Introduction to Functions and Relations

Written by Rachel J. Morris, last updated Mar 20, 2018

About

In algebra, we think of functions as something like “f(x)”, where x is the input, it’s plugged into an equation, and we get some output, f(x).

In programming, we can also define functions. These also have inputs and outputs as well.

Let’s look at another way to view functions.

Topics

1. Functions
2. Binary Relations
3. Inverses
4. Additional notation
5. More practice

1. Functions

1. Functions

Whether we're writing a function in algebra or in a computer program, functions will have inputs and outputs.

We have a **set of all possible inputs** of f, and this set is called the **domain.**

The **set of all possible outputs** of f is called the **codomain.**

Notes

Domain: Set of all possible inputs

Codomain: Set of all possible outputs.

1. Functions

While working with functions in this section, we will use this notation for a function:

$$f : A \rightarrow B$$

Where f is the function name, A is the the **domain**, and B is the **codomain**.

The function f associates one input from A with <u>one and only one</u> output in B.

The mapping between A and B is known as the **rule**. It can be a mathematical expression ("*f(x) = x2*") or even a set of ordered pairs { (a, 1), (b, 2), … }

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function f, with input from set A, & output from set B.

A function f maps some input from A to <u>one and only one</u> output from B.

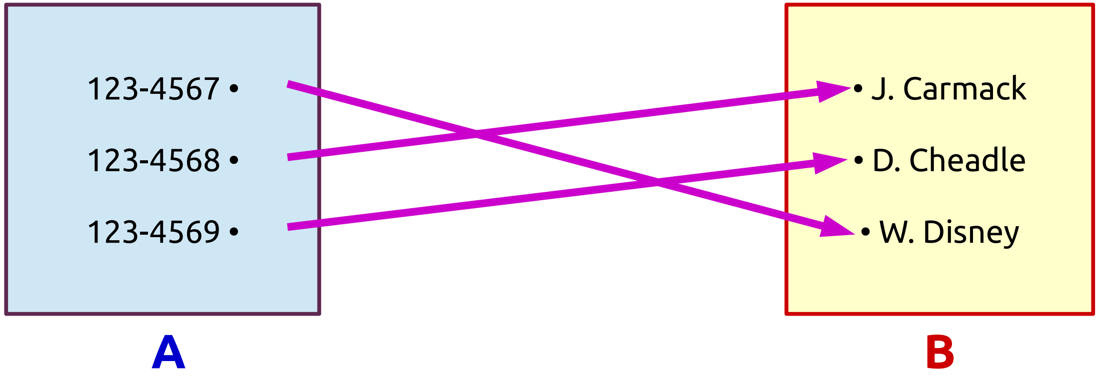
1. Functions

Example: $p : A \rightarrow B$ is a function that maps a phone-number to a single person. Each person has a unique phone number.

A = { 123-4567, 123-4568, 123-4569, etc. }

B = { J. Carmack, D. Cheadle, W. Disney, etc. }

Rule: { (123-4567, J. Carmack), (123-4568, D. Cheadle), (123-4569, W. Disney), … }



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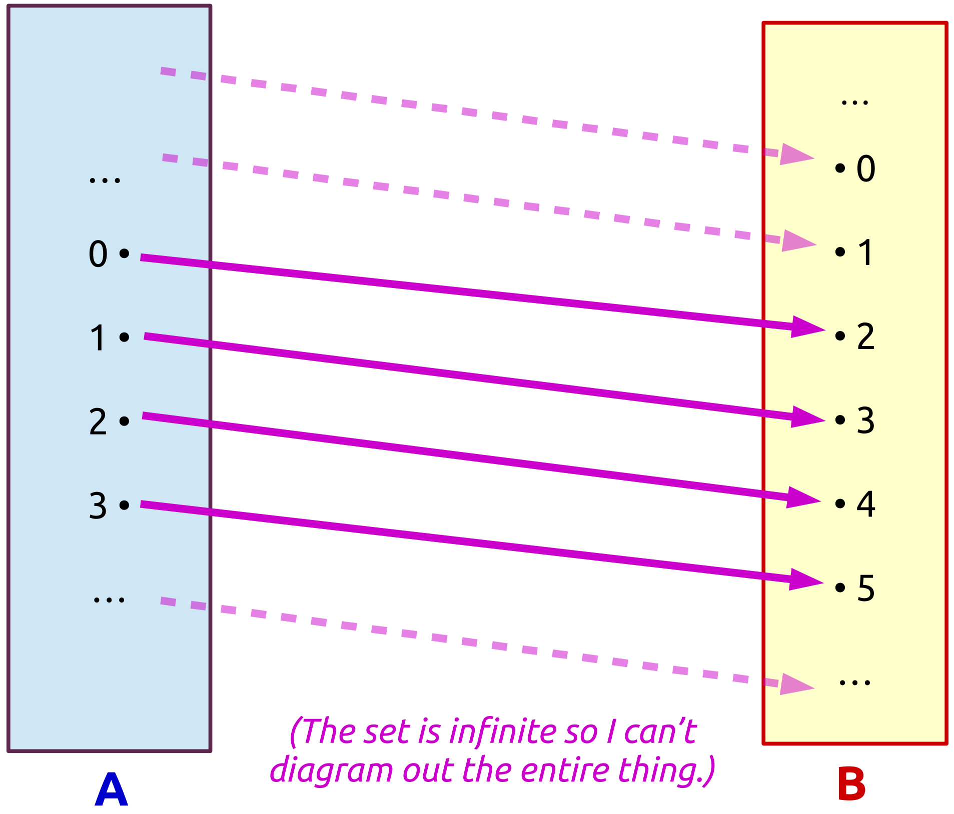
1. Functions

