Isomorphism and Planarity

Written by Rachel J. Morris, last updated Apr 2, 2018

About

Trees are a handy structure in Data Structures, and are also a part of Graph Theory.

Topics

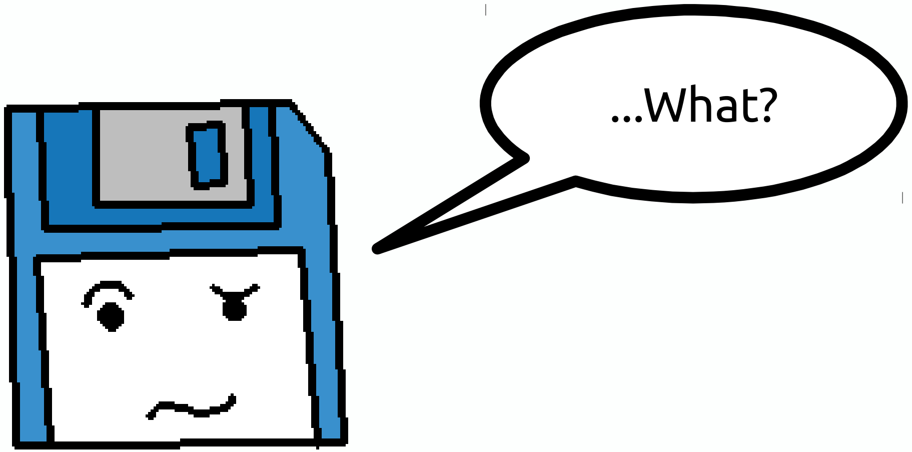
1. Isomorphism
2. Planarity
3. Spanning Tree Algorithms

Isomorphism

1. Isomorphism

Definition: Simple graphs *G* and *H* are called **isomorphic** if there is a one-to-one and onto function *f* from the nodes of *G* to the nodes of *H* such that *{v, w}* is an edge of *G* if and only if *{ f(v), f(w) }* is an edge of H.

From Discrete Mathematics, Ensley & Crawley, page 534

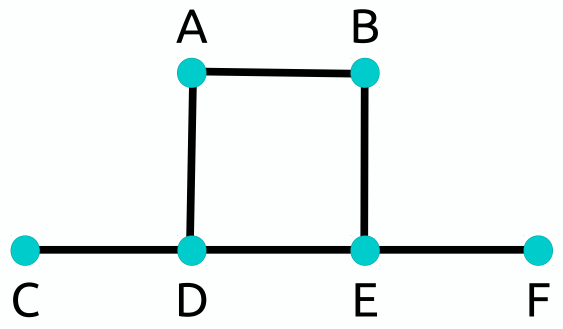


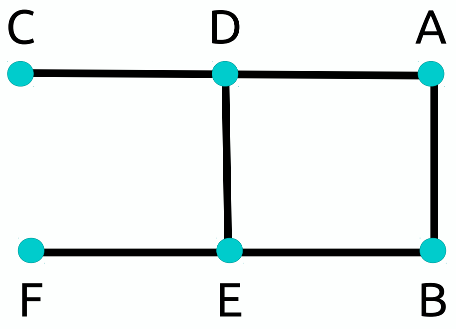
Notes

1. Isomorphism

In other words, two graphs are **isomorphic** if you can rearrange the location of the nodes to match each other.

We talk about *{v, w}* and *{f(v), f(w)}* because we think of it in terms of having a function that transforms our graph from one graph *G* to some other graph *H*.





Notes

an **isomorphism** of graphs G and H is a bijection between the vertex sets of G and H

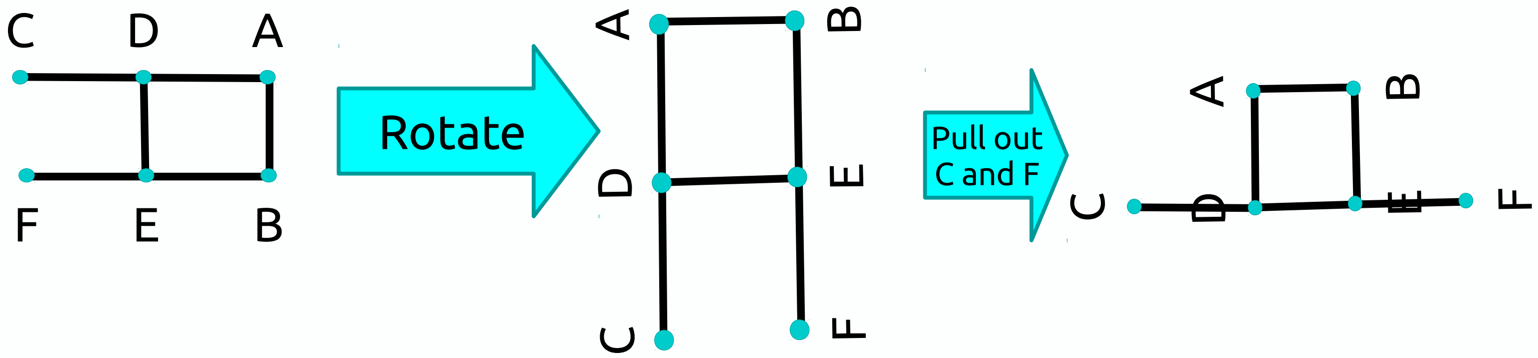
$$f : V(G) \to V(H)$$

such that any two vertices u and v of G are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H.

