3. Propositional Logic

Written by Rachel J. Morris, last updated Jun 28, 2017

About

If you've done any programming previously, these topics might seem familiar to you.

With propositional logic, we are making a statement that is either **true** or **false**, and we can modify and expand these statements with AND, OR, and NOT.

We will also be working with Truth Tables to figure out the result of a compound statement made up of two or more propositional variables.

Topics

1. Propositions
2. Logic Notation
3. Truth Tables
   And logical equivalence
4. Tautology & Contradiction

1. Propositions

In this part, we are working with propositions. **Propositions** are statements that are either <u>true</u> or <u>false</u>. You can also think of it as a “yes/no” question if phrased in question form.

<u>In English…</u>	**<u>More math-y…</u>**
“Fran has a cat.”	*“$x > 10$”*
“The cat is black.”	*“x and y are equal”*

Notes

Propositions are statements that are either true or false.

1. Propositions

A **formal proposition** is also a statement that will be **true** or **false**, but may contain AND, OR, and NOT statements, combining multiple individual propositions into a single formal proposition.

In English...	**More math-y...**
"Fran has a cat AND the cat is black"	*"x > 10 AND x < 20"*
	"x equals y AND x does NOT equal z"

Notes

Propositions are statements that are either true or false.

Formal Propositions can contain one or more propositions.

1. Propositions

So, we can build a formal proposition, like…

"The printer is offline
AND
The printer is out of paper"

The entire statement will be **true** if each proposition it is built from is also **true**, since we're using an AND to link them.

Notes

Propositions are statements that are either true or false.

Formal Propositions can contain one or more propositions.

1. Propositions

So, we can build a formal proposition, like…

“The printer is offline
AND
The printer is out of paper”

If one *part* happens to be false, like the printer is *online*, then this entire **formal proposition** will be **false**, because we can’t say that both are true at once.

Notes

Propositions are statements that are either true or false.

Formal Propositions can contain one or more propositions.

1. Propositions

So, we can build a formal proposition, like…

“The printer is offline
OR
The printer is out of paper”

If we change the AND to an OR, then it is OK for one of the parts to be **false**, as long as *at least one* proposition is **true**.

Notes

Propositions are statements that are either true or false.

Formal Propositions can contain one or more propositions.

1. Propositions

"The printer is offline
AND
The printer is out of paper"

For this **formal proposition** to be true, all propositions must also be true.

If any propositions are false, then the entire formal proposition is false.

Notes

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Formal Propositions can contain one or more propositions.

1. Propositions

"The printer is offline
AND
The printer is out of paper"

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AND
The printer is out of paper"

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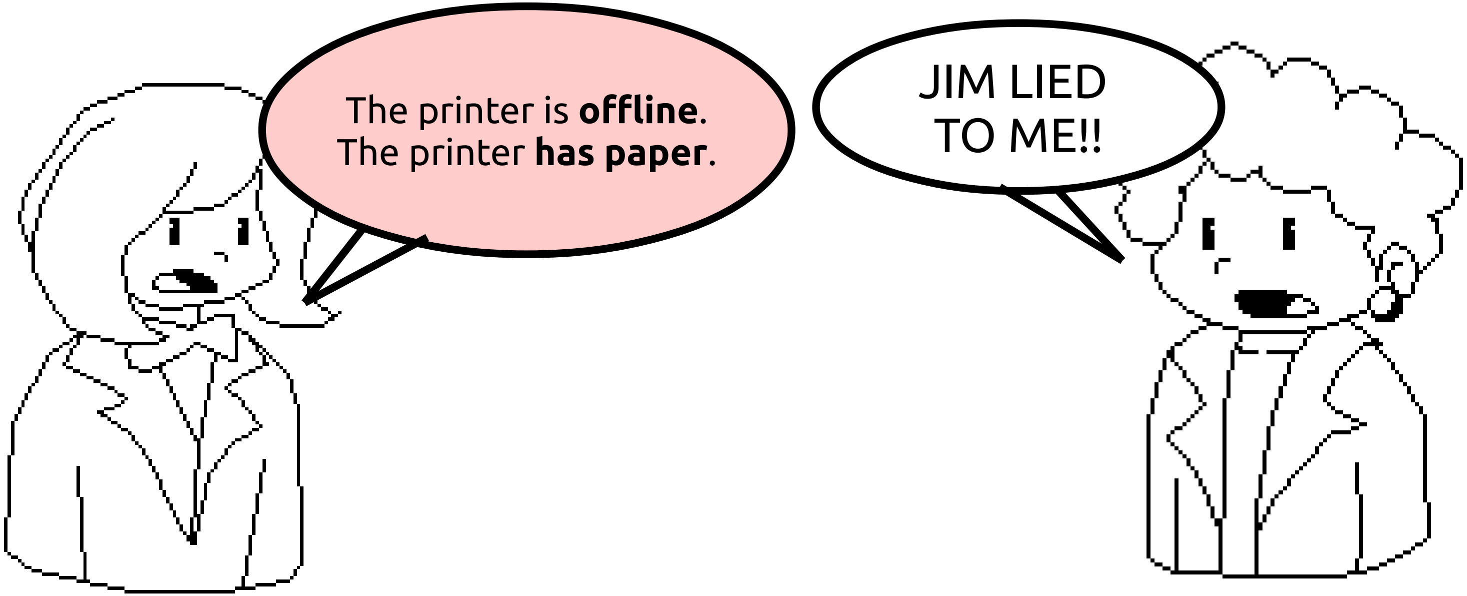