Example:

$f : \mathbb{Z} \rightarrow \mathbb{Z}$ is a function, where the rule is f(x) = x + 2



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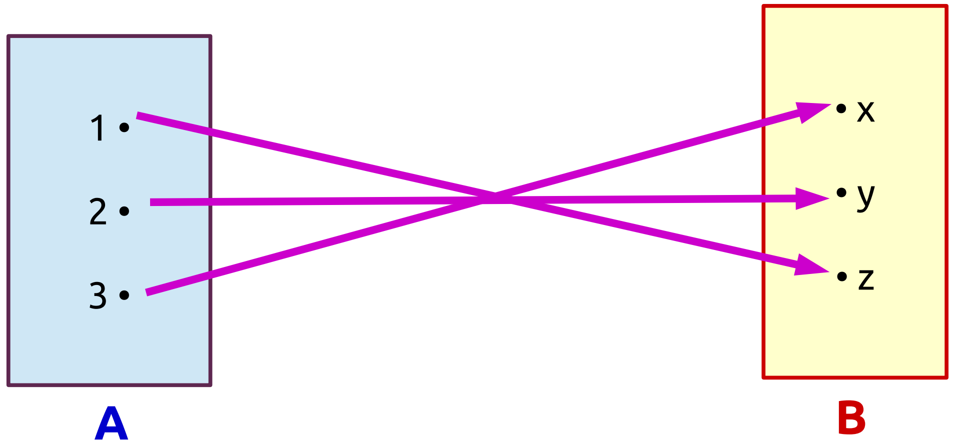
We can also specify the mapping explicitly by using a set of ordered pairs. For example:

Function: $g : A \to B$
A = { 1, 2, 3 }   B = { x, y, z }
Rule: { (1, z), (2, y), (3, x) }



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1. Functions

Practice: Diagram the following function.

Function: $h : X \rightarrow Y$
X = { 2, 4, 6, 8 }
Y = { 1, 3, 5, 7 }
Rule: { (2,7), (4,3), (6,5), (8,1) }

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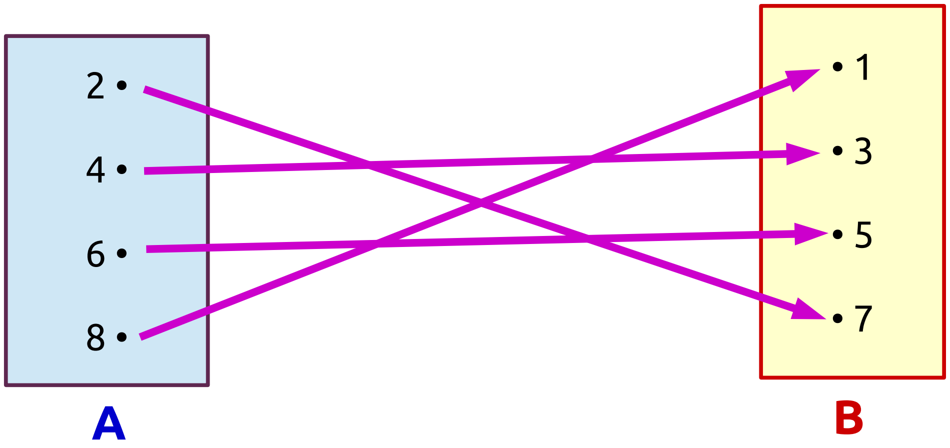

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2. Binary Relations

2. Binary Relations

"In mathematics, a binary relation on a set A is a collection of ordered pairs of elements of A. In other words, it is a subset of the Cartesian product $A^2 = A \times A$.

More generally, a binary relation between two sets A and B is a subset of $A \times B$."

From https://en.wikipedia.org/wiki/Binary_relation

In other words, a binary relation $R : A \to B$ has a **subset of** the **cartesian product A x B** as its **rule**.

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2. Binary Relations

Let's say we have
$A = \{ 1, 2 \}$ and $B = \{ a, b, c \}$,
and a relation $R : A \to B$

The Rule can be the entirety of $A \times B$:

$$\{ (1, a), (1, b), (1, c), (2, a), (2, b), (2, c) \}$$

Or it can just be a **subset** of $A \times B$:

$$\{ (1, a), (1, c), (2, b) \}$$

And this would fit the definition of a binary relation.

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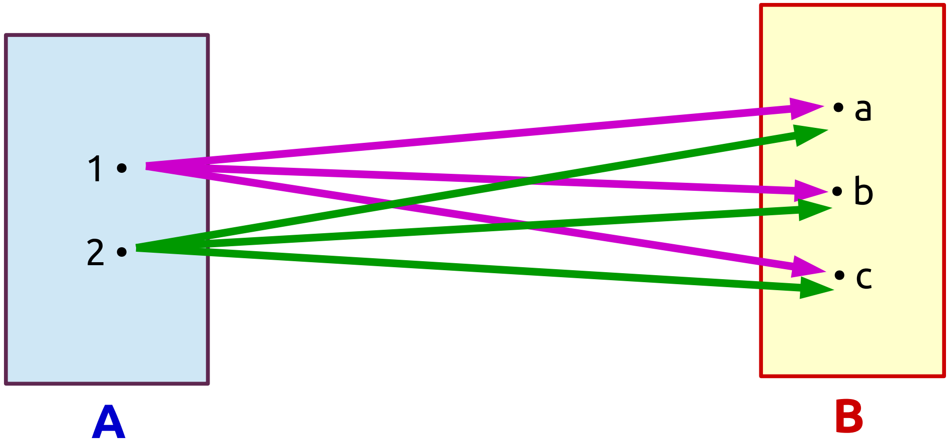
$A = \{1, 2\}$ and $B = \{a, b, c\}$, $R : A \to B$

