Basic Rules for Counting

Written by Rachel J. Morris, last updated Jan 18, 2018

About

In this section, we will cover the formula for permutations, as well as how to solve problems that as for “this <u>and</u> that” outcome, or “this <u>or</u> that” outcome, and how to use the Rule of Complements.

Topics

1. Permutations

2. Rule of Sums

3. Rule of Products

4. Rule of Complements

Permutations

1. Permutations

A permutation is written as **P(n, r)**. For Permutation problems, you need two pieces of information:

- *n*, the amount of items we have to select from
- *r*, the amount of items that we're selecting.

The formula for P(n, r) is:

$$P(n,r)=\frac{n!}{(n-r)!}$$

Notes

P(n, r):
n is # of potential items
r is # of selections

$$P(n,r)=\frac{n!}{(n-r)!}$$

1. Permutations

In cases where we have *n* items and are selecting all *n*, then we have:

P(n, n) = n!

And in cases where we have *n* items but are only selecting 1 item, we have:

P(n, 1) = n

Notes

P(n, r):
n is # of potential items
r is # of selections

$$P(n, r) = \frac{n!}{(n-r)!}$$

1. Permutations

Also remember a Permutation is when…

- Repetitions are NOT allowed
- Order DOES matter.

Notes

P(n, r):
n is # of potential items
r is # of selections

$$P(n,r)=\frac{n!}{(n-r)!}$$

1. Permutations

Example: How many ways are there to arrange five people in a line?

- What is *n*?
- What is *r*?
- What is the result?

Notes

P(n, r):
n is # of potential items
r is # of selections

$$P(n,r) = \frac{n!}{(n-r)!}$$

1. Permutations

Example: How many ways are there to arrange five people in a line?

- What is *n*? **5**
- What is *r*? **5**
- What is the result?

$$P(n,r)=P(5,5) \quad =\frac{5!}{(5-5)!}=\frac{5!}{0!}=5! \quad =5\cdot 4\cdot 3\cdot 2\cdot 1=120$$

(Note that 0! (0-factorial) equals 1.)

Notes

P(n, r):
n is # of potential items
r is # of selections

$$P(n,r)=\frac{n!}{(n-r)!}$$

Rule of Sums

2. The rule of sums

The rule of sums In combinatorics, the rule of sum or addition principle is a basic counting principle. Stated simply, it is the idea that if we have ***A*** ways of doing something and ***B*** ways of doing another thing and we can not do both at the same time, then there are ***A + B*** ways to choose one of the actions.

From https://en.wikipedia.org/wiki/Rule_of_sum

Notes

Either one thing *or* another thing:
a + b

n(A)
is notation for “the # of elements in set A”.

2. The rule of sums

Example: On the "deals" menu on PizzaWebsite.com, there are specials on 85 types of pizzas. Of these…

1. How many specials have at least 4 toppings?

# of toppings	# of available specials
2	7
3	25
4	33
5	16
6	2

Notes

Either one thing *or* another thing:
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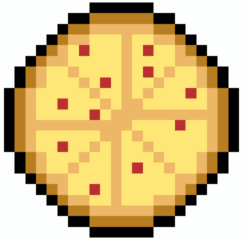
2. The rule of sums

Example: On the "deals" menu on PizzaWebsite.com, there are specials on 85 types of pizzas. Of these…

1. How many specials have at least 4 toppings?

33 + 16 + 2 = 48



# of toppings	# of available specials
2	7
3	25
4	33
5	16
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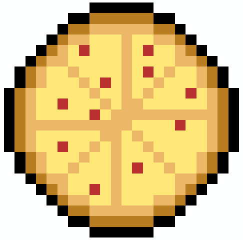
2. The rule of sums

Example: On the "deals" menu on PizzaWebsite.com, there are specials on 85 types of pizzas. Of these…

2. How many pizzas have only 1 topping?

1 topping isn't listed, so how do we figure it out?



# of toppings	# of available specials
2	7
3	25
4	33
5	16
6	2

Notes

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2. The rule of sums

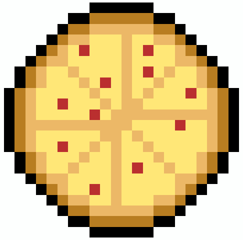
Example: On the "deals" menu on PizzaWebsite.com, there are specials on 85 types of pizzas. Of these…

2. How many pizzas have only 1 topping?

85 deals - (7 + 25 + 33 + 16 + 2) = 2 specials for 1 topping pizzas

85 is the total # of specials, minus all the specials with 2 or more toppings.

