1.2 Number Puzzles and Sequences

Written by Rachel J. Morris, last updated Jun 23, 2017

About

In this chapter, we will talk about recursive formulas and closed formulas, which we can use to describe sequences of numbers.

We will also learn about summation notation and evaluate summations.

Topics

1. Finding the next number in a sequence
2. Sequences
   a. Recursive formula
   b. Closed formula
3. Finding a sequence from a formula
4. Finding a formula from a sequence
5. Summation notation & evaluating sums

1. Finding the next number in a sequence

Scrolling on your Facebook feed you see one of these lame posts posted by an acquaintance:

> Only 5% of peple can figure this out!!12!
>
> 5, 7, 9, 11, 13, ____
>
> wat is next nuber?
> like & share if u know it!!
> but dont tell answer!! 😂

Not to be outdone by ***Linda***, you post the answer smugly:

“15, DUH!!!”

1. Finding the next number in a sequence

Only 5% of peple can figure this out!!12!

5, 7, 9, 11, 13, ___

wat is next nuber?
like & share if u know it!!
but dont tell answer!! 😂

But how do we know it's 15?

It seems almost obvious, we don't think about the steps right away.

Can we generalize the approach we take, so that we could solve this for *other* sequences of numbers?

Let's look at how we can find the pattern between a sequence of numbers...

1. Finding the next number in a sequence

In this chapter, we will be looking for these patterns.

2, 4, 6, 8, 10, ___

5, 7, 9, 11, 13, ___

2, 5, 8, 11, 14, ___

Let's start with these sequences.

These are easy – the pattern is simple addition between each item in the sequence, and the number being added stays the same each time.

1. Finding the next number in a sequence

+2 +2 +2 +2 +2

2, 4, 6, 8, 10, 12

+2 +2 +2 +2 +2

5, 7, 9, 11, 13, 15

+3 +3 +3 +3 +3

2, 5, 8, 11, 14, 17

We can look at these sequences and figure out the pattern without much inspection – we could possibly do it entirely mentally.

1. Finding the next number in a sequence

In this chapter, we will be looking for these patterns.

2, 3, 5, 8, 12, ___

These are a little harder – still addition, but the difference between each number in the sequence changes.

3, 5, 9, 15, 23, ___

However, the change in the *difference* between each number is *also* a pattern.

1. Finding the next number in a sequence

+1 +2 +3 +4 +5
2, 3, 5, 8, 12, 17

+2 +4 +6 +8 +10
3, 5, 9, 15, 23, 33

So solving this sequence might take more investigation into the differences between each number, and then seeing the pattern that arises in the *differences*.

1. Finding the next number in a sequence

1, 2, 6, 24, 120, ___

1, 3, 7, 15, 31, 63, ___

With the top sequence here, we're not *adding* between each number, but instead *multiplying*.

For the bottom one, we're adding by a different amount each time, by 2^n, with n changing for each term.

Can you figure it out?

1. Finding the next number in a sequence

In this chapter, we will be looking for these patterns.

$$\begin{array}{cccccc} & \times 2 & \times 3 & \times 4 & \times 5 & \times 6 \\ 1, & 2, & 6, & 24, & 120, & \underline{720} \end{array}$$

$$\begin{array}{ccccccc} & +2^1 & +2^2 & +2^3 & +2^4 & +2^5 & +2^6 \\ & +2 & +4 & +8 & +16 & +32 & +64 \\ 1, & 3, & 7, & 15, & 31, & 63, & \underline{127} \end{array}$$

Sometimes, we have to think in terms of multiplication, or even exponents. If the numbers increase much more quickly (not *linearly*), we are probably dealing with multiplication.

1. Finding the next number in a sequence

We can generate number sequences in code
if we know how the terms change each time...

2, 4, 6, 8, 10, ___
+2 each term

```
int a = 2;
int inc = 2;

for ( int counter = 1; counter <= 6; counter++ )
{
    Output( a );

    a = a + inc;
}
```

```
Terminal
File  Edit  View  Search  Terminal  Help

 2  4  6  8  10  12
```

2, 3, 5, 8, 12, ___
Inc + 1 each time

```
int a = 2;
int inc = 1;

for ( int counter = 1; counter <= 6; counter++ )
{
    Output( a );

    a = a + inc;
    inc = inc + 1;
}
```

```
Terminal
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 2  3  5  8  12  17
```

1. Finding the next number in a sequence

We can generate number sequences in code
if we know how the terms change each time...

