

5. Exercise: In-class exercises are meant to introduce you to a new topic and provide some practice with the new topic. Work in a team of up to 4 people to complete this exercise. You can work simultaneously on the problems, or work separate and then check your answers with each other. You can take the exercise home, score will be based on the in-class quiz the following class period. **Work out problems on your own paper** - this document just has examples and questions.

5.6 Solving Recurrence Relations

5.6.1 Review: Closed & Recursive formulas

Remember chapter 1.2...?

Back in the first week of CS 210, you were given sequences of numbers... let's say

$$3, 5, 7, 9, 11$$

...and you tasked with coming up with **closed formulas** and **recursive formulas** for these.

Closed formulas were based only on the value of n , such as...

$$a_n = 2n + 1$$

And recursive formulas were based on some starting value, a_1 , with each subsequent element being based off a previous element., such as...

$$a_1 = 3; \quad a_n = a_{n-1} + 2$$

A full class and 4.4 chapters later, we will actually show you *how* to come up with a formula given a sequence of numbers... No guesswork required.

5.6.2 Difference tables

Finding formulas: Recursive

Find a recursive formula for the sequence 2, 5, 8, 11, 14, ...

First, we will be assigning this sequence of numbers to a variable s , where n is the index (or position) in the sequence, and s_n is the element (or term) at that position.

Previously, we had our formulas begin at a_1 , but now we will be starting our sequences at index 0, like when programming.

First let's build a table with n , the index, s_n , the element at that index, and Δ_n , the difference between two elements ($\Delta_n = s_{n+1} - s_n$)

index n	0	1	2	3	4
element at n s_n	2	5	8	11	14
different between elements Δ_n	3	3	3	3	3

With the table, we can see the difference between each term (2 to 5, 5 to 8, etc...) is **3**. With this information, we can write any value s_1 in terms of the previous value:

$$\begin{array}{lcl}
 s_1 & = s_0 + \Delta_0 & | \quad (5 = 2 + 3) \\
 s_2 & = s_1 + \Delta_1 & | \quad (8 = 5 + 3) \\
 & = s_0 + (\Delta_0 + \Delta_1) & | \quad (8 = 2 + 3 + 3) \\
 s_3 & = s_2 + \Delta_2 & | \quad (11 = 8 + 3) \\
 & = s_0 + (\Delta_0 + \Delta_1 + \Delta_2) & | \quad (11 = 2 + 3 + 3 + 3) \\
 & = s_0 + \sum_{k=0}^2 \Delta_k & | \quad (11 = 2 + \sum_{k=0}^2 3)
 \end{array}$$

So given the first term being

$$s_0 = 2,$$

we can say the recursive formula is

$$s_n = s_{n-1} + \Delta_{n-1} \quad \text{or} \quad s_n = s_{n-1} + 3$$

(Continued) Finding formulas: Closed

But what about the closed formula? Well with the differences we can also describe any term as the *first term* plus the sum of the differences...

$$s_1 = s_0 + \triangle_0 \quad \left| \quad (5 = 2 + 3) \right.$$

$$\begin{aligned} s_2 &= s_1 + \triangle_1 & \left| \quad (8 = 5 + 3) \right. \\ &= s_0 + (\triangle_0 + \triangle_1) & \left| \quad (8 = 2 + 3 + 3) \right. \end{aligned}$$

$$\begin{aligned} s_3 &= s_2 + \triangle_2 & \left| \quad (11 = 8 + 3) \right. \\ &= s_0 + (\triangle_0 + \triangle_1 + \triangle_2) & \left| \quad (11 = 2 + 3 + 3 + 3) \right. \\ &= s_0 + \sum_{k=0}^2 \triangle_k & \left| \quad (11 = 2 + \sum_{k=0}^2 3) \right. \end{aligned}$$

So $s_1 = 2 + 3$, $s_2 = 2 + 3 + 3$, $s_3 = 2 + 3 + 3 + 3 \dots$
which we can write as

$$s_n = \sum_{k=0}^{n-1} (3) + 2$$

OR

$$s_n = 3 \cdot n + 2$$

Theorem 1: Fundamental Theorem of Sums and Differences

For any sequence $\{s_n\}$ with first differences $\triangle_k = s_{k+1} - s_k$, and any $n \geq 1$,

$$s_n - s_0 = \sum_{k=0}^{n-1} \triangle_k$$

or

$$s_n = \sum_{k=0}^{n-1} \triangle_k + s_0$$

Question 1

Build a difference table and find the formulas for the sequence: ¹

7, 12, 17, 22, 27, ...

Question 2

Build a difference table and find the formulas for the sequence: ²

3, 10, 17, 24, 31, ...

