3.2 More Operations on Sets

3.2.1 Cartesian Products

We can compute the Cartesian Product of two sets, such as A and B. The result will be a set of **ordered pairs**, such as (a, b), combining the elements of A and B together.

Example: For $A = \{1, 2\}$ and $B = \{4, 5, 6\}$, find $A \times B$.

	$B_1 = 4$	$B_2 = 5$	$B_3 = 6$
$A_1 = 1$	(1, 4)	(1, 5)	(1, 6)
$A_2 = 2$	(2, 4)	(2, 5)	(2, 6)

So the result is that

$$A \times B = \{(1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6)\}$$

Question 1

Given the following sets, calculate each Cartesian Product. Write it out in a table and as a set, like above.

$$A = \{1, 2\} \qquad B = \{3, 4\}$$

a. $A \times B =$

	$B_1 = 3$	$B_2 = 4$
$A_1 = 1$		
$A_2 = 2$		

b. $B \times A =$

	$A_1 = 1$	$A_2 = 2$
$B_1 = 3$		
$B_2 = 4$		

Question 2

Given the following sets, calculate each Cartesian Product. Write it out in a table and as a set.

$$A = \{x, y, z\} \qquad B = \{1, 3\}$$

a. $A \times B =$

	$B_1 = 1$	$B_2 = 3$
$A_1 = x$		
$A_2 = y$		
$A_3 = z$		

b. $B \times A =$

	$A_1 = x$	$A_2 = y$	$A_3 = z$
$B_1 = 1$			
$B_2 = 3$			

Question 3

Calculate the Cartesian Products and write out the result as a set of coordinate pairs.

$$A = \{2, 4\} \qquad B = \{1, 3\} \qquad C = \{3, 4, 5, 6\}$$

a. $A \times B$

b. $A \times C$

c. $B \times C$

d. A^2 (Hint: $A \times A$)

Question 4

With the given sets, find the intersections, unions, and differences.

$$A = \{1\} \qquad B = \{3, 5, 7\} \qquad C = \{3, 5, 9, 11\}$$

$$A \times B = \{(1, 3), (1, 5), (1, 7)\} \qquad A \times C = \{(1, 3), (1, 5), (1, 9), (1, 11)\}$$

a. $(A \times B) - (A \times C)$

b. $(A \times C) - (A \times B)$

c. $A \times (B \cup C)$

d. $(A \times (B \cup C)) \cap (A \times B)$

e. $(A \times B) \cup (A \times C)$

3.2.2 Partitions

The Partition of a set, usually denoted by S, is a set of subsets that, when combined together, form the original set.

Definition: For a set A, a partition of A is a set $S = \{S_1, S_2, S_3, ...\}$ of subsets of A, such that:

1. For all i, $S_i \neq \emptyset$; that is, each part is non-empty.
2. For all i and j, if $S_i \neq S_j$, then $S_i \cap S_j = \emptyset$; that is, different *parts* have nothing in common.
3. $S_1 \cup S_2 \cup S_3 \cup ... = A$; that is, every element in A is in some part.

Clarifications: The elements in S, such as S_i, are just sets of elements that contain elements from A. None of the elements of A can be duplicated across multiple elements from S. And, all elements of A must be present in S.

Example: Let's say we have a set $A = \{1, 2, 3, 4\}$, we could form multiple partitions, such as:

- Partition 1: $\{\{1\}, \{2\}, \{3\}, \{4\}\}$
- Partition 2: $\{\{1, 2\}, \{3, 4\}\}$
- Partition 3: $\{\{1, 2, 3\}, \{4\}\}$
- Partition 4: $\{\{1, 2, 3, 4\}\}$

Essentially, it can be any combination of subsets of whatever size, so long as all elements of A are represented in the partition.

Question 5

For the given set, write out all possible partitions of $A = \{1, 2\}$. There should be 2. Note that the *order* of the elements of the set does not matter.

Question 6

For the given set, write out all possible partitions. There should be 5.

$$B = \{1, 2, 3\}$$

1.
2.
3.
4.
5.

Question 7

Which of the following are valid partitions of the set $A = \{1, 2, 4, 8, 16, 32, 64, 128\}$? For those that are not, explain why not.

a. $\{1, 2, \{4, 8, 16\}, \{32, 64, 128\}\}$

b. $\{\{1, 16\}, \{32, 64, 2\}, \{8, 4, 16\}, \{128\}\}$

c. $\{\{1, 128\}, \{8, 4, 16\}, \{64, 2\}\}$

d. $\{\{8, 4, 2\}, \{16, 1, 128\}, \{32, 64\}\}$

Question 8

For the set $A = \{1, 2, 3, 4, 5, 6\}$, build partitions that meet the following criteria:

a. Find a partition where each part has the same size.

b. Find a partition where no two parts have the same size.

c. Find a partition that has as many parts as possible.

d. Find the partition that has as few parts as possible.

3.2.3 Power Sets

The Power Set of A is defined as $\wp(A) = \{S : S \subseteq A\}$. In other words, the Power Set is a set of all possible subsets you could build from A, including the empty set.

Example 1: Find the Power Set of $\{A\}$.

$$\wp(\{A\}) = \{\emptyset, \{A\}\}$$

Example 2: Find the Power Set of $\{A, B\}$.

$$\wp(\{A, B\}) = \{\emptyset, \{A\}, \{B\}, \{A, B\}\}$$

Example 3: Find the Power Set of $\{A, B, C\}$.

$$\wp(\{A, B, C\}) = \{\emptyset, \{A\}, \{B\}, \{C\}, \{A, B\}, \{B, C\}, \{A, C\}, \{A, B, C\}\}$$

Example 4: Find the Power Set of $\{A, B, C, D\}$.

$\wp(\{A, B, C, D\}) = \{$
$\emptyset,$
$\{A\}, \{B\}, \{C\}, \{D\},$
$\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\},$
$\{A, B, C\}, \{A, B, D\}, \{A, C, D\}, \{B, C, D\},$
$\{A, B, C, D\}$
$\}$
(Phew!)

Question 9

Find the Power Set for each.

a. $\wp(\{1, 2\}) =$

b. $\wp(\{3, 4\}) =$

c. $\wp(\{1, 2, 3\}) =$