6.6 Exercise: In-class exercises are meant to introduce you to a new topic and provide some practice with the new topic. Work in a team of up to 4 people to complete this exercise. You can work simultaneously on the problems, or work separate and then check your answers with each other. You can take the exercise home, score will be based on the in-class quiz the following class period. **Work out problems on your own paper** - this document just has examples and questions.

6.1 Matrices and Markov Chains

6.1.1 The Gambler's Ruin Problem

The Gambler's Ruin Problem

Page 478 of the book highlights a game played between two characters: Player "H" and Player "T". Each character begins with a certain amount of markers (or tokens), and they play by flipping a **coin**. Whenever one of them loses a "round", they give one marker to their opponent.

If a *heads* is flipped, then Player H wins a marker from Player T. For a *tails*, Player T wins a marker from Player H. The game is over once somebody is out of markers.

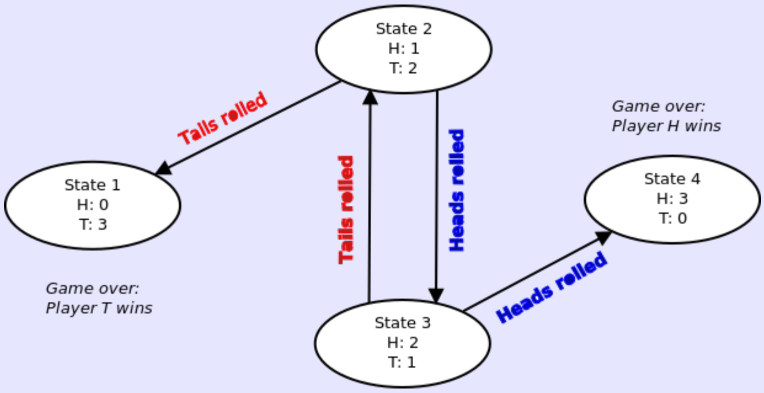
Game setup: For the game we'll be talking about in thie example, the rules are:

- There are 3 total markers (so one player will have more markers than the other)
- Once somebody is out of markers, the game is over.
- Each coin flip, the loser gives one of their markers to the other player. (One gains, one loses)

Game states: Before we start modeling the game with a matrix, let's map out all the game states. We won't worry about what is the beginning state, and each coin flip, one person gains a marker and one person loses a marker. There are four possible states in the game:

State #	H's markers	T's markers
1	0	3
2	1	2
3	2	1
4	3	0

State diagram: Given these states, and the fact that each coin flip one person loses a marker and gives it to the other, our state diagram would look like:



State change matrix: Now we will build out a matrix to show the probability of switching between states.

The matrix will be 4×4. Each **row** will be a state, and each cell in that row is the probability of going from that state to a new state.

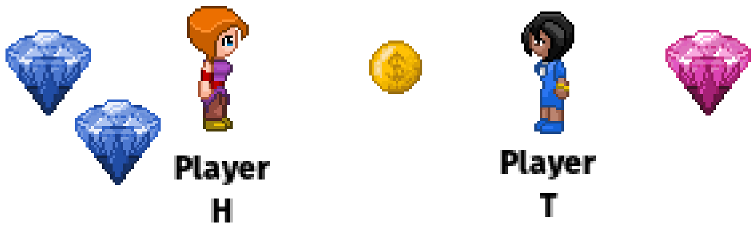
	State 1	State 2	State 3	State 4
State 1 $\rightarrow$	1	0	0	0
State 2 $\rightarrow$	1/2	0	1/2	0
State 3 $\rightarrow$	0	1/2	0	1/2
State 4 $\rightarrow$	0	0	0	1

This matrix is in the format, "from *row* state to *col* state". In Row 2, the probability of going from (State 2 $\rightarrow$ State 1) is 1/2, because the coin flip has a half chance of being heads, and a half chance of being tails.

State 1 and State 4 they are gameover states: you cannot move from State 1 to another state, so it has a 1 in that cell.

Question 1

Let's say a game starts where Player H has 2 markers and player 1 has 1 marker. This is state 3.



What is the probability of...

a. Going from State 3 to State 1?

b. Going from State 3 to State 2?

c. Going from State 3 to State 3?

d. Going from State 3 to State 4?

Transition Matrix

If you have a game with states 1 through n, then your transition matrix M is

$$M_{i,j} = Prob(\text{the game changes from state } i \text{ to state } j \text{ in one move}).$$

[a]

[a] Discrete Structures, Ensley and Crawley

Question 2

Let's say in the game there are 2 total markers instead of three.

a. What are all the states in the game?

b. Draw the transition matrix for this game.

Transition Matrix

Given the matrices M and N, where the amount of *rows* in M is the same amount of *columns* in N, we can find the product $P = M \cdot N$ where each entry at row i, column j of P is the row-column product of row i from M and column j from N. In other words, [a]

$$P_{i,j} = M_{i,1} \cdot N_{1,j} + M_{i,2} \cdot N_{2,j} + ...$$

[a] Discrete Structures, Ensley and Crawley

Question 3

Calculate the product $M \cdot M$ (aka M^2) for our original game with 3 markers.

	Col 1	Col 2	Col 3	Col 4
Row 1	1	0	0	0
Row 2	1/2	0	1/2	0
Row 3	0	1/2	0	1/2
Row 4	0	0	0	1

The result ends up being the probability that the game processes from state i to state j in **two** moves.

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