

## 1. Turning statements to implications

### Implications

This time we're exploring mathematical writing and getting introduced to proofs. This means that we are going to be working with **contrapositives** and **implications** some more in order to prove statements. To work with a statement, we turn it into an implication that we can work with mathematically.

**Example:** For every positive even integer  $n$ ,  $n + 1$  is odd.

Changing to an “if, then” statement, we can form:

If a positive integer  $n$  is even, then  $n + 1$  is odd.

### Question 1

Rewrite the following statements as “if, then” statements. They don't need to be *true* statements, we will talk about disproving statements next.

**Example:** When a positive integer  $n$  is odd, then  $n + 1$  is even.

$\forall n \in \mathbb{Z}^+$ , if  $n$  is odd, then  $n + 1$  is even.

- a. If a shape is a square, then it has four sides that are of equal length.

### Hint

Here we can define the domain of all shapes,  $S$ , and come up with a variable for some shape  $s$ . If the shape  $s$  is a square, then it will have both four equal sides, AND its sides will be equal length.

- b. All prime integers are odd. (The set of integers is  $\mathbb{Z}$ )

## 2. Counter-examples

### Counter-examples

A **counter-example** is a way to disprove a proposition. For implications, if we can keep the **hypothesis** true, and find a scenario where the **conclusion** ends up being false, then we can disprove a statement.

**Example:** If  $n$  is a prime number, then  $n$  is odd.

Can we find any examples of prime numbers that aren't odd? Yes - 2 is a prime number. So this statement is false.

### Question 2

Disprove the following statements by coming up with a counter-example:

a. For every even integer  $n$ ,  $n + 1$  is also even.

b. For every integer  $n \geq 1$ , if  $n$  is odd, then  $n^2 + 4$  is also prime.

### 3. Even and Odd

In Chapter 2, we will be working with the concept of even and odd numbers a lot, and working out proofs relating to these concepts. But, how do you actually specify that some number is even or odd symbolically?

“An integer is even if it is evenly divisible by two  
and odd if it is not even.”  
[...]

“A formal definition of an even number is that it is an integer of the form  $n = 2k$ , where  $k$  is an integer; it can then be shown that an odd number is an integer of the form  $n = 2k + 1$ .” <sup>a</sup>

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<sup>a</sup>From [https://en.wikipedia.org/wiki/Parity\\_\(mathematics\)](https://en.wikipedia.org/wiki/Parity_(mathematics))

#### Question 3

Identify the following numbers as either even or odd, by writing it as either 2 times some other integer, or 2 times some other integer plus 1.

**Example:**  $21 = 2 \cdot 10 + 1$

a. 7

b. 9

c. 15

d. 8

e. 16

f. 20

## 4. Closure properties of integers

The set of all integers is written as  $\mathbb{Z}$ .

*“A set has closure under an operation if performance of that operation on members of the set always produces a member of the same set; in this case we also say that the set is closed under the operation.”<sup>a</sup>*

- If you add two integers, the result is also an integer
- If you subtract two integers, the result is also an integer
- If you multiply two integers, the result is also an integer
- If you divide two integers, the result **may not be an integer**

We can use these properties in our proofs, by remembering that if two integers  $k$  and  $j$  are added, the result  $k + j$  is also an integer.

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<sup>a</sup>From [https://en.wikipedia.org/wiki/Closure\\_\(mathematics\)](https://en.wikipedia.org/wiki/Closure_(mathematics))

### Question 4

Identify if the result of the following operations belong to the set of integers  $\mathbb{Z}$ . Write the result as either  $\in \mathbb{Z}$  or  $\notin \mathbb{Z}$ .

a.  $2 + 8$

b.  $12 - 4$

c.  $5 * 3$

d.  $6 / 3$

e.  $5 / 2$

## 5. Proving an implication

### Simple proof

How do we prove if an implication is true? We will be studying this throughout Chapter 2...

**Example:** “The result of summing any odd integer with any even integer is an odd integer.”

We should first write this in mathematical language. Let’s define our numbers: an even number and an odd number. But how do we write these symbolically?

$$\begin{array}{ll} \text{odd integer} & \text{even integer} \\ x = 2k + 1 & y = 2j \end{array}$$

An even integer is technically a number that is **evenly divisible by 2**. If we re-word this, we can say that an even integer is “some *other* integer times 2.” (Note here that  $y$  is our even integer, and it is 2 times some other integer  $j$ .)

We know what an even integer is... and an odd integer is just one more than an even number. In this case, we define our odd integer  $x$ , as 2 times *some other integer*  $k$ , plus 1. (Again note that we’re using a different variable for  $x$ !  $j$  for  $y$  and  $k$  for  $x$ ... Make sure to not re-use the same variable.)

With these numbers defined, we can start working mathematically, translating our original statement into symbols.

$$\begin{array}{ll} x + y & \text{An odd number plus an even number...} \\ x + y = (2k + 1) + (2j) & \text{Adding definitions of odd and even...} \\ x + y = 2(k + j) + 1 & \text{Rewriting to the definition of an odd \#...} \end{array}$$

$(k + j)$  is still an integer, and the definition of an odd integer is 2 times *some integer* plus 1, proving our statement.

**Question 5**

Prove the following statements.

- a. For all integers  $n > 0$ , if  $n$  is even, then  $n^2$  is also even.

**Hint**

Begin with  $n$  is even,  $n = 2k$ , then simplifying  $n^2$  to try to get the definition of an even number,  $2 \times \text{some integer}$ .

- b. For all integers  $n > 0$ , if  $n$  is odd, then  $n^2 + n$  is even.

**Hint**

Begin with  $n$  is odd,  $n = 2k+1$ , then simplifying  $(2k+1)^2 + (2k+1)$  to try to get the definition of an even number,  $2 \times \text{some integer}$ .

**Divisibility**

We looked at the definitions for an even and odd number. Here's one more - divisibility!

An integer  $n$  is divisible by 4 if it is the result of 4 times some other integer. Symbolically,  $n = 4k$ .

We can use this definition for divisible by any number, as we need.

**Question 6**

Using the definitions of even, odd, and divisible by *some integer*, prove that the following statements are true...

**Example:** "12 is even" - rewrite as  $2(6)$ .

- a. 100 is even.
- b. 13 is odd.
- c. -13 is odd.
- d. 20 is divisible by 5.
- e. 20 is divisible by 4.
- f.  $6n$  is even.
- g.  $8n^2 + 8n + 4$  is divisible by 4.