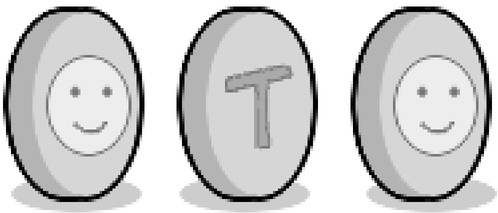
4.5 Equivalence Relations

4.5.1 Thinking about equivalence

Question 1

A person is flipping three coins, and the score that they receive is the sum of all the coins together, where Heads = 1, and Tails = 0. The minimum point value is 0 (T-T-T), and the maximum is 3 (H-H-H). List out all cooin combinations that give equivalent scores. Assume coin toss order does matter, so (T-H-H), (H-T-H), and (H-H-T) will all be separate outcomes.



Points	Coin combinations
0	(T-T-T)
1	(H-T-T), (T-H-T), (T-T-H)
2	(T-H-H), (H-T-H), (H-H-T)
3	(H-H-H)

a. How many ways are there to get 0 points? 1

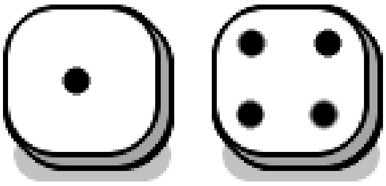
b. How many ways are there to get 1 point? 3

c. How many ways are there to get 2 points? 3

d. How many ways are there to get 3 points? 1

Question 2

A person is rolling two six-sided dice, and the score that the player receives is the sum of the two dice. The minimum point value is 2, and the maximum point value is 12. List out all dice combinations that give equivalent scores. Assume die order does matter, so list both $(2, 1)$ and $(1, 2)$.



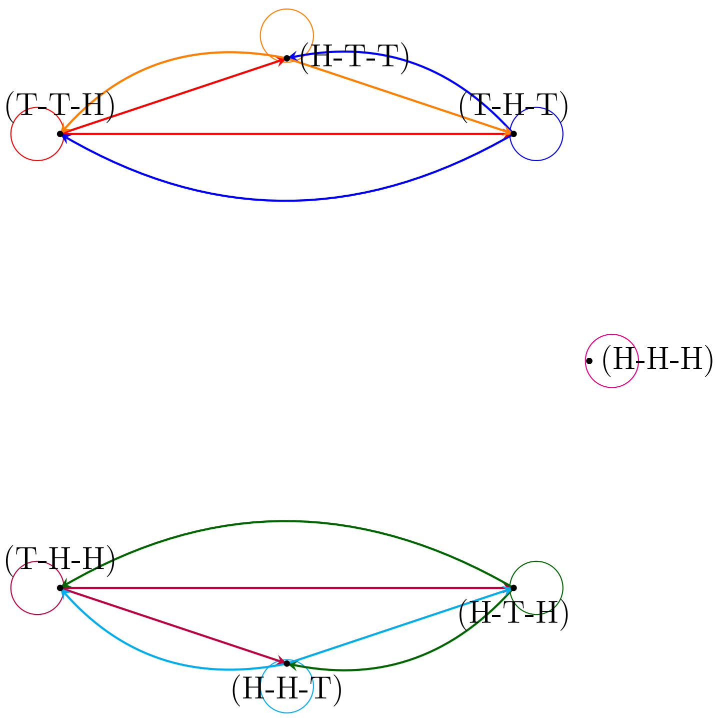
Points	Dice combinations
2	(1,1)
3	(1,2), (2,1)
4	(1,3), (3,1), (2,2)
5	(1,4), (4,1), (2,3), (3,2)
6	(1,5), (5,1), (2,4), (4,2), (3,3)
7	(1,6), (6,1), (2,5), (5,2), (3,4), (4,3)
8	(2,6), (6,2), (3,5), (5,3), (4,4)
9	(3,6), (6,3), (4,5), (5,4)
10	(4,6), (6,4), (5,5)
11	(5,6), (6,5)
12	(6,6)

4.5.2 Equivalence relations

Question 3

Again, we are playing a game where a coin is tossed 3 times. The score outcomes are $S = \{0, 1, 2, 3\}$, and the toss results are
R={(T-T-T), (T-T-H), (T-H-T), (T-H-H), (H-T-T), (H-T-H), (H-H-T), (H-H-H)}

Finish the equivalence relation diagram, where outcomes that result in the same amount of points are considered "the same". Note that each node will have a loop to itself, as it is equivalent to itself.



Symmetry

A relation R on a set A is said to be symmetric if, for all $a, b \in A$, if $(a, b) \in R$, then $(b, a) \in R$.

In terms of arrow diagrams, a symmetric relation has the property that every pair of nodes connected by an arrow is actually connected by two arrows, one in each direction. [a]

[a] Discrete Mathematics, Ensley and Crawley

Question 4

For each of the following relations given, decide if the relation is symmetric. If not, give an example to illustrate this.

a. $R_1 = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a + b \text{ is even } \}$.

