5.4 Exercise: In-class exercises are meant to introduce you to a new topic and provide some practice with the new topic. Work in a team of up to 4 people to complete this exercise. You can work simultaneously on the problems, or work separate and then check your answers with each other. You can take the exercise home, score will be based on the in-class quiz the following class period. **Work out problems on your own paper** - this document just has examples and questions.

5.4 Binary Sequences

5.4.1 Thinking algorithmically...

Question 1
How many distinguishable arrangements are there of the letters in the word "BANANA"?

a. How many B's? How many A's? How many N's?

b. How many ways are there to select a position for B? *(Specify the formula and the final answer)*

c. How many ways are there to place the A's? *(Hint: B has been placed, so there are 5 available spaces and 3 A's to place.)*

d. How many ways are there to place the N's? *(Hint: How many spaces are left? How many N's to place?)*

e. Which Rule do you use to find the final answer? What is the final answer?

Question 2
How many distinguishable arrangements are there of the word "PENNSYLVANIA"? *(Hint: You need the total amount of letters, and a count for each letter in the word!)*

5.4.2 Binary sequences

Binary sequence theorem

The number of binary sequences with r 1's and $n - r$ 0's is $C(n, r)$ or $C(n, n - r)$. [a]

This can be used for straightforward problems, like "how many binary sequences are there with x 1's and y 0's?", but we can also use it for more practical applications...!

[a] From Discrete Mathematics, Ensley and Crawley, page 409

Question 3
How many binary sequences are there with three 1's and two 0's?

a. How many 1's? $r =$

b. How many 0's? $n - r =$

c. What is the value of r?

d. How many binary sequences are there? $C(n, r) =$

Question 4
Using the Binary Sequence theorem and the Sum Rule, find the number of binary sequences that have an odd number of 1's.

Other uses of Binary Sequences

We can also use this theorem for strings that have more than just 0's and 1's...

Example 2 from the book: How many ordered lists of 10 letters, chosen from $\{m, a, t\}$, have exactly three m's?

If we think in terms of spots to fill for each letter, we can diagram it like this:

$$\underset{1}{\_\_} \quad \underset{2}{\_\_} \quad \underset{3}{\_\_} \quad \underset{4}{\_\_} \quad \underset{5}{\_\_} \quad \underset{6}{\_\_} \quad \underset{7}{\_\_} \quad \underset{8}{\_\_} \quad \underset{9}{\_\_} \quad \underset{10}{\_\_}$$

Since we have a restriction on the m's, these should be filled in first. Selecting three m's gives us $C(10, 3) = 120$ combinations.

Then, since there are no restrictions on the rest of the letters, we can select the final 7. However, we don't have 3 to choose from anymore; we wanted **exactly** 3 m's and we have already filled those. So, instead of selecting from 3 options, we only have two: $\{a, t\}$.

Since for each of the remaining, we have two options each, it will be calculated as $2 \cdot 2 \cdot 2 \cdot ...$ seven times. These remaining 2's are an **ordered list**: Order matters, and we can have duplicates (a or t).

Our final result will be $C(10, 3) \cdot 2^7$.

Formulas for each structure type

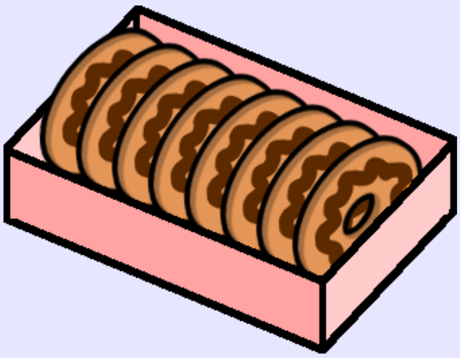
Type	Repeats allowed?	Order matters?	Formula
Ordered list of length r	yes	yes	n^r
Unordered list of length r	yes	no	$C(r + n - 1, r)$
Permutations of length r	no	yes	$P(n, r) = \frac{n!}{(n-r)!}$
Sets of length r	no	no	$C(n, r) = \frac{n!}{r!(n-r)!}$

We can also use what we've learned on application problems, where we need to group up items together.

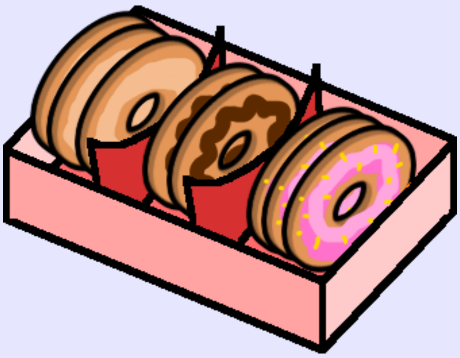
Modeling problems with binary sequences

Let's say you have a box that can store 8 donuts. If you're going to put multiple flavors in the box, you will use a separator to keep them apart.

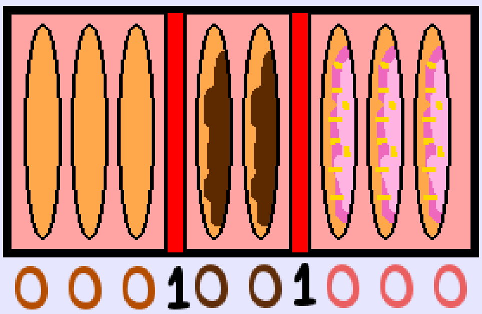
Same donuts



Different donuts



In this case, we don't care what kind of donuts are in the box; we will represent them with zeroes. The splitters, we will represent as 1. With this information, we will model our box of donuts as a binary string.



Question 5

For the following donut combinations, draw the donut box, the donuts, and the separators. Also list out the binary sequence that represents the box. Assume that the box must always have 2 separators. If a separator isn't being used, it can go at either end of the box.

a. One Chocolate, Three Strawberry, and Four Glazed

b. Four Chocolate and Four Glazed

c. Eight Glazed

Theorem
Let natural numbers n and r be given. [a]

1. The number of solutions to the equation $x_1 + ... + x_n = r$ using nonnegative integers is $C(r+n-1, r)$.
2. The number of unordered lists of length r taken from a set of size n, which repetitions allowed, is $C(r+n-1, r)$.
3. The number of bags of r pieces of fruit that can be bought at a store with n types of fruit available is $C(r+n-1, r)$.

[a] Theorem 3 from Discrete Mathematics, Ensley and Crawley, page 413

Question 6
At Darrell's Donuts, Darrell sells 5 different types of donuts. He only sells 12-donut boxes. How many different combinations of donut boxes can you build? [1]

a. How many types of donuts are there? $n =$
b. How many donuts will be selected? $r =$
c. How many separators should there be? *(Hint: It should be one less than the donut types)*
d. If donuts and separators both count as spaces in the box, how many available spaces are there total?
e. What is $r + n - 1$?
f. What is the Combination formula for finding out the total ways you can build your donut boxes?
g. What is the final answer?

Question 7
At Piper's Produce, there are 3 types of fruit for sale. How many ways can you fill a bag that can store 20 pieces of fruit?

[1] Based on Example 6 from Discrete Mathematics, Ensley and Crawley, page 413