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1. Propositions

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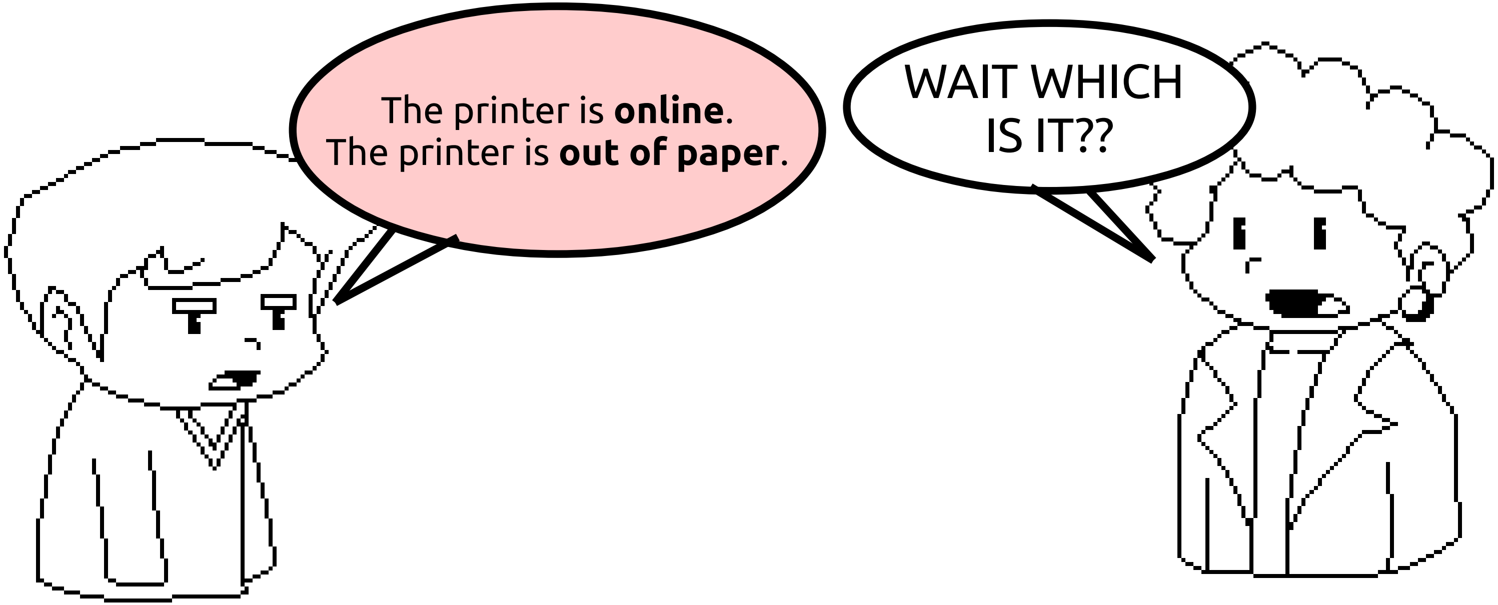