1, 2, 6, 24, 120, ___
x inc, and inc + 1 each time

```
int a = 1;
int inc = 2;

for ( int counter = 1; counter <= 6; counter++ )
{
    Output( a );

    a = a * inc;
    inc = inc + 1;
}
```

```
Terminal
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 1  2  6  24  120  720
```

But in this class, we will be interested in how to represent these number sequences with one of two types of formulas:

the **Recursive formula**
and the **Closed formula**.

Recursive Formula
(aka Recurrence Relation)

In mathematics, a recurrence relation is an equation that recursively defines a sequence or multidimensional array of values, once one or more initial terms are given: each further term of the sequence or array is defined as a function of the preceding terms.

(Wikipedia https://en.wikipedia.org/wiki/Recurrence_relation)

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With a recursive formula, we **must** specify the value of the **first term** (at $\mathbf{a_1}$), and the formula itself, where $\mathbf{a_n}$ references $\mathbf{a_{n-1}}$ to get additional items in the sequence.

2. Sequences

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$$a_1 = 2 \qquad a_n = a_{n-1} + 2$$

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$a_1 = 2$

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$$a_1 = 2 \qquad\qquad a_n = a_{n-1} + 2$$

$a_1 = 2$

$a_2 = a_{n-1} + 2 = 2 + 2 = 4$

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$$a_1 = 2 \qquad a_n = a_{n-1} + 2$$

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$a_2 = a_{n-1} + 2 = 2 + 2 = 4$

$a_3 = a_{n-1} + 2 = 4 + 2 = 6$

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With a recursive formula, we **must** specify the value of the **first term** (at $\mathbf{a_1}$), and the formula itself, where $\mathbf{a_n}$ references $\mathbf{a_{n-1}}$ to get additional items in the sequence.

$$a_1 = 2 \qquad a_n = a_{n-1} + 2$$

$a_1 = 2$
$a_2 = a_{n-1} + 2 = 2 + 2 = 4$
$a_3 = a_{n-1} + 2 = 4 + 2 = 6$
$a_4 = a_{n-1} + 2 = 6 + 2 = 8$
etc.

Closed Formula

A closed formula for a sequence is a formula where each term is described only in relation to its position in the list.

(Discrete Mathematics Mathematical Reasoning and Proof with Puzzles, Patterns, and Games by Douglas E Ensley)

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For a closed formula, the item in the sequences is calculated by using the **position**, or its **index**.

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$$a_n = 2n$$

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$a_1 = 2 \times 1 = 2$

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$a_1 = 2 \times 1 = 2$

$a_2 = 2 \times 2 = 4$

$a_3 = 2 \times 3 = 6$

Closed Formula

A closed formula for a sequence is a formula where each term is described only in relation to its position in the list.

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For a closed formula, the item in the sequences is calculated by using the **position**, or its **index**.

$$a_n = 2n$$

$a_1 = 2 \times 1 = 2$
$a_2 = 2 \times 2 = 4$
$a_3 = 2 \times 3 = 6$
$a_4 = 2 \times 4 = 8$
etc.

2. Sequences

Recursive formula	Closed formula
$a_1 = 2 \qquad a_n = a_{n-1} + 2$	$a_n = 2n$

$a_1 = 2$
$a_2 = a_{n-1} + 2 = 2 + 2 = 4$
$a_3 = a_{n-1} + 2 = 4 + 2 = 6$
$a_4 = a_{n-1} + 2 = 6 + 2 = 8$
etc.

$a_1 = 2 \times 1 = 2$
$a_2 = 2 \times 2 = 4$
$a_3 = 2 \times 3 = 6$
$a_4 = 2 \times 4 = 8$
etc.

So we can represent a sequence of numbers in different ways.

3. Finding a sequence from a formula

Let’s practice finding a sequence of numbers from an equation.

For the given formula, find the first 5 elements of the sequence.