# of toppings	# of available specials
1	2
2	7
3	25
4	33
5	16
6	2



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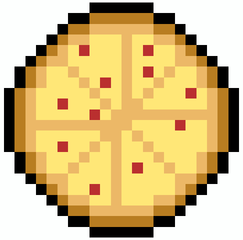
2. The rule of sums

Example: On the “deals” menu on PizzaWebsite.com, there are specials on 85 types of pizzas. Of these…

3. How many specials have at most 3 toppings?
Need to count options with 1, 2, and 3 toppings.



# of toppings	# of available specials
1	2
2	7
3	25
4	33
5	16
6	2

Notes

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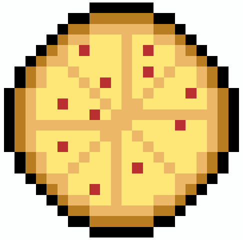
Example: On the "deals" menu on PizzaWebsite.com, there are specials on 85 types of pizzas. Of these…

3. How many specials have at most 3 toppings?
Need to count options with 1, 2, and 3 toppings.

2 + 7 + 25 = 34,
or 85 – 2 – 16 – 33 = 34

# of toppings	# of available specials
1	2
2	7
3	25
4	33
5	16
6	2



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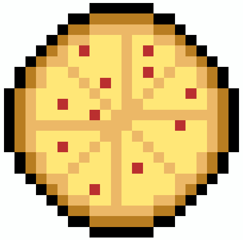
is notation for "the # of elements in set A".

2. The rule of sums

When calculating the total amount of outcomes, there could be cases of **overlap**. In this case, we need to remove the redundant cases from our final answer.

We can do this with the Rule of Sums with Overlap…



# of toppings	# of available specials
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Notes

Either one thing *or* another thing:
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2. The rule of sums

The rule of sums with overlap: If the list to count can be split into two pieces of size ***z*** and ***y***, and the pieces have z objects in common, then the original list has $x + y - z$ entries. In terms of sets, we can write this as $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ for all sets ***A*** and ***B***.

From Discrete Math Mathematical Reasoning and Proofs with Puzzles, Patterns and Games, by Ensley and Crawley, 5.2 page 391

Notes

Either one thing *or* another thing:
a + b

This or that, without duplicates:
a + b - c

2. The rule of sums

Example: Say that we're drawing cards from a deck. How many outcomes are there if we get a card value of "10" OR get a red suit?

In this case, we need to investigate both outcomes separately first:

- How many cards are there with a value of 10?

- How many cards are there that are red?

Notes

Either one thing *or* another thing:
a + b

This or that, without duplicates:
a + b - c

2. The rule of sums

A♠	2♠	3♠	4♠	5♠	6♠	7♠	8♠	9♠	10♠	J♠	Q♠	K♠
A♣	2♣	3♣	4♣	5♣	6♣	7♣	8♣	9♣	10♣	J♣	Q♣	K♣
A♥	2♥	3♥	4♥	5♥	6♥	7♥	8♥	9♥	10♥	J♥	Q♥	K♥
A♦	2♦	3♦	4♦	5♦	6♦	7♦	8♦	9♦	10♦	J♦	Q♦	K♦

- How many cards are there with a value of 10?
- How many cards are there that are red?

Notes

Either one thing *or* another thing:

a + b

This or that, without duplicates:

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2. The rule of sums

A♠	2♠	3♠	4♠	5♠	6♠	7♠	8♠	9♠	10♠	J♠	Q♠	K♠
A♣	2♣	3♣	4♣	5♣	6♣	7♣	8♣	9♣	10♣	J♣	Q♣	K♣
A♥	2♥	3♥	4♥	5♥	6♥	7♥	8♥	9♥	10♥	J♥	Q♥	K♥
A♦	2♦	3♦	4♦	5♦	6♦	7♦	8♦	9♦	10♦	J♦	Q♦	K♦

- How many cards are there with a value of 10?
  There are four
- How many cards are there that are red?