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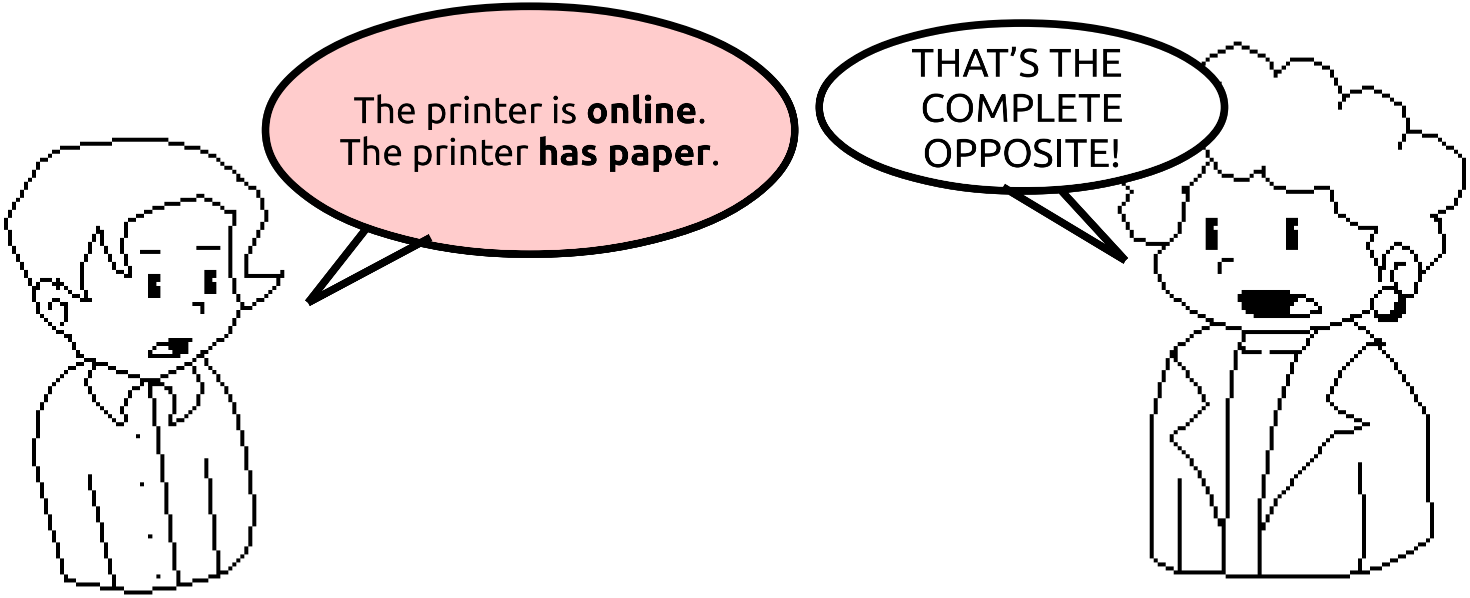


Notes

Propositions are statements that are either true or false.

Formal Propositions can contain one or more propositions.

1. Propositions

"The printer is offline
AND
The printer is out of paper"

For this **formal proposition** to be true, all propositions must also be true.

If any propositions are false, then the entire formal proposition is false.

When an **AND** combines two propositions, the **formal proposition** is only true if both propositions are true!

Notes

Propositions are statements that are either true or false.

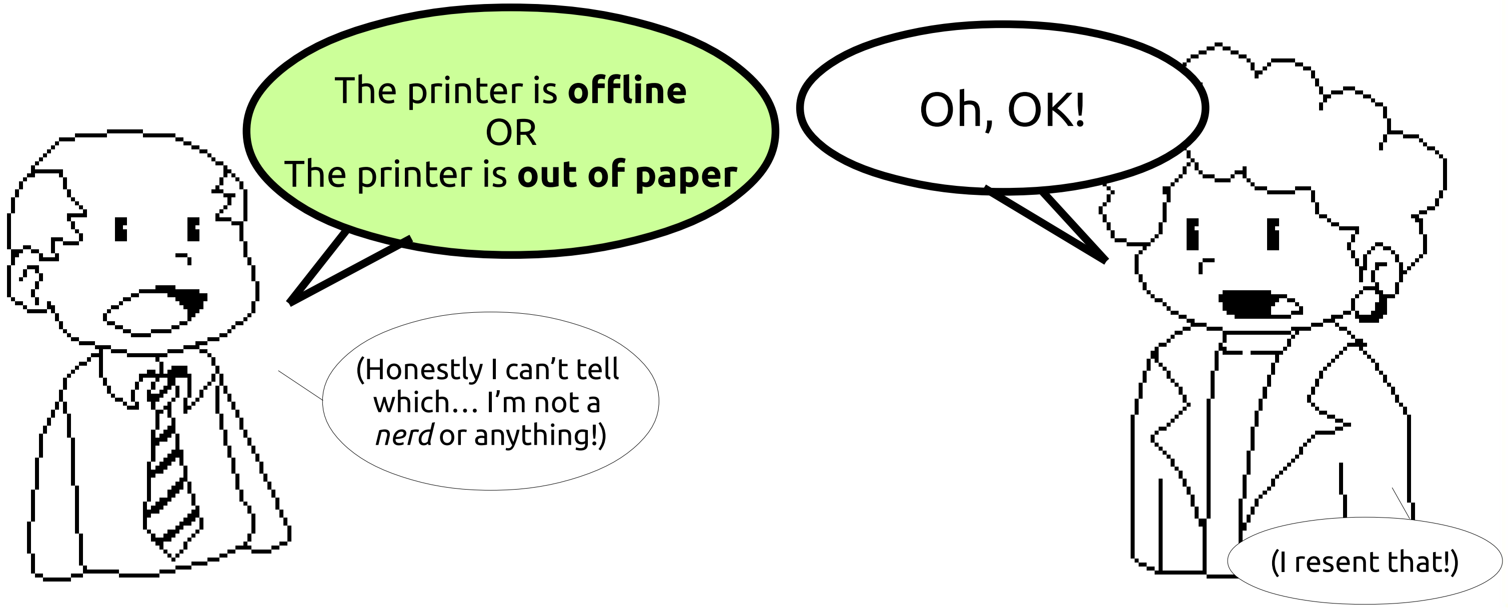
Formal Propositions can contain one or more propositions.