3. Finding a sequence from a formula

Practice 1: Find the first 5 elements of the sequence given the recursive formula

$$a_1 = 1 \qquad a_n = a_{n-1} + 3$$

3. Finding a sequence from a formula

Practice 1: Find the first 5 elements of the sequence given the recursive formula

$$a_1 = 1 \qquad a_n = a_{n-1} + 3$$

- **$a_1 = 1$**
- **$a_2 = a_{n-1} + 3 = \underline{1} + 3 = 4$**
- **$a_3 = a_{n-1} + 3 = \underline{4} + 3 = 7$**
- **$a_4 = a_{n-1} + 3 = \underline{7} + 3 = 10$**
- **$a_5 = a_{n-1} + 3 = \underline{10} + 3 = 13$**

3. Finding a sequence from a formula

Practice 2: Find the first 5 elements of the sequence given the recursive formula

$$a_1 = 1 \qquad a_n = 2 \cdot a_{n-1}$$

3. Finding a sequence from a formula

Practice 2: Find the first 5 elements of the sequence given the recursive formula

$$a_1 = 1 \qquad a_n = 2 \cdot a_{n-1}$$

- $a_1 = 1$
- $a_2 = 2 \times a_{n-1} = 2 \times \underline{1} = 2$
- $a_3 = 2 \times a_{n-1} = 2 \times \underline{2} = 4$
- $a_4 = 2 \times a_{n-1} = 2 \times \underline{4} = 8$
- $a_5 = 2 \times a_{n-1} = 2 \times \underline{8} = 16$

3. Finding a sequence from a formula

Practice 3: Find the first 5 elements of the sequence given the recursive formula

$$a_1 = 3 \qquad a_n = 3 \cdot a_{n-1} + 1$$

3. Finding a sequence from a formula

Practice 3: Find the first 5 elements of the sequence given the recursive formula

$$a_1 = 3 \qquad a_n = 3 \cdot a_{n-1} + 1$$

- $a_1 = 3$
- $a_2 = 3 \times a_{n-1} + 1 = 3 \times \underline{3} + 1 = 10$
- $a_3 = 3 \times a_{n-1} + 1 = 3 \times \underline{10} + 1 = 31$
- $a_4 = 3 \times a_{n-1} + 1 = 3 \times \underline{31} + 1 = 94$
- $a_5 = 3 \times a_{n-1} + 1 = 3 \times \underline{94} + 1 = 283$

3. Finding a sequence from a formula

Practice 4: Find the first 5 elements of the sequence given the closed formula

$$a_n = 3n$$

Practice 4: Find the first 5 elements of the sequence given the closed formula

$$a_n = 3n$$

- $a_1 = 3 \times \underline{1} = 3$
- $a_2 = 3 \times \underline{2} = 6$
- $a_3 = 3 \times \underline{3} = 9$
- $a_4 = 3 \times \underline{4} = 12$
- $a_5 = 3 \times \underline{5} = 15$

3. Finding a sequence from a formula

Practice 5: Find the first 5 elements of the sequence given the closed formula

$$a_n = 2n + 1$$

3. Finding a sequence from a formula

Practice 5: Find the first 5 elements of the sequence given the closed formula

$$a_n = 2n + 1$$

- $a_1 = 2 \times \underline{1} + 1 = 3$
- $a_2 = 2 \times \underline{2} + 1 = 5$
- $a_3 = 2 \times \underline{3} + 1 = 7$
- $a_4 = 2 \times \underline{4} + 1 = 9$
- $a_5 = 2 \times \underline{5} + 1 = 11$

3. Finding a sequence from a formula

Practice 6: Find the first 5 elements of the sequence given the closed formula

$$a_n = 2^n + 1$$

3. Finding a sequence from a formula

Practice 6: Find the first 5 elements of the sequence given the closed formula

$$a_n = 2^n + 1$$

- $A_1 = 2^{1} + 1 = 2 + 1 = 3$
- $A_2 = 2^{2} + 1 = 4 + 1 = 5$
- $a_3 = 2^{3} + 1 = 8 + 1 = 9$
- $a_4 = 2^{4} + 1 = 16 + 1 = 17$
- $a_5 = 2^{5} + 1 = 32 + 1 = 33$

4. Finding a formula from a sequence

Going from the equation to the sequence is easy, but the more challenging part is finding the formula for a given sequence of numbers.

We again have to investigate the differences between each of the terms and figure out a pattern, and then express that pattern mathematically.

4. Finding a formula from a sequence

$\overset{}{2},\overset{+2}{}\ 4,\ 6,\ 8,\ 10,\ \underline{12}$

+2 +2 +2 +2 +2
2, 4, 6, 8, 10, ___12___

+2 +2 +2 +2 +2
5, 7, 9, 11, 13, ___15___

+3 +3 +3 +3 +3
2, 5, 8, 11, 14, ___17___

Let's start with the number sequences from before – how do we turn these into formulas?

