# 6.2 The Sum Rule

**Disjoint events**
Two events are said to be **disjoint** (or *mutually exclusive* if they cannot occur simultaneously. [a]

[a] From Discrete Math by Ensley and Crawley, page 448

**Question 1** ____ / 3

For each of the experiments given below, decide if the events described are disjoint:

a. When tossing a coin four times, let $E_1$ be the event that there are exactly three heads and $E_2$ be the event that there are exactly two heads.
When tossing a coin four times, no outcome can consist of "exactly two heads" and also "exactly three heads"; hence, these two events are disjoint.

b. When choosing four cards, let $E_1$ be the event that the cards have the same value and $E_2$ be the event that the cards have the same suit.
When choosing four cards, if the four cards all have the same value, then they cannot all have the same suit; hence, these two events are disjoint.

c. When choosing a committee of three people from a club with 8 men and 12 women, let $E_1$ be the event that the committee has a woman and let $E_2$ be the event that the committee has a man.
When choosing a committee of three people from a club with 8 men and 12 women, there are many ways in which the committee can include a woman and a man, so these events are not disjoint.

**The Sum Rule, Theorem 1**

If $E_1$ and $E_2$ are disjoint events in a given experiment, then the probability that $E_1$ or $E_2$ occurs is the sum of $Prob(E_1)$ and $Prob(E_2)$. That is,

$$Prob(E_1 or E_2) = Prob(E_1) + Prob(E_2)$$

for disjoint events. [a]

[a] From Discrete Math by Ensley and Crawley, page 449

**Question 2** _____ / 3

A card is drawn from an ordinary deck of 52 cards. Show how to use the basic sum rule to find the probability that the card is...

a. An ace or a jack.

> **Hint**
> $E_1$ is the set of outcomes where you get an Ace...
> { Ace-Heart, Ace-Diamond, Ace-Spade, Ace-Club } and the probability of getting an Ace, $Prob(E_1)$ is 4 outcomes out of 52, or $\frac{1}{13}$.

$$Prob(E_1 or E_2) = Prob(E_1) + Prob(E_2) = \frac{1}{13} + \frac{1}{13} = \frac{2}{13}$$

b. A diamond or a black jack, queen, or king card.

$E_1$ is the outcomes where you get a diamond, and $E_2$ is the outcomes where you get a black Jack, King, or Queen. These sets are disjoint, so...

$$Prob(E_1 or E_2) = Prob(E_1) + Prob(E_2) = \frac{1}{4} + \frac{6}{52} = \frac{19}{52}$$

c. An even number value or a red jack, queen, or king card.

$E_1$ is the outcomes where the card has an even numbered value, and $E_2$ is the set of outcomes with a red Jack, King, or Queen. These sets are disjoint, so...

$$Prob(E_1 or E_2) = Prob(E_1 + Prob(E_2) = \frac{5}{13} + \frac{6}{52} = \frac{1}{2}$$

**The General Sum Rule, Theorem 2**

If $E_1$ and $E_2$ are any events in a given experiment, then the probability that $E_1$ or $E_2$ occurs is given by

$$Prob(E_1 or E_2) = Prob(E_1) + Prob(E_2) - Prob(E_1 and E_2)$$

If $E_1$ and $E_2$ are disjoint, then $E_1 \cap E_2 = \emptyset$, so $Prob(E_1 and E_2) = 0$. [a]

[a] From Discrete Math by Ensley and Crawley, page 450

**Question 3** ______ / 1

What is the probability that when a pair of dice are rolled, either (at least) one die shows a 5 or the dice sum to 8?

**What is the sample size?** $S =$ 36

| | | | | | |
|---|---|---|---|---|---|
| (1,1) | (1,2) | (1,3) | (1,4) | (1,5) | (1,6) |
| (2,1) | (2,2) | (2,3) | (2,4) | (2,5) | (2,6) |
| (3,1) | (3,2) | (3,3) | (3,4) | (3,5) | (3,6) |
| (4,1) | (4,2) | (4,3) | (4,4) | (4,5) | (4,6) |
| (5,1) | (5,2) | (5,3) | (5,4) | (5,5) | (5,6) |
| (6,1) | (6,2) | (6,3) | (6,4) | (6,5) | (6,6) |

**How many outcomes $n(E_1)$ are there where you get either 5 for the first die, 5 for the second die, or 5 for both dice? Write out the set of $E_1$.**

$E_1 = \{(5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,5), (4,5), (3,5), (2,5), (1,5)\}$

$n(E_1) = 11$

**How many outcomes $n(E_2)$ are there where the two dice sum to 8? Write out the set of $E_2$.**

$E_2 = \{(2,6), (6,2), (3,5), (5,3), (4,4)\}$

$n(E_2) = 5$

**What is the amount of overlap $n(E_1 \quad AND \quad E_2)$? Write out this set.**

$n(E_1 \quad AND \quad E_2) = 2$

**Use the General Sum Rule to find the probability that you will get either at least one die showing a 5, OR the dice sum to 8.**