1. Propositions

"The printer is offline
OR
The printer is out of paper"

For this **formal proposition** to be true, at least one proposition must be true.

The formal proposition will only be false if *all propositions* are false.



Notes

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OR
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Propositions are statements that are either true or false.

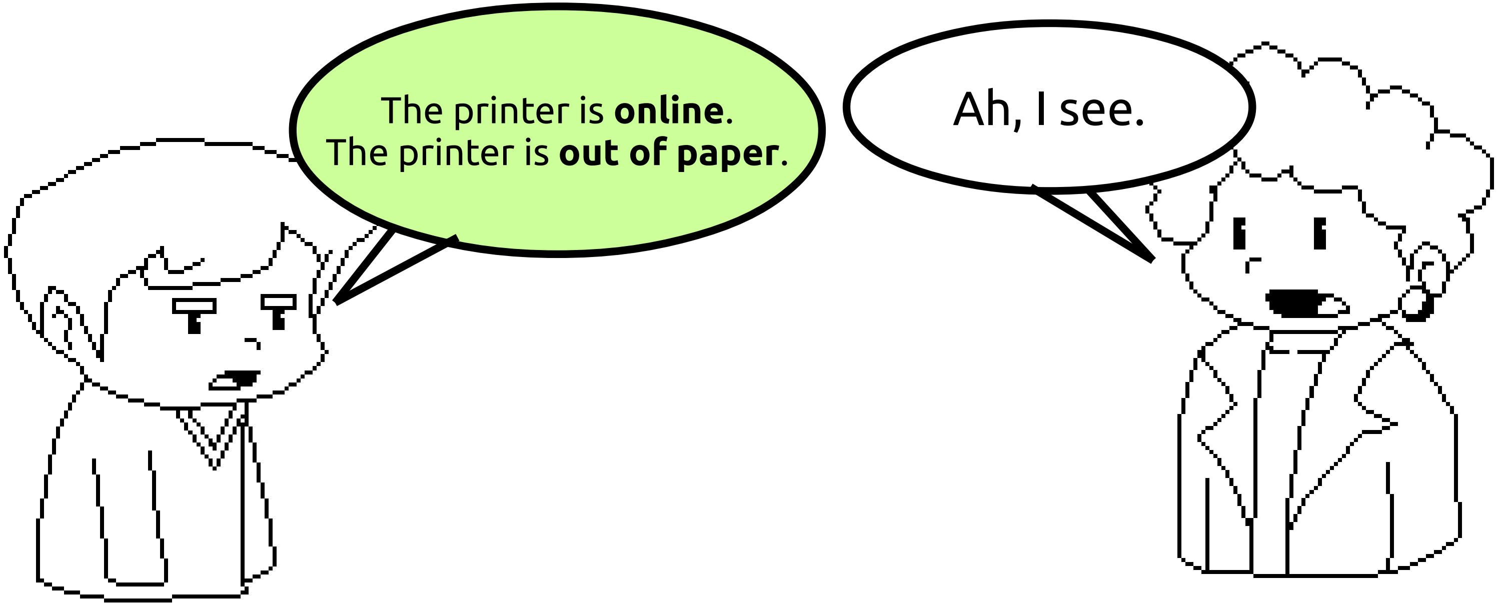
Formal Propositions can contain one or more propositions.

1. Propositions

"The printer is offline
OR
The printer is out of paper"

For this **formal proposition** to be true, at least one proposition must be true.

The formal proposition will only be false if *all propositions* are false.



Notes

Propositions are statements that are either true or false.