4. Finding a formula from a sequence

+2 +2 +2 +2 +2

2, 4, 6, 8, 10, 12

+2 +2 +2 +2 +2

5, 7, 9, 11, 13, 15

+3 +3 +3 +3 +3

2, 5, 8, 11, 14, 17

For a closed formula, let's look at how each term in the sequence relates to its **position** in the sequence.

4. Finding a formula from a sequence

+2 +2 +2 +2 +2
2, 4, 6, 8, 10, 12

+2 +2 +2 +2 +2
5, 7, 9, 11, 13, 15

+3 +3 +3 +3 +3
2, 5, 8, 11, 14, 17

For a closed formula, let's look at how each term in the sequence relates to its **position** in the sequence.

- $a_1 = 2$
- $a_2 = 4$
- $a_3 = 6$
- $a_4 = 8$
- $a_5 = 10$
- $a_6 = 12$

4. Finding a formula from a sequence

+2 +2 +2 +2 +2
2, 4, 6, 8, 10, ___12___

+2 +2 +2 +2 +2
5, 7, 9, 11, 13, ___15___

+3 +3 +3 +3 +3
2, 5, 8, 11, 14, ___17___

Sometimes seeing the pattern isn't too hard.

- $a_1 = 2 = 2 \times 1 = 2 \times n$
- $a_2 = 4 = 2 \times 2 = 2 \times n$
- $a_3 = 6 = 2 \times 3 = 2 \times n$
- $a_4 = 8 = 2 \times 4 = 2 \times n$
- $a_5 = 10 = 2 \times 5 = 2 \times n$
- $a_6 = 12 = 2 \times 6 = 2 \times n$

$$\boxed{a_n = 2n}$$

4. Finding a formula from a sequence

+2	+2	+2	+2	+2	
2,	4,	6,	8,	10,	12

+2	+2	+2	+2	+2	
5,	7,	9,	11,	13,	15

+3	+3	+3	+3	+3	
2,	5,	8,	11,	14,	17

But sometimes seeing the pattern can take a bit more analysis.

- $a_1 = 2$
- $a_2 = 5$
- $a_3 = 8$
- $a_4 = 11$
- $a_5 = 14$
- $a_6 = 17$

4. Finding a formula from a sequence

$+2$ $+2$ $+2$ $+2$ $+2$

2, 4, 6, 8, 10, _12_

$+2$ $+2$ $+2$ $+2$ $+2$

5, 7, 9, 11, 13, _15_

$+3$ $+3$ $+3$ $+3$ $+3$

2, 5, 8, 11, 14, _17_

For this sequence, we know that there's a difference of 3 between each term, but it isn't exactly 3, 6, 9, 12, etc. What do we offset it by?

- $a_1 = 2 \quad = 3 \times 1 +/- ???$
- $a_2 = 5 \quad = 3 \times 2 +/- ???$
- $a_3 = 8 \quad = 3 \times 3 +/- ???$
- $a_4 = 11 \quad = 3 \times 4 +/- ???$
- $a_5 = 14 \quad = 3 \times 5 +/- ???$
- $a_6 = 17 \quad = 3 \times 6 +/- ???$

4. Finding a formula from a sequence

$+2$ $+2$ $+2$ $+2$ $+2$		

2, 4, 6, 8, 10, _12_

+2 +2 +2 +2 +2
5, 7, 9, 11, 13, _15_

+3 +3 +3 +3 +3
2, 5, 8, 11, 14, _17_

This is just the closed formula for it, though the recursive formula would be much easier!

- $a_1 = 2 \quad = 3 \times 1 - 1 \quad = 3 \times n - 1$
- $a_2 = 5 \quad = 3 \times 2 - 1 \quad = 3 \times n - 1$
- $a_3 = 8 \quad = 3 \times 3 - 1 \quad = 3 \times n - 1$
- $a_4 = 11 \quad = 3 \times 4 - 1 \quad = 3 \times n - 1$
- $a_5 = 14 \quad = 3 \times 5 - 1 \quad = 3 \times n - 1$
- $a_6 = 17 \quad = 3 \times 6 - 1 \quad = 3 \times n - 1$

$$\boxed{a_n = 3n - 1}$$

4. Finding a formula from a sequence

We just need to identify the **first term** (easy!) and then the rest of the terms based on the previous term.