Notes

Either one thing *or* another thing:
a + b

This or that, without duplicates:
a + b - c

2. The rule of sums

A♠	2♠	3♠	4♠	5♠	6♠	7♠	8♠	9♠	10♠	J♠	Q♠	K♠
A♣	2♣	3♣	4♣	5♣	6♣	7♣	8♣	9♣	10♣	J♣	Q♣	K♣
A♥	2♥	3♥	4♥	5♥	6♥	7♥	8♥	9♥	10♥	J♥	Q♥	K♥
A♦	2♦	3♦	4♦	5♦	6♦	7♦	8♦	9♦	10♦	J♦	Q♦	K♦

- How many cards are there with a value of 10?
  There are four
- How many cards are there that are red?
  There are 26

Notes

Either one thing *or* another thing:

a + b

This or that, without duplicates:

a + b - c

2. The rule of sums

Example: Say that we're drawing cards from a deck. How many outcomes are there if we get a card value of "10" OR get a red suit?

- How many cards are there with a value of 10? **4**

10♠	10♣	10♥	10♦

- How many cards are there that are red? **26**

A♥	2♥	3♥	4♥	5♥	6♥	7♥	8♥	9♥	10♥	J♥	Q♥	K♥
A♦	2♦	3♦	4♦	5♦	6♦	7♦	8♦	9♦	10♦	J♦	Q♦	K♦

Notes

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- How many cards are there that are red? **26**

A♥	2♥	3♥	4♥	5♥	6♥	7♥	8♥	9♥	10♥	J♥	Q♥	K♥
A♦	2♦	3♦	4♦	5♦	6♦	7♦	8♦	9♦	10♦	J♦	Q♦	K♦

- How many cards both have a value of 10 <u>AND</u> are red?

Notes

Either one thing *or* another thing:

a + b

This or that, without duplicates:

a + b - c

2. The rule of sums

Example: Say that we're drawing cards from a deck. How many outcomes are there if we get a card value of "10" OR get a red suit?

- How many cards are there with a value of 10? **4**

10♠ 10♣ 10♥ 10♦

- How many cards are there that are red? **26**

A♥ 2♥ 3♥ 4♥ 5♥ 6♥ 7♥ 8♥ 9♥ 10♥ J♥ Q♥ K♥

A♦ 2♦ 3♦ 4♦ 5♦ 6♦ 7♦ 8♦ 9♦ 10♦ J♦ Q♦ K♦

- How many cards both have a value of 10 <u>AND</u> are red? **2**

10♥ 10♦

Notes

Either one thing *or* another thing:

a + b

This or that, without duplicates:

a + b - c

2. The rule of sums

Example: Say that we're drawing cards from a deck. How many outcomes are there if we get a card value of "10" OR get a red suit?

- How many cards are there with a value of 10? **4**

10♠	10♣	10♥	10♦

- How many cards are there that are red? **26**

A♥	2♥	3♥	4♥	5♥	6♥	7♥	8♥	9♥	10♥	J♥	Q♥	K♥
A♦	2♦	3♦	4♦	5♦	6♦	7♦	8♦	9♦	10♦	J♦	Q♦	K♦

- How many cards both have a value of 10 AND are red? **2**

10♥	10♦

Using the rule of sums with overlap, what's the answer?

Notes

Either one thing *or* another thing:

a + b

This or that, without duplicates:

a + b - c

2. The rule of sums

Example: Say that we're drawing cards from a deck. How many outcomes are there if we get a card value of "10" OR get a red suit?

- How many cards are there with a value of 10? **4**

10♠	10♣	10♥	10♦

- How many cards are there that are red? **26**

A♥	2♥	3♥	4♥	5♥	6♥	7♥	8♥	9♥	10♥	J♥	Q♥	K♥
A♦	2♦	3♦	4♦	5♦	6♦	7♦	8♦	9♦	10♦	J♦	Q♦	K♦

- How many cards both have a value of 10 AND are red? **2**

10♥	10♦

Using the rule of sums with overlap, what's the answer? **4 + 26 – 2 = 28**

Notes

Either one thing *or* another thing:

a + b

This or that, without duplicates:

a + b - c

Rule of Products

3. The rule of products

The rule of products In combinatorics, the rule of product or multiplication principle is a basic counting principle (a.k.a. the fundamental principle of counting). Stated simply, it is the idea that if there are a ways of doing something and b ways of doing another thing, then there are $a \cdot b$ ways of performing both actions.

From https://en.wikipedia.org/wiki/Rule_of_product

Notes

Doing one thing *and* another thing:
a x b

Doing *either* one thing *or* another thing:
a + b

n(A)
is notation for "the # of elements in set A".

