

**Team name:**

Please write down all people in your team.

- |    |    |
|----|----|
| 1. | 2. |
| 3. | 4. |
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**Grading**

Question	Score	Max
1		4
2		4
3		2
Total		10

## 6.6 Matrices and Markov Chains

### The Gambler's Ruin Problem

Page 478 of the book highlights a game played between two characters: Player “H” and Player “T”. Each character begins with a certain amount of markers (or tokens), and they play by flipping a **coin**. Whenever one of them loses a “round”, they give one marker to their opponent.



If a *heads* is flipped, then Player H wins a marker from Player T. For a *tails*, Player T wins a marker from Player H. The game is over once somebody is out of markers.

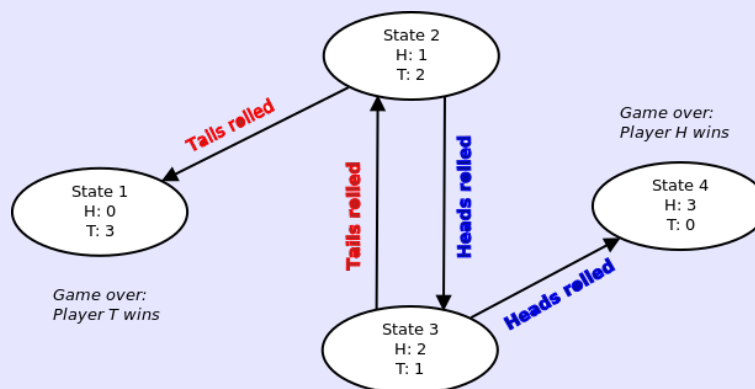
**Game setup:** For the game we'll be talking about in this example, the rules are:

- There are 3 total markers (so one player will have more markers than the other)
- Once somebody is out of markers, the game is over.
- Each coin flip, the loser gives one of their markers to the other player. (One gains, one loses)

**Game states:** Before we start modeling the game with a matrix, let's map out all the game states. We won't worry about what is the beginning state, and each coin flip, one person gains a marker and one person loses a marker. There are four possible states in the game:

State #	H's markers	T's markers
1	0	3
2	1	2
3	2	1
4	3	0

**State diagram:** Given these states, and the fact that each coin flip one person loses a marker and gives it to the other, our state diagram would look like:



**State change matrix:** Now we will build out a matrix to show the probability of switching between states.

The matrix will be  $4 \times 4$ . Each **row** will be a state, and each cell in that row is the probability of going from that state to a new state.

	State 1	State 2	State 3	State 4
State 1 →	1	0	0	0
State 2 →	1/2	0	1/2	0
State 3 →	0	1/2	0	1/2
State 4 →	0	0	0	1

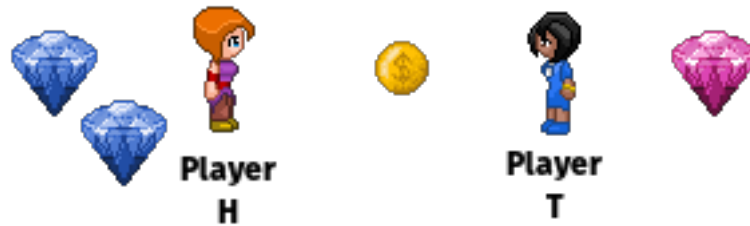
This matrix is in the format, “from *row* state to *col* state”. In Row 2, the probability of going from (State 2 → State 1) is 1/2, because the coin flip has a half chance of being heads, and a half chance of being tails.

State 1 and State 4 they are gameover states: you cannot move from State 1 to another state, so it has a 1 in that cell.

**Question 1**

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Let's say a game starts where Player H has 2 markers and player 1 has 1 marker. This is state 3.



What is the probability of...

- Going from State 3 to State 1?
- Going from State 3 to State 2?
- Going from State 3 to State 3?
- Going from State 3 to State 4?

**Transition Matrix**

If you have a game with states 1 through  $n$ , then your transition matrix  $M$  is

$$M_{i,j} = \text{Prob}(\text{the game changes from state } i \text{ to state } j \text{ in one move}).$$

<sup>a</sup>

<sup>a</sup>Discrete Structures, Ensley and Crawley

**Question 2**

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Let's say in the game there are 2 total markers instead of three.

a. What are all the states in the game?

b. Draw the transition matrix for this game.

**Transition Matrix**

Given the matrices  $M$  and  $N$ , where the amount of *rows* in  $M$  is the same amount of *columns* in  $N$ , we can find the product  $P = M \cdot N$  where each entry at row  $i$ , column  $j$  of  $P$  is the row-column product of row  $i$  from  $M$  and column  $j$  from  $N$ . In other words,

$$P_{i,j} = M_{i,1} \cdot N_{1,j} + M_{i,2} \cdot N_{2,j} + \dots$$

<sup>a</sup>

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<sup>a</sup>Discrete Structures, Ensley and Crawley

**Question 3**

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Calculate the product  $M \cdot M$  (aka  $M^2$ ) for our original game with 3 markers.

	Col 1	Col 2	Col 3	Col 4
Row 1	1	0	0	0
Row 2	1/2	0	1/2	0
Row 3	0	1/2	0	1/2
Row 4	0	0	0	1

The result ends up being the probability that the game processes from state  $i$  to state  $j$  in **two** moves.

$$M_{1,1}^2 = \begin{matrix} M_{1,1} \cdot M_{1,1} + & M_{1,2} \cdot M_{2,1} + & M_{1,3} \cdot M_{3,1} + & M_{1,4} \cdot M_{4,1} \\ 1 \cdot 1 & 0 \cdot 1/2 & 0 \cdot 0 & 0 \cdot 0 \end{matrix} = 1$$

$$M_{1,2}^2$$

$$M_{1,3}^2$$

$$M_{1,4}^2$$

$$M_{2,1}^2$$

$$M_{2,2}^2$$

$$M_{2,3}^2$$

$$M_{2,4}^2$$

$$M_{3,1}^2 = \begin{matrix} M_{3,1} \cdot M_{1,1} + & M_{3,2} \cdot M_{2,1} + & M_{3,3} \cdot M_{3,1} + & M_{3,4} \cdot M_{4,1} \\ 0 \cdot 1 & 1/2 \cdot 1/2 & 0 \cdot 0 & 1/2 \cdot 0 \end{matrix} = 1/4$$

$$M_{3,2}^2$$

$$M_{3,3}^2$$

$$M_{3,4}^2$$

$$M_{4,1}^2$$

$$M_{4,2}^2$$

$$M_{4,3}^2$$

$$M_{4,4}^2$$

Draw the matrix  $M^2$ :

**Theorem 1**

If  $M$  is an  $n \times n$  transition matrix reflecting the one-move transition probabilities for states 1 through  $n$  of a game, then for any integer  $k \geq 1$ , the entry in row  $i$ , column  $j$  of the Matrix  $M^k$  is the probability of the game moving from state  $i$  to state  $j$  in  $k$  moves. <sup>a</sup>

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<sup>a</sup>Discrete Structures, Ensley and Crawley