If the rule is all of A x B,
{ (1, a), (1, b), (1, c), (2, a), (2, b), (2, c) },
it the relation will look like this:



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Practice: Given A = { 1, 2 }, B = { a, b, c }, and the relation S with the rule: { (1, a), (2, b), (1, c) }, diagram the relation.

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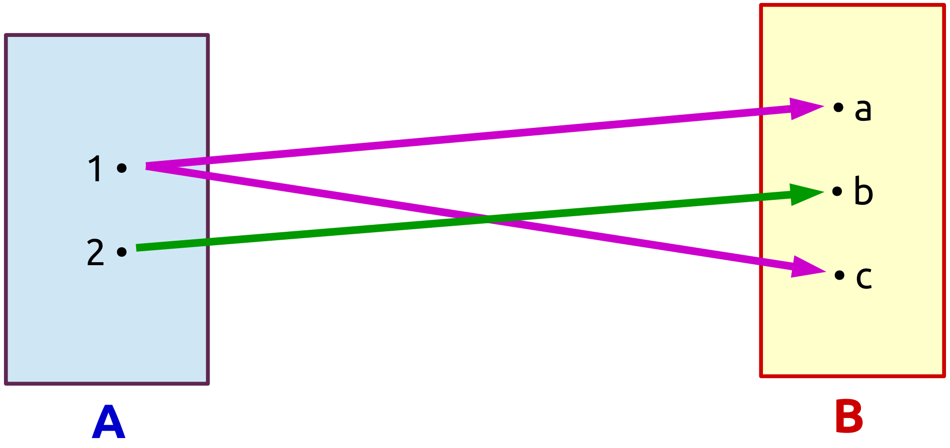

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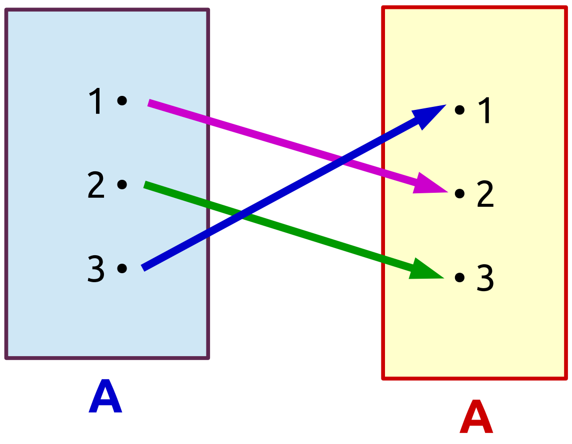
2. Binary Relations

If a binary relation has the same set as the domain and codomain, we can diagram it with the arrow diagram…



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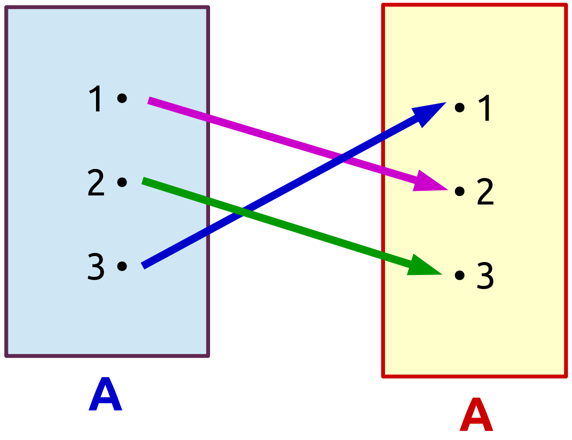
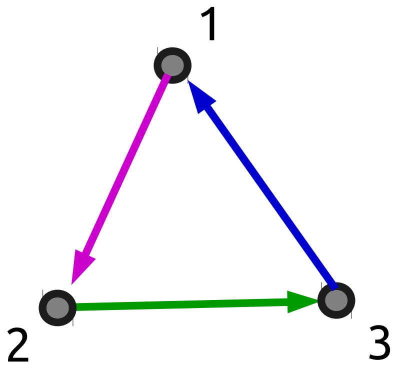


Or with a graph like this:



Diagramming like this will come in handy later with more complex relations.

