Practice problems

Question 1: Direct proofs Chapter 2.1 10%

Prove the following statements using a **direct proof**. Make sure to write the **final answer** in terms of the appropriate definition.

1. If n is even, then $n^2 - n$ is even.
2. The product of two odd integers is always odd.

Question 2: Division theorem Chapter 2.2 10%

Fill in the blanks in the style of the **division theorem**.

> For all integers a and b (with $b > 0$), there is an integer q and an integer r such that:
> 1. $a = b \cdot q + r$ and
> 2. $0 \le r < b$.

1. $20 =$ _____ $\cdot 6 +$ _____
2. $-13 =$ _____ $\cdot 6 +$ _____
3. $(3k^2 + 8) =$ _____ $\cdot 3 +$ _____

Question 3: Modulus Chapter 2.2 10%

Solve the following modulus problems

1. 73 mod 6
2. −15 mod 2

Question 4: Proof by induction Chapter 2.3 10%

Show that the sequence defined by the recursive formula

$$a_k = a_{k-1} + 2 \text{ , where } a_1 = 2, \text{ and for } k \ge 2$$

is equivalently described by the closed formula $a_n = 2n$

Question 5: Proof by induction Chapter 2.3 10%

Use induction to prove that $\sum_{i=1}^{n}(2i-1) = n^2$ for each $n \geq 1$

Question 6: Deriving a recursive formula from a sum Chapter 2.4 10%

Consider the sum $\sum_{i=1}^{n}(3i+2)$

Use s_n to denote this sum. Find a recursive description of s_n.

Question 7: Proof by contradiction Chapter 2.5 10%

Use **proof by contradiction** to prove the following statement:

If $n^2 - 1$ is divisible by 5, then n is not divisible by 5.

Question 8: Numerical representations Chapter 2.6 10%

Write the following numbers as the sum of multiples of powers of the base. ***Do not simplify!***

1. $(\texttt{246})_{10} =$
2. $(\texttt{0100 1001})_2 =$
3. $(\texttt{F00D})_{16} =$

Question 9: Binary ↔ Hexadecimal Chapter 2.6 10%

HEX	0	1	2	3	4	5	6	7
BINARY	0000	0001	0010	0011	0100	0101	0110	0111
HEX	8	9	A	B	C	D	E	F
BINARY	1000	1001	1010	1011	1100	1101	1110	1111

Using the table, convert directly between base-2 and base-16.

1. Convert $(\texttt{ 1111 0000 0000 1101 })_2$ to base-16
2. Convert $(\texttt{ CAF3 })_{16}$ to base-2

Question 10: Binary $\leftrightarrow$ Decimal Chapter 2.6 10%

Using an algorithm, convert between base-2 and base-10.

1. Convert $(\texttt{ 75 })_{10}$ to base-2
2. Convert $(\texttt{ 0101 1010 })_2$ to base-10

Answer key

Solution: Question 1

1. If n is even, then $n^2 - n$ is even.
   1. $n = 2k$
   2. $n^2 - n \Rightarrow (2k)^2 - 2k$
   3. $4k^2 - 2k$
   4. $2(2k^2 - k)$

2. The product of two odd integers is always odd.
   1. Integer 1: $n = 2k + 1$ Integer 2: $m = 2j + 1$.
   2. $n \cdot m \Rightarrow (2k + 1)(2j + 1)$
   3. $4kj + 2k + 2j + 1$
   4. $2(2kj + k + j) + 1$

Solution: Question 2

1. $20 = 3 \cdot 6 + 2$
2. $-13 = -3 \cdot 6 + 5$
3. $(3k^2 + 8) = (k^2 + 2) \cdot 3 + 2$

Solution: Question 3

1. 73 mod 6 = 3
2. -15 mod 2 = 1

Solution: Question 4

1. Check first term:
   Recursive: $a_1 = 2$ Closed: $a_1 = 2(1)$
2. Find a_{m-1} with closed formula:
   $a_{m-1} = 2(m-1) = 2m - 2$
3. Plug a_{m-1} into the recursive formula:
   $a_m = a_{m-1} + 2$
   $a_m = 2m - 2 + 2$
   $a_m = 2m$
4. This matches the closed formula given, so we have proved it.

