2.1 Mathematical Writing

2.1.1 Turning statements to implications

Implications

This time we're exploring mathematical writing and getting introduced to proofs. This means that we are going to be working with **contrapositives** and **implications** some more in order to prove statements. To work with a statement, we turn it into an implication that we can work with mathematically.

Example: For every positive even integer n, $n + 1$ is odd.
Changing to an "if, then" statement, we can form:
If a positive integer n is even, then $n + 1$ is odd.

Question 1
Rewrite the following statements as "if, then" statements. They don't need to be *true* statements, we will talk about disproving statements next.

a. When a positive integer n is odd, then $n + 1$ is even.

b. All squares have four equal sides.

Hint
Think of representing the square as a variable, and the length of a side as a variable.

c. All prime numbers are odd.

2.1.2 Counter-examples

Counter-examples

A **counter-example** is a way to disprove a proposition. For implications, if we can keep the **hypothesis** true, and find a scenario where the **conclusion** ends up being false, then we can disprove a statement.

Example: If n is a prime number, then n is odd.

Can we find any examples of prime numbers that aren't odd? Yes - 2 is a prime number. So this statement is false.

Question 2

Disprove the following statements by coming up with a counter-example:

a. For every even integer n, $n + 1$ is also even.

b. If n is a non-negative integer, then $n! > n$, where $n!$ is n-factorial.

c. For every integer $n \geq 1$, if n is an odd prime, then $n^2 + 4$ is also prime.

2.1.3 Even, Odd, and Divisibility

In Chapter 2, we will be working with the concept of even and odd numbers a lot, and working out proofs relating to these concepts. But, how do you actually specify that some number is even or odd symbolically?

"An integer is even if it is evenly divisible by two
and odd if it is not even."
[...]
"A formal definition of an even number is that it is an integer of the form $n = 2k$, where k is an integer; it can then be shown that an odd number is an integer of the form $n = 2k + 1$." [a]

[a] From https://en.wikipedia.org/wiki/Parity_(mathematics)

Question 3
Identify the following numbers as either even or odd, by writing it as either 2 times some other integer, or 2 times some other integer plus 1.

Example: $21 = 2 \cdot 10 + 1$

a. 7

b. 9

c. 15

d. 8

e. 16

f. 20

Divisibility

We looked at the definitions for an even and odd number. Here's one more - divisibility!

An integer n is divisible by 4 if it is the result of 4 times some other integer. Symbolically, $n = 4k$.

We can use this definition for divisible by any number, as we need.

Question 4

Using the definitions of even, odd, and divisible by *some integer*, prove that the following statements are true...

Example: "12 is even" - rewrite as 2(6).

a. 100 is even.

b. 13 is odd.

c. -13 is odd.

d. 20 is divisible by 5.

e. 20 is divisible by 4.

f. $6n$ is even.

g. $8n^2 + 8n + 4$ is divisible by 4.

2.1.4 Closure properties of integers

The set of all integers is written as $\mathbb{Z}$.

"A set has closure under an operation if performance of that operation on members of the set always produces a member of the same set; in this case we also say that the set is closed under the operation." [a]

- If you add two integers, the result is also an integer
- If you subtract two integers, the result is also an integer
- If you multiply two integers, the result is also an integer
- If you divide two integers, the result **may not be an integer**

We can use these properties in our proofs, by remembering that if two integers k and j are added, the result $k + j$ is also an integer.

[a] From https://en.wikipedia.org/wiki/Closure_(mathematics)

Question 5
Identify if the result of the following operations belong to the set of integers $\mathbb{Z}$. Write the result as either $\in \mathbb{Z}$ or $\notin \mathbb{Z}$.

a. 2 + 8

b. 12 - 4

c. 5 * 3

d. 6 / 3

e. 5 / 2

2.1.5 Proving an implication

Simple proof

How do we prove if an implication is true? We will be studying this throughout Chapter 2...

Example: "The result of summing any odd integer with any even integer is an odd integer."

We should first write this in mathematical language. Let's define our numbers: an even number and an odd number. But how do we write these symbolically?

odd integer	even integer
$x = 2k + 1$	$y = 2j$

An even integer is technically a number that is **evenly divisible by 2**. If we re-word this, we can say that an even integer is "some *other* integer times 2." (Note here that y is our even integer, and it is 2 times some other integer j.)

We know what an even integer is... and an odd integer is just one more than an even number. In this case, we define our odd integer x, as 2 times *some other integer* k, plus 1. (Again note that we're using a different variable for x! j for y and k for x... Make sure to not re-use the same variable.)

With these numbers defined, we can start working mathematically, translating our original statement into symbols.

$x + y$	An odd number plus an even number...
$x + y = (2k + 1) + (2j)$	Adding definitions of odd and even...
$x + y = 2(k + j) + 1$	Rewriting to the definition of an odd #...

