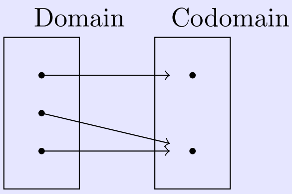
7.3 Exercise: In-class exercises are meant to introduce you to a new topic and provide some practice with the new topic. Work in a team of up to 4 people to complete this exercise. You can work simultaneously on the problems, or work separate and then check your answers with each other. You can take the exercise home, score will be based on the in-class quiz the following class period. **Work out problems on your own paper** - this document just has examples and questions.

7.3 Isomorphism and Planarity

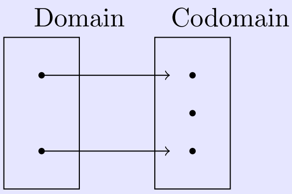
7.3.1 Review: Functions

Properties of Functions

- **Onto:** A function is **onto** if every element of the codomain has at least one element in the domain pointing to it. (Every output is attainable via at least one input.)
- **One-to-one:** A function is **one-to-one** if none of the elements in the codomain is the output from two *different* inputs from the domain.



Onto but not one-to-one

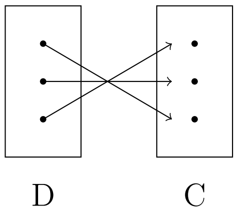


One-to-one but not onto

Question 1

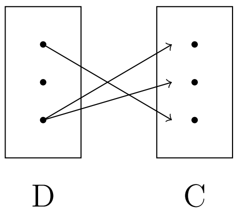
Identify the properties for the following graphs.

a.



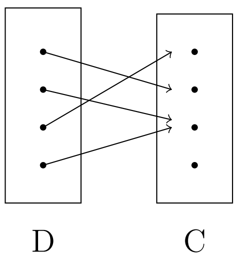
Is it Onto?
Is it One-to-one?

b.



Is it Onto?
Is it One-to-one?

c.



Is it Onto?
Is it One-to-one?

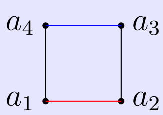
7.3.2 Isomorphism

Isomorphism

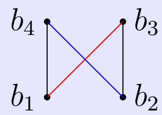
Simple graphs G and H are called **isomorphic** if there is a one-to-one and onto function f from the nodes of G to the nodes of H such that $\{v, w\}$ is an edge of G if and only if $\{f(v), f(w)\}$ is an edge of H. The function f is called an isomorphism. Hence, an isomorphism is simply a **rule** associating nodes that preserves the edges joining the nodes. [a]

In other words, two graphs are isomorphic if they're essentially the same graph, even if the vertices are in different positions.

Example:



G



H

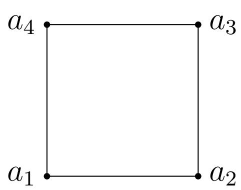
These are isomorphic... imagine taking a_2 and a_3 from graph G and physically flipping them with the edges still connected. In this case, our mapping is... $a_1 \to b_1$ $a_2 \to b_3$ $a_3 \to b_2$ $a_4 \to b_4$.

[a] Discrete Mathematics, Ensley and Crawley

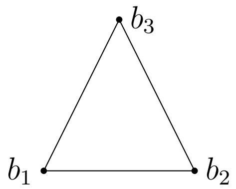
Question 2

Redraw the following graphs by moving the vertices around, but keeping the edges connected.

a.



b.



Properties of isomorphic graphs

Two graphs that are isomorphic to one another must have...: [a]

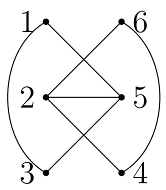
- The same number of nodes
- The same number of edges
- The same number of nodes of any given degree.
- The same number of cycles.
- The same number of cycles of any given size.

[a] Discrete Mathematics, Ensley and Crawley

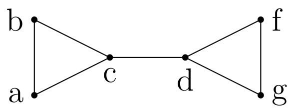
Question 3

Given the following two graphs...

G



H



a. Write out all edges for both graphs.