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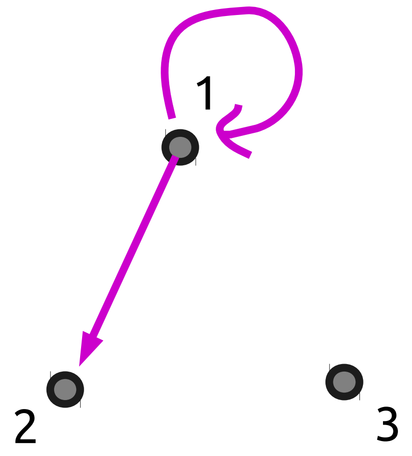
2. Binary Relations

Practice: Finish drawing the diagram for the following relation.
A = { 1, 2, 3 }
R : A → A
Rule: { (1,1), (1,2), (1,3), (2,3), (3,2) }



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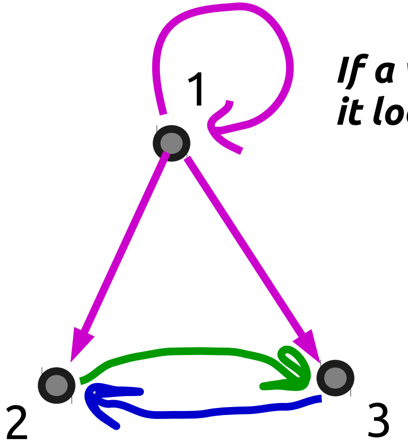

If a value points to itself, it looks like a loop.

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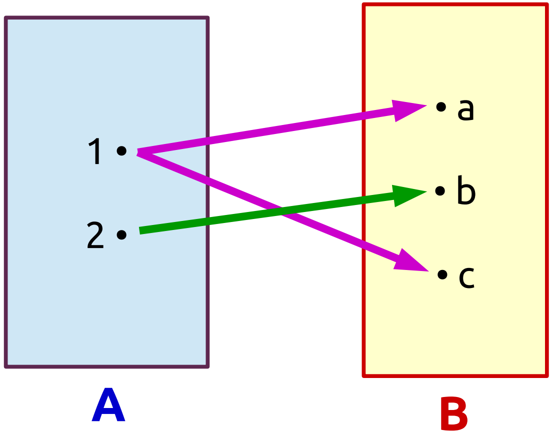
A relation *R* has a subset of *A x B* as its rule.

2. Binary Relations

Remember that a function must map each value from the set A to <u>one and only one</u> value from the set B.

A relation doesn't have this restriction.

Valid relation



Invalid function

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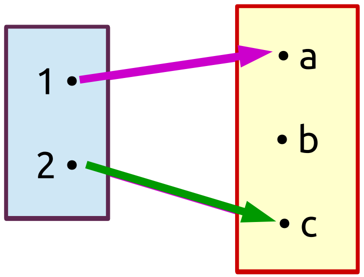
2. Binary Relations

Practice: Identify whether the following relations are also functions.

1



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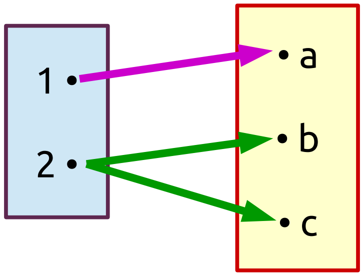

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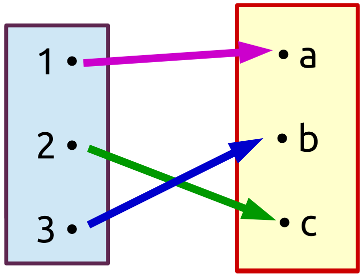

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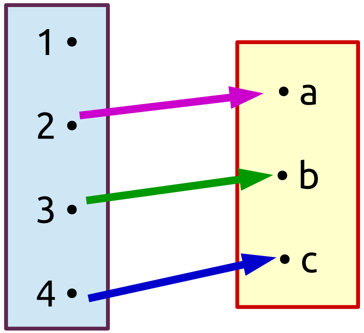

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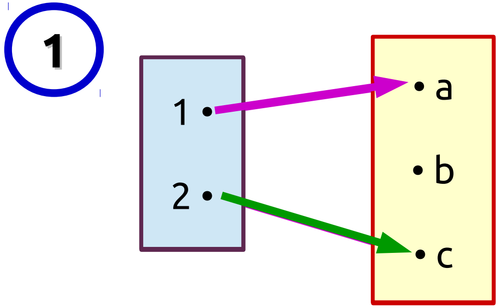
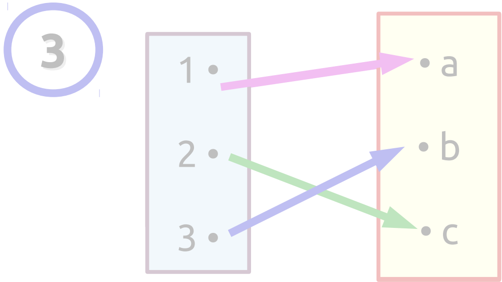
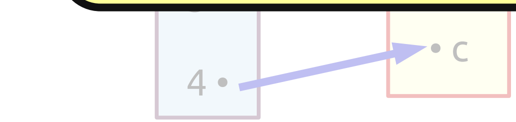


This is a valid function; each input from A maps to one and only one output in B.

It is OK that the output "b" doesn't have anything mapped to it.





