5. Implications

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About

In programming, we work with “if, then” statements a lot to decide how program flow branches.

In Discrete Math, we use a hypothesis and a conclusion to make a statement in the form,
“if the hypothesis is true, then the conclusion is true.”

This type of statement is called an implication. Implications and their negations have logic associated with them, which we can model with truth tables.

Topics

1. Implications
2. Truth Tables of implications
3. Negations of implications
4. Contrapositive, Converse, & Inverse

1. Implications

"If, then" statements are an integral part of programming, where we execute a set of instructions only *if* some condition evaluates to true.

```
if ( taller_than_48_inches )
{
    admitted_on_coaster = true;
}
```

With implications in Discrete Math, we identify two propositions; one is the **hypothesis** and one is the **conclusion**, and the implication is of the form,

"If the hypothesis is true, then the conclusion is true."

Notes

1. Implications

An implication is written in the form:

$$p \rightarrow q$$

Which can be read as, *"If p, then q"*, or *"p implies q"*.

In this case, **p** on the left of the arrow is the **hypothesis**, and **q** on the right side of the arrow is the **conclusion**.

In this case, we are using two **propositions**, p and q.

Notes

An implication is of the form
$p \rightarrow q$

1. Implications

So we could create an implication by specifying our propositions and building the implication…

t is the proposition, "the guest is over 48 inches tall"

r is the proposition, "the guest may ride the coaster"

$$\mathbf{t} \rightarrow \mathbf{r}$$

"If <u>the guest over 48 inches tall</u>,
then <u>the guest may ride the coaster</u>."

Notes

An implication is of the form
$p \rightarrow q$

1. Implications

Each side of the implication can be a formal proposition as well, combining multiple propositions together.

a is the proposition, “Bob is 21 or over”,

b is the proposition, “Bob can drink a beer”,

s is the proposition, “Bob can drink a soda”.

$$a \rightarrow (b \lor s)$$

“If Bob is over 21,
then bob can drink a beer, or bob can drink a soda.”

Notes

An implication is of the form
$p \rightarrow q$

1. Implications

Each side of the implication can be a formal proposition as well, combining multiple propositions together.

p is the proposition, "the printer has paper"

o is the proposition, "the printer is out of order"

n is the proposition, "anyone can print a document"

$$(p \land \neg o) \rightarrow n$$

*"If the printer has paper and the printer is **not** out of order, then anyone can print a document."*

Notes

An implication is of the form
$p \rightarrow q$

1. Implications

We can also use predicates with implications, to make an "if, then" statement about something in general.

O(x) is the predicate, "x ends with a 0"

E(x) is the predicate, "x is even"

O(x) → E(x)

"If x ends with a 0,
then x is even"

Notes

An implication is of the form
p → q

1. Implications

<u>Practice 1:</u> Write the following "if, then" statement symbolically as an implication.

"If <u>we can dance</u> and <u>your friends don't dance</u>, then <u>your friends are not my friends</u>"

Propositions:

w: We can dance **d: Your friends dance**

f: Your friends are my friends

Notes

An implication is of the form
$p \rightarrow q$

Practice 1: Write the following "if, then" statement symbolically as an implication.

"If <u>we can dance</u> and <u>your friends don't dance</u>, then <u>your friends are not my friends</u>"

Propositions:

w: We can dance **d: Your friends dance**

f: Your friends are my friends

$$(w \land \neg d) \to \neg f$$

Notes

An implication is of the form

$p \to q$

2. Truth tables of implications

An implication, generally, is in the form:

“if ***the hypothesis is true***,
then ***the conclusion is true***”

How can we tell if an implication is true or false?

What makes an implication false?

Let’s think of this in terms of science experiments.

Notes

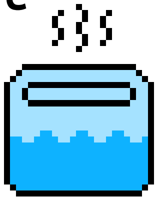
2. Truth tables of implications

Let’s say that we’ve come up with a scientific hypothesis that we want to test…

“If you watch a pot of water, then the water will never boil.”

The hypothesis is, “You watch a pot of water”, and the conclusion is “the water will never boil”.

We are going to perform an experiment to test this hypothesis.