¹From Jim Van Horn's POGIL exercises

²From Discrete Mathematics, 5.6 exercise 1a, Ensley and Crawley

5.6.3 Complex sequences

Complex sequences

Sometimes, the difference between each term in a sequence is not the same; maybe the difference is 3, then 4, then 5, and so on. In this case, the difference itself also has a difference. In this case, that “difference-of-the-differences” is known as the *second difference*, whereas the difference between the terms themselves is the *first difference*.

Example: Find a closed formula for the sequence 6, 11, 19, 30, 44.

As previously, we can start by writing out a table of the index n , the term s_n , and the difference Δ_n ...

n	0	1	2	3	4
s_n	6	11	19	30	44
Δ_n	5	8	11	14	?

Once we’ve figured out Δ_n , we can see that it isn’t constant each time, so we can’t apply the same techniques as before. Instead, let’s expand the table to have a fourth row: the difference of the differences. We will use a triangle again to symbolize “difference”, but we will add a number to it, so Δ_n^1 is the difference between terms, and Δ_n^2 is the difference of those differences.

n	0	1	2	3	4
s_n	6	11	19	30	44
Δ_n^1	5	8	11	14	?
Δ_n^2	3	3	3

Ahh, can we maybe apply what we learned last time! Solving this will actually mean that we’re working recursively.

Let’s look at this closer...

n	0	1	2	3	4
s_n	6	11	19	30	44
Δ_n^1	5	8	11	14	?
Δ_n^2	3	3	3

$$\Delta_n^k = \Delta_{n+1}^{k-1} - \Delta_n^{k-1}$$

(Continued) Complex sequences

$$\begin{array}{lcl}
\Delta_1^1 & = \Delta_0^1 + \Delta_0^2 & | (8 = 5 + 3) \\
\Delta_2^1 & = \Delta_1^1 + \Delta_1^2 & | (11 = 5 + 3 + 3) \\
& = \Delta_0^1 + (\Delta_0^2 + \Delta_1^2) & | (14 = 5 + 3 + 3) \\
\Delta_3^1 & = \Delta_2^1 + \Delta_2^2 & | (11 = 8 + 3) \\
& = \Delta_0^1 + (\Delta_0^2 + \Delta_1^2 + \Delta_2^2) & | (11 = 2 + 3 + 3 + 3)
\end{array}$$

So we can use the same strategy to find values of Δ_n^1 ...

$$\Delta_n^1 = \Delta_0^1 + \sum_{k=0}^{n-1} (\Delta_k^2)$$

$$\Delta_n^1 = 5 + \sum_{k=0}^{n-1} (3)$$

$$\Delta_n^1 = 3n + 5$$

Finding s_n

After we have an equation for the first level difference, we can then repeat the Theorem to find s_n ...

$$s_n = s_0 + \sum_{k=0}^{n-1} (\Delta_k^1)$$

$$s_n = 6 + \sum_{k=0}^{n-1} (3k + 5)$$

And simplifying the sum...

$$s_n = 6 + \sum_{k=0}^{n-1} (5) + 3 \sum_{k=0}^{n-1} (k)$$

But how do we find the value of $\sum_{k=0}^{n-1} (k)$?

Proposition 1 from Chapter 2.3

$$\sum_{i=1}^n (i) = \frac{n(n+1)}{2}$$

Or, rewritten for our use-case:

$$\sum_{k=0}^{n-1} (k) = \frac{n(n-1)}{2}$$

(Continued) Complex sequences

So, continuing to simplify, we have:

$$s_n = 6 + 5n + 3 \frac{n(n-1)}{2}$$

$$s_n = 6 + \frac{5n \cdot 2}{2} + \frac{3n(n-1)}{2}$$

$$s_n = \frac{10n + 3n^2 - 3n}{2} + 6$$

$$s_n = \frac{3}{2}n^2 + \frac{7}{2}n + 6$$

And that's the final answer.

Question 3

Build a difference table and find the **closed formula** for the sequence: ³

1, 3, 8, 16, 27, 41, ...

³From Discrete Mathematics, 5.6 exercise 1c, Ensley and Crawley

5.6.4 Review Theorems

Definition of the k th level difference at index n

$$\Delta_n^k = \Delta_{n+1}^{k-1} - \Delta_n^{k-1}$$

Theorem 1 (Revisited)

$$s_n = \sum_{k=0}^{n-1} \Delta_k^1 + s_0$$

Theorem 2:

$$s_n = s_0 + \sum_{i=1}^k \Delta_0^i \cdot C(n, i)$$

Proposition 1 from Chapter 2.3

$$\sum_{k=0}^{n-1} \binom{n-1}{k} = \frac{n(n-1)}{2}$$