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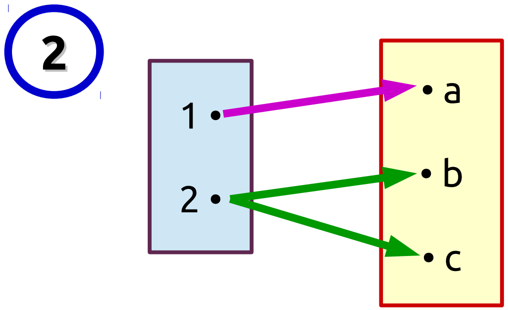
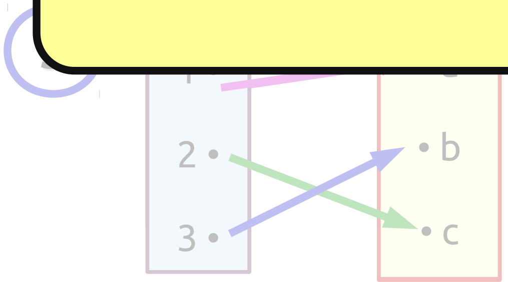
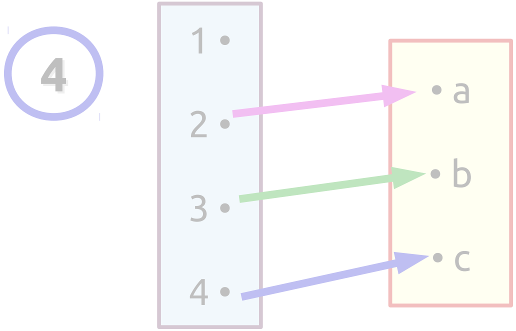
2. Binary Relations

Practice: Identify whether the following relations are also functions.

This is <u>not</u> a function because the input value "2" maps to two different outputs, "b" and "c".







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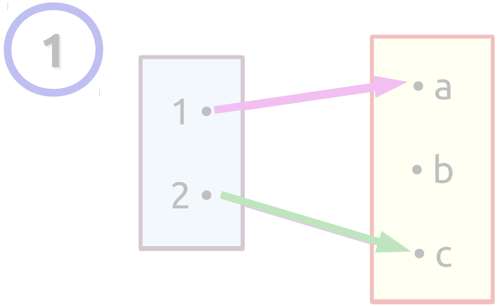

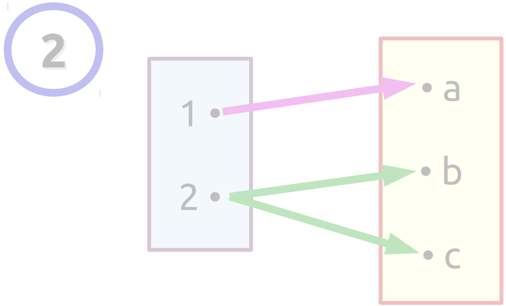

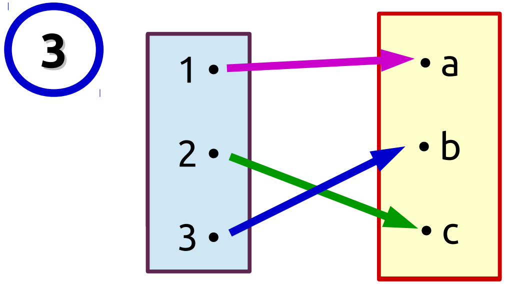

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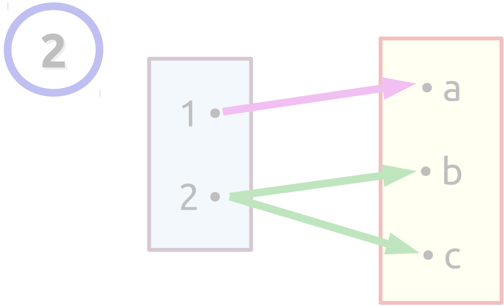
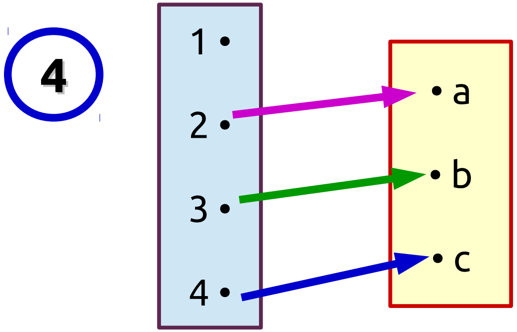
Practice: Identify whether the following relations are also functions.

This is <u>not</u> a valid function because the input value "1" is not being mapped to anything.

While there can be elements in the codomain not mapped *to*, everything in the domain must be mapped *to something*.





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3. Inverses

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We can also take the inverse of a function or a relation.

Given some relation $R : A \to B$,
the inverse is $R^{-1} : B \to A$

We also reverse the mappings, so a map from input x to output y (x, y)

would become (y, x) in the inverse.

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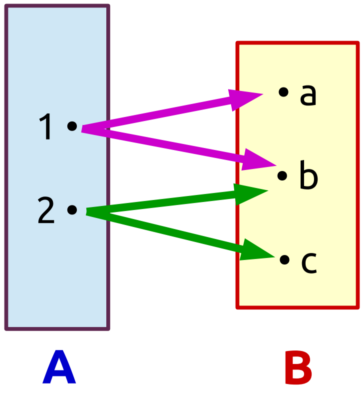
3. Inverses

For example, say we have…

$R : A \to B$
A = { 1, 2 }   B = { a, b, c }
Rule: { (1, a), (1, b), (2, b), (2, c) }



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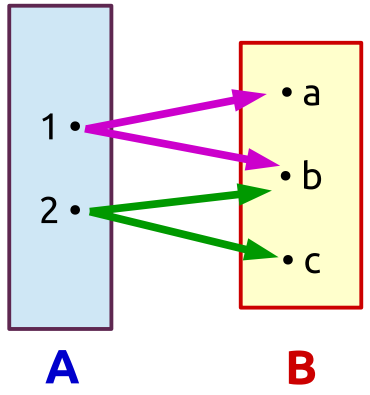
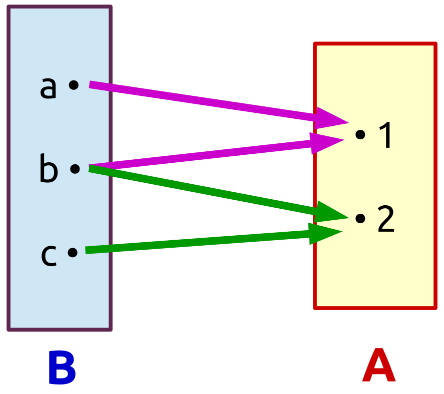


Its inverse will be...