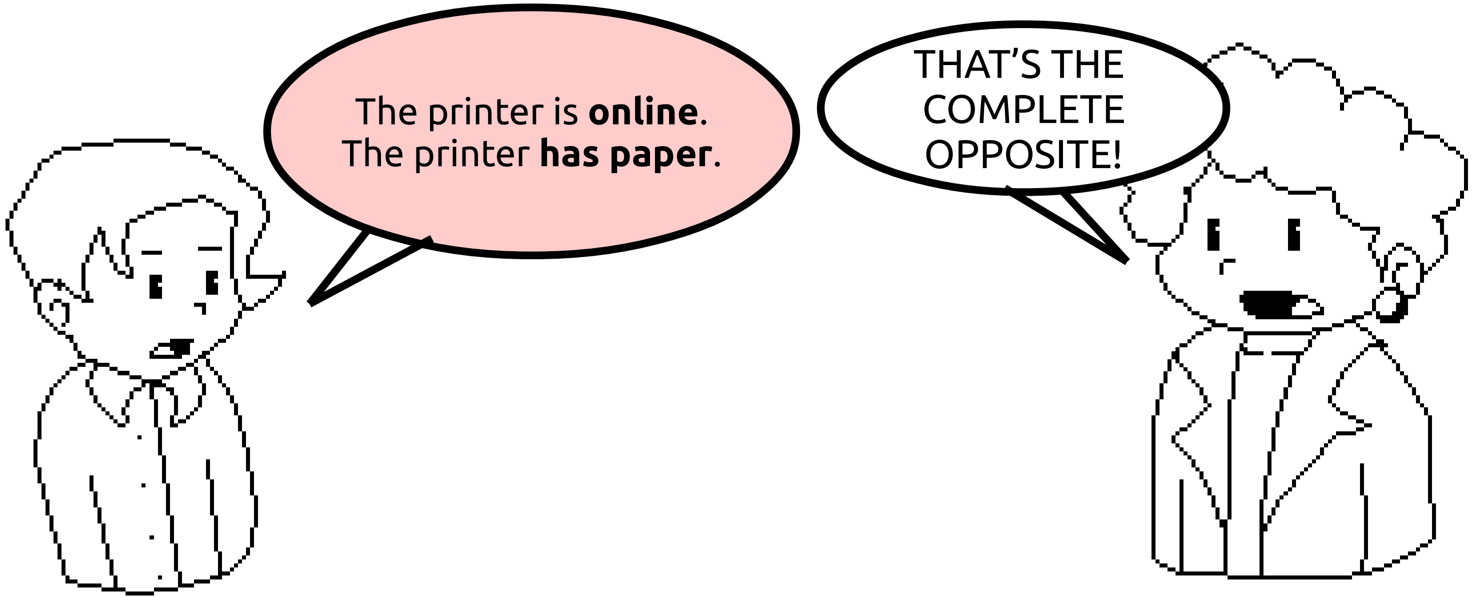
Formal Propositions can contain one or more propositions.

1. Propositions

"The printer is offline
OR
The printer is out of paper"

For this **formal proposition** to be true, at least one proposition must be true.

The formal proposition will only be false if *all propositions* are false.



Notes

Propositions are statements that are either true or false.

Formal Propositions can contain one or more propositions.

1. Propositions

"The printer is offline
OR
The printer is out of paper"

For this **formal proposition** to be true, at least one proposition must be true.

The formal proposition will only be false if *all propositions* are false.

When an **OR** combines two propositions, the **formal proposition** is true if one or both of the propositions are true!

Notes

Propositions are statements that are either true or false.

Formal Propositions can contain one or more propositions.

2. Logic notation

Notes

When working with propositions, we will use propositional logic notation to represent **AND, OR,** and **NOT.**

We will also shorten propositional statements into **propositional variables**

2. Logic notation

∧ AND	∨ OR	¬ NOT (negation)

When we are creating a **formal proposition** from several other propositions, we specify our **propositional variables**, what they represent, and write our formal proposition with symbols.

Notes

∧ AND
∨ OR
¬ NOT

2. Logic notation

∧ AND	∨ OR	¬ NOT (negation)

When we are creating a **formal proposition** from several other propositions, we specify our **propositional variables**, what they represent, and write our formal proposition with symbols.

p: The printer is out of paper
o: The printer is offline

p ∧ o: ***the printer is out of paper AND the printer is offline.***

Notes

∧ AND
∨ OR
¬ NOT

2. Logic notation

Practice 1: Given the following propositions:

p: I am a pirate *g: I drink grog*

Write the following symbolically:

1. I am a pirate AND I drink grog
2. I am a pirate AND I don’t drink grog
3. Either I am a pirate OR I don’t drink grog

Notes

∧	AND
∨	OR
¬	NOT

2. Logic notation

Practice 1:

Given the following propositions:

p: I am a pirate *g: I drink grog*

Write the following symbolically:

1. I am a pirate AND I drink grog

   $p \wedge g$

2. I am a pirate AND I don't drink grog

   $p \wedge \neg g$

3. Either I am a pirate OR I don't drink grog

   $p \vee \neg g$

Notes

$\wedge$	AND
$\vee$	OR
$\neg$	NOT

2. Logic notation

∧ AND	∨ OR	¬ NOT (negation)

If we prepend a **negation symbol** ¬ to a propositional variable, the result of this formal proposition is the opposite of the proposition on its own.

p: The printer is out of paper

¬p: The printer is NOT out of paper

Notes

∧ AND
∨ OR
¬ NOT

2. Logic notation

$\wedge$ AND	$\vee$ OR	$\neg$ NOT (negation)

If we have a formal proposition using AND:

$p \wedge q$ *p AND q*

And we negate it:

$\neg(p \wedge q)$ *NOT (p AND q)*

The result is:

