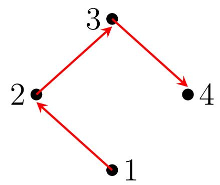
4.4 Properties of Relations

4.4.1 Relations

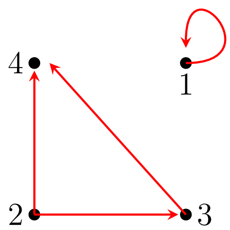
Question 1

Draw the arrows for the following relations:

R



S

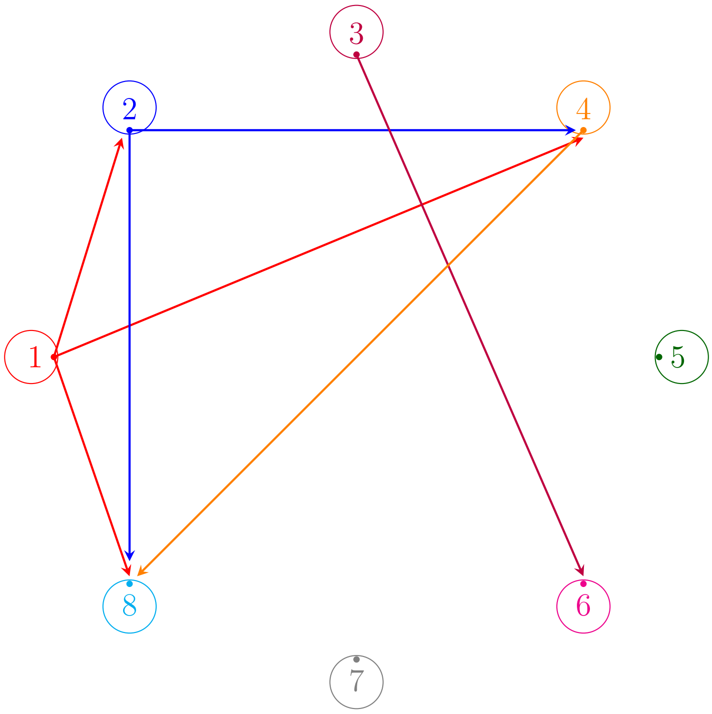


Set A is $A = \{1, 2, 3, 4\}$.
Relation R: $R : A \to A$, with the rule: $\{(1, 2), (2, 3), (3, 4)\}$
Relation S: $S : A \to A$, with the rule: $\{(1, 1), (2, 3), (2, 4), (3, 4)\}$

Question 2

Complete the arrow diagram for the relation on $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Also identify the properties of each. Give reasons why the relation is transitive, intransitive, neither, symmetric, antisymmetric, neither, and/or reflexive.

$R_1 = \{$ (1,1), (1,2), (1,4), (1,8), (2,2), (2,4), (2,8), (3,3), (3,6), (4,4), (4,8) (5,5), (6,6), (7,7), (8,8) $\}$



☐ Reflexive? ☐ Irreflexive? ☐ Neither? Why?

It is transitive - All elements of A point back to itself (every node has a loop).

☐ Symmetric? ☐ Antisymmetric? ☐ Neither? Why?

It is antisymmetric - No arrows go in both directions.

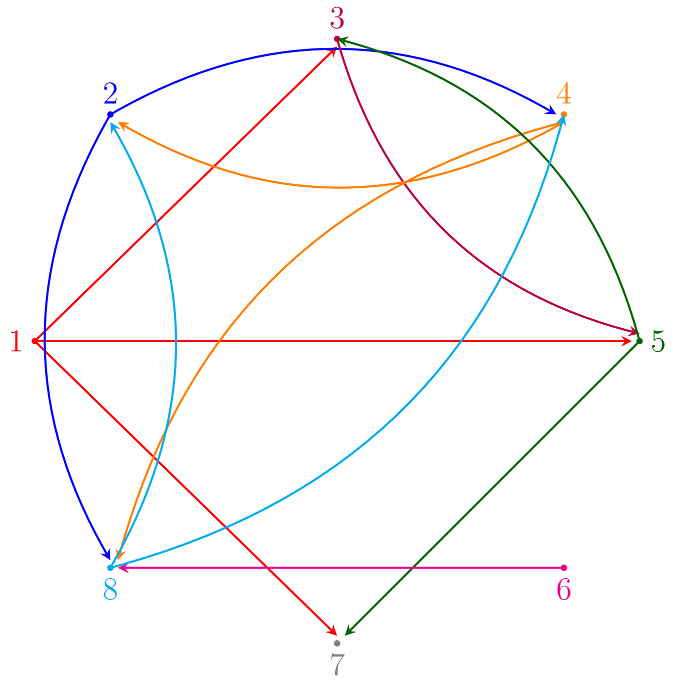
☐ Transitive? Why?

