Basic Rules for Counting

Written by Rachel J. Morris, last updated Jan 9, 2018

About

In this section, we will cover the formula for permutations, as well as how to solve problems that as for “this <u>and</u> that” outcome, or “this <u>or</u> that” outcome, and how to use the Rule of Complements.

Topics

1. Permutations
2. Rule of Sums
3. Rule of Products
4. Rule of Complements

Permutations

1. Permutations

A permutation is written as **P(n, r)**. For Permutation problems, you need two pieces of information:

- *n*, the amount of items we have to select from
- *r*, the amount of items that we're selecting.

The formula for P(n, r) is:

$$P(n,r)=\frac{n!}{(n-r)!}$$

Notes

P(n, r):
n is # of potential items
r is # of selections

$$P(n,r)=\frac{n!}{(n-r)!}$$

1. Permutations

In cases where we have *n* items and are selecting all *n*, then we have:

P(n, n) = n!

And in cases where we have *n* items but are only selecting 1 item, we have:

P(n, 1) = n

Notes

P(n, r):

n is # of potential items

r is # of selections

$$P(n,r)=\frac{n!}{(n-r)!}$$

1. Permutations

Also remember a Permutation is when…

- Repetitions are NOT allowed
- Order DOES matter.

Notes

P(n, r):
n is # of potential items
r is # of selections

$$P(n,r)=\frac{n!}{(n-r)!}$$

1. Permutations

Example: How many ways are there to arrange five people in a line?

- What is *n*?
- What is *r*?
- What is the result?

Notes

P(n, r):

n is # of potential items

r is # of selections

$$P(n,r)=\frac{n!}{(n-r)!}$$

1. Permutations

Example: How many ways are there to arrange five people in a line?

- What is *n*? **5**
- What is *r*? **5**
- What is the result?

$$P(n,r)=P(5,5) \quad =\frac{5!}{(5-5)!}=\frac{5!}{0!}=5! \quad =5\cdot4\cdot3\cdot2\cdot1=120$$

(Note that 0! (0-factorial) equals 1.)

Notes

P(n, r):
n is # of potential items
r is # of selections

$$P(n,r)=\frac{n!}{(n-r)!}$$

Rule of Sums

2. The rule of sums

The rule of sums In combinatorics, the rule of sum or addition principle is a basic counting principle. Stated simply, it is the idea that if we have ***A*** ways of doing something and ***B*** ways of doing another thing and we can not do both at the same time, then there are ***A + B*** ways to choose one of the actions.

From https://en.wikipedia.org/wiki/Rule_of_sum

Notes

Either one thing *or* another thing:
a + b

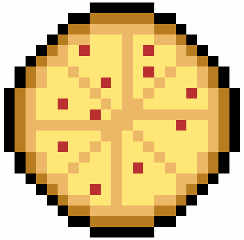
n(A)
is notation for “the # of elements in set A”.

2. The rule of sums

Example: On the “deals” menu on PizzaWebsite.com, there are specials on 85 types of pizzas. Of these…

- Specials on 7 varieties of pizzas with 2 topping.
- Specials on 25 varieties of pizzas with 3 toppings.
- Specials on 33 varieties of pizzas with 4 toppings.
- Specials on 16 varieties of pizzas with 5 toppings.
- Specials on 2 varieties of pizzas with 6 toppings.

1. How many specials have at least 4 toppings?



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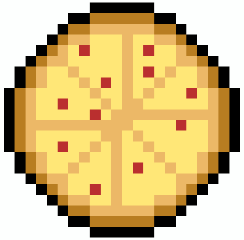
- Specials on 2 varieties of pizzas with 6 toppings.

1. How many specials have at least 4 toppings?

33 + 16 + 2 = 48



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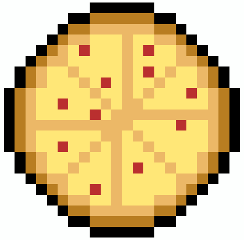
2. How many pizzas have only 1 topping?



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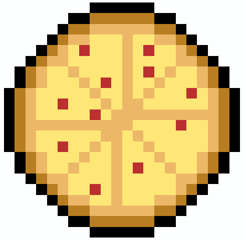
- Specials on 16 varieties of pizzas with 5 toppings.
- Specials on 2 varieties of pizzas with 6 toppings.

2. How many pizzas have only 1 topping?

85 - (7 + 25 + 33 + 16 + 2) = 2

85 is the total # of specials, minus all the specials with 2 or more toppings.



