Markov Spring 2025 Homework 4

Due Feb. 26 on Gradescope

Plotting and reporting.

- **Document.** Submit a single document that includes all your figures.

- **Figures.** Include clean, labeled plots (axes, units, parameters) with consistent limits across panels. No photos of screens are accepted, although clear screenshots embedded in or appended to your document are Ok. Calculate key results such as empirical means, variances, and probabilities with 3–4 significant digits.

- **Code.** You don't need to include code in your written homework. Instead, create a github account if you don't already have one, create a public repository labeled "Markov2026", and upload your code there, ready to run. Label the code as "HW1_problem4.m", for example. Include the name of your account and repository in the homework. You can use any programming language you want (excel is strongly discouraged, though).

- **Math.** Accompany your calculations with clear and complete sentences. Discuss your results.

Problems

1. **Gambler's Ruin with Retirement.** A gambler starts with i dollars. Every time step, they win one dollar with probability $p > 0$, lose one dollar with probability $q > 0$, or decide to retire with probability $s > 0$, with $p + q + s = 1$.

   (a) Find the probability that they retire before losing all their money.

   (b) Find the expected amount of money the player has when the game ends (the player retires or loses all their money).

   (c) Set $q = 0.4$, $p = 0.35$, $s = 0.25$. Simulate this game $100,000$ times starting with 10 dollars and calculate the expected value in (b) numerically. Compare your numerical answer with the theoretical answer.

   Hint: In this case you will have only one boundary condition. You can get another condition by requiring that your variables behave reasonably as $i \to \infty$.

2. **Greedy Management.** A company has 5 coffee machines for all its employees. Every week, each one of the machines breaks down with probability $1/10$, independently of the others. When all the machines are damaged, the employees complain and 5 new machines are installed next week.

   (a) Construct a Markov Chain to describe this process, where the states are the number of operating coffee machines. Specify its transition probability matrix.

   (b) Find the expected time until all the newly installed machines fail.

   (c) Find the probability that only one machine is working in a randomly chosen week (in the long run).

3. **Long-term transition probabilities.** $\longrightarrow$ **Skip**

Consider a Markov chain with state space $S = \{1, 2, 3, 4, 5\}$ and probability transition matrix

$$p = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2/3 & 0 & 1/3 & 0 \\ 1/8 & 1/4 & 5/8 & 0 & 0 \\ 0 & 1/6 & 0 & 5/6 & 0 \\ 1/3 & 0 & 1/3 & 0 & 1/3 \end{bmatrix} \tag{1}$$

Determine $\lim_{n\to\infty} p^n(i,j)$ for each of the following combinations of $i, j \in S$.

(a) $i \in S$ and $j \in \{3, 5\}$.

(b) $i = 1$ and $j \in \{1, 2, 4\}$.

(c) $i \in \{2, 4\}$ and $j = 1$.

(d) $i, j \in \{2, 4\}$.

(e) $i = 3, 5$ and $j = 1$.

(f) $i = 3, 5$ and $j = 2$.

(g) $i = 3, 5$ and $j = 4$.

4. **Convergence to the stationary distribution.** Consider the two-state Markov Chain with probability transition matrix

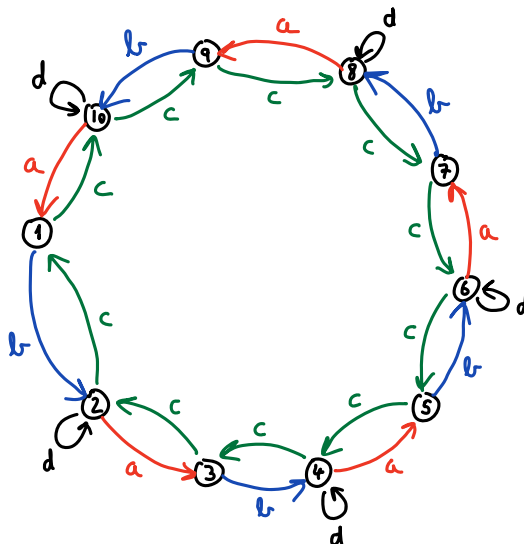$$p = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}, \tag{2}$$

where $0 < a < 1$, $0 < b < 1$. In class we showed that the stationary distribution is

$$\pi_1 = \frac{b}{a+b}, \qquad \pi_2 = \frac{a}{a+b}.$$

In this problem we will show that, in fact, $q_n(i) = P(X_n = i)$ converges exponentially fast to π_i.

(a) Set up the equations $[q_{n+1}(1), q_{n+1}(2)] = [q_n(1), q_n(2)]p$.

(b) Introduce new variables x_n and y_n that measure the deviation of $q_n(1)$ and $q_n(2)$ from the stationary distribution, $q_n(1) = \pi_1 + x_n$, $q_n(2) = \pi_2 + y_n$. Rewrite and simplify the equations from (a) in terms of these new variables and get equations for x_{n+1} and y_{n+1} in terms of x_n and y_n.

(c) Solve the equations you found in (c) and show that x_n and y_n decay to zero exponentially.

5. **Ring.** Consider the Markov Chain shown below, where $0 < a < 1$, $0 < b < 1$, $0 < c < 1$, $0 < d < 1$.

(a) Find relationships between a, b, c, and d.

(b) Find the stationary distribution of the Markov Chain and express it only in terms of d. (Hint: since the even nodes all behave identically, and same for the odd nodes, you only need to set up equations for π_{even} and π_{odd}, and make sure they are properly normalized.)

6. **Matrix equations for hitting time.** Consider a finite irreducible Markov Chain with states $1, 2, 3, \ldots, N - 1, N$ and probability transition matrix p. Define

$$q_i = E(\text{number of steps until reaching state } N \text{ starting from state } i).$$

(a) Write down the equation satisfied for q_i for $i = 1, 2, \ldots, N - 1$.

(b) If $\mathbf{q} = [q_1, q_2, \ldots, q_{N-1}]^T$, find an equation for $\mathbf{q}$ of the form

$$\mathbf{q} = B\mathbf{z},$$

where B and $\mathbf{z}$ are a matrix and a vector that depend on p.

7. **(For APPM 5560/5100 students only.)** Consider a random walk on the integers. The state space S is $\mathbb{N} = \{\ldots, -2, -1, -0, 1, 2, \ldots\}$. The state of the Markov Chain evolves as

$$X_{n+1} = \begin{cases} X_n + 1, & \text{with probability } p, \\ X_n - 1, & \text{with probability } 1 - p, \end{cases} \tag{3}$$

Suppose that $X_0 = 0$. As seen in class, the expected number of times that 0 is revisited is given by

$$E_0[N_0] = \sum_{n=1}^{\infty} p^n(0, 0).$$

(a) By counting the number of ways in which the Markov chain can return to 0 and calculating their probabilities, find $p^n(0, 0)$.

(b) Using an appropriate convergence test, show that the series converges if $p \neq 1/2$. What does this say about the state 0?

(c) Using an appropriate convergence test, show that the series diverges for the case $p = 1/2$. For this, you can use Stirling's formula:

$$\lim_{n \to \infty} \frac{n!}{\sqrt{2\pi n}(n/e)^n} = 1. \tag{4}$$

What can you conclude about the state 0?

$\boxed{\textcolor{red}{\textbf{END}}}$