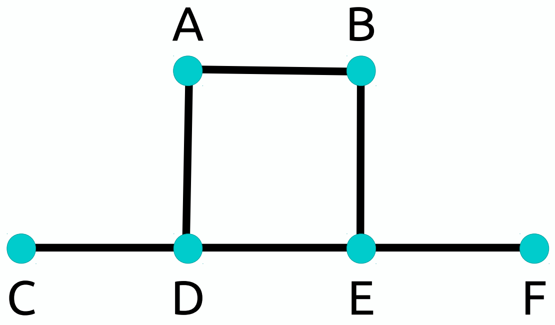
(From https://en.wikipedia.org/wiki/Graph_isomorphism)

1. Isomorphism

Transforming this graph...



Into this graph:



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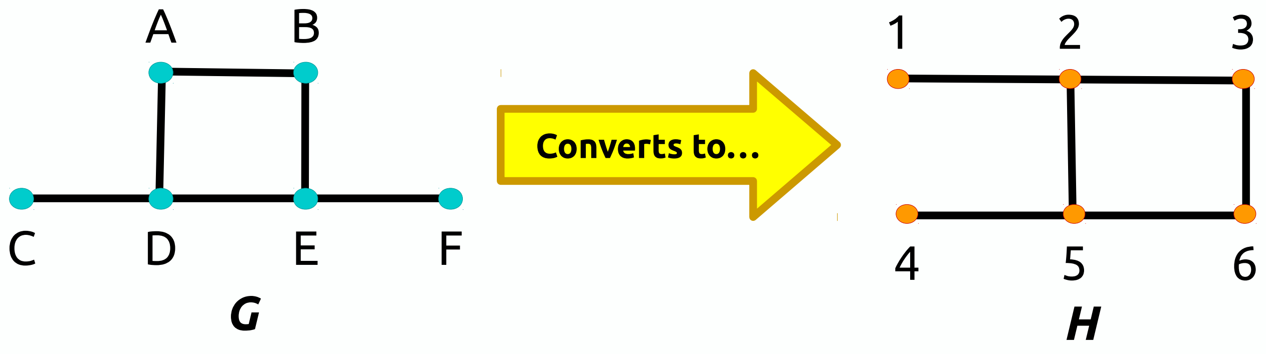
1. Isomorphism

The two graphs that are related don’t need to have the same vertex names, either. It just has to have some sort of relation where a vertex from G is “equivalent” to a vertex from H.

Nodes in G	A	B	C	D	E	F
Nodes in H	3	6	1	2	5	4



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1. Isomorphism

We can also write these transformations as:

$A \mapsto 3$, $B \mapsto 6$, $C \mapsto 1$, $D \mapsto 2$, $E \mapsto 5$, $F \mapsto 4$

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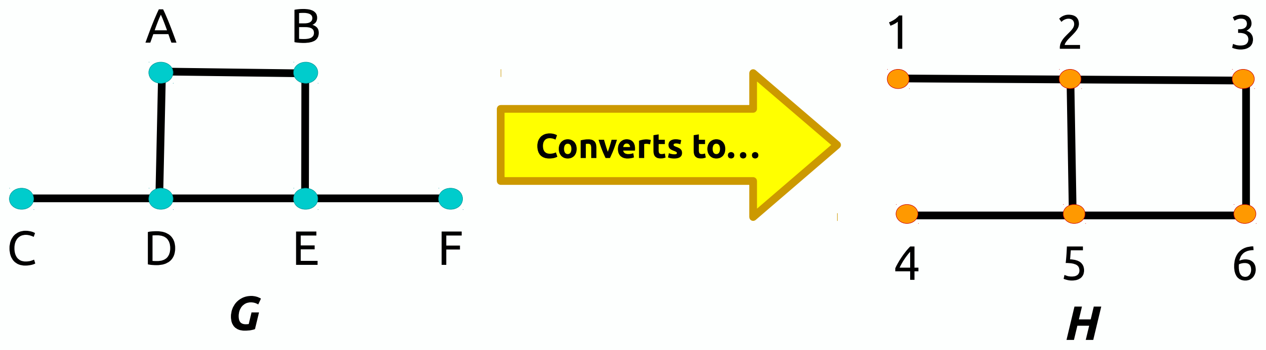