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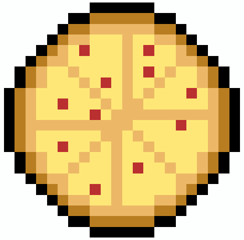
- Specials on 33 varieties of pizzas with 4 toppings.
- Specials on 16 varieties of pizzas with 5 toppings.
- Specials on 2 varieties of pizzas with 6 toppings.
- **Specials on 2 varieties of pizzas with 1 topping.**

3. How many specials have at most 3 toppings?
Need to count options with 1, 2, and 3 toppings.



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2. The rule of sums

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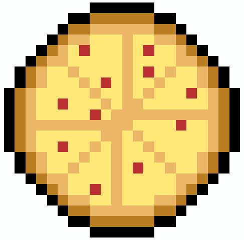
- Specials on 33 varieties of pizzas with 4 toppings.
- Specials on 16 varieties of pizzas with 5 toppings.
- Specials on 2 varieties of pizzas with 6 toppings.
- **Specials on 2 varieties of pizzas with 1 topping.**

3. How many specials have at most 3 toppings?
Need to count options with 1, 2, and 3 toppings.

2 + 7 + 25 = 34, or 85 – 2 – 16 – 33 = 34



Notes

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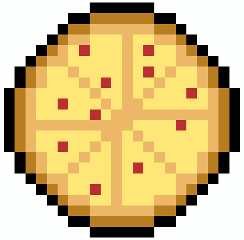
2. The rule of sums

When calculating the total amount of outcomes, there could be cases of **overlap**. In this case, we need to remove the redundant cases from our final answer.

We can do this with the Rule of Sums with Overlap…



Notes

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2. The rule of sums

The rule of sums with overlap: If the list to count can be split into two pieces of size ***z*** and ***y***, and the pieces have z objects in common, then the original list has $x + y - z$ entries. In terms of sets, we can write this as $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ for all sets ***A*** and ***B***.

From Discrete Math Mathematical Reasoning and Proofs with Puzzles, Patterns and Games, by Ensley and Crawley, 5.2 page 391

Notes

Either one thing *or* another thing:

a + b

This or that, without duplicates:

a + b - c

2. The rule of sums

Example: Say that we're drawing cards from a deck. How many outcomes are there if we get a card value of "10" OR get a red suit?

In this case, we need to investigate both outcomes separately first:

- How many cards are there with a value of 10?

- How many cards are there that are red?

10♠

10♣

10♥

10♦

Notes

Either one thing *or* another thing:
a + b

This or that, without duplicates:
a + b - c

2. The rule of sums

Example: Say that we're drawing cards from a deck. How many outcomes are there if we get a card value of "10" OR get a red suit?

In this case, we need to investigate both outcomes separately first:

- How many cards are there with a value of 10?
  There are four suits for each value, so 4.
- How many cards are there that are red?
  Each suit has 13 cards, so 26.

10♠

10♣

10♥

10♦

Notes

Either one thing *or* another thing:
a + b

This or that, without duplicates:
a + b - c

2. The rule of sums

Example: Say that we're drawing cards from a deck. How many outcomes are there if we get a card value of "10" OR get a red suit?

Then we have to figure out the overlap

- How many cards are red AND have a 10?

10♠

10♣

10♥

10♦

a	***b***	***c***
Red cards: 26	"10" cards: 4	"10" AND red cards:

Notes

Either one thing *or* another thing:
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This or that, without duplicates:
a + b - c

2. The rule of sums

Example: Say that we're drawing cards from a deck. How many outcomes are there if we get a card value of "10" OR get a red suit?

Then we have to figure out the overlap

- How many cards are red AND have a 10?
  There are two red "10"s, so 2.

10♠

10♣

10♥

10♦

a	***b***	***c***
Red cards: 26	"10" cards: 4	"10" AND red cards: 2

Notes

Either one thing *or* another thing:
a + b

This or that, without duplicates:
a + b - c

2. The rule of sums

Example: Say that we're drawing cards from a deck. How many outcomes are there if we get a card value of "10" OR get a red suit?

Now we can compute the result using the rule of sums with overlap. ***c*** is the overlap, and we subtract it from ***a + b***.

a + b – c **= 26 + 4 – 2** **= 28**

10♠

10♣

10♥

10♦

a	***b***	***c***
Red cards: 26	"10" cards: 4	"10" AND red cards: 2

Notes

Either one thing *or* another thing:
a + b

This or that, without duplicates:
a + b - c

Rule of Products

3. The rule of products

The rule of products In combinatorics, the rule of product or multiplication principle is a basic counting principle (a.k.a. the fundamental principle of counting). Stated simply, it is the idea that if there are ***a*** ways of doing something and ***b*** ways of doing another thing, then there are ***a* · *b*** ways of performing both actions.