$$\overset{+2}{}2, \overset{+2}{}4, \overset{+2}{}6, \overset{+2}{}8, \overset{+2}{}10, \underline{12}$$

+2 +2 +2 +2 +2

2, 4, 6, 8, 10, __12__

+2 +2 +2 +2 +2

5, 7, 9, 11, 13, __15__

+3 +3 +3 +3 +3

2, 5, 8, 11, 14, __17__

- $a_1 = 2$
- $a_2 = 5 \quad = a_1 + 3 \quad = a_{n-1} + 3$
- $a_3 = 8 \quad = a_2 + 3 \quad = a_{n-1} + 3$
- $a_4 = 11 \quad = a_3 + 3 \quad = a_{n-1} + 3$
- $a_5 = 14 \quad = a_4 + 3 \quad = a_{n-1} + 3$
- $a_6 = 17 \quad = a_5 + 3 \quad = a_{n-1} + 3$

$$a_1 = 2 \qquad a_n = 3n - 1$$

4. Finding a formula from a sequence

$$\overset{+1}{2,}\ \overset{+2}{3,}\ \overset{+3}{5,}\ \overset{+4}{8,}\ \overset{+5}{12,}\ \underline{17}$$

$$\overset{\times 2}{1,}\ \overset{\times 3}{2,}\ \overset{\times 4}{6,}\ \overset{\times 5}{24,}\ \overset{\times 6}{120,}\ \underline{720}$$

With a recursive formula, we can also use *n* (the index) as part of the formula as well.

- $a_1 = 2$
- $a_2 = 3 \quad = a_1 + 1 \quad = a_{n-1} + (n-1)$
- $a_3 = 5 \quad = a_2 + 2 \quad = a_{n-1} + (n-1)$
- $a_4 = 8 \quad = a_3 + 3 \quad = a_{n-1} + (n-1)$
- $a_5 = 12 \quad = a_4 + 4 \quad = a_{n-1} + (n-1)$
- $a_6 = 17 \quad = a_5 + 5 \quad = a_{n-1} + (n-1)$

4. Finding a formula from a sequence

+1 +2 +3 +4 +5

2, 3, 5, 8, 12, 17

x2 x3 x4 x5 x6

1, 2, 6, 24, 120, 720

Let's look at some sequences that don't just increase by the same amount each time.

4. Finding a formula from a sequence

+1 +2 +3 +4 +5

2, 3, 5, 8, 12, 17

x2 x3 x4 x5 x6

1, 2, 6, 24, 120, 720

Again, for either formula, first write out all the terms with their positions in the sequence.

- $a_1 = 2$
- $a_2 = 3$
- $a_3 = 5$
- $a_4 = 8$
- $a_5 = 12$
- $a_6 = 17$

4. Finding a formula from a sequence

$$\overset{+1}{2,}\ \overset{+2}{3,}\ \overset{+3}{5,}\ \overset{+4}{8,}\ \overset{+5}{12,}\ \underline{\ 17\ }$$

$$\overset{x2}{1,}\ \overset{x3}{2,}\ \overset{x4}{6,}\ \overset{x5}{24,}\ \overset{x6}{120,}\ \underline{\ 720\ }$$

For recursive, how does each term relate to the previous one?

- $a_1 = 2$
- $a_2 = 3 \qquad = a_1 + 1$
- $a_3 = 5 \qquad = a_2 + 2$
- $a_4 = 8 \qquad = a_3 + 3$
- $a_5 = 12 \qquad = a_4 + 4$
- $a_6 = 17 \qquad = a_5 + 5$

Here we can see that the difference increases each time.

4. Finding a formula from a sequence

$\overset{+1}{2,}\ \overset{+2}{3,}\ \overset{+3}{5,}\ \overset{+4}{8,}\ \overset{+5}{12,}$ _*17*_

x2 x3 x4 x5 x6

1, 2, 6, 24, 120, _*720*_

With a recursive formula, we can also use *n* (the index) as part of the formula as well.