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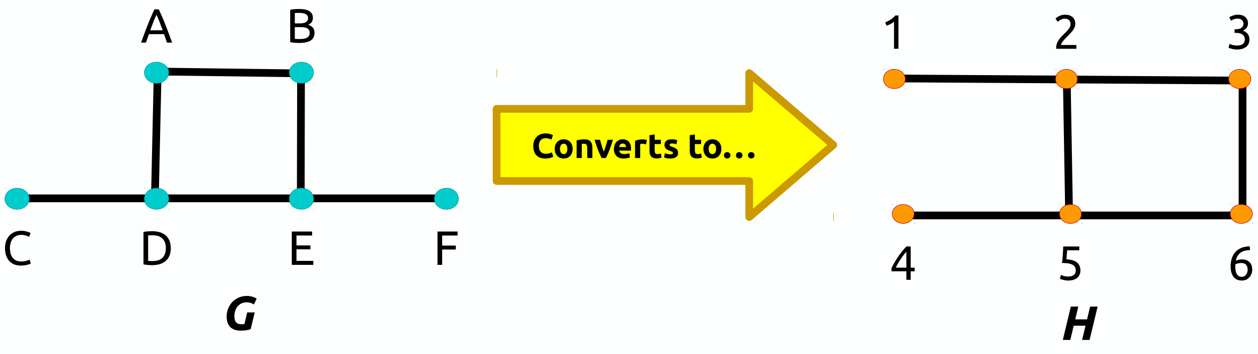
1. Isomorphism

We can also investigate the edges between two nodes, and show the relationships.

Edges in G	{C,D}	{A,D}	{A,B}	{B,E}	{E,F}	{D,E}
Edges in H	{1,2}	{3,2}	{3,6}	{6,5}	{5,4}	{2,5}



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1. Isomorphism

Proposition 1: Two graphs that are isomorphic to one another must have…

1) The same # of nodes

2) The same # of edges

3) The same # of nodes of any given degree

4) The same # of cycles

5) The same # of cycles of any given size

From Discrete Mathematics, Ensley & Crawley, page 535

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Planarity

2. Planarity

Definitions:

1. A simple, connected graph is called **planar** if there is a way to draw it (on a plane) so that no edges cross.

2. A graph is called **bipartite** if its set of nodes can be partitioned into two disjoint sets S_1 and S_2 so that every edge in the graph has one endpoint in S_1 and one endpoint in S_2.

From Discrete Mathematics, Ensley & Crawley, page 536

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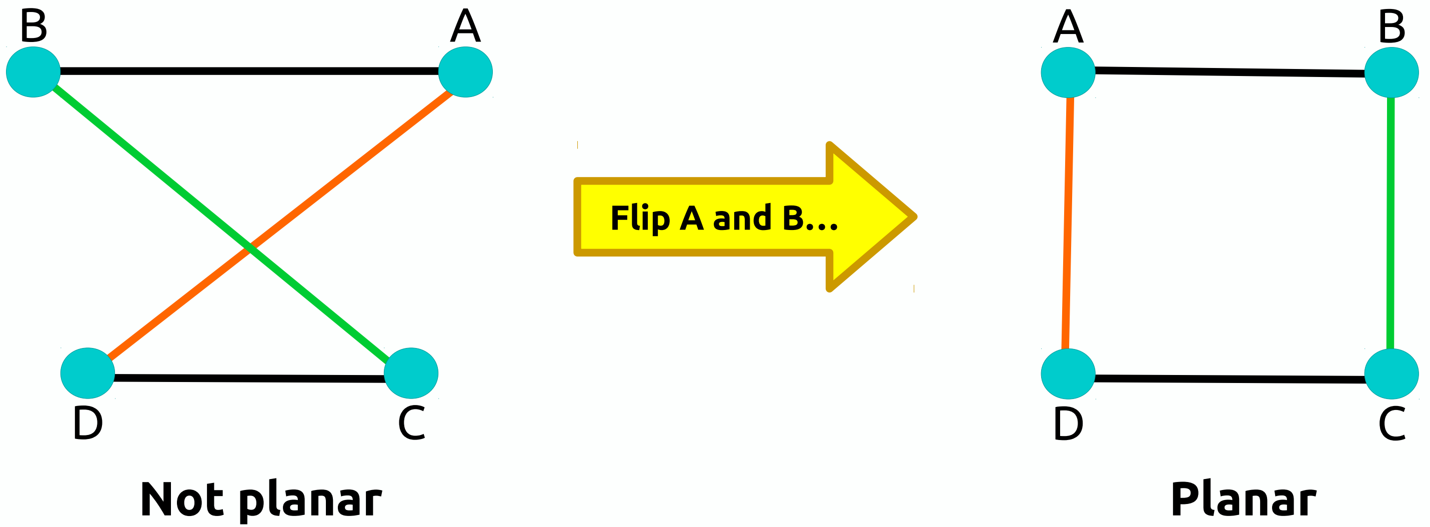
Planar: A simple, connected graph is called **planar** if there is a way to draw it (on a plane) so that no edges cross.