Notes

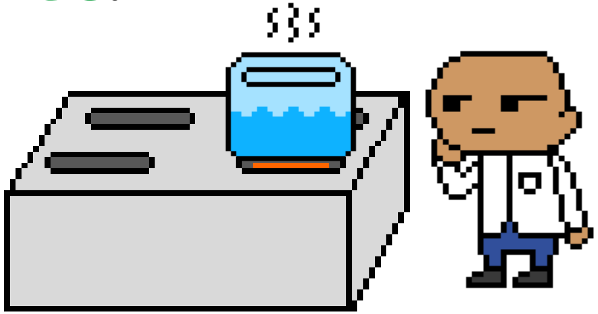
“if ***the hypothesis is true***, then ***the conclusion is true***”

2. Truth tables of implications

"If you watch a pot of water, then the water will never boil."

To test the hypothesis, we heat up a pot of water and watch it.

If the water, indeed, never boils, then the **hypothesis** and the **conclusion** were both true, and the entire **implication is true**.



Notes

"if ***the hypothesis is true***, then ***the conclusion is true***"

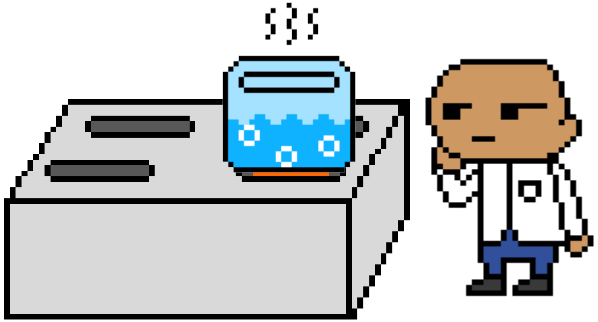
If the hypothesis is **true**, and the conclusion is **true**, then the implication is **true**.

2. Truth tables of implications

"If you watch a pot of water, then the water will never boil."

To test the hypothesis, we heat up a pot of water and watch it.

If the water begins boiling, then while our **hypothesis** was true, the **conclusion is false**, so the **entire implication is false**.



Notes

"if ***the hypothesis is true***, then ***the conclusion is true***"

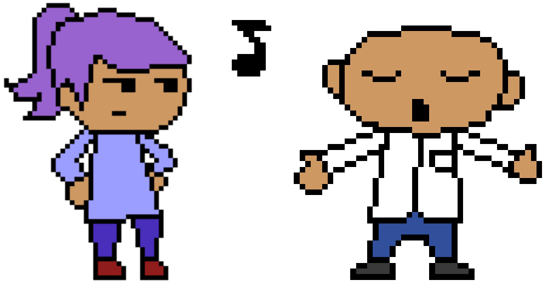
If the hypothesis is **true**, and the conclusion is **true**, then the implication is **true**.

If the hypothesis is **true**, and the conclusion is **false**, then the implication is **false**.

"If you watch a pot of water, then the water will never boil."

However, let's say we ***don't*** watch the pot of water – the hypothesis is ***false***.

What does it mean if we're not watching the pot, and the conclusion is **true** – the pot never boils?



Notes

"if ***the hypothesis is true***, then ***the conclusion is true***"

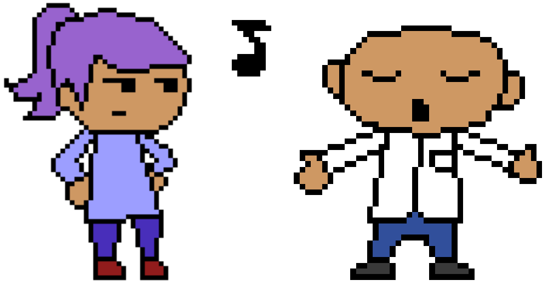
If the hypothesis is **true**, and the conclusion is **true**, then the implication is **true**.

If the hypothesis is **true**, and the conclusion is **false**, then the implication is **false**.

2. Truth tables of implications

"If you watch a pot of water, then the water will never boil."

Well, we haven't proven our implication, and we haven't *disproven it, either*. The logical result of the implication, "if you watch a pot of water, then the water will never boil", is going to be ***true***.



Notes

"if ***the hypothesis is true***, then ***the conclusion is true***"

If the hypothesis is **true**, and the conclusion is **true**, then the implication is **true**.