$\neg p \vee \neg q$ *NOT p OR NOT q*

Notes

$\wedge$ AND
$\vee$ OR
$\neg$ NOT

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

2. Logic notation

$\wedge$ AND

$\vee$ OR

$\neg$ NOT (negation)

If we have a formal proposition using OR:

$$p \vee q$$

p OR q

And we negate it:

$$\neg(p \vee q)$$

NOT (p OR q)

The result is:

$$\neg p \wedge \neg q$$

NOT p AND NOT q

Notes

$\wedge$ AND
$\vee$ OR
$\neg$ NOT

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

Practice 2:

Given the following propositions:
p: I am a pirate *g: I drink grog*
Write the following in English:

1. $\neg(p \wedge g)$

2. $\neg(p \vee g)$

Notes

$\wedge$ AND
$\vee$ OR
$\neg$ NOT

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

Practice 2: Given the following propositions:

p: I am a pirate *g: I drink grog*

Write the following in English:

1. $\neg(p \wedge g)$ It is not true that… I am a pirate AND I drink grog.

   $\equiv \neg p \vee \neg q$ I am NOT a pirate OR I DON'T drink grog.

2. $\neg(p \vee g)$ It is not true that… I am a pirate OR I drink grog.

   $\equiv \neg p \vee \neg q$ I am NOT a pirate AND I DON'T drink grog.

Notes

$\wedge$ AND
$\vee$ OR
$\neg$ NOT

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

2. Logic notation

Wait, but how do we know that these are actually **logically equivalent??**

$$\neg(p \land q) \equiv \neg p \lor \neg q$$

$$\neg(p \lor q) \equiv \neg p \land \neg q$$

We know because we can **diagram** all the possible states using a **truth table**.

Two different **formal propositions** are **logically equivalent** if their outcomes are the same for all possible states of **p** and **q** (or however many propositional variables there are...)

Notes

$\land$	AND
$\lor$	OR
$\neg$	NOT

$\neg(p \land q) \equiv \neg p \lor \neg q$

$\neg(p \lor q) \equiv \neg p \land \neg q$

3. Truth Tables

p
True
False

The truth table of a single propositional variable is very simple… it can only be true or false!

Notes

$\wedge$	AND
$\vee$	OR
$\neg$	NOT

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

3. Truth Tables

p
True
False

The truth table of a single propositional variable is very simple… it can only be true or false!

For a **formal proposition**, we start by writing out all possible combinations of values for each **propositional variable.**

Remember the coin flips? But here we use **true** and **false** instead of heads and tails…

Notes

$\wedge$	AND
$\vee$	OR
$\neg$	NOT

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

3. Truth Tables

p	q
True	True
True	False
False	True
False	False

If we’re going to be working with a formal proposition like $p \wedge q$, then first we need to write out all combinations of p and q together.

Each variable can be **true** or **false**, but we have four outcomes:

1. True, True
2. True, False
3. False, True
4. False False

Notes

$\wedge$ AND
$\vee$ OR
$\neg$ NOT

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

3. Truth Tables

p	q	p∧q
True	True	?
True	False	?
False	True	?
False	False	?

After we've written out all the **combinations of propositional variables**,

then we write out the **formal proposition** itself.

Notes

∧ AND
∨ OR
¬ NOT

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

3. Truth Tables

p	q	p∧q
True	True	True
True	False	False
False	True	False
False	False	False

Remember that for a statement with AND, the entire formal proposition can only be true if *all propositions are true*.

Otherwise, the entire formal proposition will be *false*.

Notes

∧ AND
∨ OR
¬ NOT

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

3. Truth Tables

p	q	p∧q
True	True	True
True	False	False
False	True	False
False	False	False

Remember that for a statement with AND, the entire formal proposition can only be true if *all propositions are true*.

Otherwise, the entire formal proposition will be *false*.

p	q	p∨q
True	True	True
True	False	True
False	True	True
False	False	False

And with a statement with OR, the entire formal proposition will be *true* if *at least one proposition is true*.

It can only be *false* if all propositions are *false*.

Notes

∧ AND
∨ OR
¬ NOT

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

3. Truth Tables

p	¬p
True	False
False	True

Let’s not forget the negation!

A negation can work on a single propositional variable, and the result is the opposite of whatever that variable’s value is!

Notes

∧ AND
∨ OR
¬ NOT

p	q	p∧q
True	True	True
True	False	False
False	True	False
False	False	False

p	q	p∨q
True	True	True
True	False	True
False	True	True
False	False	False

3. Truth Tables

Using these truth tables, we can build out truth tables for more complex formal propositions, including proving that two formal propositions are **logically equivalent**.

