students to try to write "x is even" mathematically. They usually write it wrong, so don't! We will cover definitions of *even* and *odd* numbers in another section.**

Notes

For all: $\forall$

There exists: $\exists$

Domain:
Set of input values

$x \in D$
x is an element of D

3. Counter-examples

If we have a quantified statement of the form:

$$\forall x \in D, P(x)$$

If we can find *at least one x* that makes P(x) false, then we can disprove the entire quantified statement with this **counter-example**.

We can only use a counter-example to disprove a statement that specifies “for all”, but we only need one example to do so.

Notes

You can use a single counter-example to disprove a quantified statement that uses “for all”, if you can find an example that makes the predicate false.

3. Counter-examples

So, let's say we have the domain:

$D = \{ 1, 3, 5, 7, 9, 10 \}$

the predicate:

$P(x)$ is "x is odd"

and the quantified statement:

$\forall x \in D, P(x)$

If we inspect the domain D,
we can see that not *all* elements are odd – 10 isn't.

Therefore, we can use 10 as the **counter-example** to show that the quantified statement $\forall x \in D, P(x)$ is **false**.

Notes

You can use a single counter-example to disprove a quantified statement that uses "for all", if you can find an example that makes the predicate false.

Practice 5: Find a counter-example for the following.

Given the domain D = { 3, 4, 6, 12 },
P(x) is the predicate, “x^2 is even”

$$\forall x \in D, P(x)$$

Notes

You can use a single counter-example to disprove a quantified statement that uses “for all”, if you can find an example that makes the predicate false.

3. Counter-examples

Practice 5: Find a counter-example for the following.

Given the domain D = { 3, 4, 6, 12 },
P(x) is the predicate, "x^2 is even"

$$\forall x \in D, P(x)$$

This is false for P(3); P(3) → 3^2 → 9

Notes

You can use a single counter-example to disprove a quantified statement that uses "for all", if you can find an example that makes the predicate false.

4. Negations of Predicates

If we come across a quantified statement that is false, we can say that the *negation* of that statement is true.

D = { 1, 2, 4, 6, 8, 10 }

E(x) is the predicate, “x is even”

Notes

4. Negations of Predicates

For example, let's look at this quantified statement again:

$\forall x \in D, E(x),$ with $D = \{ 1, 2, 4, 6, 8, 10 \}$
and E(x) is the predicate, "x is even"

Not every input *x* in the domain *D* makes the predicate E(x) result to true, so we **cannot** state that it is "true for all elements of x in D".

But what do we get if we negate it?

Notes

4. Negations of Predicates

For example, let's look at this quantified statement again:

$$\forall x \in D, E(x), \quad \text{with} \quad D = \{ 1, 2, 4, 6, 8, 10 \}$$

and E(x) is the predicate, "x is even"

$\neg(\forall x \in D, E(x))$: **It is not true that… For all x in D, x is even.**

Or,

$\exists x \in D, \neg E(x)$: **There exists some x in D, such that x is not even.**

Notes

$\neg(\forall x \in D, P(x))$
$\equiv \quad \exists x \in D, \neg P(x)$

4. Negations of Predicates

Likewise, let's look at a statement that uses "there exists".

$$\exists x \in D, O(x), \qquad \text{with} \quad D = \{ 2, 4, 6, 8, 10 \}$$
$$\text{and} \quad O(x) \text{ is the predicate, "x is odd"}$$

We can see that in the set, *none of the elements are odd!* – they're all even. So what is the negation of this quantified statement?

Notes

$\neg(\forall x \in D, P(x))$
$\equiv \quad \exists x \in D, \neg P(x)$

4. Negations of Predicates

Likewise, let's look at a statement that uses "there exists".

$\exists x \in D, O(x)$, with $D = \{ 2, 4, 6, 8, 10 \}$
and $O(x)$ is the predicate, "x is odd"

$\neg(\exists x \in D, O(x))$: It is not true that…
There exists some x in D,
such that x is odd.

Or,

$\forall x \in D, \neg O(x)$: For all x in D, x is not odd.

Notes

$\neg(\forall x \in D, P(x))$
$\equiv \quad \exists x \in D, \neg P(x)$

$\neg(\exists x \in D, P(x))$
$\equiv \quad \forall x \in D, \neg P(x)$

4. Negations of Predicates

To summarize…

$$\neg(\forall x \in D, P(x)) \equiv \exists x \in D, \neg P(x)$$

It is not true that… (for all x in D, P(x))
≡ There exists some x in D, such that NOT P(x).

$$\neg(\exists x \in D, P(x)) \equiv \forall x \in D, \neg P(x)$$

It is not true that… (there exists some x in D, such that P(x))
≡ For all x in D, NOT P(x).

Notes

$\neg(\forall x \in D, P(x))$
$\equiv \quad \exists x \in D, \neg P(x)$

$\neg(\exists x \in D, P(x))$
$\equiv \quad \forall x \in D, \neg P(x)$

4. Negations of Predicates

Practice 6: Find the negation of the following:

1. $\exists x \in D, P(x)$

2. $\forall x \in D, \neg Q(x)$

Notes

$\neg(\forall x \in D, P(x))$
$\equiv \quad \exists x \in D, \neg P(x)$

$\neg(\exists x \in D, P(x))$
$\equiv \quad \forall x \in D, \neg P(x)$

4. Negations of Predicates

Practice 6: Find the negation of the following:

1. $\exists x \in D, P(x)$

   $\neg (\exists x \in D, P(x))$

   $= \forall x \in D, \neg P(x)$

2. $\forall x \in D, \neg Q(x)$

   $\neg(\forall x \in D, \neg Q(x))$

   $= \exists x \in D, \neg (\neg Q(x))$

   $= \exists x \in D, Q(x)$

Notes

$\neg(\forall x \in D, P(x))$

$\equiv \quad \exists x \in D, \neg P(x)$

$\neg(\exists x \in D, P(x))$

$\equiv \quad \forall x \in D, \neg P(x)$

5. Common numerical sets

Throughout this class, we will be working with sets, but it is impossible to list all the elements of a set like, say,

“The set of all numbers”.

