2.5 Proof by contradiction

2.5.1 Review practice

Question 1

For the statement, "if $n\%3 = 1$, then $n\%9 \neq 5$", where % stands for "modulus"...

a. What is the hypothesis p? $n\%3 = 1$

b. What is the conclusion q? $n\%9 \neq 5$

c. Using $\neg(p \rightarrow q) \equiv p \wedge \neg q$, write out the negation of this implication in English. $n\%3 = 1$ and $n\%9 = 5$

This negation can be used to build the counter-example, which we will use for the proof by contradiction.

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2.5.2 Proof by contradiction

Prove by contradiction: If n^2 is even, then n is even

Step 1: Identify the hypothesis and conclusion: Here, our **hypothesis** is "n^2 is even", and our **conclusion** is "n is even".

Step 2: Identify the negation: The negation of an implication is **not** an implication. A counter-example of this would be if we have $p \wedge \neg q$, or "n^2 is even and n is odd".

Step 3: Build a counter-example: Our counter-example is the scenario where the hypothesis is true and the conclusion is false... or in other words, the negation. **Counter-example:** "n^2 is even and n is odd"

Step 4: Write the hypothesis & conclusion symbolically: (For our counter-example implication)
$n^2 = 2k$ (some even integer) $n = 2j + 1$ (some odd integer)

Step 5: Write equation: Using the statement, we are going to turn this into an equation. n is odd, and n^2 is even, so if we square the odd n to get the even n^2, we would have... to get the even n^2, we would have... $(2j + 1)^2 = 2k$

Step 6: Simplify until we have a contradiction:
$(2j + 1)^2 = 2k$
$\Rightarrow \quad 4j^2 + 4j + 1 = 2k$
$\Rightarrow \quad 1 = 2k - 4j^2 - 4j$
$\Rightarrow \quad \frac{1}{2} = k - 2j^2 - 2j$

Since k and j are both integers, through the closure property of integers (+, -, and $\times$ results in an integer), we can show that $k - 2j^2 - 2j$ results in something that is *not an integer* – this is a contradiction. It shows that our counter-example is **false**, and no counter-example can exist.

Question 2

Prove by contradiction: If n^2 is odd, then n is odd.

Step 1: Identify the hypothesis and conclusion:
Hypothesis p: n^2 is odd
Conclusion q: n is odd

Step 2: Identify the negation (counter-example):

p: n^2 is odd AND $\neg q$: n is even

Step 3: Write the hypothesis & conclusion symbolically:
(Make sure you use different variables for n^2 and n.)

(p) $n^2 = 2k + 1$

$(\neg q)$ $n = 2j$

Step 5: Write equation: Set the equation for n-squared equal to the equation for n^2.
$2k + 1 = (2j)^2$

Step 6: Simplify until we have a contradiction:
$2k + 1 = 4j^2$
$\Rightarrow$ $1 = 4j^2 - 2k$
$\Rightarrow$ $\frac{1}{2} = 2j^2 - k$
Result: The result is a fraction, not an integer, therefore no counter-example exists.

Question 3

Use proof by contradiction to explain why it is impossible for a number n to be of the form $5k+3$ and of $5j+1$ for integers k and j.

Hint
$n = 5k+3$ is one statement, and $n = 5j+1$ is the other statement, so $5k+3 = 5j+1$ is your starting point.

This isn't in an implication form, so we begin at Step 5...

Step 5: Write equation: $5k+3 = 5j+1$

Step 6: Simplify until we have a contradiction:

$3 - 1 = 5j - 5k$

$\Rightarrow \quad 2 = 5(j-k)$

$\Rightarrow \quad \frac{2}{5} = j - k$

Result: Since j is an integer and k is an integer, $j - k$ must also be an integer. Here, we get a fraction, so the counter-example is invalid.