5.5 Exercise: In-class exercises are meant to introduce you to a new topic and provide some practice with the new topic. Work in a team of up to 4 people to complete this exercise. You can work simultaneously on the problems, or work separate and then check your answers with each other. You can take the exercise home, score will be based on the in-class quiz the following class period. **Work out problems on your own paper** - this document just has examples and questions.

5.5 Recursive Counting

Remember Chapter 2.3...?

Back in CS 210, we did proofs by using induction. With induction, our goal was to show that some statement was true for the first value, a_1, and then for all values going up through a_{m-1}. Let's look at the steps for this sort of problem.

Exercise 2.3, 3a from textbook Show that the sequence defined by $a_k = a_{k-1} + 4; a_1 = 1$ for $k \geq 2$ is equivalently described by the closed formula $a_n = 4n - 3$.

Step 1: Check values for both formulas, for a_1:

Recursive: $a_1 = 1$ (provided) Closed: $a_1 = 4(1) - 3 = 1$

They match, so we can continue.

Step 2: Rewrite the recursive formula in terms of m: $a_m = a_{m-1} + 4$

Step 3: Find the equation for a_{m-1} through the recursive formula:

$a_n = 4n - 3$

$=> a_{m-1} = 4(m-1) - 3$

$=> a_{m-1} = 4m - 7$

Step 4: Plug a_{m-1} into the recursive formula from step 2, and simplify.

$a_m = a_{m-1} + 4$

$=> a_m = (4m - 7) + 4$

$=> a_m = 4m - 3$

PROOF: $a_m = 4m - 3$ and the closed formula $a_n = 4n - 3$ match, so the closed formula and recursive formula are equivalent.

This section is quite small and just highlighting how we can use our previous proofs with Permutations.

5.5.1 Recursive counting

Proving for recurrence relations

Prove by induction and the recurrence relation [a]

$P(n, r) = n \cdot P((n-1), (r-1))$ with $P(n, 0) = 1$

$$\text{that } P(n, n) = n! \text{ for all } n \geq 0.$$

Step 1: Check that it works for $n = 1$:

$$\begin{aligned} P(1,1) &= 1 \cdot P(0,0) \\ &= \frac{0!}{0!} = \frac{1}{1} = 1; \\ 1! &= 1 \end{aligned}$$

This step is OK! Moving on...

Step 2: Plug in n for r in the original statement.

$$\begin{aligned} P(n,n) &= n \cdot P((n-1),(n-1)) \\ &= n \cdot \frac{(n-1)!}{((n-1)-(n-1))!} \\ &= n \cdot \frac{(n-1)!}{(n-1-n+1)!} \\ &= n \cdot \frac{(n-1)!}{(n-n)!} \\ &= n \cdot \frac{(n-1)!}{(0)!} \\ &= n \cdot \frac{(n-1)!}{1} \\ &= n \cdot (n-1)! \\ &= n! \end{aligned}$$

(n times $(n-1)!$ is equivalent to just having $n!$)

[a] From 5.5 Exercise 4, Discrete Mathematics by Ensley and Crawley

Review: Permutation

$$P(n,r) = \frac{n!}{(n-r)!}$$

Question 1

Use the recurrence relation

$$P(n,r) = n \cdot P((n-1),(r-1))$$

(with $P(n,0) = 1$ for all $n \geq 0$) to prove each of the following: [1]

a. For all $n \geq 1$, $P(n,1) = n$

b. For all $n \geq 2$, $P(n,2) = n \cdot (n-1)$

c. For all $n \geq 3$, $P(n,3) = n \cdot (n-1) \cdot (n-2)$

[1] From 5.5 Exercise 5, Discrete Mathematics by Ensley and Cralwey