From https://en.wikipedia.org/wiki/Rule_of_product

Notes

Doing one thing *and* another thing:
a x b

Doing *either* one thing *or* another thing:
a + b

n(A)
is notation for “the # of elements in set A”.

3. The rule of products

Example: You have to order 3 pizzas for a group event, and there are 4 types of pizzas.

1. How many different ways can you select 3 pizzas, with no restrictions?

Notes

Doing one thing *and* another thing:
a x b

Doing *either* one thing *or* another thing:
a + b

n(A)
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3. The rule of products

Example: You have to order 3 pizzas for a group event, and there are 4 types of pizzas.

1. How many different ways can you select 3 pizzas, with no restrictions?

4 x 4 x 4 = 64

Notes

Doing one thing *and* another thing:

a x b

Doing *either* one thing *or* another thing:

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n(A)

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3. The rule of products

Example: You have to order 3 pizzas for a group event, and there are 4 types of pizzas.

1. How many different ways can you select 3 pizzas, with no restrictions?

$4 \times 4 \times 4 = 64$

2. How many different was can you select 3 pizzas, without a repeat?

Notes

Doing one thing *and* another thing:

a x b

Doing *either* one thing *or* another thing:

a + b

n(A)

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3. The rule of products

Example: You have to order 3 pizzas for a group event, and there are 4 types of pizzas.

1. How many different ways can you select 3 pizzas, with no restrictions?

4 x 4 x 4 = 64

2. How many different was can you select 3 pizzas, without a repeat?

4 x 3 x 2 = 24

Notes

Doing one thing *and* another thing:

a x b

Doing *either* one thing *or* another thing:

a + b

n(A)

is notation for “the # of elements in set A”.

Rule of Complements

4. The rule of complements

If there are ***x*** objects and ***y*** of those objects have a particular property, then the number of those objects that do <u>not</u> have that particular property is $\boldsymbol{x - y}$. In terms of sets, using ***U*** for the universal set, we can write this as $\boldsymbol{n(A') = n(U) - n(A)}$ for all sets ***A*** with elements from ***U***.

From Discrete Math Mathematical Reasoning and Proofs with Puzzles, Patterns and Games, by Ensley and Crawley, 5.2 page 390

Notes

Doing one thing *and* another thing:
a x b

Doing *either* one thing *or* another thing:
a + b

n(A)
is notation for "the # of elements in set A".

4. The rule of complements

Example 7 from the book: How many five-digit numbers use distinct digits from {0, …, 6}?

___ ___ ___ ___ ___

Notes

Doing one thing *and* another thing:
a x b

Doing *either* one thing *or* another thing:
a + b

n(A)
is notation for "the # of elements in set A".

4. The rule of complements

Example 7 from the book: How many five-digit numbers use distinct digits from {0, …, 6}?

0	1	2	3	4	5	6

Notes: A number cannot have a leading 0. Also in a list of 0, …, 6 we have 7 options to choose from.

So for the first position, we have 6 options – not 7 – since we can only select between 1 and 6.

1-6

___ ___ ___ ___ ___

6 options

Notes

Doing one thing *and* another thing:
a x b

Doing *either* one thing *or* another thing:
a + b

n(A)
is notation for "the # of elements in set A".

4. The rule of complements

Example 7 from the book: How many five-digit numbers use distinct digits from {0, …, 6}?

0	1	2	3	4	5	6

After the first digit, we have 6 options to choose from; 0 through 6, except for whatever was selected for the first number.

1-6	0-6 except 1st digit.			
___	___	___	___	___
6 options	6 options			

Notes

Doing one thing *and* another thing:
a x b

Doing *either* one thing *or* another thing:
a + b

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is notation for "the # of elements in set A".

4. The rule of complements

Example 7 from the book: How many five-digit numbers use distinct digits from {0, …, 6}?

0	1	2	3	4	5	6

And from there we keep going down: 5 options, then 4, then 3…

1-6	0-6 except 1st digit.	…	…	…
6 options	6 options	5 options	4 options	3 options

Notes

Doing one thing *and* another thing:

a x b

Doing *either* one thing *or* another thing:

a + b

n(A)

is notation for "the # of elements in set A".