   G: {2, 5}

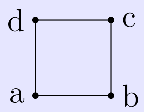
   H: {d, c}

b. For each edge from G, write out what edge in H corresponds to it.
   Example: {2,5} $\rightarrow$ {d, c}

7.3.3 Adjacency matrix

We can also use a matrix to list out which vertices are adjacent to which other vertices in order to help us figure out if two graphs are isomorphic.

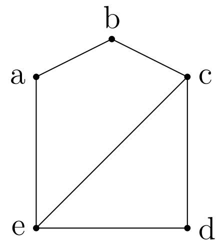
Example:



	a	b	c	d
a	0	**1**	0	**1**
b	**1**	0	**1**	0
c	0	**1**	0	**1**
d	**1**	0	**1**	0

a is adjacent to *b* and *d*, so in the *a* row we have 1's under the *b* and *d* columns.

Question 4
For the following graph...



a. Finish the adjacency matrix:

	a	b	c	d	e
a					
b					
c					
d					
e					

b. Fill out the degrees of each:

a	b	c	d	e

7.3.4 Planarity

Planarity

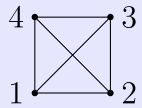
1. A simple, connected graph is called **planar** if there is a way to draw it (on a plane) so that no edges cross (i.e., they can only meet at a node). We will call "drawing" of a graph on a plane surface with no edge-crossings an **embedding** of the graph in the plane.

2. A graph is called **bipartite** if its set of nodes can be partitioned into two disjoint sets S_1 and S_2 so that every edge in the graph has one endpoint in S_1 and one endpoint in S_2.

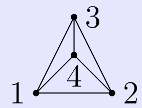
3. The **complete graph** on n nodes, denoted by K_n, is the simple graph with nodes $\{1, ..., n\}$ and an edge between every pair of distinct nodes.

4. The **complete bipartite graph** on n, m nodes, denoted by $K_{n,m}$, is the simple bipartite graph with nodes $S_1 = \{a_1, a_2, ..., a_n\}$ and $S_2 = \{b_1, b_2, ..., b_m\}$ and with edges connecting each node in S_1 to every node in S_2. [a]

Example: Let's redraw the graph K_4 so it has no overlapping edges.



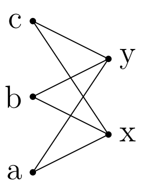
K_3



K_3 redrawn

[a] Discrete Mathematics, Ensley and Crawley

Question 5
Redraw the following graph, $K_{3,2}$, so that no edges are overlapping.



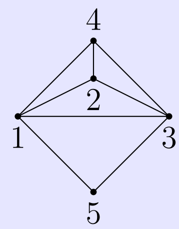
Faces

For a planar graph G embedded in the plane, a **face** of the graph is a region of the plane created by the drawing. Since the plane is an unbounded surface, every embedding of a finite planar graph will have exactly one **unbound** face. [a]

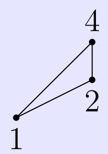
Unbound (external) face: Think of the external face as the "canvas" that all other faces are painted on to. Or, if you were viewing a silhouette of the drawing, you would only see the unbounded face - the sum of all the faces.

Example:

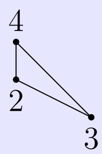
For the drawing, identify the faces by giving the cycle that creates each face, and highlight the unbounded face.



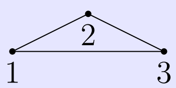
Faces:



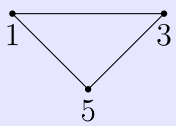
1, 2, 4, 1



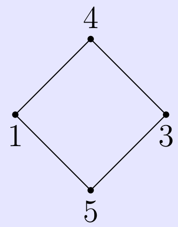
2, 3, 4, 2



1, 2, 3, 1



1, 3, 5, 1



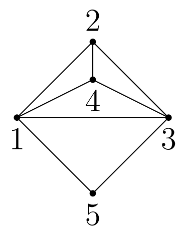
1, 4, 3, 5, 1
(unbounded)

[a] Discrete Mathematics, Ensley and Crawley

Question 6

For both graphs, draw out each of its **faces**, then write out all the cycles bordering faces and identify the unbounded cycle.

a.



b.