5. Exercise: In-class exercises are meant to introduce you to a new topic and provide some practice with the new topic. Work in a team of up to 4 people to complete this exercise. You can work simultaneously on the problems, or work separate and then check your answers with each other. You can take the exercise home, score will be based on the in-class quiz the following class period. **Work out problems on your own paper** - this document just has examples and questions.

5.6 Solving Recurrence Relations

5.6.1 Review: Recursive formulas

Remember chapter 1.2...?

Back in the first week of CS 210, you were given sequences of numbers... let's say

$$3, 5, 7, 9, 11$$

...and you tasked with coming up with **closed formulas** and **recursive formulas** for these.

Closed formulas were based only on the value of n, such as...

$$a_n = 2n + 1$$

And recursive formulas were based on some starting value, a_1, with each subsequent element being based off a previous element., such as...

$$a_1 = 3; \qquad a_n = a_{n-1} + 2$$

A full class and 4.4 chapters later, we will actually show you *how* to come up with a formula given a sequence of numbers... No guesswork required.

5.6.2   Difference tables

Finding formulas: Recursive

Find a recursive formula for the sequence $2, 5, 8, 11, 14, ...$

First, we will be assigning this sequence of numbers to a variable s, where n is the index (or position) in the sequence, and s_n is the element (or term) at that position.

Previously, we had our formulas begin at a_1, but now we will be starting our sequences at index 0, like when programming.

First let's build a table with n, the index, s_n, the element at that index, and $\triangle_n$, the difference between two elements ($\triangle_n = s_{n+1} - s_n$)

index n	0	1	2	3	4
element at n s_n	2	5	8	11	14
different between elements $\triangle_n$	3	3	3	3	3

With the table, we can see the difference between each term (2 to 5, 5 to 8, etc...) is **3**. With this information, we can write any value s_1 in terms of the previous value:

$$s_1 \quad = s_0 + \triangle_0 \qquad\qquad \Big| \ (5 = 2 + 3)$$

$$
\begin{aligned}
s_2 \quad &= s_1 + \triangle_1 \\
&= s_0 + (\triangle_0 + \triangle_1)
\end{aligned}
\qquad
\begin{aligned}
&(8 = 5 + 3) \\
&(8 = 2 + 3 + 3)
\end{aligned}
$$

$$
\begin{aligned}
s_3 \quad &= s_2 + \triangle_2 \\
&= s_0 + (\triangle_0 + \triangle_1 + \triangle_2) \\
&= s_0 + \textstyle\sum_{k=0}^{2} \triangle_k
\end{aligned}
\qquad
\begin{aligned}
&(11 = 8 + 3) \\
&(11 = 2 + 3 + 3 + 3) \\
&(11 = 2 + \textstyle\sum_{k=0}^{2} 3)
\end{aligned}
$$

So given the first term being
$$s_0 = 2,$$
we can say the recursive formula is
$$s_n = s_{n-1} + \triangle_{n-1} \qquad \text{or} \qquad s_n = s_{n-1} + 3$$

(Continued) Finding formulas: Closed

But what about the closed formula? Well with the differences we can also describe any term as the *first term* plus the sum of the differences...

$$s_1 \quad = s_0 + \triangle_0 \qquad\qquad\qquad \big|\ (5 = 2 + 3)$$

$$s_2 \quad = s_1 + \triangle_1 \qquad\qquad\qquad (8 = 5 + 3)$$
$$ = s_0 + (\triangle_0 + \triangle_1) \qquad\quad (8 = 2 + 3 + 3)$$

$$s_3 \quad = s_2 + \triangle_2 \qquad\qquad\qquad\ (11 = 8 + 3)$$
$$ = s_0 + (\triangle_0 + \triangle_1 + \triangle_2)\ \ (11 = 2 + 3 + 3 + 3)$$
$$ = s_0 + \sum_{k=0}^{2} \triangle_k \qquad\qquad (11 = 2 + \sum_{k=0}^{2} 3)$$