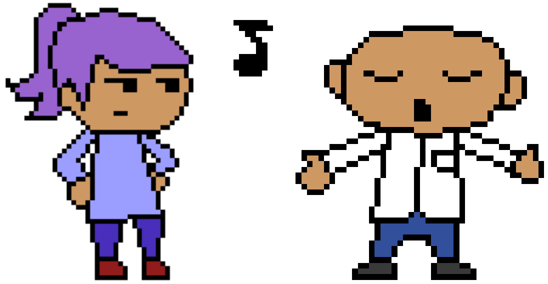
If the hypothesis is **true**, and the conclusion is **false**, then the implication is **false**.

If the hypothesis is **false**, and the conclusion is **true**, then the implication is **true**.

2. Truth tables of implications

"If you watch a pot of water, then the water will never boil."

Maybe that seems weird logically, but think of it as, "the implication still holds – we haven't disproven it. This is because we didn't even do the hypothesis *correctly*."



Notes

"if ***the hypothesis is true***, then ***the conclusion is true***"

If the hypothesis is **true**, and the conclusion is **true**, then the implication is **true**.

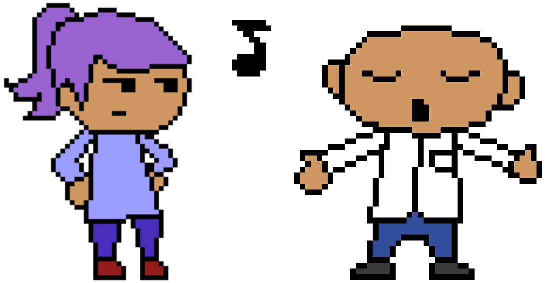
If the hypothesis is **true**, and the conclusion is **false**, then the implication is **false**.

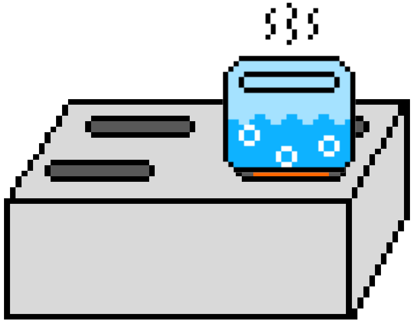
If the hypothesis is **false**, and the conclusion is **true**, then the implication is **true**.

2. Truth tables of implications

"If you watch a pot of water, then the water will never boil."

Similarly, if we're not watching the pot of water (**hypothesis is false**), but it begins boiling (**conclusion is false**), the result of this **implication is *also true*.**





Notes

"if ***the hypothesis is true***, then ***the conclusion is true***"

If the hypothesis is **true**, and the conclusion is **true**, then the implication is **true**.

If the hypothesis is **true**, and the conclusion is **false**, then the implication is **false**.

If the hypothesis is **false**, and the conclusion is **true**, then the implication is **true**.

If the hypothesis is **false**, and the conclusion is **false**, then the implication is **true**.

2. Truth tables of implications

We can only say our implication is **false** if we actually disprove it by making sure the **hypothesis is true** and the **conclusion turns out to be false**.

p	q	p → q
True	True	True
True	False	False
False	True	True
False	False	True

Notes

"if ***the hypothesis is true***, then ***the conclusion is true***"

If the hypothesis is **true**, and the conclusion is **true**, then the implication is **true**.

If the hypothesis is **true**, and the conclusion is **false**, then the implication is **false**.

If the hypothesis is **false**, and the conclusion is **true**, then the implication is **true**.

If the hypothesis is **false**, and the conclusion is **false**, then the implication is **true**.

2. Truth tables of implications

If the **hypothesis is false**, no matter what the conclusion is, the **implication is true**.

p	q	p → q
True	True	True
True	False	False
False	True	True
False	False	True

Notes

"if ***the hypothesis is true***, then ***the conclusion is true***"

If the hypothesis is **true**, and the conclusion is **true**, then the implication is **true**.

If the hypothesis is **true**, and the conclusion is **false**, then the implication is **false**.

If the hypothesis is **false**, and the conclusion is **true**, then the implication is **true**.

If the hypothesis is **false**, and the conclusion is **false**, then the implication is **true**.

2. Truth tables of implications

When we're working with an implication as part of a quantified statement,

$$\forall x, P(x) \rightarrow Q(x)$$

The statement will only be false if we can find a counter-example. In other words, there must exist *at least one x* such that the hypothesis will be false.

Notes

"if ***the hypothesis is true***, then ***the conclusion is true***"

If the hypothesis is **true**, and the conclusion is **true**, then the implication is **true**.

