6.4 Expected value in games of chance

Definition

For a given probability experiment, let X be a random variable whose possible values come from the set of numbers $x_1, ..., x_n$. Then the **expected value of** X, denoted by $E[X]$, is the sum

$$(x_1) \cdot Prob(X = x_1) + (x_2) \cdot Prob(X = x_2) + ... + (x_n) \cdot Prob(X = x_n)$$

This is sometimes called the *average value* of the random variable, thinking of the average of the values X takes on over many repetitions of the experiment. [a]

Example 2 Suppose I have a "loaded" die for which the probability of a 6 appearing is $\frac{1}{2}$, while the probability of each of the other faces appearing is $\frac{1}{10}$. What is the expected value on one roll? Compare to the expected value of a fair die.

For the loaded die:

$$E[X] = (1)(\frac{1}{10}) + (2)(\frac{1}{10}) + (3)(\frac{1}{10}) + (4)(\frac{1}{10}) + (5)(\frac{1}{10}) + (6)(\frac{1}{2})$$

$$= \frac{1}{10}(15) + \frac{1}{2}(6) = 4.5$$

For the fair die:

$$E[X] = (1)(\frac{1}{6}) + (2)(\frac{1}{6}) + (3)(\frac{1}{6}) + (4)(\frac{1}{6}) + (5)(\frac{1}{6}) + (6)(\frac{1}{6})$$

$$= \frac{1}{6}(21) = 3.5$$

[a] From Discrete Math by Ensley and Crawley, page 467

Question 1 _____ / 3

Suppose you pay \$2 each time to play the following game: Two dice are rolled, and you win \$5 for each 6 that comes up. Do you expect to win more than you pay if you play many, many times?

Let X represent the amount of money you win in one play of the game. So, you can win either \$0, \$5, or \$10, so the values are $\{0, 5, 10\}$.

What is $Prob(X = 0)$?

What is $Prob(X = 5)$?

What is $Prob(X = 10)$?

Then, $E[X] = 0 \cdot Prob(X = 0) + 5 \cdot Prob(X = 5) + 10 \cdot (X = 10)$

Question 2 _____ / 3

Suppose that the payoff for the game (in question 1) is \$5 for rolling one 6, and \$25 for rolling two 6's. Now is it worth \$2 to play?

6.4.1 Expectation in Bernoulli trials

Theorem 1
Suppose an experiment consists of the independent repetition of a trial n times, and the probability of that trial's individual success is p each time it is performed. If X denotes the number of successful trials in this experiment, then $E[X] = n \cdot p$. [a]

Practice Problem 3 Use the definition of expected value to show that the average number of results of heads in an experiment consisting of tossing a coin three times is 1.5

So, our X will be the number of successful trials (heads tossed). The coin is tossed three times, so X takes on the values from the set $\{0, 1, 2, 3\}$. By the definition of the expected value,

$E[X] = 0 \cdot Prob(X = 0) + 1 \cdot Prob(X = 1) + 2 \cdot Prob(X = 2) + 3 \cdot Prob(X = 3)$
$= 0 \cdot C(3, 0) \cdot (1/2)^3 + 1 \cdot C(3, 1) \cdot (1/2)^3 + 2 \cdot C(3, 2) \cdot (1/2)^3 + 3 \cdot C(3, 3) \cdot (1/2)^3$
$= (0 + 3 + 6 + 3) \cdot (1/8) = 1.5$

And using the theorem, $n = 3$, $p = 1/2$...
$E[X] = 3 \cdot (1/2) = (3/2) = 1.5$

[a] From Discrete Math by Ensley and Crawley, page 469