- $a_1 = 2$
- $a_2 = 3 \qquad = a_1 + 1 \qquad = a_1 + (\underline{2}-1)$
- $a_3 = 5 \qquad = a_2 + 2 \qquad = a_2 + (\underline{3}-1)$
- $a_4 = 8 \qquad = a_3 + 3 \qquad = a_3 + (\underline{4}-1)$
- $a_5 = 12 \qquad = a_4 + 4 \qquad = a_4 + (\underline{5}-1)$
- $a_6 = 17 \qquad = a_5 + 5 \qquad = a_5 + (\underline{6}-1)$

4. Finding a formula from a sequence

$$\overset{+1}{2,}\ \overset{+2}{3,}\ \overset{+3}{5,}\ \overset{+4}{8,}\ \overset{+5}{12,}\ \underline{\mathit{17}}$$

$$\overset{x2}{1,}\ \overset{x3}{2,}\ \overset{x4}{6,}\ \overset{x5}{24,}\ \overset{x6}{120,}\ \underline{\mathit{720}}$$

With a recursive formula, we can also use *n* (the index) as part of the formula as well.

- $a_1 = 2$
- $a_2 = 3 \quad = a_1 + 1 \quad = a_{n-1} + (n-1)$
- $a_3 = 5 \quad = a_2 + 2 \quad = a_{n-1} + (n-1)$
- $a_4 = 8 \quad = a_3 + 3 \quad = a_{n-1} + (n-1)$
- $a_5 = 12 \quad = a_4 + 4 \quad = a_{n-1} + (n-1)$
- $a_6 = 17 \quad = a_5 + 5 \quad = a_{n-1} + (n-1)$

$$a_1 = 2 \qquad a_n = a_{n-1} + n - 1$$

4. Finding a formula from a sequence

$2, \overset{+1}{3}, \overset{+2}{5}, \overset{+3}{8}, \overset{+4}{12}, \overset{+5}{\underline{17}}$

$1, \overset{x2}{2}, \overset{x3}{6}, \overset{x4}{24}, \overset{x5}{120}, \overset{x6}{\underline{720}}$

With a recursive formula, we can also use *n* (the index) as part of the formula as well.

- $a_1 = 2$
- $a_2 = 3 \quad = a_1 + 1 \quad = a_{n-1} + (n-1)$
- $a_3 = 5 \quad = a_2 + 2 \quad = a_{n-1} + (n-1)$
- $a_4 = 8 \quad = a_3 + 3 \quad = a_{n-1} + (n-1)$
- $a_5 = 12 \quad = a_4 + 4 \quad = a_{n-1} + (n-1)$
- $a_6 = 17 \quad = a_5 + 5 \quad = a_{n-1} + (n-1)$

$$a_1 = 2 \qquad a_n = a_{n-1} + n - 1$$

4. Finding a formula from a sequence

+1 +2 +3 +4 +5
2, 3, 5, 8, 12, _17_

x2 x3 x4 x5 x6
1, 2, 6, 24, 120, _720_

And along the same lines when it comes to multiplication and using *n* in the recursive formula.

- $a_1 = 1$
- $a_2 = 2$
- $a_3 = 6$
- $a_4 = 24$
- $a_5 = 120$
- $a_6 = 720$

4. Finding a formula from a sequence

$+1 \quad +2 \quad +3 \quad +4 \quad +5$

2, 3, 5, 8, 12, _17_

$\times 2 \quad \times 3 \quad \times 4 \quad \times 5 \quad \times 6$

1, 2, 6, 24, 120, _720_

And along the same lines when it comes to multiplication and using *n* in the recursive formula.

- $a_1 = 1$
- $a_2 = 2 \quad = a_1 \times 2 \quad = a_{n-1} \times n$
- $a_3 = 6 \quad = a_2 \times 3 \quad = a_{n-1} \times n$
- $a_4 = 24 \quad = a_3 \times 4 \quad = a_{n-1} \times n$
- $a_5 = 120 \quad = a_4 \times 5 \quad = a_{n-1} \times n$
- $a_6 = 720 \quad = a_5 \times 6 \quad = a_{n-1} \times n$

$$a_1 = 1 \qquad a_n = n \cdot a_{n-1}$$

4. Finding a formula from a sequence

Even if you can find a pattern for a sequence of numbers, it is not always practical to find the closed formula...

$$2, \overset{+1}{3}, \overset{+2}{5}, \overset{+3}{8}, \overset{+4}{12}, \overset{+5}{\underline{17}}$$

$$a_n = \frac{1}{2}\left(n^2 - n + 4\right)$$

But finding the recursive formula is usually much easier.