It is transitive - All two-arrow paths also have a direct arrow between the endpoints.

Question 3

Complete the arrow diagram for the relation on $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Also identify the properties of each. Give reasons why the relation is transitive, intransitive, neither, symmetric, antisymmetric, neither, and/or reflexive.

R_2 = { (1,1), (1,3), (1,5), (1,7) (2,2), (2,4), (2,8),
(3,3), (3,5), (3,7), (4,2), (4,4), (4,8), (5,3), (5,7),
(6,6), (6,8), (8,2), (8,4), (8,8) }



☐ Reflexive? ☐ Irreflexive? ☐ Neither? Why?

It is neither – there are some loops, but not on all (reflexive) and not on none (irreflexive).

☐ Symmetric? ☐ Antisymmetric? ☐ Neither? Why?

It is neither - some arrows go in both directions, but not all (symmetric), and not none (antisymmetric). e.g., (2,8) and (8,2) exist.

☐ Transitive? Why?

It is transitive - not; 6 goes to 8, and 8 goes to other places, but 6 doesn't go to other locations.

Recap

- **Reflexive:** $(a, a) \in R$ for all $a \in A$
- **Irreflexive:** $(a, a) \notin R$ for all $a \in A$
- **Antisymmetric:** for all $a, b \in A$, if $a \neq b$ and $(a, b) \in R$, then $(b, a) \notin R$
- **Transitive:** if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$

Question 4

Given the relation, $R_1 = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a + b \text{ is even}\}$.

a. This relation is **reflexive**. Find an example to illustrate why.

> a and b are both in the set of integers. We are checking to see if the result of (a, a) is always in the relation R_1... so, if you plug in (a, a) into the relation, is the output still "is even"?

Since $a + a = 2a$ is always even, we know that $(a, a) \in R_1$ for all $a \in \mathbb{Z}$. Hence, R_1 is reflexive (and not irreflexive).

b. This relation is **symmetric**. Find an example to illustrate why.

> Find some (a, b) and (b, a) that are both in the relation. If you can, it's symmetric.

Since $(1, 3) \in R_1$ and $(3, 1) \in R_1$, R_1 is symmetric.

Question 5

Let C be the set of all cats who have ever lived. For each of the following relations on the set C, decide if the given is reflexive irreflexive, transitive, or antisymmetric. Some of these can satisfy more than one property. Give explanations on how you decided each of these.

a. $R_1 = \{(a, b) \in C \times C : a \text{ is a child of } b\}$

   - Reflexive - Is $(a, a) \in C$ for all a valid? no; a cat cannot be its own parent
   - Irreflexive - Is $(a, a) \notin C$ for all a valid? yes; a cat cannot be its own parent
   - Transitive - Is there some $(a, b) \in C$ and $(b, c) \in C$? [1] no; if "Cat A" is a child of "Cat B", and "Cat B" is a child of "Cat C", then "Cat A" cannot also be a child of "Cat C".
   - Antisymmetric - Is $(a, b) \in C$ and $(b, a) \notin C$ valid? yes; if "Cat A" is a child of "Cat B", then "Cat B" cannot be a child of "Cat A".

b. $R_2 = \{(a, b) \in C \times C : a \text{ is a descendant of } b\}$

   - Reflexive - Is $(a, a) \in C$ for all a valid? no; same as above
   - Irreflexive - Is $(a, a) \notin C$ for all a valid? yes; same as above
   - Transitive - Is there some $(a, b) \in C$ and $(b, c) \in C$? yes; f "Cat A" is a descendant of "Cat B" and "Cat B" is a descendant of "Cat C", then "Cat A" is also a descendant of "Cat C".
   - Antisymmetric - Is $(a, b) \in C$ and $(b, a) \notin C$ valid? yes; same as above

[1] Just assume a cat isn't going to mate with its child. : |