$(k + j)$ is still an integer, and the definition of an odd integer is 2 times *some integer* plus 1, proving our statement.

Question 6

Prove the following statements.

a. For all integers $n > 0$, if n is even, then n^2 is also even.

> **Hint**
> Begin with n is even, $n = 2k$, then simplifying 2 to try to get the definition of an even number, $2 \times$ some integer.

b. For all integers $n > 0$, if n is odd, then $n^2 + n$ is even.

> **Hint**
> Begin with n is odd, $n = 2k+1$, then simplifying $(2k+1)^2+(2k+1)$ to try to get the definition of an even number, $2 \times$ some integer.

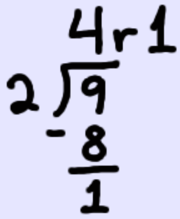
2.2 Proofs about numbers

2.2.1 More definitions

Modulus

"In computing, the modulo operation finds the remainder after division of one number by another (sometimes called modulus).

Given two positive numbers, a (the dividend) and n (the divisor), a modulo n (abbreviated as a mod n) is the remainder of the Euclidean division of a by n." [a]



9 mod 2 = 1

If we're dividing a by b, the result is a quotient q. If we're calculating a mod b, the result is the remainder r.

We can also write this out as:
$a = b \cdot q + r$, where $0 \leq r < b$, and q and r are the only two integers that will satisfy the equation.

[a]From https://en.wikipedia.org/wiki/Modulo_operation

Rational numbers

In mathematics, a rational number is any number that can be expressed as the quotient or fraction p/q of two integers, a numerator p and a non-zero denominator q.[1] Since q may be equal to 1, every integer is a rational number. [a]

The set of rational numbers is written as $\mathbb{Q}$.

[a]From https://en.wikipedia.org/wiki/Rational_number

Question 1
Solve the following modulus problems.

Example: Solve 13 mod 5
13 / 5 = 2, 13 mod 5 = 3, $13 = 5 \cdot 2 + 3$

a. 9 mod 7

b. 5 mod 2

c. 15 mod 3

d. -7 mod 2

Question 2
Prove the following propositions:

a. If a divides b and a divides c, then a divides $b + c$. [1]

> Start with: $b = ak$ and $c = aj$ and calculate $b + c$.

b. If a divides b and c divides d, then ac divides bd. [2]

> Start with: $b = ak, d = cj$ and calculate bd.

[1]From Discrete Mathematics by Ensley and Crawley
[2]From Discrete Mathematics by Ensley and Crawley

2.3 Mathematical induction

2.3.1 Intro practice

Question 1

Plug in the following values to the given predicate, and state whether it results in a true proposition, or a false one. $P(n)$ is "$n^2 + 1$ is prime".

a. $P(1) =$

b. $P(3) =$

c. $P(9) =$

Question 2

For the recursive formula $a_k = a_{k-1} + 4$, $a_1 = 1$, find the values for the following.

a. $a_1 =$

b. $a_2 =$

c. $a_3 =$

d. $a_{m-1} =$ (Anywhere you see k, plug in $m - 1$.)

Question 3

For the closed formula $a_n = 4n - 3$, find the values for the following.

a. $a_1 =$

b. $a_3 =$

c. $a_5 =$

d. $a_{m-1} =$ (Anywhere you see n, plug in $m - 1$.)

2.3.2 Recursive / Closed formula equivalence

Question 3a from the textbook
Show that the sequence defined by $a_k = a_{k-1} + 4; a_1 = 1$ for $k \geq 2$ is equivalently described by the closed formula $a_n = 4n - 3$.

Step 1: Check a_1 for both formulas.
Recursive: $a_1 = 1$ (provided); Closed: $a_1 = 4(1) - 3 = 1$ ✓OK

Step 2: Rewrite the recursive formula in terms of m:

$$a_m = a_{m-1} + 4$$

Step 3: Find the equation for a_{m-1} via the closed formula:

$$a_n = 4n - 3; \qquad a_{m-1} = 4(m-1) - 3; \qquad a_{m-1} = 4m - 7$$

Step 4: Plug a_{m-1} back into the recursive formula and simplify.

$$a_m = a_{m-1} + 4; \qquad \rightarrow \qquad a_m = (4m - 7) + 4$$

$$a_m = 4m - 7 + 4; \qquad a_m = 4m - 3$$

Looking back at the closed formula, $a_n = 4n - 3$, our result from Step 4 and this match, so we have proven that, for all values $k \geq 2$, the recursive formula and closed formula give the same sequence.

Next you will solve these types of proofs, so follow the steps. We will go over several types of proofs this chapter, and they will be on the exam, so make sure you can tell the different *types* of proofs apart!

Question 4

Show that the sequence defined by $a_n = a_{n-1} + 2; a_1 = 5$ for $k \geq 2$ is equivalently described by the closed formula, $a_n = 2n + 3$.