3. The rule of products

Example: You have to order food for a group event. There are 5 types of pizza to choose from, and 3 types of ice cream to choose from. Your guests only want 1 of each type of pizza.

How many ways can you select 3 pizzas AND 1 ice cream?

Notes

Doing one thing *and* another thing:

a x b

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How many ways can you select 3 pizzas AND 1 ice cream?

Pizza Plate A		Pizza Plate B		Pizza Plate C	Ice Cream Bowl
5 options	x	4 options	x	3 options	

Pizza choices: **P(5, 3)** = **5 x 4 x 3 = 60**

Notes

Doing one thing *and* another thing:
a x b

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Pizza choices: **P(5, 3)** = **5 x 4 x 3 = 60**
Ice cream choices: **P(3, 1)** = **3**

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5 options	x	4 options	x	3 options	3 options

Pizza choices: **P(5, 3)** = 5 x 4 x 3 = 60
Ice cream choices: **P(3, 1)** = 3
Combinations of pizza AND ice cream: **P(5, 3) + P(3, 1) = 60 x 3 = 180**

Notes

Doing one thing *and* another thing:
a x b

Doing *either* one thing *or* another thing:
a + b

n(A)
is notation for "the # of elements in set A".

Rule of Complements

4. The rule of complements

If there are ***x*** objects and ***y*** of those objects have a particular property, then the number of those objects that do <u>not</u> have that particular property is ***x – y***. In terms of sets, using ***U*** for the universal set, we can write this as ***n(A') = n(U) – n(A)*** for all sets ***A*** with elements from ***U***.

From Discrete Math Mathematical Reasoning and Proofs with Puzzles, Patterns and Games, by Ensley and Crawley, 5.2 page 390

Notes

Doing one thing *and* another thing:
a x b

Doing *either* one thing *or* another thing:
a + b

n(A)
is notation for "the # of elements in set A".

4. The rule of complements

Essentially, if there are n total options, and x options have property-X, then $n - x$ options do NOT have property-X…

Notes

Doing one thing *and* another thing:
a x b

Doing *either* one thing *or* another thing:
a + b

n(A)
is notation for "the # of elements in set A".

4. The rule of complements

Example: A pet store sells red and green squeaky toys. If there are 10 total toys and 7 are red, how many are green?

Notes

Doing one thing *and* another thing:
a x b

Doing *either* one thing *or* another thing:
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4. The rule of complements

Example: A pet store sells red and green squeaky toys. If there are 10 total toys and 7 are red, how many are green?

10 total = 7 red + x green

10 – 7 = x

x = 3

3 total green toys

Notes

Doing one thing *and* another thing:
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Doing *either* one thing *or* another thing:
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4. The rule of complements

Example: How many ways are there to re-arrange the letters in the word "RAT"?

- How many ways and in "T"?
- How many ways DON'T end in "T"?

R A T

Notes

Doing one thing *and* another thing:
a x b

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a + b

n(A)
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4. The rule of complements

Example: How many ways are there to re-arrange the letters in the word “RAT”?

- How many ways and in “T”?
- How many ways DON’T end in “T”?

For the first letter, we will have 3 options…

Letter 1	Letter 2	Letter 3

T

R A

Notes

Doing one thing *and* another thing:

a x b

Doing *either* one thing *or* another thing:

a + b

n(A)

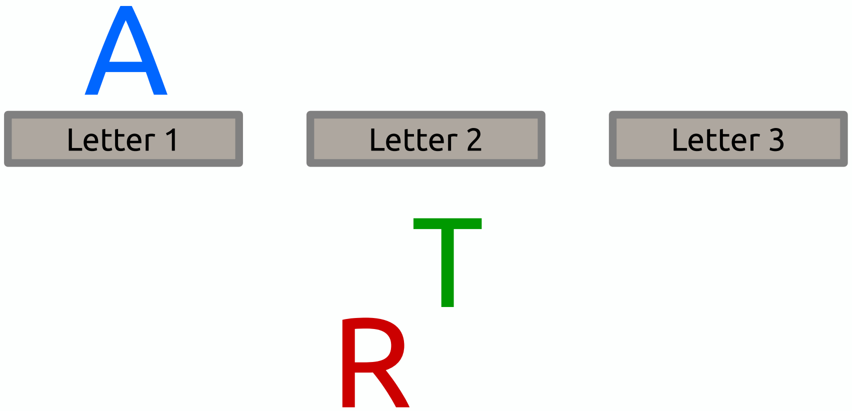
is notation for “the # of elements in set A”.