If $a + b$ is even, then is $b + a$ also even?

yes

b. $R_2 = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a + 2b \text{ is even } \}$.

For two numbers (a, b), if $a + 2b$ is even, then is (b, a) (or $b + 2a$) also even?

no. Example: (4,2) vs. (2,4), and (2,1) vs. (1,2).

Review: Properties of a relation
Let R be a binary relation on set A.

Reflexive: R is said to be reflexive if $(a, a) \in R$ for all $a \in A$. In terms of the arrow diagram, this means that **every node has a loop**.

Irreflexive: A relation R on set A is irreflexive if, for all $a \in A$, $(a, a) \notin R$. On the arrow diagram, this means **there are no loops.**

Symmetric: R is called symmetric if for all $a, b \in A$, if $a \neq b$ and $(a, b) \in R$, then $(b, a) \in R$. In terms of the arrow diagram, this means that **every arrow goes in both directions**.

Antisymmetric: R is called antisymmetric if for all $a, b \in A$, if $a \neq b$ and $(a, b) \in R$, then $(b, a) \notin R$. In terms of the arrow diagram, this means that **arrows only go in one direction**.

Transitive: R is transitive if, whenever $(a, b) \in R$ and $(b, c) \in R$, it must also be the case that $(a, c) \in R$. In terms of the arrow diagram, this means that **whenever you can follow two arrows to get from node a to node c, you can also get there along a single arrow**.

[a]

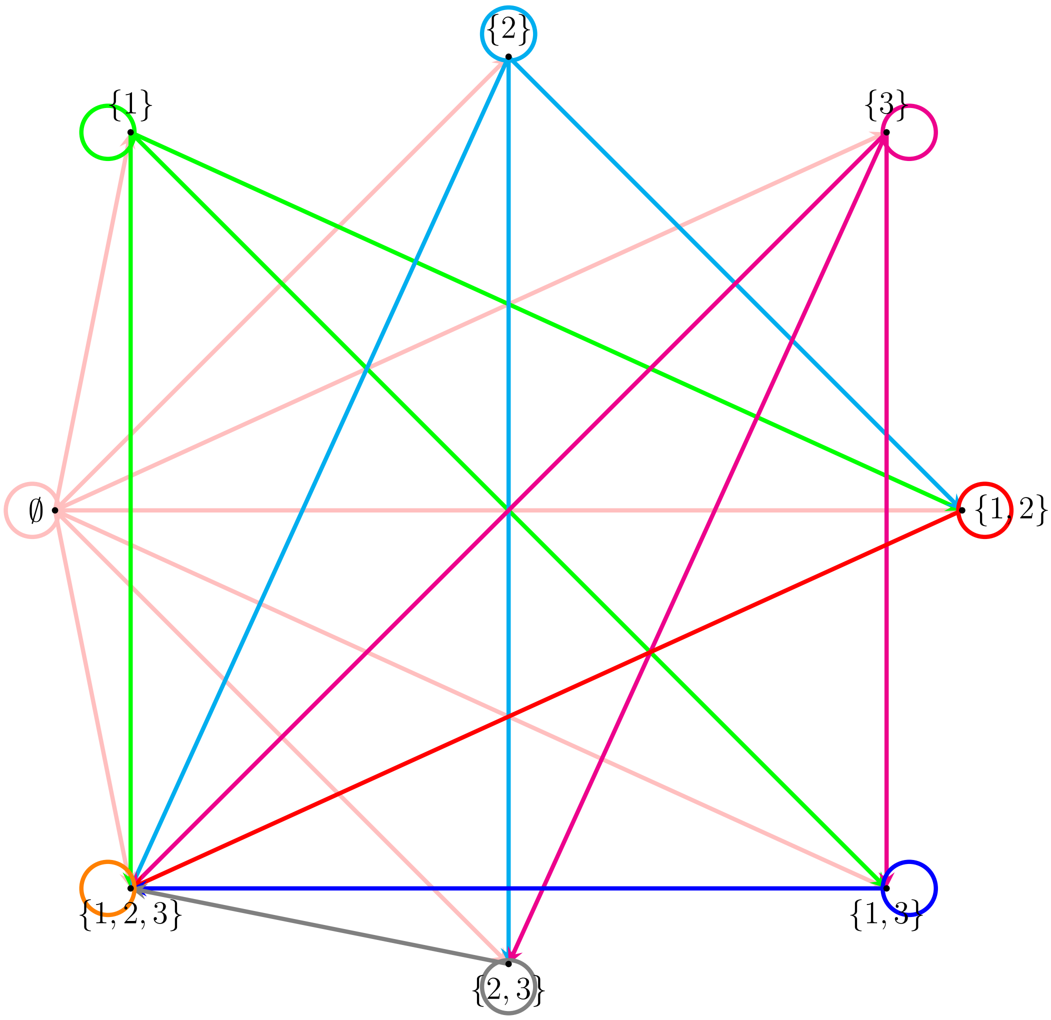
[a]Discrete Mathematics, Ensley and Crawley

Question 5

Let $S = \{1, 2, 3\}$. For each of the following relations on $\wp(S)$, draw the arrow diagram identify its properties: Reflexive, Transitive, and/or Symmetric.

