2.4 More about induction

2.4.1 Sums as recursive sequences

Last time, we were proving that a sum and a closed formula were equivalent. Now we will use induction to prove that a sum and a recursive sequence are equivalent. Again, we have a set of steps that you'll need to follow in order to solve these.

Example 1 from the textbook Consider the sum $\sum_{i=1}^{n}(2i-1)$, which is the same as $1 + 3 + 5 + ... + (2n-1)$. Use the notation s_n to denote this sum. Find a recursive description of s_n.

Step 1: Find the first term, s_1: We solve for s_1, or in other words, $\sum_{i=1}^{1}(2i-1)$.
$\sum_{i=1}^{1}(2i-1) = (2 \cdot 1 - 1) = 1 \qquad s_1 = 1$

Step 2: Restate the result of s_n as s_{n-1} plus the final term: Similar to last time, we were rewriting some $\sum_{i=1}^{n}$ as the sum up until $n-1$, plus the final term. Here, we're using s_n to represent this sum from $i = 1$ to n, so we can rewrite it as:

$$s_n = s_{n-1} + (2n-1)$$

So we have found an equation and the first term, and we can show that the recursive formula that is equivalent is:

$$s_1 = 1 \qquad s_n = s_{n-1} + 2n - 1$$

Question 1

Consider the sum $\sum_{i=1}^{n}(3n^2)$. Use the notation s_n to denote this sum. Find a recursive description of s_n.

Step 1: Find s_1 $s_1 = \sum_{i=1}^{1}(2n^2) = 2(1)^2 = 2$

Step 2: Rewrite s_n in terms of s_{n-1} plus final term $s_n = s_{n-1} + 3n^2$

So, the recursive formula is:

$s_1 = 2$	$s_n = s_{n-1} + 3n^2$

Question 2

Consider the sum $\sum_{i=1}^{n}(2^{i-1}+1)$. Use the notation s_n to denote this sum. Find a recursive description of s_n.

Step 1: Find s_1 $s_1 = \sum_{i=1}^{1}(2^{i-1}+1) = 2^0 + 1 = 2$

Step 2: Rewrite s_n in terms of s_{n-1} plus final term $s_n = s_{n-1}+2^{n-1}+1$

So, the recursive formula is:

$s_1 = 2$	$s_n = s_{n-1} + 2^{n-1} + 1$

2.4.2 More proofs by induction

Back in the early parts of Chapter 2, we were proving that items were divisible by some number, or even, or odd. If we want to do this for a formula, we can use induction to show that the statement is true for *all* values of n.

Example 6 from the book Show that $n^3 + 2n$ is divisible by 3 for all positive integers n.

Step 1: Rephrase as a function: $D(n) = n^3 + 2n$

Step 2: Check proposition for $D(1)$: Make sure that $D(1)$ is divisible by 3. $D(1) = 1^3 + 2 \cdot 1 = 3$ ✓

Next, we assume that we have shown that the proposition holds for all numbers from $D(1)$ to $D(m-1)$, as part of our inductive proof. (We could also check $D(2)$, $D(3)$, etc.but we will just say we've checked $m-1$ values...)

Step 3: Write out $D(m-1)$ and simplify:
$D(m-1) = (m-1)^3 + 2(m-1) = m^3 - 3m^2 + 3m - 1 + 2m - 2$
Note that we're not completely adding like terms here.

Step 4: Rewrite so that $D(m)$ is part of the equation:
We want to reorganize our terms of $D(m-1)$ to include $D(m)$, which is $m^3 + 2m$.

$D(m-1) = (m^3 + 2m) - 3m^2 + 3m - 3.$

Step 5: Rewrite with $D(m)$: $D(m-1) = D(m) - 3m^2 + 3m - 3.$

Step 6: Solve for $D(m)$: $D(m) = D(m-1) + 3m^2 - 3m + 3$

Step 7: Replace $D(m-1)$... From an earlier step, we said that we have proven the proposition for $D(1)$ through $D(m-1)$. This means that we "have shown" that $D(m-1)$ is divisible by 3, or is some integer times 3. We can then write $D(m-1)$ as $3k$ in our equation instead...

$D(m) = 3k + 3m^2 - 3m + 3$

Step 8: Factor out common terms to prove: Finally, to show it is divisible by 3, we factor out the 3 in the equation:

$D(m) = 3(k + m^2 - m + 1$

Question 3
Use induction to prove that for each integer $n \geq 1$, $2n$ is even.

Step 1: Rephrase as a function:
$D(m) = 2m$

Step 2: Check proposition for $D(1)$:
$D(1) = 2(1) = 2$ ✓

Step 3: Write out $D(m-1)$ and simplify:
$D(m-1) = 2(m-1) = 2m - 2$

Step 4: Rewrite so that $D(m)$ is part of the equation:
$D(m-1) = 2m - 2$

Step 5: Rewrite with $D(m)$:
$D(m-1) = D(m) - 2$

Step 6: Solve for $D(m)$:
$D(m) = D(m-1) + 2$

Step 7: Replace $D(m-1)$:
$D(m) = 2k + 2$

Step 8: Factor out common terms to prove:
$D(m) = 2(k+1)$

Question 4

Use induction to prove that for each integer $n \geq 1$, $4n + 1$ is odd.

Step 1: Rephrase as a function:

$D(m) = 4m + 1$

Step 2: Check proposition for $D(1)$:

$D(1) = 4(1) + 1 = 5$ ✓

Step 3: Write out $D(m-1)$ and simplify:

$D(m-1) = 4(m-1) + 1 = 4m - 4 + 1$

Step 4: Rewrite so that $D(m)$ is part of the equation:

$D(m-1) = (4m + 1) - 4$

Step 5: Rewrite with $D(m)$:

$D(m-1) = D(m) - 4$

Step 6: Solve for $D(m)$:

$D(m) = D(m-1) + 4$

Step 7: Replace $D(m-1)$:

$D(m) = (2k + 1) + 4 = 2k + 4 + 1$

Step 8: Factor out common terms to prove:

$D(m) = 2(k + 2) + 1$

Question 5
Use induction to prove that for each integer $n \geq 1$, $n^2 - n$ is even.

Step 1: Rephrase as a function:
$D(m) = m^2 - m$

Step 2: Check proposition for $D(1)$:
$D(1) = 1^2 - 1 = 0$ ✓
(This will be 0, but according to $n = 2k$, $2(0)$ is "even"?)

Step 3: Write out $D(m-1)$ and simplify:
$D(m-1) = (m-1)^2 - (m-1) = m^2 - 2m + 1 - m + 1$

Step 4: Rewrite so that $D(m)$ is part of the equation:
$D(m-1) = (m^2 - m) - 2m + 2$

Step 5: Rewrite with $D(m)$:
$D(m-1) = D(m) - 2m + 2$

Step 6: Solve for $D(m)$:
$D(m) = D(m-1) + 2m - 2$

Step 7: Replace $D(m-1)$:
$D(m) = 2k + 2m - 2$

Step 8: Factor out common terms to prove:
$D(m) = 2(k + m - 1)$