Homework 3

Quan Nguyen

February 2026

1 Acceptance–Rejection with Optimized Proposal

a) We will begin by dividing the two equations resulting in

$$\begin{aligned}
\frac{f(x)}{g_a(x)} &= \frac{x(1+x)e^{-x}}{3a^2xe^{-ax}} && (1)\\
&= \frac{(1+x)e^{(a-1)x}}{3a^2} && (2)\\
&= \frac{(1+x)}{3a^2}\frac{1}{e^{-(a-1)x}} && (3)\\
& && (4)
\end{aligned}$$

Make a note that the rate paramemter must be positive and that x is always positive or 0. Hence $\frac{(1+x)}{3a^2}$ is always positive and $e^{(a-1)x}$ is also positive. If we take the limit of the quotient, it is an undetermined form that can be converted using L'H resulting in

$$\frac{1}{3a^2}\frac{1-a}{e^{-(a-1)x}} \tag{5}$$

In order for the equation to not be unbounded, $a-1<0$ to keep the exponential value in the denominator. Since the equation is continuous and $x=0$ does not diverge the equation must contain an absolute maximum if $0<a<1$.

b)

We will now find some value of a where x is maximized by taking the derivative and solving for 0.

$$\begin{aligned}
\frac{dc}{dx} &= \frac{d}{dx}\frac{(1+x)e^{(a-1)x}}{3a^2} && (6)\\
&= \frac{e^{(a-1)x}}{3a^2} + \frac{(1+x)(a-1)e^{(a-1)x}}{3a^2} && (7)\\
&= \frac{1}{3a^2}(a+ax-x)e^{(a-1)x} && (8)\\
\implies 0 &= (a+ax-x) && (9)\\
0 &= a+x(a-1) && (10)\\
-a &= x(a-1) && (11)\\
x &= \frac{a}{1-a} && (12)
\end{aligned}$$

Note $\lim_{x\to\infty} h_a(x)=0$ and $h_a(0)=\frac{1}{3a^2}>0$. Since there is only one critical point, the function is always positive, and it is continuous, the critical point must be a maximum. To calculate the actual maximum, simply substitute.

$$\begin{aligned}
c(a) &= \frac{(1+\frac{a}{1-a})e^{(a-1)\frac{a}{1-a}}}{3a^2} && (13)\\
&= \frac{\frac{1}{1-a}}{3a^2e^a} && (14)\\
&= \frac{1}{3a^2e^a(1-a)} && (15)\\
& && (16)
\end{aligned}$$

c)

This is an optimization question. Furthermore $1/c(a)$ is the lowest probability of acceptence hence we should look to maximize it with some value a. Simply take the derivative and solve for 0.

$$
\begin{aligned}
\frac{d}{da}3a^2e^a(1-a) &= 6ae^a(1-a)+3a^2e^a(1-a)-3a^2e^a && (17)\\
&= (6a-6a^2+3a^2-3a^3-3a^2)e^a && (18)\\
&= 3(2a-2a^2-a^3)e^a && (19)\\
\implies 0 &= -a^3-2a^2+2a && (20)\\
0 &= -a(a^2+2a-2) && (21)\\
\implies a &= 0, \frac{-2\pm\sqrt{4+8}}{2} && (22)\\
&= 0, -1+\sqrt{3}, -1-\sqrt{3} && (23)\\
& && (24)
\end{aligned}
$$

However since $0 < a < 1$ the only viable value for a is $a = \sqrt{3}-1 \approx .7321$. To make sure this value does maximize $1/c(x)$ we must take the second derivative.

$$
\begin{aligned}
\frac{d}{da}3(2a-2a^2-a^3)e^a|_{a=\sqrt{3}-1} &= 3(2a-2a^2-a^3)e^a+3(2-4a-3a^2)e^a|_{a=\sqrt{3}-1} && (25)\\
&= 3(2-4a-3a^2)e^a|_{a=\sqrt{3}-1} && (26)\\
&= -15.81899 < 0 && (27)
\end{aligned}
$$

Since the second derivative is negative, the point $a = \sqrt{3}-1$ is concave down and is hence the maximum.

d)

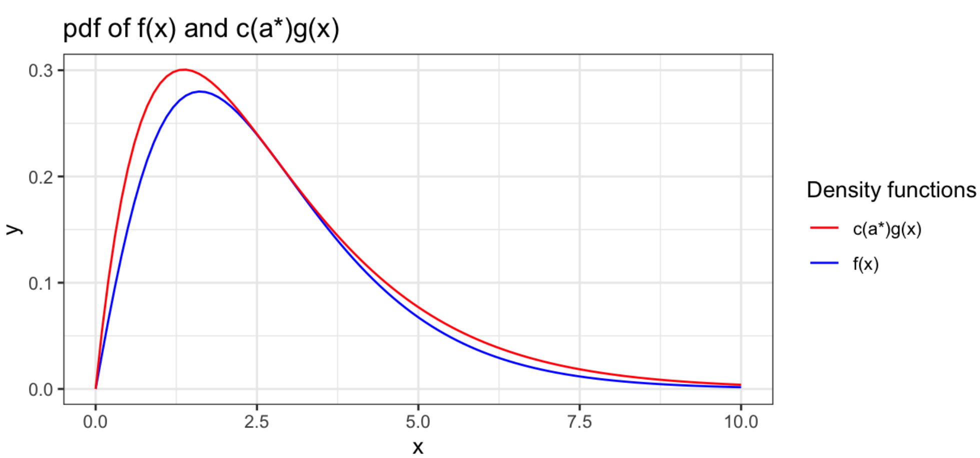

Figure 1: Graph showing the pdf for $f(x)$ and $c(a^*)g_{a^*}(x)$

e) If there is no program that can sample from the exponential distribution, use inverse sampling by calculating the CDF of the exponential distribution:

$$
\begin{aligned}
F(X) &= \int_0^x a^*e^{-a^*x}dx && (28)\\
&= -e^{-a^*x}|_0^x && (29)\\
&= 1-e^{-a^*x} && (30)\\
& && (31)
\end{aligned}
$$

Then we take the inverse function which is

$$x = -\frac{\ln(1-k)}{a^*} \quad (32)$$

To sample from the exponential distribution, sample $k \sim U(0,1)$ and plug it into the inverse function above.
However to calculate $g_{a^*}(x)$. Use inverse sampling method twice to yield x_1 and x_2. Sum those two values to obtain a sample from the gamma distribution $g_{a^*}(x)$

2 Cell Cycles

a)
First we will define states:
1. G is state 1
2. S is state 2
3. D is state 3
Hence we get a probability transition matrix of

$$p = \begin{bmatrix} \frac{9}{10} & \frac{1}{10} & 0 \\ 0 & \frac{7}{8} & \frac{1}{8} \\ \frac{2}{5} & 0 & \frac{3}{5} \end{bmatrix} \tag{33}$$

b) By modeling it with a geometric distribution, we can set p as the probability that the cell leaves to state S which is $p = 1/10$. The expected value for a geometric distribution is $1/p = 10$. Hence the expected time the cell stays in the growing state is 10,