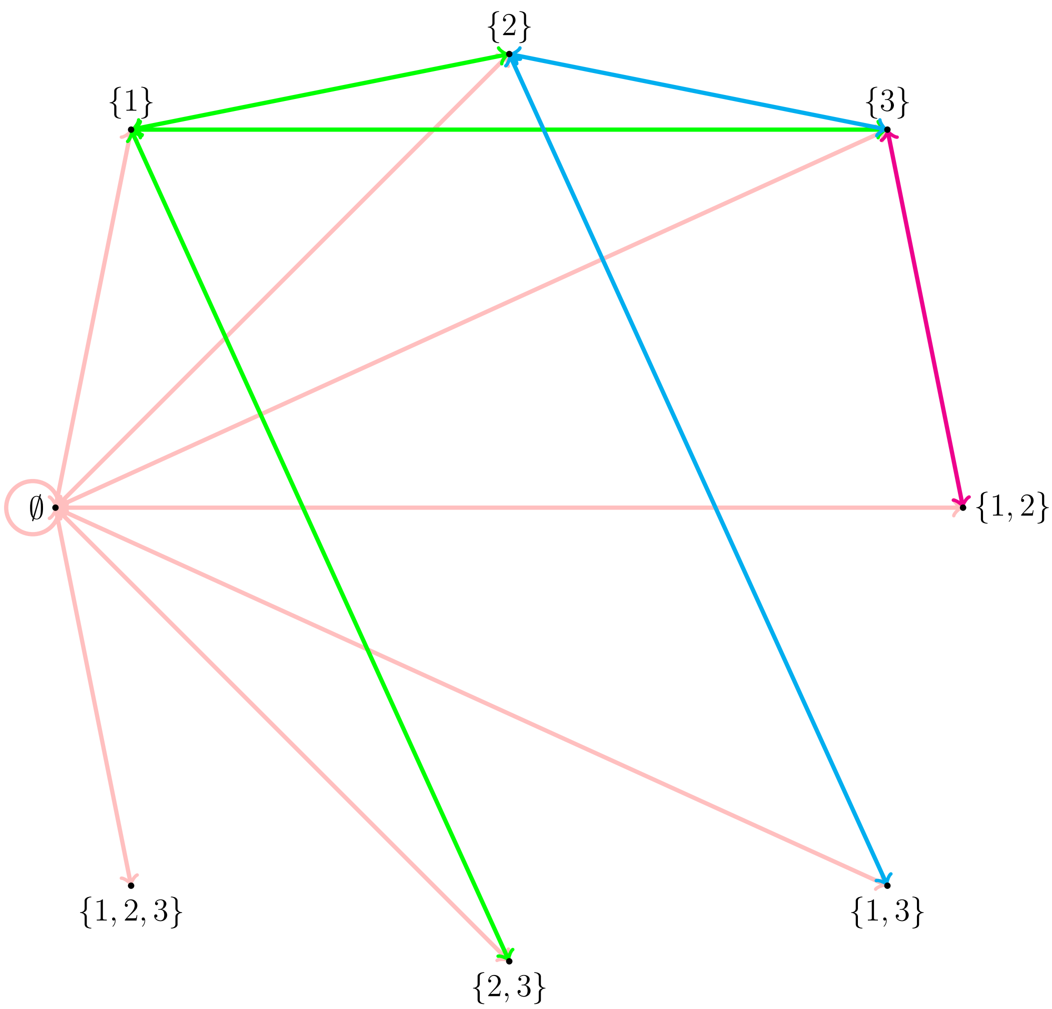
a. $R_1 = \{(A, B) \in \wp(S) \times \wp(S) : A \subseteq B\}$
Hint: Two sets A and B are "equivalent" if A is a subset (or equal to) B. The empty set is considered a subset of everything.

It is reflexive and transitive, but not symmetric.
Example: ({1}, {1,2}) $\in R_1$ but ({1,2}, {1}) is not. (This is antisymmetric.)


{2}
{1}
{3}
∅
{1, 2}
{1, 2, 3}
{1, 3}
{2, 3}

b. $R_2 = \{(A, B) \in \wp(S) \times \wp(S) : A \cap B = \emptyset\}$
Hint: Two sets A and B are "equivalent" if they have nothing in common. The empty set is also considered to not have anything in common with any other set, including itself.

It is symmetric but not reflexive: $(\{1\}, \{1\}) \notin R$, and not transitive: $(\{1\},\{3\}) \in R$, but $(\{1\}, \{1\}) \notin R$.



c. $R_3 = \{(A, B) \in \wp(S) \times \wp(S) : n(A) = n(B)\}$
Hint: Two sets A and B are "equivalent" if they have the same amount of elements.

It is reflexive, symmetric, and transitive.