If the hypothesis is **true**, and the conclusion is **false**, then the implication is **false**.

If the hypothesis is **false**, and the conclusion is **true**, then the implication is **true**.

If the hypothesis is **false**, and the conclusion is **false**, then the implication is **true**.

3. Negations of implications

Previously, we saw that if a proposition or a quantified statement was **false**, we could also say that the *negation* of that statement is **true**.

We can also negate our implications.

Notes

Implication

p	q	p → q
True	True	True
True	False	False
False	True	True
False	False	True

3. Negations of implications

Let's say the doctor tells you,

"If you exercise more, you will sleep better"

p is the proposition, "you exercise more"
q is the proposition, "you will sleep better"

The only way the doctor's implication is false is if you exercise more, but don't sleep better.

If you don't exercise, you can't judge the sleep outcome, whether you end up sleeping better or not.

Notes

Implication

p	q	p → q
True	True	True
True	False	False
False	True	True
False	False	True

3. Negations of implications

"If you exercise more, you will sleep better"

p is the proposition, "you exercise more"
q is the proposition, "you will sleep better"

If the implication is **false**, we can figure out the negation simply by using the truth table:

p	q	p → q	¬(p → q)
True	True	True	False
True	False	False	True
False	True	True	False
False	False	True	False

Notes

Implication

p	q	p → q
True	True	True
True	False	False
False	True	True
False	False	True

3. Negations of implications

However, if we want to simplify **¬(p → q)**, what would it be?
The negation isn't **¬p → ¬q**, or any implication...

p	q	¬(p → q)
True	True	False
True	False	True
False	True	False
False	False	False

≠

p	q	¬p → ¬q
True	True	True
True	False	True
False	True	False
False	False	True

Notes

Implication

p	q	p → q
True	True	True
True	False	False
False	True	True
False	False	True

3. Negations of implications

The implication **$\neg(p \to q)$** is actually equivalent to **$p \wedge \neg q$**.
(Hypothesis **p** is true, and conclusion **q** is not-true.)

p	q	$\neg(p \to q)$
True	True	False
True	False	True
False	True	False
False	False	False

$\equiv$

p	q	$p \wedge \neg q$
True	True	False
True	False	True
False	True	False
False	False	False

This is important to remember, as it mixes up a lot of students when they're not paying attention!

The negation of an implication **is not an implication!**

Notes

Implication

p	q	$p \to q$
True	True	True
True	False	False
False	True	True
False	False	True

$\neg(p \to q)$
$\equiv p \wedge \neg q$

3. Negations of implications

Practice 2: Find the negation of the following implications.

1. $p \rightarrow (q \wedge r)$

2. $\forall x \in D, P(x) \rightarrow Q(x)$

Notes

Implication

p	q	p → q
True	True	True
True	False	False
False	True	True
False	False	True

$\neg(p \rightarrow q)$
$\equiv p \wedge \neg q$

3. Negations of implications

Practice 2: Find the negation of the following implications.

1. $p \to (q \land r)$

   $\equiv \neg(p \to (q \land r))$

   $\equiv p \land \neg(q \land r)$

   $\equiv p \land \neg q \lor \neg r$

2. $\forall x \in D, P(x) \to Q(x)$

   $\equiv \neg(\forall x \in D, P(x) \to Q(x))$

   $\equiv \exists x \in D, P(x) \land \neg(Q(x))$

Notes

Implication

p	q	p → q
True	True	True
True	False	False
False	True	True
False	False	True

$\neg(p \to q)$
$\equiv p \land \neg q$

4. Contrapositive, converse, and inverse

Sometimes, it can be helpful to *reframe* an implication in other ways.

If we look at an implication and its contrapositive, they will be logically equivalent to each other.

And the inverse of an implication, and the converse of the same implication, will also be logically equivalent.