$R^{-1} : B \to A$
A = { 1, 2 }   B = { a, b, c }
Rule: { (a, 1), (b, 1), (b, 2), (c, 2) }



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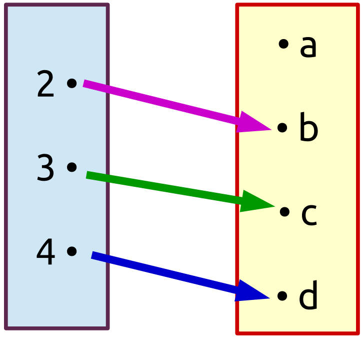
3. Inverses

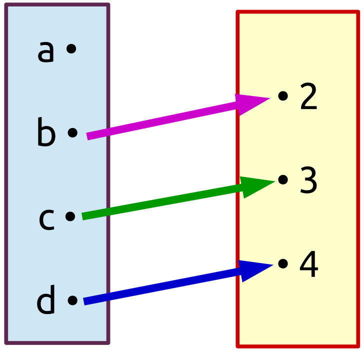
Note that the inverse of a function will not always be a function...

Valid function



f

Invalid function



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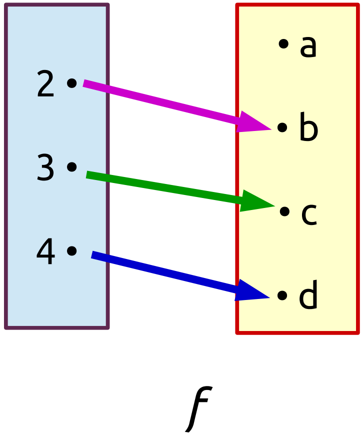

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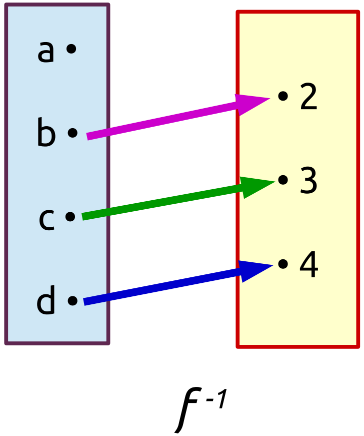

This is <u>not</u> a valid function because the input value “a” is not being mapped to anything.

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3. Inverses

Practice: Diagram the following relation and its inverse.

Relation: R : A → B   A = { 1, 2, 3 }   B = { x, y, z }
Rule: { (1, x), (1, y), (1, z), (2, x), (2, y), (3, z) }

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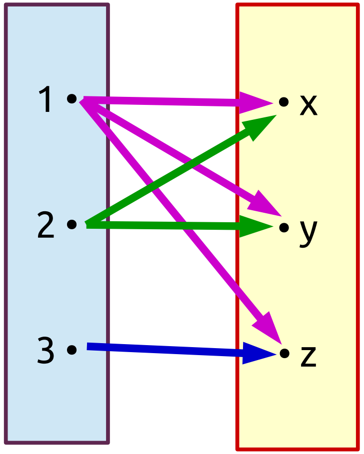
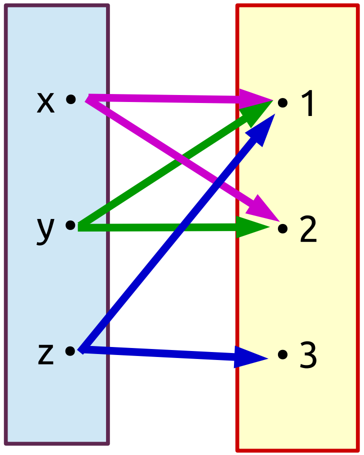


R



R^{-1}

Relation: R^{-1} : A → B   A = { 1, 2, 3 }   B = { x, y, z }
Rule: { (x, 1), (y, 1), (z, 1), (x, 2), (y, 2), (z, 3) }

Notes

Domain: Set of all possible inputs

Codomain: Set of all possible outputs.

$f : A \to B$
function f, with input from set A, & output from set B.

$R : A \to B$
A relation *R* has a subset of *A x B* as its rule.

4. Additional Notation

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1

For a function $f : A \rightarrow B$, writing out $\boldsymbol{(x, y) \in f}$ means The input $\boldsymbol{x}$ *(from A)* is mapped to the output $\boldsymbol{y}$ *(from B)* in the rules of the function f.

The value x is from A, so

$$\mathbf{x \in A}$$

The value y is from B, so

$$\mathbf{y \in B}$$

And the mapping from x to y exists for the function, so

$$\boldsymbol{(x, y) \in f}$$

Notes

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$R : A \rightarrow B$
A relation R has a subset of $A \times B$ as its rule.

5. More practice

5. More practice

Practice: Diagram the relation R : A → B

Where A = $\wp(\{1,2,3\})$

and B = { 0, 1, 2, 3 }

With the rule (x, y) ∈ R if n(x) = y.