4. Finding an equation from a sequence

Practice 7: Find the closed formula and recursive formula for the following sequence of numbers.

$$\overset{+2}{}\ \overset{+2}{}\ \overset{+2}{}\ \overset{+2}{}\ \overset{+2}{}$$

5, 7, 9, 11, 13, __15__

Practice 7: Find the closed formula and recursive formula for the following sequence of numbers.

+2 +2 +2 +2 +2

5, 7, 9, 11, 13, 15

- $a_1 = 5$ $= 2 \times 1 + 3 = 2n + 3$
- $a_2 = 7$ $= 2 \times 2 + 3 = 2n + 3$
- $a_3 = 9$ $= 2 \times 3 + 3 = 2n + 3$
- $a_4 = 11$ $= 2 \times 4 + 3 = 2n + 3$
- $a_5 = 13$ $= 2 \times 5 + 3 = 2n + 3$

$$\boxed{a_n = 2n + 3}$$

5. Summation notation & evaluating sums

Now that we understand how to get elements of a sequence from a closed formula, we can use this knowledge to evaluate sums.

If you're familiar with programming, sums are basically like building a for loop from 1 to some value *n*, and summing all the elements in the sequence step-by-step.

5. Summation notation & evaluating sums

The *general* form of a sum looks like this:

$$\sum_{k=1}^{n} a_k$$

Below the sigma is our index variable and starting value (usually 1).

Above the sigma is the ending value
(n will be replaced by a different integer)

And a_k is the closed formula that we use to get any given term at position k, and add it on to our running total.

5. Summation notation & evaluating sums

So if we want to evaluate a sum…

$$\sum_{k=1}^{5}(2k+1)$$

We need terms a_1, a_2, a_3, a_4, and a_5 to find the sum.

The result will be

$$a_1+a_2+a_3+a_4+a_5$$

5. Summation notation & evaluating sums

So if we want to evaluate a sum…

$$\sum_{k=1}^{5}(2k+1)=a_1+a_2+a_3+a_4+a_5=3+5+7+9+11$$

And then we used the closed formula $a_k=2k+1$ to find each term.

$a_1 = 2 \times 1 + 1 = 3$

$a_2 = 2 \times 2 + 1 = 5$

$a_3 = 2 \times 3 + 1 = 7$

$a_4 = 2 \times 4 + 1 = 9$

$a_5 = 2 \times 5 + 1 = 11$

5. Summation notation & evaluating sums

So if we want to evaluate a sum…

$$\sum_{k=1}^{5}(2k+1)=3+5+7+9+11 \quad \mathbf{= 35}$$

And then we just add each term.

5. Summation notation & evaluating sums

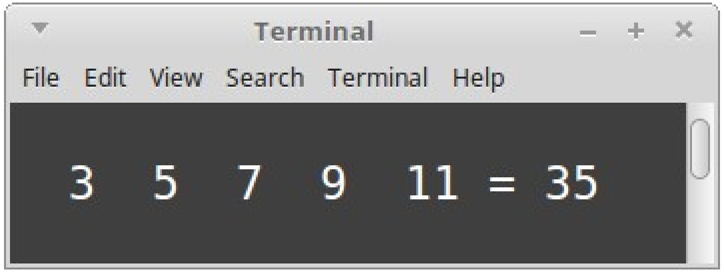
$$\sum_{k=1}^{5}(2k+1)=3+5+7+9+11 \quad \mathbf{= 35}$$

We can also write a program
to evaluate sums with a for loop!

```
int start = 1;
int end = 5;
int sum = 0;

for ( int k = start; k <= end; k++ )
{
    Output( 2 * k + 1 );
    sum += ( 2 * k + 1 );
}

OutputResult( sum );
```



5. Summation notation & evaluating sums

Practice 8: Evaluate the following sum $\sum_{k=1}^{5} \left(2^k\right)$

$$= 2^1 + 2^2 + 2^3 + 2^4 + 2^5$$

$$= 2 + 4 + 8 + 16 + 32$$

$$= 62$$

Conclusion

Maybe on the job you will have to derive a formula given some data set of numbers… or maybe not.

Either way, the practice of analyzing data and coming up with a solution, in general, is a useful skill to have.

The rest of Chapter 1 of our book relates more to propositional logic – items that evaluate to *true* or *false* (like if statements!)

So if you're not feeling too enthusiastic about these sequences, don't worry – finding formulas for sequences is only in Chapter 1.2.