4. The rule of complements

Example: How many ways are there to re-arrange the letters in the word “RAT”?

- How many ways and in “T”?
- How many ways DON’T end in “T”?

For the second letter, there will be two options…



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4. The rule of complements

Example: How many ways are there to re-arrange the letters in the word "RAT"?

- How many ways and in "T"?
- How many ways DON'T end in "T"?

For the third letter, there will be one option…

A	T	
Letter 1	Letter 2	Letter 3
		R

Notes

Doing one thing *and* another thing:
a x b

Doing *either* one thing *or* another thing:
a + b

n(A)
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- How many ways and in “T”?
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For the third letter, there will be one option…

A	T	R
Letter 1	Letter 2	Letter 3

Notes

Doing one thing *and* another thing:
a x b

Doing *either* one thing *or* another thing:
a + b

n(A)
is notation for “the # of elements in set A”.

4. The rule of complements

Example: How many ways are there to re-arrange the letters in the word "RAT"? 6

- How many ways and in "T"?
- How many ways DON'T end in "T"?

For the third letter, there will be one option…

A T R

Letter 1 Letter 2 Letter 3

3 x 2 x 1 = 6 ways to re-arrange the letters.

Notes

Doing one thing *and* another thing:

a x b

Doing *either* one thing *or* another thing:

a + b

n(A)

is notation for "the # of elements in set A".

4. The rule of complements

Example: How many ways are there to re-arrange the letters in the word “RAT”? 6

- How many ways and in “T”?
- How many ways DON’T end in “T”?

How many ways end in “T”?

1. RAT	**4. ATR**
2. RTA	**5. TAR**
3. ART	**6. TRA**

Notes

Doing one thing *and* another thing:
a x b

Doing *either* one thing *or* another thing:
a + b

n(A)
is notation for “the # of elements in set A”.

4. The rule of complements

Example: How many ways are there to re-arrange the letters in the word "RAT"? **6**

- How many ways and in "T"? **2**
- How many ways DON'T end in "T"?

How many ways end in "T"?
There are 2.

1.	**RAT**	**4. ATR**
2.	**RTA**	**5. TAR**
3.	**ART**	**6. TRA**

asdf

Notes

Doing one thing *and* another thing:
a x b

Doing *either* one thing *or* another thing:
a + b

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4. The rule of complements

Example: How many ways are there to re-arrange the letters in the word "RAT"? 6

- How many ways and in "T"? 2
- How many ways DON'T end in "T"?

How many ways DON'T end in "T"? We can use the Rule of complements…

1.	**RAT**	**4. ATR**
2.	**RTA**	**5. TAR**
3.	**ART**	**6. TRA**

Notes

Doing one thing *and* another thing:
a x b

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4. The rule of complements

Example: How many ways are there to re-arrange the letters in the word "RAT"? **6**

- How many ways and in "T"? **2**
- How many ways DON'T end in "T"?

How many ways DON'T end in "T"? We can use the Rule of complements…

1.	**RAT**	**4. ATR**
2.	**RTA**	**5. TAR**
3.	**ART**	**6. TRA**

TotalArrangements = EndsWithT + DoesntEndWithT

Notes

Doing one thing *and* another thing:
a x b

Doing *either* one thing *or* another thing:
a + b

n(A)
is notation for "the # of elements in set A".

4. The rule of complements

Example: How many ways are there to re-arrange the letters in the word "RAT"? 6

- How many ways and in "T"? 2
- How many ways DON'T end in "T"?

How many ways DON'T end in "T"? We can use the Rule of complements…

1. RAT	**4. ATR**
2. RTA	**5. TAR**
3. ART	**6. TRA**

TotalArrangements = EndsWithT + DoesntEndWithT

6 = 2 + x

6 – 2 = x

x = 4

Notes

Doing one thing *and* another thing:

a x b

Doing *either* one thing *or* another thing:

a + b

n(A)

is notation for "the # of elements in set A".

Conclusion

These principles will be used all through Chapter 5, as well as in some form in Chapter 6 once we're figuring probability.

Make sure to pay attention to when a problem asks for "this **and** that", or "this **or** that", as that will tip you off on which rule to use.