4. The rule of complements

Example 7 from the book: How many five-digit numbers use distinct digits from {0, …, 6}?

0	1	2	3	4	5	6

So the result, without any restrictions, is

6 x 6 x 5 x 4 x 3 = 2,160

1-6	0-6 except 1st digit.	…	…	…
6 options	6 options	5 options	4 options	3 options

Notes

Doing one thing *and* another thing:
a x b

Doing *either* one thing *or* another thing:
a + b

n(A)
is notation for "the # of elements in set A".

4. The rule of complements

Example 7 from the book: How many five-digit numbers use distinct digits from {0, …, 6}? **2,160**

- Of these, how many are odd?

___ ___ ___ ___ ___

Notes

Doing one thing *and* another thing:
a x b

Doing *either* one thing *or* another thing:
a + b

n(A)
is notation for "the # of elements in set A".

4. The rule of complements

Example 7 from the book: How many five-digit numbers use distinct digits from {0, …, 6}? **2,160**

- Of these, how many are odd?

The part of the number that affects whether it's even or odd is the right-most digit. So we will build out the number again, but going from right-to-left.

___ ___ ___ ___ ___

Notes

Doing one thing *and* another thing:
a x b

Doing *either* one thing *or* another thing:
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4. The rule of complements

Example 7 from the book: How many five-digit numbers use distinct digits from {0, …, 6}? **2,160**

- Of these, how many are odd?

For the right-most number, we have the options: 1, 3, 5.

___ ___ ___ ___ ___

1,3,5

3 options

Notes

Doing one thing *and* another thing:

a x b

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4. The rule of complements

Example 7 from the book: How many five-digit numbers use distinct digits from {0, …, 6}? **2,160**

- Of these, how many are odd?

Next, we do the left-most digit again: 1 through 6, except what was selected for the right-most digit.

1-6 except right-most digit				1,3,5
___	___	___	___	___
5 options				3 options

Notes

Doing one thing *and* another thing:
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4. The rule of complements

Example 7 from the book: How many five-digit numbers use distinct digits from {0, …, 6}? **2,160**

- Of these, how many are odd?

Then we can place the rest of the digits.

1-6 except right-most digit	0-6 except taken digits	…	…	1,3,5
5 options	5 options	4 options	2 options	3 options

Notes

Doing one thing *and* another thing:
a x b

Doing *either* one thing *or* another thing:
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4. The rule of complements

Example 7 from the book: How many five-digit numbers use distinct digits from {0, …, 6}? **2,160**

- Of these, how many are odd?

So the result is **5 x 5 x 4 x 3 x 3 = 900**

1-6 except right-most digit	0-6 except taken digits	…	…	1,3,5
5 options	5 options	4 options	3 options	3 options

Notes

Doing one thing *and* another thing:
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4. The rule of complements

Example 7 from the book: How many five-digit numbers use distinct digits from {0, …, 6}? **2,160**

- Of these, how many are odd? **900**
- How many are even?

…	…	…	…	0,2,4,6
___	___	___	___	___

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4. The rule of complements

Example 7 from the book: How many five-digit numbers use distinct digits from {0, …, 6}? **2,160**

- Of these, how many are odd? **900**
- How many are even?

How we can use the **rule of complements** to solve this. We know the total amount of digits, and the amount of odd digits.

If a number isn't odd, then it's even.

Notes

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4. The rule of complements

Example 7 from the book: How many five-digit numbers use distinct digits from {0, …, 6}? **2,160**

- Of these, how many are odd? **900**
- How many are even?

We can calculate the amount of even numbers with:

Even numbers = 2,160 – 900 = 1,260

Notes

Doing one thing *and* another thing:
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4. The rule of complements

Example 7 from the book: How many five-digit numbers use distinct digits from {0, …, 6}? **2,160**

- Of these, how many are odd? **900**
- How many are even? **1,260**

You can also validate this by figuring out how many even numbers in the same way we did odd numbers, except the right-most digit can be **0, 2, 4,** or **6**, whereas for odd it can be **1, 3,** or **5**.

Notes

Doing one thing *and* another thing:
a x b

Doing *either* one thing *or* another thing:
a + b

n(A)
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Conclusion

These principles will be used all through Chapter 5, as well as in some form in Chapter 6 once we're figuring probability.

Make sure to pay attention to when a problem asks for "this **and** that", or "this **or** that", as that will tip you off on which rule to use.