2. Planarity



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Planar: A simple, connected graph is called **planar** if there is a way to draw it (on a plane) so that no edges cross.

2. Planarity

Definitions:

3. The **complete graph** on n nodes, denoted by $\boldsymbol{K_n}$, is the simple graph with nodes $\{1, \ldots, n\}$ and an edge between every pair of distinct nodes.

4. A **complete bipartite graph** on n,m nodes, denoted by $\boldsymbol{K_{n,m}}$, is the simple bipartite graph with nodes $S_1 = \{a_1, a_2, \ldots, a_n\}$ and $S_2 = \{b_1, b_2, \ldots, b_m\}$ and with edges connecting each node in S_1 to every node in S_2.

From Discrete Mathematics, Ensley & Crawley, page 536

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2. Planarity

Example: Draw the diagram K_4.

- Should have nodes {1, 2, 3, 4}.
- Each node is connected to every other node.

From Discrete Mathematics, Ensley & Crawley, page 537

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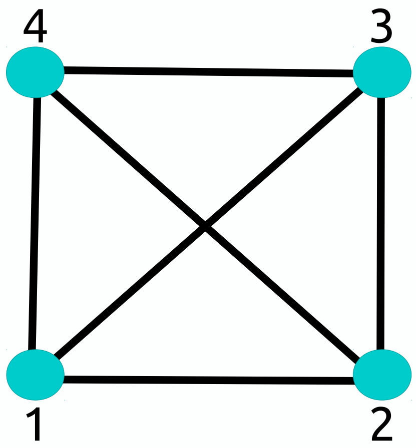
Example: Draw the diagram K_4.



From Discrete Mathematics, Ensley & Crawley, page 537

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2. Planarity

Example: Draw the diagram $K_{3,2}$.

- $S_1 = \{ 1, 2, 3 \}$
- $S_2 = \{ 1, 2 \}$
- Each node from S_1 is connected to every node in S_2.

From Discrete Mathematics, Ensley & Crawley, page 537

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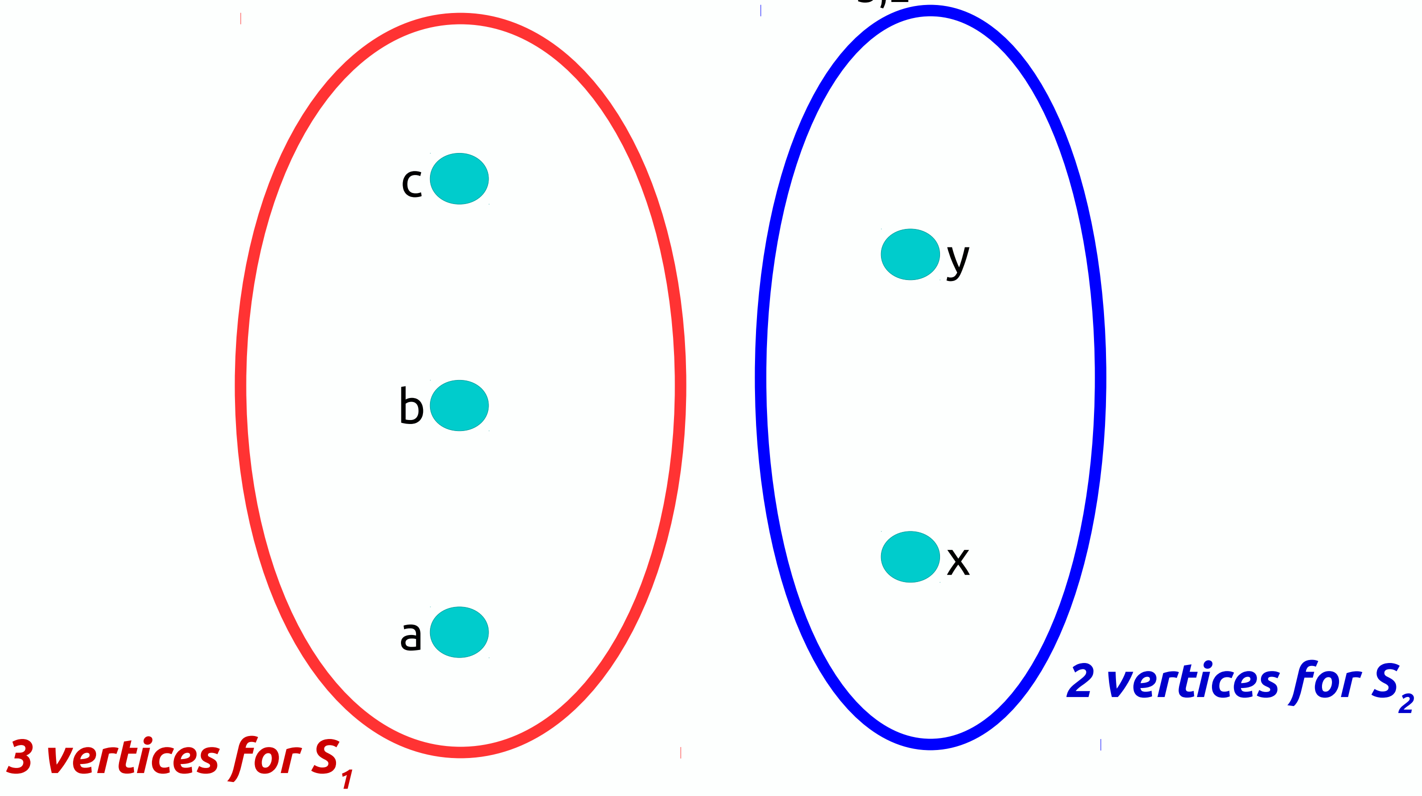

