Homework 1

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See https://github.com/nguyencquan/Markov2026 for code used in simulations.

1 Power Law Sampling

Assume we have some distribution

$$\begin{cases} Cx^{-\gamma} & x \geq x_0, \\ 0, & x < x_0 \end{cases} \tag{1}$$

a) Where $x_0 > 0$ and $\gamma > 3$
A valid pdf must have a total probability of 1, which is calculated below:

$$1 = \int_{-\infty}^{\infty} Cx^{-\gamma}dx = C\int_{x_0}^{\infty} x^{-\gamma}dx = \frac{C}{x^{\gamma-1}(1-\gamma)}|_{x_0}^{\infty} = \frac{C}{x_0^{\gamma-1}(\gamma-1)} \tag{2}$$

$$C = x_0^{\gamma-1}(\gamma-1) \tag{3}$$

b) First calculating the cummulative distribution we take a step from equation 2 except take the integral from x_0 to x instead.

$$1 = \int_{-\infty}^{\infty} Cx^{-\gamma}dx = C\int_{x_0}^{x} x^{-\gamma}dx = \frac{C}{x^{\gamma-1}(1-\gamma)}|_{x_0}^{x} = \frac{C}{x_0^{\gamma-1}(\gamma-1)} + \frac{C}{x^{\gamma-1}(1-\gamma)} \tag{4}$$

Substituting in C

$$F(x) = \begin{cases} 1 - \frac{x_0^{\gamma-1}}{x^{\gamma-1}}, & x \geq x_0 \\ 0, & x < x_0 \end{cases} \tag{5}$$

To calculate the inverse, note we can do root math since $x_0 > 0$

$$p = 1 - \frac{x_0^{\gamma-1}}{x^{\gamma-1}} \tag{6}$$

$$\frac{x_0^{\gamma-1}}{x^{\gamma-1}} = 1 - p \tag{7}$$

$$\frac{x_0}{x} = (1-p)^{\frac{1}{\gamma-1}} \tag{8}$$

$$x = \frac{x_0}{(1-p)^{\frac{1}{\gamma-1}}} \tag{9}$$

c) Also we can note that we can use $U(0,1)$ without $1-p$ to speed it up since their distribution is exactly the same.
1. Sample from a normal distribution $U(0,1)$
2. substitue value into p for $\frac{x_0}{p^{\frac{1}{\gamma-1}}}$

d)

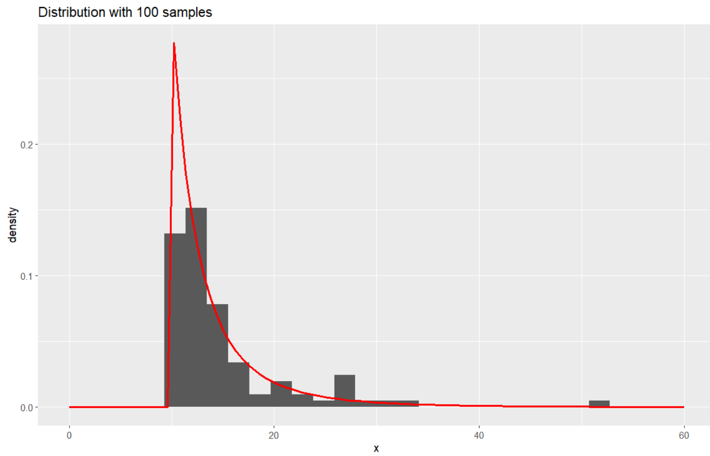

Figure 1: Graph of pdf and random sampling of 100 points, given $\gamma = 4$ and $x_0 = 10$

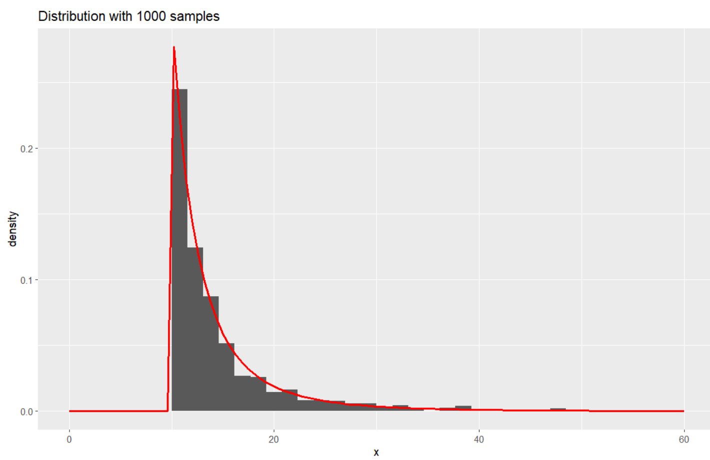

Figure 2: Graph of pdf and random sampling of 1000 points, given $\gamma = 4$ and $x_0 = 10$