Notes

p	¬p
True	False
False	True

p	q	p∧q
True	True	True
True	False	False
False	True	False
False	False	False

p	q	p∨q
True	True	True
True	False	True
False	True	True
False	False	False

Let's show that $\neg(p \wedge q) \equiv \neg p \vee \neg q$

We will start with the table for *p AND q*…

p	q	p∧q
True	True	True
True	False	False
False	True	False
False	False	False

Notes

p	¬p
True	False
False	True

p	q	p∧q
True	True	True
True	False	False
False	True	False
False	False	False

p	q	p∨q
True	True	True
True	False	True
False	True	True
False	False	False

3. Truth Tables

Let's show that $\neg(p \wedge q) \equiv \neg p \vee \neg q$

We will start with the table for *p AND q*…

Then, knowing that the negation of a proposition will give us the *opposite*, we can write out $\neg(p \wedge q)$

p	q	$p \wedge q$	$\neg(p \wedge q)$
True	True	True	False
True	False	False	True
False	True	False	True
False	False	False	True

Notes

p	$\neg p$
True	False
False	True

p	q	$p \wedge q$
True	True	True
True	False	False
False	True	False
False	False	False

p	q	$p \vee q$
True	True	True
True	False	True
False	True	True
False	False	False

Let's show that $\neg(p \wedge q) \equiv \neg p \vee \neg q$

Now maybe $\neg p \vee \neg q$ is hard to do in our head, so let's break it down and do $\neg p$ and $\neg q$ first…

p	q	p∧q	¬(p∧q)	¬p	¬q
True	True	True	False	False	False
True	False	False	True	False	True
False	True	False	True	True	False
False	False	False	True	True	True

Notes

p	¬p
True	False
False	True

p	q	p∧q
True	True	True
True	False	False
False	True	False
False	False	False

p	q	p∨q
True	True	True
True	False	True
False	True	True
False	False	False

3. Truth Tables

Let's show that $\neg(p \wedge q) \equiv \neg p \vee \neg q$

And now we'll figure out $\neg p \vee \neg q$ using the $\neg p$ and $\neg q$ columns.

(Remember: for an OR statement, if one proposition is true, then the entire formal proposition is true!)

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
True	True	True	False	False	False	False
True	False	False	True	False	True	True
False	True	False	True	True	False	True
False	False	False	True	True	True	True

Notes

p	$\neg p$
True	False
False	True

p	q	$p \wedge q$
True	True	True
True	False	False
False	True	False
False	False	False

p	q	$p \vee q$
True	True	True
True	False	True
False	True	True
False	False	False

3. Truth Tables

And from building out the truth table to show the results of $\neg(p \wedge q)$ and $\neg p \vee \neg q$ we can show that they are logically equivalent, which is denoted by $\equiv$

p	q	p∧q	¬(p∧q)	¬p	¬q	¬p∨¬q
True	True	True	False	False	False	False
True	False	False	True	False	True	True
False	True	False	True	True	False	True
False	False	False	True	True	True	True

Notes

p	¬p
True	False
False	True

p	q	p∧q
True	True	True
True	False	False
False	True	False
False	False	False

p	q	p∨q
True	True	True
True	False	True
False	True	True
False	False	False

3. Truth Tables

<u>Practice 3:</u> Build out the truth table for $p \vee \neg q$

Notes

p	¬p
True	False
False	True

p	q	p∧q
True	True	True
True	False	False
False	True	False
False	False	False

p	q	p∨q
True	True	True
True	False	True
False	True	True
False	False	False

Practice 3: Build out the truth table for $p \vee \neg q$

p	q	¬q	p ∨ ¬q
True	True	False	True
True	False	True	True
False	True	False	False
False	False	True	True

Notes

p	¬p
True	False
False	True

p	q	p∧q
True	True	True
True	False	False
False	True	False
False	False	False

p	q	p∨q
True	True	True
True	False	True
False	True	True
False	False	False

Practice 4: Build out the truth table for
$p \wedge \neg q$

Notes

p	¬p
True	False
False	True

p	q	p∧q
True	True	True
True	False	False
False	True	False
False	False	False

p	q	p∨q
True	True	True
True	False	True
False	True	True
False	False	False

Practice 4: Build out the truth table for $p \land \neg q$

p	q	¬q	p ∧ ¬q
True	True	False	False
True	False	True	True
False	True	False	False
False	False	True	False

Notes

p	¬p
True	False
False	True

p	q	p∧q
True	True	True
True	False	False
False	True	False
False	False	False

p	q	p∨q
True	True	True
True	False	True
False	True	True
False	False	False

3. Truth Tables

Practice 5: Build out the truth table in order to prove that $\neg(p \lor q) \equiv \neg p \land \neg q$

Notes

p	¬p
True	False
False	True

p	q	p∧q
True	True	True
True	False	False
False	True	False
False	False	False

p	q	p∨q
True	True	True
True	False	True
False	True	True
False	False	False

3. Truth Tables

Practice 5: Build out the truth table in order to prove that $\neg(p \lor q) \equiv \neg p \land \neg q$

p	q	$p \lor q$	$\neg(p \lor q)$	$\neg p$	$\neg q$	$\neg p \land \neg q$
True	True	True	False	False	False	False
True	False	True	False	False	True	False
False	True	True	False	True	False	False
False	False	False	True	True	True	True