Solution: Question 5

1. Check first 3 terms:

   (a) $n = 1$:
   Left-hand side: $\sum_{i=1}^{1}(2i-1) = (2(1)-1) = 1$
   Right-hand side: $1^2 = 1$ ✓

   (b) $n = 2$:
   Left-hand side: $\sum_{i=1}^{2}(2i-1) = (2(1)-1) + (2(2)-1)$
   $= 1 + 3 = 4$
   Right-hand side: $2^2 = 4$ ✓

   (c) $n = 3$:
   Left-hand side: $\sum_{i=1}^{3}(2i-1) = (2(1)-1) + (2(2)-1) + (2(3)-1)$
   $= 1 + 3 + 5 = 9$
   Right-hand side: $3^2 = 9$ ✓

2. Rewrite the sum:
   $\sum_{i=1}^{m}(2i-1) = \sum_{i=1}^{m-1}(2i-1) + (2m-1)$

3. Find $\sum_{i=1}^{m-1}(2i-1)$ using the proposition:
   $\sum_{i=1}^{n}(2i-1) = n^2$
   $\sum_{i=1}^{m-1}(2i-1) = (m-1)^2$
   $\sum_{i=1}^{m-1}(2i-1) = m^2 - 2m + 1$

4. Plug $\sum_{i=1}^{m-1}(2i-1)$ into the sum from (2):
   $\sum_{i=1}^{m}(2i-1) = \sum_{i=1}^{m-1}(2i-1) + (2m-1)$
   $\sum_{i=1}^{m}(2i-1) = m^2 - 2m + 1 + 2m - 1$ $\sum_{i=1}^{m}(2i-1) = m^2$

5. This matches the original proposition, so we have proved it.

Solution: Question 6

1. Rewrite the sum:
   $\sum_{i=1}^{m}(3i+2) = \sum_{i=1}^{m-1}(3i+2) + 2m + 2$
   $s_m = s_{m-1} + 2m + 2$

2. Find s_1:
   $s_1 = \sum_{i=1}^{1}(3i+2) = 3(1) + 2 = 5$

3. Write out together:
   $s_m = s_{m-1} + 2m + 2 \qquad s_1 = 5$

Solution: Question 7

1. Hypothesis: $n^2 - 1$ is divisible by 5.
   Conclusion: n is not divisible by 5.

2. Negation:
   $n^2 - 1$ is divisible by 5 and n IS divisible by 5.

3. Symbolically:
   $n^2 - 1 = 5k \qquad n = 5j$

4. Equation:
   $(5j)^2 - 1 = 5k$

5. Simplify until contradiction:
   $25j^2 - 5k = 1 \qquad \rightarrow 5(5j^2 - k) = 1 \qquad \rightarrow 5j^2 - k = \frac{1}{5}$

Solution: Question 8

1. $\texttt{(246)}_{10} = 2 \cdot 10^2 + 4 \cdot 2^1 + 6 \cdot 2^0$

2. $\texttt{(0100 1001)}_2 = 1 \cdot 2^6 + 1 \cdot 2^3 + 1 \cdot 2^0$

3. $\texttt{(F00D)}_{16} = 15 \cdot 16^3 + 13 \cdot 16^0$

Solution: Question 9

1. Convert (`1111 0000 0000 1101`)$_2$ to base-16
   = F 0 0 D

2. Convert (`CAF3`)$_{16}$ to base-2
   = 1100 1010 1111 0011

Solution: Question 10

1. Convert (`75`)$_{10}$ to base-2

   (a) 75 / 2 = 37 r 1
   (b) 37 / 2 = 18 r 1
   (c) 18 / 2 = 9 r 0
   (d) 9 / 2 = 4 r 1
   (e) 4 / 2 = 2 r 0
   (f) 2 / 2 = 1 r 0
   (g) 1 / 2 = 0 r 1

   = 0100 1011

2. Convert (`0101 1010`)$_2$ to base-10

   (a) $= 1 \cdot 2^6 + 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^1 = 90$