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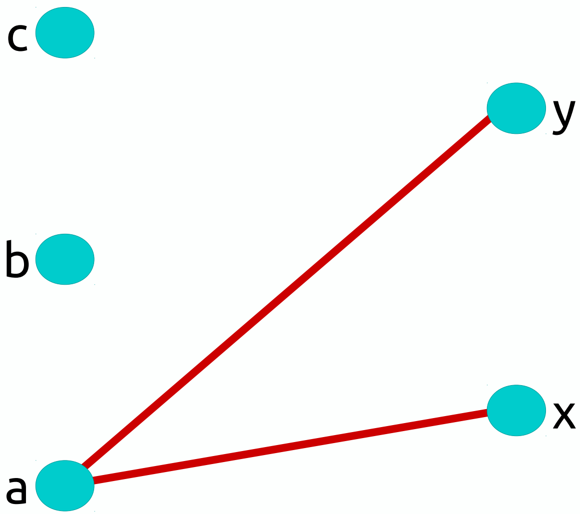

For each node in S_1, it is connected to every node in S_2.

From Discrete Mathematics, Ensley & Crawley, page 537

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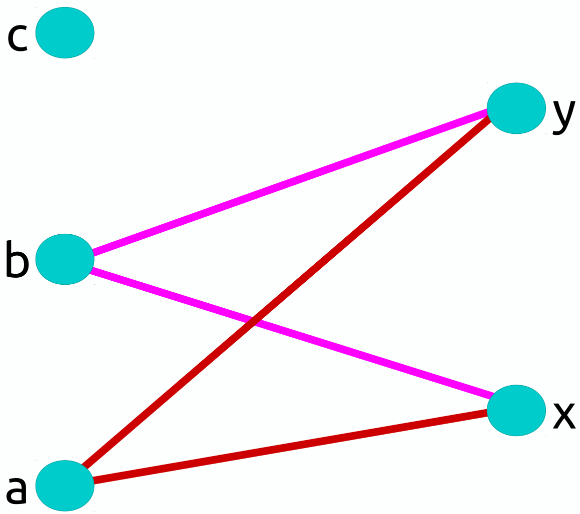

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2. Planarity

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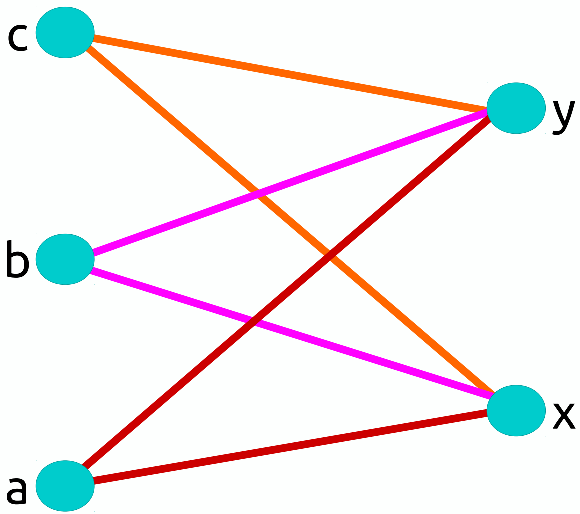

No connections in-between nodes from S_1 and themselves,

or from S_2 and themselves.

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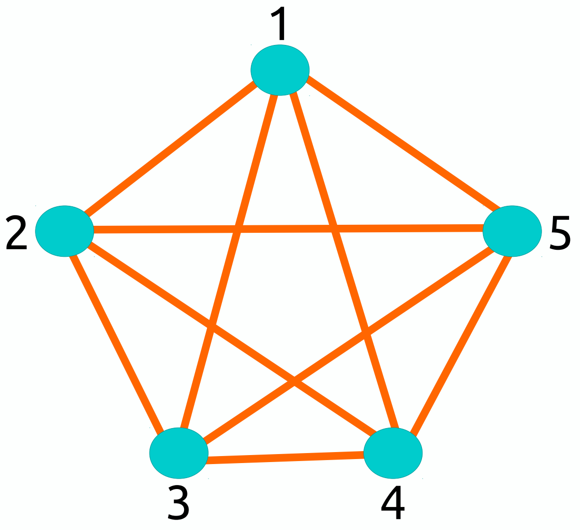
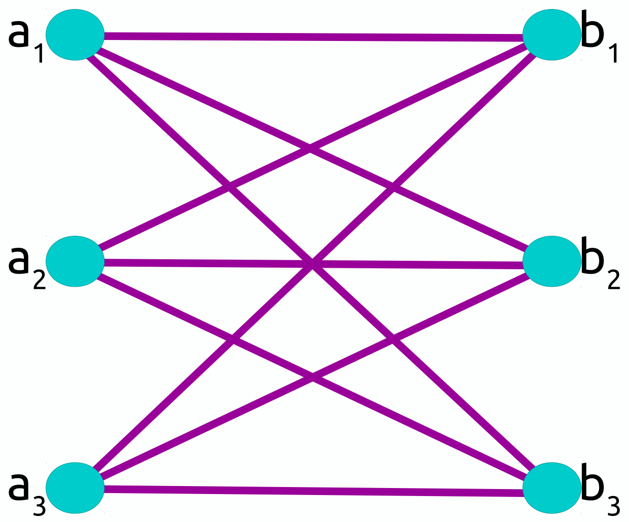
2. Planarity

The only two (base) nonplanar graphs are K_5 and $K_{3,3}$.



K_5



$K_{3,3}$

From Discrete Mathematics, Ensley & Crawley, page 537

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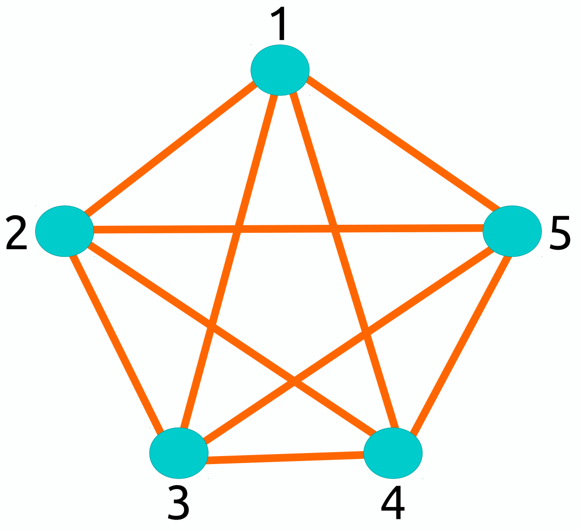
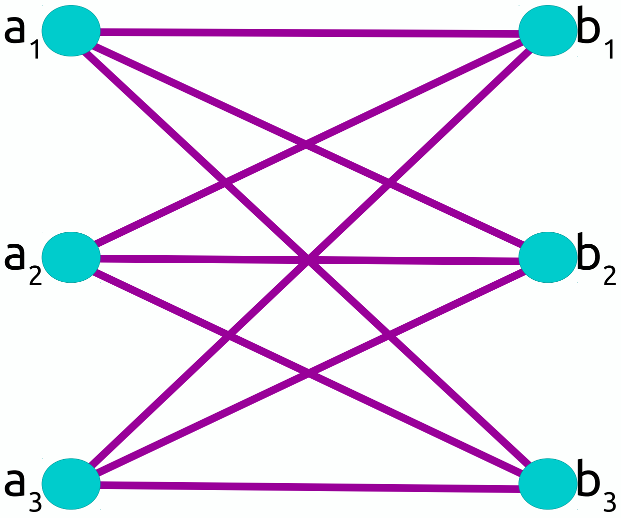
2. Planarity

Theorem 3: A graph G is planar if and only if it contains no “copies” of $K_{3,3}$ or K_5 as subgraphs.



$\mathbf{K_5}$



$\mathbf{K_{3,3}}$

From Discrete Mathematics, Ensley & Crawley, page 538

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2. Planarity

Definition: For a planar graph G embedded in the plane, a **face** of the graph is a region of the plane created by the drawing. Since the plane is an unbounded surface, every embedding of a finite planar graph will have exactly one unbound face.

From Discrete Mathematics, Ensley & Crawley, page 540

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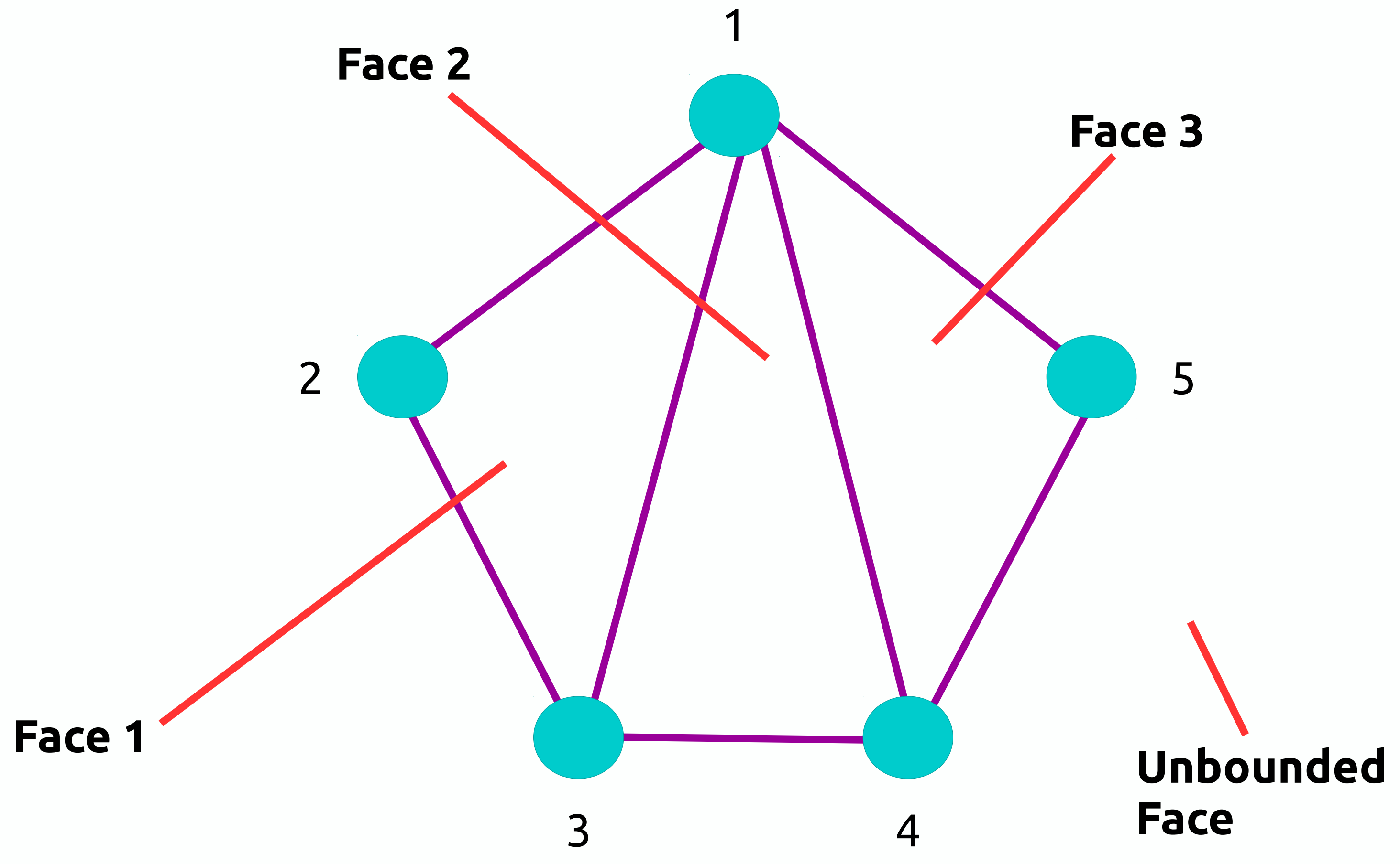

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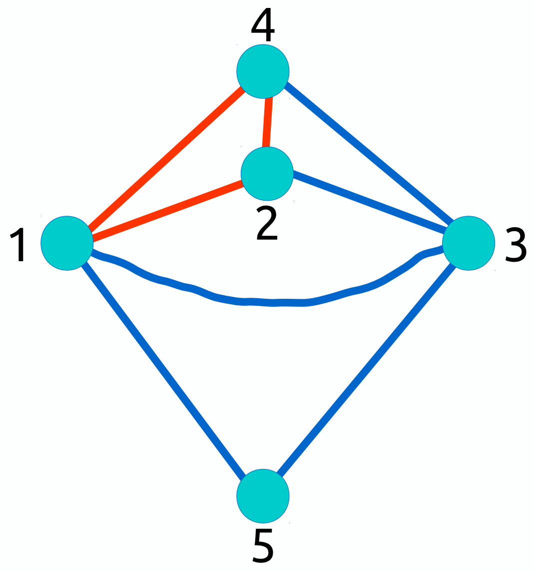

Faces:

1, 2, 4, 1

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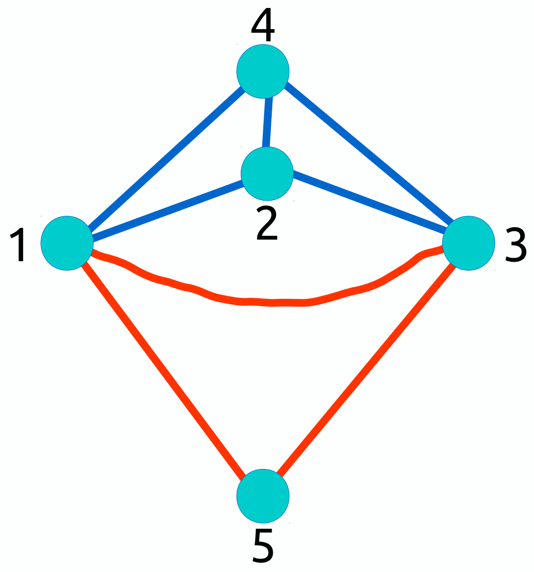

Faces:

1, 2, 4, 1

1, 3, 5, 1

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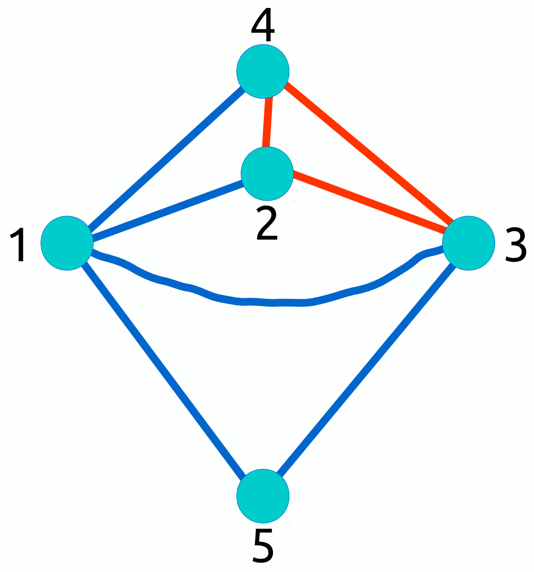

Faces:

1, 2, 4, 1

1, 3, 5, 1

2, 3, 4, 2

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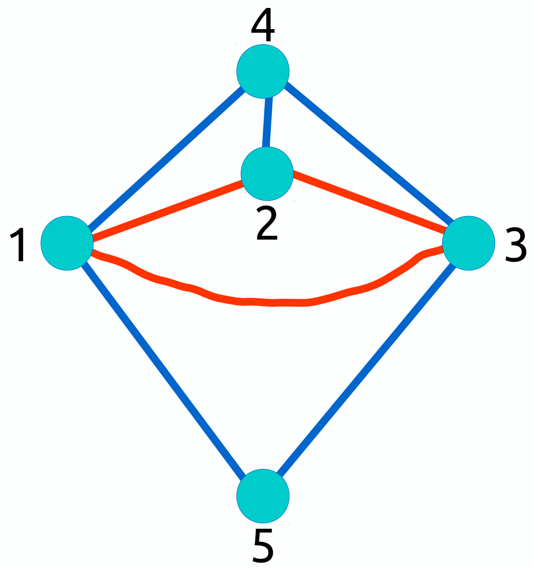

Faces:

1, 2, 4, 1

1, 3, 5, 1

2, 3, 4, 2

1, 2, 3, 1

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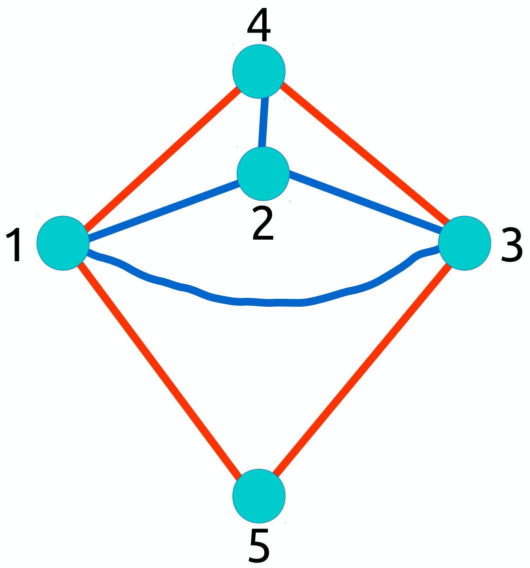
2. Planarity



Faces:

1, 2, 4, 1

1, 3, 5, 1

2, 3, 4, 2

1, 2, 3, 1

1, 4, 3, 5, 1 (Unbounded)

From Discrete Mathematics, Ensley & Crawley, page 541

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Conclusion

Graphs…