Hints: First, A is the power-set of { 1, 2, 3 }. This means, expanded out, all its elements are:

{ {}, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3} }

Notes

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$f : A \to B$
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A relation *R* has a subset of *A* x *B* as its rule.

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Practice: Diagram the relation R : A → B

Where A = $\wp(\{1,2,3\})$ = { {}, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3} }

and B = { 0, 1, 2, 3 }

With the rule (x, y) ∈ R if n(x) = y.

Hints: Second, the rule (x, y) ∈ R if n(x) = y means that the relation exists if n(x) (the size of the input set x) matches the y value.

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5. More practice

Practice: Diagram the relation R : A → B
Where A = $\wp(\{1,2,3\})$ = { {}, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3} }
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With the rule (x, y) ∈ R if n(x) = y.

Hints: Second, the rule (x, y) ∈ R if n(x) = y means that the relation exists if n(x) (the size of the input set x) matches the y value.

If we choose some arbitrary input value x from the set A, we will have a set like {1}, or {1, 3}, or {1, 2, 3}.

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function f, with input from set A, & output from set B.

$R : A \rightarrow B$
A relation *R* has a subset of *A x B* as its rule.

5. More practice

Practice: Diagram the relation $R : A \rightarrow B$
Where $A = \wp(\{1, 2, 3\})$ = { {}, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3} }
and B = { 0, 1, 2, 3 }
With the rule $(x, y) \in R$ if $n(x) = y$.

Hints: Second, the rule $(x, y) \in R$ if $n(x) = y$ means that the relation exists if n(x) (the size of the input set x) matches the y value.

The relationship between the value x and some output y exists if n(x) = y.

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Where A = $\wp(\{1,2,3\})$ = { {}, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3} }
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Hints: Second, the rule (x, y) ∈ R if n(x) = y means that the relation exists if n(x) (the size of the input set x) matches the y value.

So if we choose the input x = {1, 3}, it has a relationship with the output y = 2.

The size of n({1, 3}) is 2.

Notes

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5. More practice

Practice: Diagram the relation R : A → B

Where A = $\wp(\{1,2,3\})$ = { {}, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3} }

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With the rule (x, y) ∈ R if n(x) = y.

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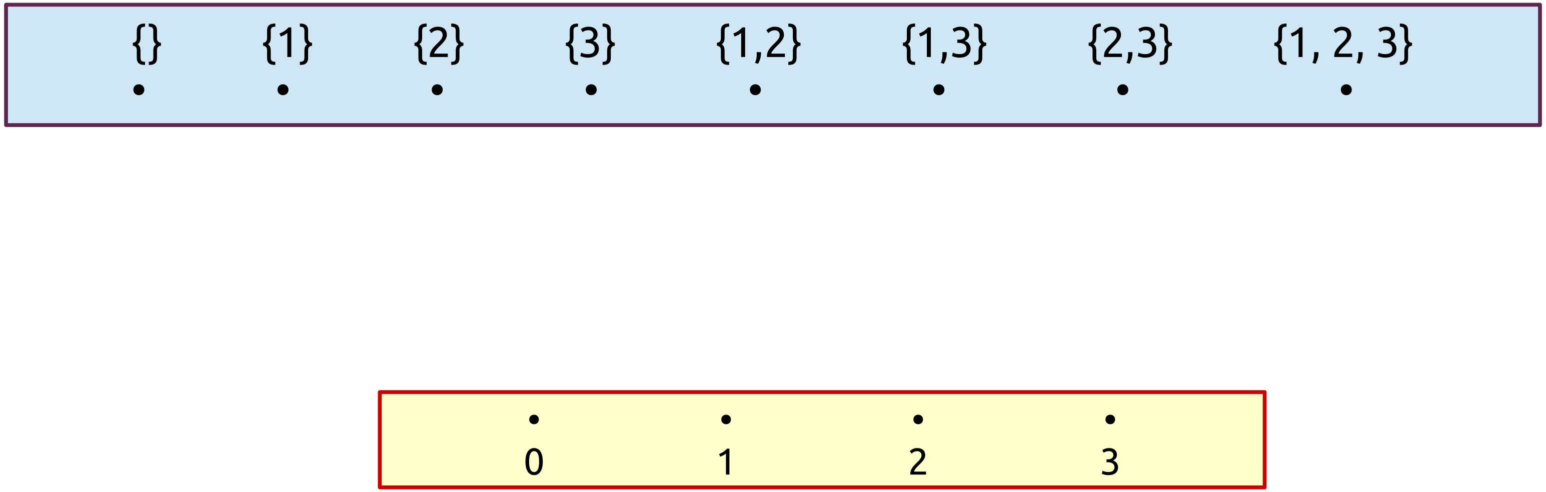
Practice: Diagram the relation $R : A \rightarrow B$

Where $A = \wp(\{1,2,3\})$ = { {}, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3} }

and B = { 0, 1, 2, 3 }

With the rule $(x, y) \in R$ if $n(x) = y$.

So, given this information, try to finish this diagram:



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5. More practice

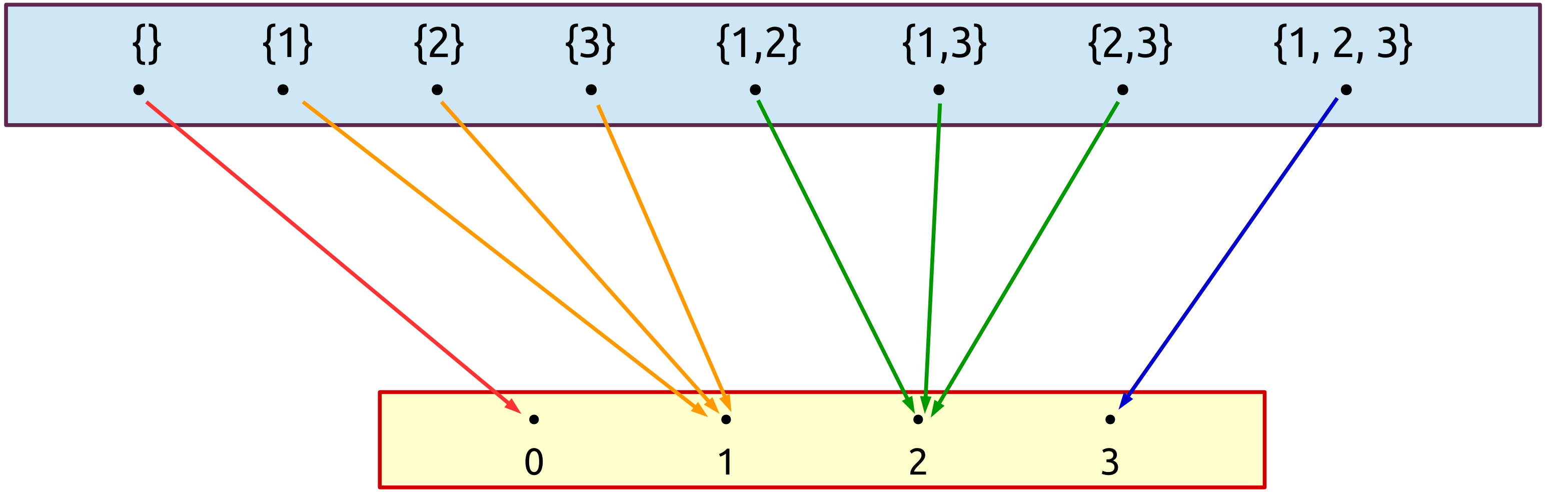
Practice: Diagram the relation R : A → B

Where A = $\wp(\{1,2,3\})$ = { {}, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3} }

and B = { 0, 1, 2, 3 }

With the rule (x, y) ∈ R if n(x) = y.

So, given this information, try to finish this diagram:



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A relation R has a subset of A x B as its rule.

5. More practice

Practice: Diagram the relation $R : A \to A$

Where $A = \{ 1, 2, 3, 4, 5, 6 \}$

With the rule $(x, y) \in R$ if $x - y$ is a positive even integer.

Notes

Domain: Set of all possible inputs

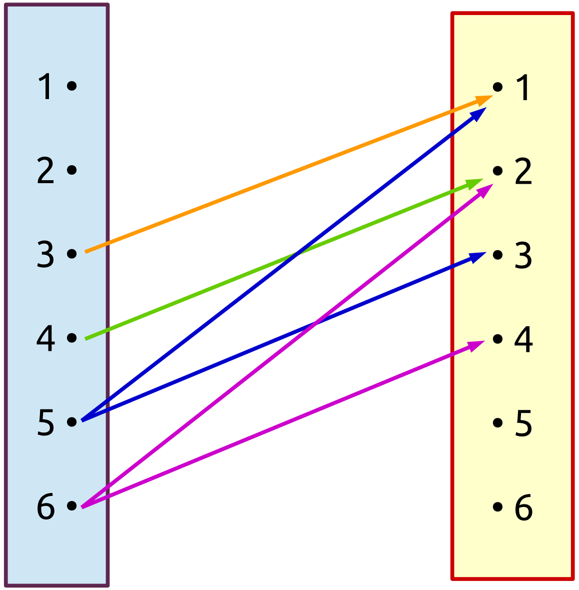
Codomain: Set of all possible outputs.

$f : A \to B$
function f, with input from set A, & output from set B.

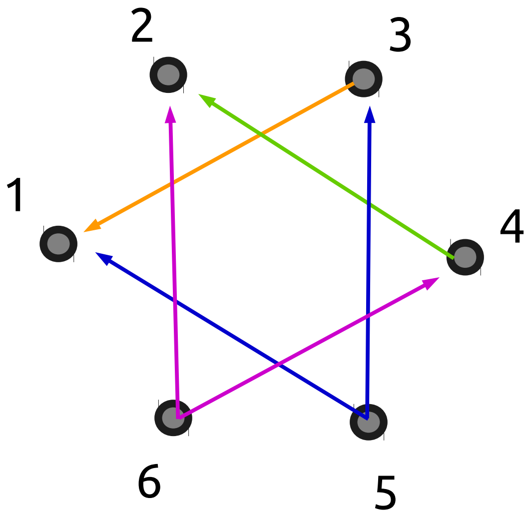
$R : A \to B$
A relation R has a subset of $A \times B$ as its rule.

5. More practice

Practice: Diagram the relation R : A → A
Where A = { 1, 2, 3, 4, 5, 6 }
With the rule (x, y) ∈ R if x – y is a positive even integer.



OR



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A relation *R* has a subset of *A x B* as its rule.

Conclusion

Make sure you understand these core concepts before continuing to the next topics.

Next time we will talk about properties of functions and properties of relations.