Step 1: Check a_1 for both formulas.

Step 2: Rewrite the recursive formula in terms of m:

Step 3: Find the equation for a_{m-1} via the closed formula:

Step 4: Plug a_{m-1} back into the recursive formula and simplify.

Exponent rules
Power rule: $(a^m)^n = a^{mn}$
Negative exponent rule: $a^{-n} = \frac{1}{a^n}$
Product rule: $a^m \cdot a^n = a^{m+n}$
Quotient rule: $\frac{a^m}{a^n} = a^{m-n}$

Question 5
Show that the sequence defined by $a_k = 2 \cdot a_{k-1} + 1; a_1 = 1$ for $k \geq 2$ is equivalently described by the closed formula, $a_n = 2^n - 1$

Step 1: Check a_1 for both formulas.

Step 2: Rewrite the recursive formula in terms of m:

Step 3: Find the equation for a_{m-1} via the closed formula:

Step 4: Plug a_{m-1} back into the recursive formula and simplify.

Question 6

Show that the sequence defined by $b_k = 4 \cdot b_{k-1} + 3, b_1 = 3$ for $k \geq 2$, is equivalently described by the closed formula $b_n = 2^{2n} - 1$.

Step 1: Check a_1 for both formulas.

Step 2: Rewrite the recursive formula in terms of m:

Step 3: Find the equation for a_{m-1} via the closed formula:

Step 4: Plug a_{m-1} back into the recursive formula and simplify.

2.3.3 Summation / Closed formula equivalence

Another type of proof we will do is to show that, for some n plugged into a sum and into a closed formula for that value.

Question 8a from the textbook

Use induction to prove the proposition. As part of the proof, verify the statement for $n = 1$, $n = 2$, and $n = 3$. $\sum_{i=1}^{n}(2i-1) = n^2$ for each $n \geq 1$.

Step 1: Show that the proposition is true for 1, 2, and 3.

i **value**	$\sum_{i=1}^{n}(2i-1)$	n^2	
$i = 1$	$\sum_{i=1}^{1}(2i-1)$	1^2	
	$= (2 \cdot 1 - 1) = 1$	$= 1$	✓
$i = 2$	$\sum_{i=1}^{2}(2i-1)$	2^2	
	$= (2 \cdot 1 - 1) + (2 \cdot 2 - 1)$		
	$= (1) + (3) = 4$	$= 4$	✓
$i = 3$	$\sum_{i=1}^{3}(2i-1)$	3^2	
	$= 1 + 3 + (2 \cdot 3 - 1)$		
	$= 1 + 3 + 5 = 9$	$= 9$	✓

Step 2: Rewrite the summation as $\sum_{i=1}^{m-1}(2i-1) + (2m-1)$:

$\sum_{i=1}^{m}(2i-1) = \sum_{i=1}^{m-1}(2i-1) + (2m-1)$

Step 3: Find an equation for $\sum_{i=1}^{m-1}$ via the proposition:

$\sum_{i=1}^{n}(2i-1) = n^2$ ... $\sum_{i=1}^{m-1}(2i-1) = (m-1)^2$

Step 4: Plug $\sum_{i=1}^{m-1}$ into the summation from step (2):

$\sum_{i=1}^{m}(2i-1) = \sum_{i=1}^{n-1}(2i-1) + (2m-1) \qquad = (m-1)^2 + (2m-1)$
$\sum_{i=1}^{m}(2i-1) = m^2 - 2m + 1 + 2m - 1$
$\sum_{i=1}^{m}(2i-1) = m^2$ ✓

We get the same form as the original proposition, proving our statement.

Question 7

Use induction to prove

$$\sum_{i=1}^{n}(2i+4) = n^2 + 5n$$

for each $n \geq 1$.

Step 1: Show that the proposition is true for 1, 2, and 3.

i **value**	$\sum_{i=1}^{n}(2i+4)$	n^2+5n
$i = 1$		
$i = 2$		
$i = 3$		

Step 2: Rewrite the summation:
As the sum from $i = 1$ to $m - 1$, plus the final term $(2m + 4)$.

Step 3: Find an equation for $\sum_{i=1}^{m-1}$ via the proposition:

Step 4: Plug $\sum_{i=1}^{m-1}$ into the summation from step 2 and simplify:

Question 8

Use induction to prove that for every positive integer n,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Step 1: Show that the proposition is true for 1, 2, and 3.

i **value**	$\sum_{i=1}^{n} i$	$\frac{n(n+1)}{2}$
$i = 1$		
$i = 2$		
$i = 3$		

Step 2: Rewrite the summation:
As the sum from $i = 1$ to $m - 1$, plus the final term m.

Step 3: Find an equation for $\sum_{i=1}^{m-1}$ via the proposition:

Step 4: Plug $\sum_{i=1}^{m-1}$ into the summation from step 2 and simplify: