Exam 2 Chapter 6 preview

Chapter 6.1

Probability
Given an experiment with a sample space S of equally likely outcomes and an event E, the *probability of the event* (denoted by $Prob(E)$) is the ratio of the number of successful outcomes to the total number of outcomes: $Prob(E) = \frac{n(E)}{n(S)}$

Complements Given an event E,
$Prob(E) + Prob(\bar{E}) = 1$ Where $\bar{E}$ is the complement of the event E.

Question 1 Fill out the following table. The event is that we are drawing a playing card from a standard deck of 52 cards.

Outcome	$n(E)$	$n(S)$	$Prob(E)$
The card is a Jack.			
The card is a Red Jack.			
The card is a King or a Queen.			
The card is either a Diamond or a Jack.			

Question 2 Two cards are selected out of a standard deck. Find the probability that...

a. The first card has a value of King.

b. The first card is is a suit of Hearts and the second card is a suit of Diamonds.

c. Both cards have the same suit.

Question 3 Two dice are rolled: One red, one green. What is the probability that the sum of the dice is 5?

Chapter 6.2

Disjoint events
Two events are said to be **disjoint** (or *mutually exclusive* if they cannot occur simultaneously.

The General Sum Rule If E_1 and E_2 are any events in a given experiment, then the probability that E_1 or E_2 occurs is given by

$$Prob(E_1 or E_2) = Prob(E_1) + Prob(E_2) - Prob(E_1 and E_2)$$

If E_1 and E_2 are disjoint, then $E_1 \cap E_2 = \emptyset$, so $Prob(E_1 and E_2) = 0$.

Independent events Two events are said to be **independent** if the occurrence of one event is not influenced by the occurred (or nonoccurrence) of the other event.

The General Product Rule If E_1 and E_2 are any events in a given experiment, then the probability that both E_1 and E_2 occur is given by
$Prob(E_1 and E_2) = Prob(E_2) \cdot Prob(E_1|E_2)$
$= Prob(E_1) \cdot Prob(E_2|E_1)$

The probability of E_1 given E_2 Given events E_1 and E_2 for some experiment, we define the probability of E_1 given E_2, denoted by $Prob(E_1|E_2)$, as the probability that E_1 happens given that E_2 occurs. Note that if E_1 and E_2 are independent, then $Prob(E_1|E_2) = Prob(E_1)$.

Question 4 Identify whether the following events are disjoint or not.

a. When drawing two cards from a standard deck, with event 1 being getting at least one ace, and event 2 being getting at least one club.

b. When rolling two six-sided dice, with event 1 being getting at least one 5, and event 2 being getting a sum of 8.

c. When drawing two cards from a standard deck, with event 1 being getting two aces, and event 2 being getting two clubs.

Question 5 Identify whether the following events are independent or not.

a. When dealing two cards from a standard deck, with event 1 being getting an Ace as the first card, and event 2 being getting a 10, Jack, Queen, or King as the second card.

b. When rolling a red die and a green die, with event 1 being getting a red 5, and event 2 being getting a sum of 8.

c. When rolling a red die and a green die, with event 1 being getting a red 5, and event 2 being getting a green 6.

Question 6 Use the General Sum Rule to solve the following.
What is the probability that three cards chosen from a standard deck of cards consist of either three face cards (i.e., cards having a value of Jack, Queen, or King) or three cards of the same suit?

Question 7 For the following, find $Prob(E_1|E_2)$.
When selecting two cards, what is the probability that (E_1) the first card is an Ace and (E_2)the second card is a 10, Jack, Queen, or King?

Question 8 Use the General Product Rule to solve the following.
Two marbles are chosen from a bag containing three red, five white, and eight green marbles. What is the probability that both are red?

Chapter 6.3

Probability of having exactly k successes Given a simple experiment, called a **Bernoulli trial**, and an event that occurs with a probability p, if the trial is repeated independently n times, then the probability of having exactly k successes is $C(n,k) \cdot p^k \cdot (1-p)^{n-k}$

Question 9 What is the probability that in seven rolls of a six-sided die, the result of 1 appears exactly five times?

Chapter 6.4

Expected (average) value For a given probability experiment, let X be a random variable whose possible values come from the set of numbers $x_1, ..., x_n$. Then the **expected value of** X, denoted by $E[X]$, is the sum $E[X] = (x_1) \cdot Prob(X = x_1) + (x_2) \cdot Prob(X = x_2) + ... + (x_n) \cdot Prob(X = x_n)$

Expectation in Bernoulli trials Suppose an experiment consists of the independent repetition of a trial n times, and the probability of that trial's individual success is p each time it is performed. If X denotes the number of successful trials in this experiment, then $E[X] = n \cdot p$.

Question 10 If you roll two dice many times and record the sum of the two dice in each case, what do you expect will be the average of all these sums?

Question 11 Suppose five cards are drawn from a standard deck. What is the expected number of aces among the five cards?

Answer key

Question 1

Outcome	$n(E)$	$n(S)$	$Prob(E)$
The card is a Jack.	4	52	4/52
The card is a Red Jack.	2	52	2/52
The card is a King or a Queen.	4+4	52	8/52
The card is either a Diamond or a Jack.	13+4-1	52	16/52

Question 2

a. The first card has a value of King.

- There are 4 kings in a deck of 52 cards, so the probability of getting a king as the first card is $\frac{4}{52}$.
- Then, we don't care about the value of the next card, but we have 51 cards left to choose from, so selecting the second card is $\frac{51}{51}$.
- Therefore, the probability of getting a King as the first card is: $\frac{4}{52} \cdot \frac{51}{51}$

b. The first card is is a <u>suit</u> of Hearts and the second card is a <u>suit</u> of Diamonds.

- There are 13 Heart cards in a deck of 52, so the probability of the first card being a Heart is $\frac{13}{52}$.
- There are 13 Diamond cards in a deck of 52. However, we've already chosen the first card, so we are selecting the second card out of a deck of 51. Therefore, our probability is $\frac{13}{51}$.
- Getting both a Heart AND a Diamond results in the probability $\frac{13}{52} \cdot \frac{13}{51}$.

c. Both cards have the same <u>suit</u>.

- For the first card, we don't care *what* it is... it could be anything! So the probability of getting any card out of the deck is $\frac{52}{52}$.
- For the second card, it must have the same suit! There are 13 cards of each type of suit in the deck, but we've already taken ONE card of the first card's suit, so now the probability of getting a card with the same suit is $\frac{12}{51}$.
- Therefore, the probability of getting two cards of the same suit is $\frac{52}{52} \cdot \frac{12}{51}$.

Question 3

- To get a sum of five, we can roll the following: (1, 4), (4, 1), (2, 3), (3, 2). Therefore, there are four possibilities.
- We are rolling two dice, so there are $6 \cdot 6 = 36$ total outcomes.
- Therefore, the probability of getting something that sums to 5 is $\frac{4}{36}$ or $\frac{1}{9}$.

Question 4

a. When drawing two cards from a standard deck, with event 1 being getting at least one ace, and event 2 being getting at least one club. NOT DISJOINT

b. When rolling two six-sided dice, with event 1 being getting at least one 5, and event 2 being getting a sum of 8. NOT DISJOINT

c. When drawing two cards from a standard deck, with event 1 being getting two aces, and event 2 being getting two clubs. DISJOINT

Question 5

a. When dealing two cards from a standard deck, with event 1 being getting an Ace as the first card, and event 2 being getting a 10, Jack, Queen, or King as the second card. NOT INDEPENDENT

b. When rolling a red die and a green die, with event 1 being getting a red 5, and event 2 being getting a sum of 8. NOT INDEPENDENT

c. When rolling a red die and a green die, with event 1 being getting a red 5, and event 2 being getting a green 6. INDEPENDENT

Question 6

- EXPERIMENT: We are selecting 3 cards from a deck of 52, so our sample space size, $n(S)$ is $C(52,3)$.
- EVENT 1: There are $3 \cdot 4 = 12$ face cards in a deck of 52. To select three of these cards, we would use $C(12,3)$.
  So the probability of E_1 occurring is $\frac{C(12,3)}{C(52,3)}$
- EVENT 2: There are 13 cards of each suit and 4 suits, so to select three of these cards, we would use $4 \cdot C(13,3)$.
  So the probability of E_2 occurring is $\frac{4 \cdot C(13,3)}{C(52,3)}$
- EVENT 1 AND EVENT 2: We need to account for the overlap. The overlap would occur if we drew three face cards with the same suit. This means $C(3,3)$. However, since there are four suits, we multiply it by 4. The probability of E_1 AND E_2 occuring is $\frac{4 \cdot C(3,3)}{C(52,3)}$ or $\frac{4}{C(52,3)}$.

- GENERAL SUM RULE:
  $Prob(E_1 or E_2) = Prob(E_1) + Prob(E_2) - Prob(E_1 and E_2)$
  $Prob(E_1 or E_2) = \frac{C(12,3)}{C(52,3)} + \frac{4 \cdot C(13,3)}{C(52,3)} - \frac{4}{C(52,3)} = \frac{1360}{22100} \approx 0.0615$

Question 7

- E_1 is selecting an Ace as the first card. There are 4 aces in a deck of 52, so $Prob(E_1) = \frac{4}{52}$
- E_2 is selecting a 10, Jack, Queen, or King as the 2nd card. There are 4 of each of these cards, so 16 total cards out of 52. However, since we've already selected the 1st card, we are selecting the second card out of a deck of 51. So, $Prob(E_2|E_1) = \frac{16}{51} \approx 0.314$.

Question 8

- There are a total of $3 + 5 + 8 = 16$ marbles.
- EVENT 1: The probability of getting a red marble as the first item is $Prob(E_1) = \frac{3}{16}$.
- EVENT 2: We now have one less red marble, and one less marble overall, so our probability $Prob(E_2|E_1) = \frac{2}{15}$.
- EVENT 1 AND EVENT 2: The General Product Rule is...

  $$Prob(E_1 and E_2) = Prob(E_2) \cdot Prob(E_1|E_2) = Prob(E_1) \cdot Prob(E_2|E_1)$$

  $$Prob(E_1 and E_2) = \frac{3}{16} \cdot \frac{2}{15} = \frac{6}{240} = \frac{1}{40}$$

Question 9

- All we need to do is find n, k, and p.
- $n = 7$ $\quad k = 5$ $\quad p = (1/6)$
- $C(n,k) \cdot p^k \cdot (1-p)^{n-k}$

  $C(7,5) \cdot (1/6)^5 \cdot (5/6)^2 = \frac{175}{93{,}312} \approx 0.001875$

Question 10

- First, we need to figure out what all the possible sums are! We cannot have a value below 2 (because, at minimum, we will get two 1's), and we cannot have a value above 12 (because, at maximum, we will get two 6's). What is the probability of getting each sum value between 2 and 12?

SUM	OUTCOMES	PROBABILITY
2	(1,1)	1/36
3	(1,2), (2,1)	2/36
4	(1,3), (3,1), (2,2)	3/36
5	(1,4), (4,1), (2,3), (3,2)	4/36
6	(5,1), (5,1), (4,2), (2,4), (3,3)	5/36
7	(6,1), (1,6), (5,2), (2,5), (4,3), (3,4)	6/36
8	(6,2), (2,6), (5,3), (3,5), (4,4)	5/36
9	(6,3), (3,6), (5,4), (4,5)	4/36
10	(6,4), (4,6), (5,5)	3/36
11	(6,5), (5,6)	2/36
12	(6,6)	1/36

- Now that we have that, the formula for the Expected Value is: $E[X] = (x_1) \cdot Prob(X = x_1) + (x_2) + ... + (x_n) \cdot Prob(X = x_n)$

  $E[X] = \frac{1}{36}(2) + \frac{2}{36}(3) + \frac{3}{36}(4) + \frac{4}{36}(5) + \frac{5}{36}(6) + \frac{6}{36}(7) + \frac{5}{36}(8) + \frac{4}{36}(9) + \frac{3}{36}(10) + \frac{2}{36}(11) + \frac{1}{36}(12) = 7$

Question 11

- We need to get the probability of all possible outcomes. In choosing 5 cards, the options for amount of Aces are 0, 1, 2, 3, or 4.
- We are selecting 5 cards out of a deck of 52, so our Sample Space size is $n(S) = C(52, 5)$.
- Now we can find the probabilities...

ACES	PROBABILITY OF ACE	PROBABILITY OF OTHER
0	C(0,4) / C(52,5)	C(48,5) / C(52,5)
1	C(1,4) / C(52,5)	C(48,4) / C(52,5)
2	C(2,4) / C(52,5)	C(48,3) / C(52,5)
3	C(3,4) / C(52,5)	C(48,2) / C(52,5)
4	C(4,4) / C(52,5)	C(48,1) / C(52,5)

- Expected value is... $E[X] = (x_1) \cdot Prob(X = x_1) + (x_2) + ... + (x_n) \cdot Prob(X = x_n)$

$$E[X] = (1)\frac{C(4,1) \cdot C(48,4)}{C(52,5)} + (2)\frac{C(4,2) \cdot C(48,3)}{C(52,5)} + (3)\frac{C(4,3) \cdot C(48,2)}{C(52,5)} + (4)\frac{C(4,4) \cdot C(48,1)}{C(52,5)}$$

$$= \frac{5}{13} \approx 0.385$$