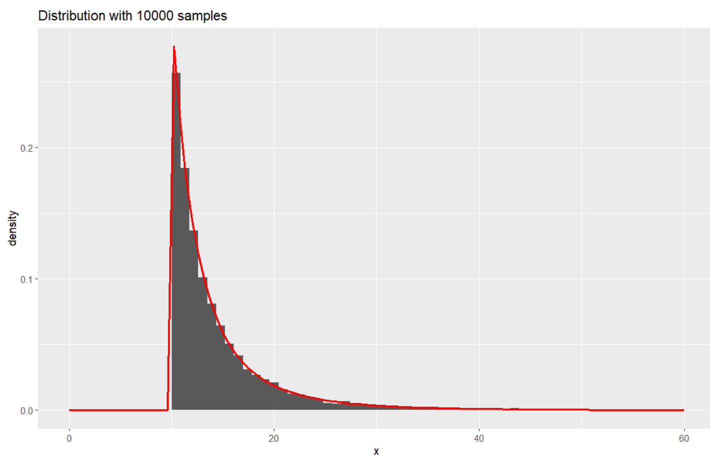

Figure 3: Graph of pdf and random sampling of 10000 points, given $\gamma = 4$ and $x_0 = 10$

2 Gamma Sampling

Suppose we have $X \sim Gamma(2, 1)$ with pdf

$$f(x) = \begin{cases} xe^{-x}, & x > 0 \\ 0, & x \leq 0 \end{cases} \tag{10}$$

a) To calculate the CDF we take the integral of the pdf from 0 to x. Using integration by part with $u = x$ and $dv = e^{-x}$

$$\int_{-\infty}^{x} f(x)dx = \int_{0}^{x} xe^{-x}dx = -xe^{-x} + \int_{0}^{x} e^{-x}dx \tag{11}$$
$$= -e^{-x}(x+1)|_0^x \tag{12}$$
$$= 1 - e^{-x}(x+1) \tag{13}$$

Hence we have

$$F(x) = \begin{cases} 1 - e^{-x}(x+1), & x > 0 \\ 0, & x \leq 0 \end{cases} \tag{14}$$

b I will be using Newtons method to find the inverse of the CDF. The method will guess a root at one and runs 5 times then repeated until the root is positive. There is a sanity check where if the root stays negative an error occurs. Fortunately the code works. The solution to the CDF will be calcualted by solving the root of $F(x) - u = 0$. The function took 5.1287 seconds to run.

c To solve for the optimal value for c, I will use the inequality:

$$C\frac{1}{2}e^{-\frac{x}{2}} \geq xe^{-x} \tag{15}$$
$$C \geq 2xe^{\frac{-x}{2}} = h(x) \tag{16}$$
$$\tag{17}$$

To solve for the critical point of the equation, take the derivative and solve for the roots

$$h'(x) = 2e^{\frac{-x}{2}} - xe^{\frac{-x}{2}} \tag{18}$$
$$= e^{\frac{-x}{2}}(2-x) = 0 \tag{19}$$
$$\tag{20}$$

Since the derivative is a strictly decreasing function and is positive when $x < 2$, the critical value is the point where $h(x)$ is the absolute maximum yielding

$$C \geq h(2) = 4/e \tag{21}$$

Since efficiency is determined by $1/C$ the smallest possible value of C to maximize efficiency is $4/e$.
Also note that the given distribution $g(x)$ is aa pdf for an exponential distribution with $\lambda = 1/2$ Using an acceptace rejection algorithm, it took 3.1873 seconds to complete.
d After running the simulation code, the experiment was completed in 0.0872 seconds.
f The fastest algorithm is summing the exponential distributions, then it is the acceptance-rejection criterion, and the slowest is using the root solver. Note the root solver doesn't even give very accurate results and takes even more time if I want it to be more accurate.
g