Notes

4. Contrapositive, converse, and inverse

Given some implication:

$$p \rightarrow q$$

The **contrapositive** is of the form:

$$\neg q \rightarrow \neg p$$

p	q	p → q	¬p	¬q	¬q → ¬p
True	True	True	False	False	True
True	False	False	False	True	False
False	True	True	True	False	True
False	False	True	True	True	True

Notes

Given p → q,

Contrapositive:
¬q → ¬p

4. Contrapositive, converse, and inverse

So for the implication,

"If <u>you like SHINee</u>,
then <u>you like Korean Pop</u>"

The contrapositive is:

"If <u>you don't like Korean Pop</u>,
then <u>you don't like SHINee</u>"

Notes

Given $p \to q$,

Contrapositive:
$\neg q \to \neg p$

4. Contrapositive, converse, and inverse

Given some implication:

$$p \to q$$

The **converse** is of the form:

$$q \to p$$

(This is not equivalent to $p \to q$,
but I've included both in the truth table for comparison:)

p	q	$p \to q$	$q \to p$
True	True	True	True
True	False	False	True
False	True	True	False
False	False	True	True

Notes

Given $p \to q$,

Contrapositive:
$\neg q \to \neg p$

Converse:
$q \to p$

4. Contrapositive, converse, and inverse

So for the implication,

"If you are a rich man,
then you don't have to work hard"

The converse is:

"If you don't have to work hard,
then you are a rich man"

Notes

Given $p \rightarrow q$,

Contrapositive:
$\neg q \rightarrow \neg p$

Converse:
$q \rightarrow p$

4. Contrapositive, converse, and inverse

Given some implication:

$$p \rightarrow q$$

The **inverse** is of the form:

$$\neg p \rightarrow \neg q$$

(The inverse and the converse are logically equivalent):

p	q	$p \rightarrow q$	$q \rightarrow p$	$\neg p \rightarrow \neg q$
True	True	True	True	True
True	False	False	True	True
False	True	True	False	False
False	False	True	True	True

Notes

Given $p \rightarrow q$,

Contrapositive:
$\neg q \rightarrow \neg p$

Converse:
$q \rightarrow p$

Inverse:
$\neg p \rightarrow \neg q$

4. Contrapositive, converse, and inverse

So for the implication,

"If <u>you collect 100 coins</u>,
then <u>you get an extra life</u>"

The inverse is:

"If <u>you don't collect 100 coins</u>,
then <u>you don't get an extra life</u>"

Notes

Given $p \to q$,

Contrapositive:
$\neg q \to \neg p$

Converse:
$q \to p$

Inverse:
$\neg p \to \neg q$

4. Contrapositive, converse, and inverse

Later on, we will work with ***proofs by contrapositive***, when proving the vanilla form of an implication is difficult.

For now, it is just good to know the logic for the contrapositive, converse, and inverse of an implication.

Notes

Given $p \rightarrow q$,

Contrapositive:
$\neg q \rightarrow \neg p$

Converse:
$q \rightarrow p$

Inverse:
$\neg p \rightarrow \neg q$

4. Contrapositive, converse, and inverse

Practice 3:

Find the contrapositive, converse, and inverse of the following implication in English.

$$p \rightarrow q$$

"If pineapple is good, then pineapple belongs on pizza"

Where **p** is "pineapple is good" and **q** is "pineapple belongs on pizza"

1. **Contrapositive:** $\neg q \rightarrow \neg p$
2. **Converse:** $q \rightarrow p$
3. **Inverse:** $\neg p \rightarrow \neg q$

Notes

Given $p \rightarrow q$,

Contrapositive:
$\neg q \rightarrow \neg p$

Converse:
$q \rightarrow p$

Inverse:
$\neg p \rightarrow \neg q$

4. Contrapositive, converse, and inverse

Practice 3: Find the contrapositive, converse, and inverse of the following implication in English.

$p \to q$

"If pineapple is good, then pineapple belongs on pizza"

Where **p** is "pineapple is good" and **q** is "pineapple belongs on pizza"

1. **Contrapositive:** $\neg q \to \neg p$
   "If pineapple doesn't belong on pizza, then pineapple isn't good."
2. **Converse:** $q \to p$
   "If pineapple belongs on pizza, then pineapple is good."
3. **Inverse:** $\neg p \to \neg q$
   "If pineapple is not good, then pineapple does not belong on pizza."

Notes

Given $p \to q$,

Contrapositive:
$\neg q \to \neg p$

Converse:
$q \to p$

Inverse:
$\neg p \to \neg q$

Conclusion

Remember the truth tables for implications and the negation of an implication, as these two things tend to trip up students!

By covering propositions, predicates, quantified statements, and implications, we have covered the foundation of logic. You brain should now be wired like a `robot`. Please be a benevolent robot.

But overall, this will help you understand the way we use logic in our computer programs to define the program's flow.