Notes

p	$\neg p$
True	False
False	True

p	q	$p \land q$
True	True	True
True	False	False
False	True	False
False	False	False

p	q	$p \lor q$
True	True	True
True	False	True
False	True	True
False	False	False

4. Tautology & Contradiction

Sometimes, we may get a formal proposition where, no matter what the states of the propositions it is made up of, all outcomes turn out **true**, or all outcomes turn out **false.**

When all outcomes are **true**, the formal proposition is a **tautology**.

When all outcomes are **false**, the formal proposition is a **contradiction**.

Notes

p	¬p
True	False
False	True

p	q	p∧q
True	True	True
True	False	False
False	True	False
False	False	False

p	q	p∨q
True	True	True
True	False	True
False	True	True
False	False	False

4. Tautology & Contradiction

As a simple example, let's say we have a formal proposition $p \vee \neg p$ p OR NOT p…

Notes

p	¬p
True	False
False	True

p	q	p∧q
True	True	True
True	False	False
False	True	False
False	False	False

p	q	p∨q
True	True	True
True	False	True
False	True	True
False	False	False

4. Tautology & Contradiction

As a simple example, let's say we have a formal proposition $p \vee \neg p$ p OR NOT p…

When we draw out the truth table, we can see that all results are **True**. This is a **tautology.**

p	¬p	p ∨ ¬p
True	False	True
False	True	True

Notes

p	¬p
True	False
False	True

p	q	p∧q
True	True	True
True	False	False
False	True	False
False	False	False

p	q	p∨q
True	True	True
True	False	True
False	True	True
False	False	False

4. Tautology & Contradiction

Or how about p AND NOT p? $p \wedge \neg p$

Notes

p	¬p
True	False
False	True

p	q	p∧q
True	True	True
True	False	False
False	True	False
False	False	False

p	q	p∨q
True	True	True
True	False	True
False	True	True
False	False	False

4. Tautology & Contradiction

Or how about p AND NOT p? $p \wedge \neg p$

All outcomes are **false**, so this is a **contradiction**.

p	¬p	p ∧ ¬p
True	False	False
False	True	False

Notes

p	¬p
True	False
False	True

p	q	p∧q
True	True	True
True	False	False
False	True	False
False	False	False

p	q	p∨q
True	True	True
True	False	True
False	True	True
False	False	False

4. Tautology & Contradiction

Sometimes, a formal proposition may end up being a tautology or a contradiction.

Later on, we will do some **proofs by contradiction**, to prove that a statement is true, because the opposite would be a contradiction.

p	¬p	p ∨ ¬p
True	False	True
False	True	True

p	¬p	p ∧ ¬p
True	False	False
False	True	False

Notes

p	¬p
True	False
False	True

p	q	p∧q
True	True	True
True	False	False
False	True	False
False	False	False

p	q	p∨q
True	True	True
True	False	True
False	True	True
False	False	False

Finally...

We can build out truth tables that have even more propositional variables, and the table may get confusing to keep track of.

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Finally...

We can build out truth tables that have even more propositional variables, and the table may get confusing to keep track of.

p	q	r
T	T	T
T	T	F
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The rule of thumb is to start at all-true, and then change the right-most variable.

Finally...

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p	q	r
T	T	T
T	T	F
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The rule of thumb is to start at all-true, and then change the right-most variable.

Then, go to the next column, change it, and repeat the column to the right.

Finally...

We can build out truth tables that have even more propositional variables, and the table may get confusing to keep track of.

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

The rule of thumb is to start at all-true, and then change the right-most variable.

Then, go to the next column, change it, and repeat the column to the right.

And move from right-to-left.

Finally…

You should notice a pattern…

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

The right-most column changes value at every-other-row

Finally…

You should notice a pattern…

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

The 2nd column goes by 2's

The right-most column changes value at every-other-row

Finally…

You should notice a pattern…

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

The 3rd column goes by 4's

The 2nd column goes by 2's

The right-most column changes value at every-other-row

Finally…

You should notice a pattern…

So keep to the pattern! It makes it really hard to grade things if they're not in order!

ಠ_ಠ

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

The 3rd column goes by 4's

The 2nd column goes by 2's

The right-most column changes value at every-other-row

Conclusion

Next time we will be working with propositional logic more, but using predicates.

It is similar to what we did this time, but instead of just "p" and "q", we have "P(x)" and "Q(x)", where *x* is an input variable, and whether "P(x)" evaluates to *true* or *false* depends on what the input value *x* is.

We will also explore the usage of "For All" and "There Exists" when we are building quantified statements.