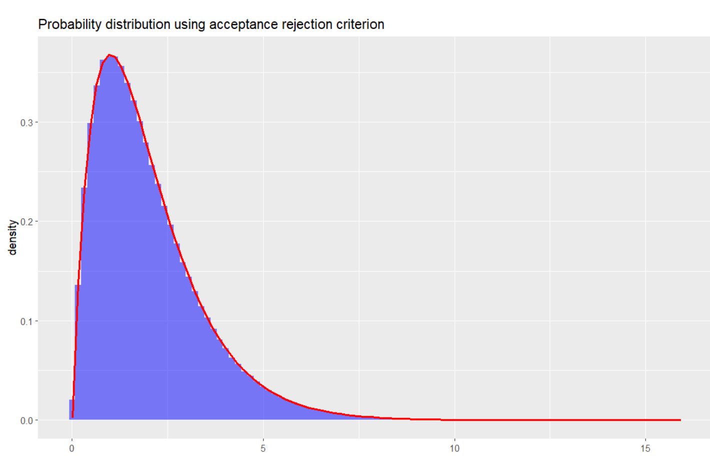

Figure 4: Graph for the pdf of xe^{-x} using acceptance rejection criterion.

3 Mixture Sampling to Polynomial Sampling

a To show that these two distributions are the same, we will be calculating the CDF of Y. Note the main proof follows the law of total probability:

$$Y(y) = P(Y \leq y) = \sum_{j=1}^{N} P(X_J \leq y | J = j) P(J = j) \tag{22}$$

$$= \sum_{j=1}^{N} P(X_j \leq y) P(J = j) \tag{23}$$

$$= \sum_{j=1}^{N} P(X_j \leq y) a_j \tag{24}$$

$$= \sum_{j=1}^{N} \int_0^y f_j(x) dx a_j \tag{25}$$

$$= \sum_{j=1}^{N} \int_0^y f_j(x) a_j dx \tag{26}$$

$$= \int_0^y \sum_{j=1}^{N} a_j f_j(x) dx \tag{27}$$

$$= \int_0^y f(x) dx = F(y) \tag{28}$$

b To solve for the CDF, I will be taking the integral of f_j from 0 to x.

$$\int_0^x f_j(x) dx = \int_0^x (j+1) x^j dx == x^{j+1}|_0^x = x^{j+1} \tag{29}$$

To sample from this distribution I will use inverse sampling. The inverse of the CDF is $x = u^{\frac{1}{j+1}}$. Hence the pseudo code will be.
1. Generate a random value $u \sim U(0,1)$
2. return $u^{\frac{1}{j+1}}$

c I will be finding a way to setup $f(z)$ Using the sum that is given above. Since the highest power of $f(z)$ is 5, $N = 5$ resulting in the following sum:

$$\sum_{j=1}^{5} a_j f_j(z) = a_1 2z + a_2 3z^2 + a_3 4z^3 + a_4 5z^4 + a_5 6z^5 \tag{30}$$

Furthermore $a_1 = \frac{1}{2}, a_5 = \frac{1}{2}$ while all other a_j equal 0. Using the fact from question a due to the law of total probability, we can do conditional sampling.
1. Sample $j \sim U(0,1)$
2. if $j \geq .5$ $j = 1$ else $j = 5$
3. generate $u \sim U(0,1)$
4. return $u^{\frac{1}{j+1}}$

4 Gaussian Sampling

The pdf of a standard normal distribution is shown below.

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \tag{31}$$

Note it is not possible to conver this integral to a CDF. Hence Acceptance-rejection will be used. I will be using the pmf $g(x)$ of an exponential distribution with $\lambda = 1$ since

$$cg(x) \geq f(x) \tag{32}$$

$$ce^{-x} \geq \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \tag{33}$$

$$c\sqrt{2\pi}e^{-x} \geq e^{\frac{-x^2}{2}} \tag{34}$$

$$c\sqrt{2\pi}e^{\frac{x^2-2x}{2}} \geq 1 \tag{35}$$

$$c\sqrt{2\pi}e^{\frac{x(x-2)}{2}} \geq 1 \tag{36}$$

$$\tag{37}$$

Note how there are two critical points for $(x(x-2))$ at $x = 0, 2$. By taking the derivative, there is one critical point at at $x = 1$ and it is the absolute minimum as this is a concave up quadratic curve. Hence the value of x that would minimize the left side of the inequality would be $x = 2$ yielding $\sqrt{2\pi}e^{-1/2} = .6577$. However we can set c to $\sqrt{\frac{e}{2\pi}} = 1.5205$ to satisfy the inequality. Also note the gaussian distribution is symmetrical hence:

1. Choose a random $x \sim Exp(1)$
2. Choose a random $u \sim U(0, 1)$
3. if $u \geq \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} / exp(\frac{x^2-2x-1}{2})$ continue to 4 else go back to 1
4. Choose a random $l \sim U(0, 1)$
5. If $l \geq .5$ return x else return -x