So $s_1 = 2 + 3,$ $\qquad s_2 = 2 + 3 + 3,$ $\qquad s_3 = 2 + 3 + 3 + 3...$
which we can write as

$$s_n = \sum_{k=0}^{n-1} (3) + 2$$

OR

$$s_n = 3 \cdot n + 2$$

Question 1
TODO PRACTICE

TODO: Explain this thing

Theorem 1: Fundamental Theorem of Sums and Differences

For any sequence $\{s_n\}$ with first differences $\triangle_k = s_{k+1} - s_k$, and any $n \geq 1$,

$$s_n - s_0 = \sum_{k=0}^{n-1} \triangle_k$$

5.6.3   Complex sequences

Complex sequences

Sometimes, the difference between each term in a sequence is not the same; maybe the difference is 3, then 4, then 5, and so on. In this case, the difference itself also has a difference. In this case, that "difference-of-the-differences" is known as the *second difference*, whereas the difference between the terms themselves is the *first difference*.

Example: Find a closed formula for the sequence 6, 11, 19, 30, 44.

As previously, we can start by writing out a table of the index n, the term s_n, and the difference $\triangle_n$...

n	0	1	2	3	4
s_n	6	11	19	30	44
$\triangle_n$	5	8	11	14	?

Once we've figured out $\triangle_n$, we can see that it isn't constant each time, so we can't apply the same techniques as before. Instead, let's expand the table to have a fourth row: the difference of the differences. We will use a triangle again to symbolize "difference", but we will add a number to it, so $\triangle_n^1$ is the difference between terms, and $\triangle_n^2$ is the difference of those differences.

n	0	1	2	3	4
s_n	6	11	19	30	44
$\triangle_n^1$	5	8	11	14	?
$\triangle_n^2$	3	3	3	...	...

Ahh, can we maybe apply what we learned last time! Solving this will actually mean that we're working recursively.

Let's look at this closer...

n	0	1	2	3	4
s_n	6	11	19	30	44
$\triangle_n^1$	5	8	11	14	?
$\triangle_n^2$	3	3	3	...	...

$$\triangle_n^k = \triangle_{n+1}^{k-1} - \triangle_n^{k-1}$$

(Continued) Complex sequences

$$
\begin{array}{ll}
\triangle_1^1 = \triangle_0^1 + \triangle_0^2 & (8 = 5 + 3) \\
\triangle_2^1 = \triangle_1^1 + \triangle_1^2 & (11 = 5 + 3 + 3) \\
\quad = \triangle_0^1 + (\triangle_0^2 + \triangle_1^2) & (14 = 5 + 3 + 3) \\
\triangle_3^1 = \triangle_2^1 + \triangle_2^2 & (11 = 8 + 3) \\
\quad = \triangle_0^1 + (\triangle_0^2 + \triangle_1^2 + \triangle_2^2) & (11 = 2 + 3 + 3 + 3)
\end{array}
$$

So we can use the same strategy to find values of $\triangle_n^1$...

Theorem 1 (Revisited)

For any sequence $\{s_n\}$ with first differences $\triangle_k = s_{k+1} - s_k$, and any $n \geq 1$,

$$
s_n - s_0 = \sum_{k=0}^{n-1} \triangle_k
$$

Definition For a given sequence of numbers $\{s_n\}$ beginning with s_0 and an integer $k \geq 2$, we define the sequence $\{\triangle_n^k\}$ of kth differences by the rule

$$
\triangle_n^k = \triangle_{n+1}^{k-1} - \triangle_n^{k-1}
$$

for all $n \geq 0$. Hence, the first term in the sequence of kth differences is $\triangle_0^k = \triangle_1^{k-1} - \triangle_0^{k-1}$. For consistency, we refer to the sequence of first differences as $\{\triangle_n^1\}$ instead of $\{\triangle_n\}$ in this context.

Theorem 2:

For a sequence $\{s_n\}$ whose k^{th} differences are constant, for all $n \geq 0$,

$$
s_n = s_0 + \sum_{i=1}^{k} \triangle_0^i \cdot C(n, i)
$$

Question 2
TODO PRACTICE