So, we have some special symbols to denote common numerical sets.

Notes

5. Common numerical sets

$\mathbb{R}$ The set of all **real numbers**

This will be pretty much any number you can think of, including those with infinitely repeating decimals. I*maginary numbers* (like $\sqrt{-1}$) are not included.

Notes

$\mathbb{R}$ The set of all **real numbers**

5. Common numerical sets

$\mathbb{R}$ The set of all **real numbers**

This will be pretty much any number you can think of, including those with infinitely repeating decimals. I*maginary numbers* (like $\sqrt{-1}$) are not included.

$\mathbb{Q}$ The set of all **rational numbers**

Any number that can be expressed as a fraction, including whole numbers, as they can be written as *n/1*, as well as fractions that become infinitely repeating decimals like 1/3.

Notes

$\mathbb{R}$ The set of all **real numbers**

$\mathbb{Q}$ The set of all **rational numbers**

5. Common numerical sets

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This will be pretty much any number you can think of, including those with infinitely repeating decimals. I*maginary numbers* (like $\sqrt{-1}$) are not included.

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Any number that can be expressed as a fraction, including whole numbers, as they can be written as *n/1*, as well as fractions that become infinitely repeating decimals like 1/3.

$\mathbb{Z}$ The set of all **integers**

A number that can be written without a fractional component; whole numbers, including 0 and negative numbers.

Notes

$\mathbb{R}$ The set of all **real numbers**

$\mathbb{Q}$ The set of all **rational numbers**

$\mathbb{Z}$ The set of all **integers**

5. Common numerical sets

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Any number that can be expressed as a fraction, including whole numbers, as they can be written as *n/1*, as well as fractions that become infinitely repeating decimals like 1/3.

$\mathbb{Z}$ The set of all **integers**

A number that can be written without a fractional component; whole numbers, including 0 and negative numbers.

$\mathbb{N}$ The set of all **natural numbers**

Usually thought of as “counting numbers”, these are whole numbers, including 0, but excluding any negative numbers.

Notes

$\mathbb{R}$ The set of all **real numbers**

$\mathbb{Q}$ The set of all **rational numbers**

$\mathbb{Z}$ The set of all **integers**

$\mathbb{N}$ The set of all **natural numbers**

5. Common numerical sets

Additionally, we can specify “≥ 0” or “+” to further restrict a set to “greater than or equal to 0”, or “positive”.

Set	Only positive elements	Only elements 0 or greater
$\mathbb{R}$	$\mathbb{R}^+$	$\mathbb{R}^{\geq 0}$
$\mathbb{Q}$	$\mathbb{Q}^+$	$\mathbb{Q}^{\geq 0}$
$\mathbb{Z}$	$\mathbb{Z}^+$	$\mathbb{Z}^{\geq 0}$
$\mathbb{N}$	$\mathbb{N}^+$	$\mathbb{N}^{\geq 0}$

Notes

$\mathbb{R}$ The set of all **real numbers**

$\mathbb{Q}$ The set of all **rational numbers**

$\mathbb{Z}$ The set of all **integers**

$\mathbb{N}$ The set of all **natural numbers**

6. Predicates with multiple variables

Finally, we might also want to write a quantified statement that includes multiple variables. Instead of one input variable, we will have two:

$$P(x, y) \text{ is the predicate, "} y + 1 = x \text{".}$$

But when we add another variable, we have to make sure to add another quantifier for it.

$$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, P(x,y)$$

"For all integers x, there exists some integer y, such that y + 1 = x."

Notes

6. Predicates with multiple variables

When we have the negation of a quantified statement that includes multiple variables, each part of the statement will be negated:

$$\neg(\ \forall x \in \mathbb{Z},\ \exists y \in \mathbb{Z},\ P(x,y)\ )$$

$$\equiv\ \neg(\ \forall x \in \mathbb{Z}\ ),\ \neg(\ \exists y \in \mathbb{Z}\ ),\ \neg(\ P(x,y)\ )$$

$$\equiv\ \exists x \in \mathbb{Z},\ \forall y \in \mathbb{Z},\ \neg P(x,y)$$

Notes

Practice 7: Negate the following quantified statement, and specify whether the negation or the original was true.

$\forall a \in \mathbb{Z}, \forall b \in \mathbb{Z}$, P(x)
P(x) is the predicate, “$a + b \in \mathbb{Z}$”

Noes

Practice 7: Negate the following quantified statement, and specify whether the negation or the original was true.

$\forall a \in \mathbb{Z}, \forall b \in \mathbb{Z}, P(x)$

$P(x)$ is the predicate, "$a + b \in \mathbb{Z}$"

*"For all integers **a** and **b**, a + b is also an integer."* **True**

$$\neg(\forall a \in \mathbb{Z}, \forall b \in \mathbb{Z}, P(x))$$

$$\equiv \exists a \in \mathbb{Z}, \exists b \in \mathbb{Z}, \neg P(x)$$

False

"There exists an integer a and an integer b, such that a + b is not an integer."

Noes

Conclusion

That was a lot of information, so make sure to practice to gain a better grasp of how predicates and quantified statements work!

Next time we will be talking about *implications*, which are essentially “if, then” statements. In discrete math, we can write these either with propositions or with predicates.