$\frac{11}{36} + \frac{5}{36} - \frac{2}{36} = \frac{7}{18}$

## 6.3 The Product Rule

**Independent events**

Two events are said to be **independent** if the occurrence of one event is not influenced by the occurred (or nonoccurrence) of the other event. [a]

[a] From Discrete Math by Ensley and Crawley, page 451

**Question 4** ____ / 2

For each of the following experiments given below, decide if the events described are independent:

a. When rolling a 6-sided die four times, let $E_1$ be the event that the first two rolls sum to 7 and let $E_2$ be the event that the last two rolls sum to 10.
The events are independent. The first two rolls have no influence on the last two rolls.

b. When choosing a committee of three dogs from a club of 8 corgis and 12 labradors, let $E_1$ be the event that the committee has a labrador and let $E_2$ be the event that the committee has a corgi.
The events are not independent. The probability of the committee having a corgi ($E_2$) is different when $E_1$ occurs than it is when $E_1$ does not occur. Specifically, if $E_1$ does not occur, then $E_2$ happens for sure (i.e., its probability is 1), and if $E_1$ does occur, then $E_2$ is not guaranteed to happen (i.e., probability is less than 1).

**The Product Rule, Theorem 3**

If $E_1$ and $E_2$ are independent events in a given experiment, then the probability that both $E_1$ and $E_2$ occur is the product of $Prob(E_1$ and $Prob(E_2$. That is,

$$Prob(E_1 and E_2) = Prob(E_1) \cdot Prob(E_2)$$

for independent events. [a]

[a]From Discrete Math by Ensley and Crawley, page 452

**Question 5** ____ / 2

Suppose I have a "loaded" die for which the probability of a 6 appearing is $\frac{1}{2}$, while the probability of each of the other faces appearing is $\frac{1}{10}$. What is the probability of getting a 5 and then a 6 on two tosses of the loaded die?

First identify $E_1$ and $E_2$. These events are independent, so you can use the Product Rule to find $Prob(E_1 and E_2)$.

The events $E_1$, "getting a 5 on the first toss", and $E_2$, "getting a 6 on the second toss", are independent, so by the product rule,

$$Prob(E_1 and E_2) = Prob(E_1) \cdot Prob(E_2) = \frac{1}{10} \cdot \frac{1}{2} = \frac{1}{20}$$

**The probability of $E_1$ given $E_2$**

Given events $E_1$ and $E_2$ for some experiment, we define the probability of $E_1$ given $E_2$, denoted by $Prob(E_1|E_2)$, as the probability that $E_1$ happens given that $E_2$ occurs. Note that if $E_1$ and $E_2$ are independent, then $Prob(E_1|E_2) = Prob(E_1)$. [a]

[a]From Discrete Math by Ensley and Crawley, page 452

**The General Product Rule, Theorem 4**

If $E_1$ and $E_2$ are any events in a given experiment, then the probability that both $E_1$ and $E_2$ occur is given by

$$Prob(E_1 and E_2) = Prob(E_2) \cdot Prob(E_1|E_2)$$
$$= Prob(E_1) \cdot Prob(E_2|E_1)$$

Note that if $E_1$ and $E_2$ are independent, then this says the same thing as Theorem 3. [a]

[a] From Discrete Math by Ensley and Crawley, page 453

**Question 6** ____ / 1

Two marbles are chosen from a bag containing three red, five white, and eight green marbles, so there are 16 total marbles. What is the probability that both are red?

Here, the event $R_1$ is "the first marble is red", and the event $R_2$ is "the second marble is red".

What is $Prob(R_1)$? $\frac{3}{16}$

Since $R_2$ depends on $R_1$ occurring, after $R_1$ occurs, there are 15 marbles left. One red marble has been selected, so there are 2 red marbles left.

What is $Prob(R_2|R_1)$? $\frac{2}{15}$

With this information, what is $Prob(R_1 and R_2)$?

$Prob(R_1) = \frac{3}{16}$, $Prob(R_2) = \frac{2}{15}$.
$Prob(R_1 and R_2) = Prob(R_1) \cdot Prob(R_2|R_1)$
$= \frac{6}{240} = \frac{1}{40}$

**Question 7** ____ / 1

Two marbles are chosen from a bag containing three red, five white, and eight green marbles, so there are 16 total marbles. What is the probability that one is white and one is green?

Let's say we have the events $W_1$ (White first), $W_2$ (White second), $G_1$ (Green first), and $G_2$ (Green second), so we can get our result in two ways: with $(W_1, G_2)$ **OR** with$(G_1, W_2)$, so you can calculate the result as

$$Prob(W_1 and G_2) + Prob(G_1 and W_2)$$

$W_1$ will be white as the first, and $G_2$ will be green as the second.

$Prob(W_1 and G_1 + Prob(G_1 and W_2) =$
$Prob(W_1) \cdot Prob(G_2|W_1) + Prob(G_1) \cdot Prob(W_2|G_1)$
$= \frac{5}{16} \cdot \frac{8}{15} + \frac{8}{16} \cdot \frac{5}{15} = \frac{1}{3}$