Homework 1

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January 2026

1 Airplane seat overselling

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a

Looking at this question, the binomial distribution represents the theoretical distribution since we want to count the amount of sucesses within a certain amount of samples. However, if we count a passenger missing a flight being a success with probability $p = .02$, with a large sample of $n = 100$ and an expected value of $np = 2$. We can approximate it with a poisson distribution where $\lambda = np = 2$

The probability of not having enough seats occurs when the event of only 0 or 1 passengers misses their flight.

$$P(X \leq 1) = P(X = 0) + P(X = 1) = \frac{2^0 e^{-2}}{0!} + \frac{2^1 e^{-2}}{1!} = .4060 \tag{1}$$

There is a .4169 probability that there will not be enough seats on the plane.

b

Binomial distribution pdf is $\binom{n}{k}(1-p)^{n-k}p^k$ where $n = 100$ and $p = .02$

$$P(X = 0) = \binom{100}{0}(1-p)^{100}p^0 = 1 * .98^{100} = .1326 \tag{2}$$

$$P(X = 1) = \binom{100}{1}(1-p)^{99}p^1 = 100 * .98^{99} * .02 = .2707 \tag{3}$$

$$P(X \leq 1) = P(X = 0) + P(X = 1) = .4033 \tag{4}$$

$$\tag{5}$$

The approximation using the poisson distribution is slightly larger then the binomial distribution but is overall pretty close.

c

This is a conditional probabiliy where given the plane has all its passenger, there are exactly two no shows. The probability of 2 no shows and the plane being full can be modeled with a binomial distribution in equation (6). The probability that every seat is filled occurs when there are two or less no shows which is modeled in equation (7). The conditional probability is calculated in equation (8). The results indicate a probability of .4040 that an airline will not need to reimburse passengers given the plan is full.

$$P(X \geq 2 \cap X \leq 2) = P(X = 2) = \binom{100}{2}(1-p)^{98}p^2 = \frac{100!}{2!98!}.98^{98}.2^2 = .2734 \tag{6}$$

$$P(X \leq 2) = P(X = 2) + P(X \leq 1) = .2734 + .4033 = .6767 \tag{7}$$

$$P(X \geq 2 \cap x \leq 2 | X \leq 2) = \frac{P(X = 2)}{P(X \leq 2)} = \frac{.2734}{.6767} = .4040 \tag{8}$$

2 Joint distributions

$$f(x,y) = \begin{cases} Cxy & 0 \le y \le x, 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

a

We need to find some c where the joint cummulative density equals 1 when the entire sample space is integrated.

$$\begin{align}
1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Cxy dR = 0 + \int_0^1 \int_0^x Cxydydx \tag{10}\\
= \frac{1}{2}\int_0^1 Cx^3dx \tag{11}\\
= \frac{1}{2}\frac{1}{4}cx^4|_0^1 = C\frac{1}{8} \tag{12}\\
\implies c = 8 \tag{13}
\end{align}$$

Hence $C = 8$ for the joint distribution

b

The joint distribution is not independent. By looking at it, when x, is 0, the probability of get 0 for y is 100%, but clearly when x is some other value, the probability of y being 0 is not 1. This can be proven more rigorously by showing the product of the marginal distribution does not equal the joint distribution.

$$\begin{align}
f_x = \int_0^x 8xydy = 4x^3 \tag{14}\\
f_y = \int_y^1 8xydx = 4(y - y^3) \tag{15}\\
f_x * f_y = 8(x^3y - x^3y^3) \ne 8xy \tag{16}
\end{align}$$

c

We will setup a cdf using $P(Z \le z)$

$$\begin{align}
F(z) = P(Z \le z) = P(Y/X \le z) \tag{17}\\
= P(Y \le zX) \tag{18}\\
= \int_0^1 \int_0^{xz} 8xydydx \tag{19}\\
= \int_0^1 4x(xz)^2dx \tag{20}\\
= x^4z^2|_0^1 = z^2 \tag{21}
\end{align}$$

However to convert this to a valid pdf, we need to take the derivative and calculate the bounds. Since $0 \le y \le x$, this implies $0 \le y/x \le 1$ and $0 \le z \le 1$ being the bounds for z. Hence by taking the derivative of (21) and defining the bounds we get a pdf of

$$f(z) = \begin{cases} 2z & 0 \le z \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

3 Numerical evaluation of integrals via Monte Carlo simulation

Consider the following equation

$$I = \int_1^\infty \frac{1}{1+x^6} dx \tag{23}$$

a

I will be using $u = \frac{1}{x}$ and $du = \frac{-1}{x^2} dx$ for the substitution yielding:

$$\int_1^0 \frac{1}{1+\frac{1}{u^6}} * -x^2 du \tag{24}$$

$$\int_0^1 \frac{1}{u^2 + \frac{1}{u^4}} du \tag{25}$$

$$\int_0^1 \frac{u^4}{u^6+1} du \tag{26}$$

$$\tag{27}$$

Note that $u^6 + 1 \geq 1$. This implies the following:

$$\frac{u^4}{u^6+1} \leq u^4 \tag{28}$$

$$\int_0^1 \frac{u^4}{u^6+1} du \int_0^1 \leq u^4 du = \frac{u^5}{5}|_0^1 = \frac{1}{5} \tag{29}$$

Hence the integral cannot be greater then 1. It also cannot be less then 0 since $u^4, u^6 \geq 0$ for any $u \in [0, 1]$.

b

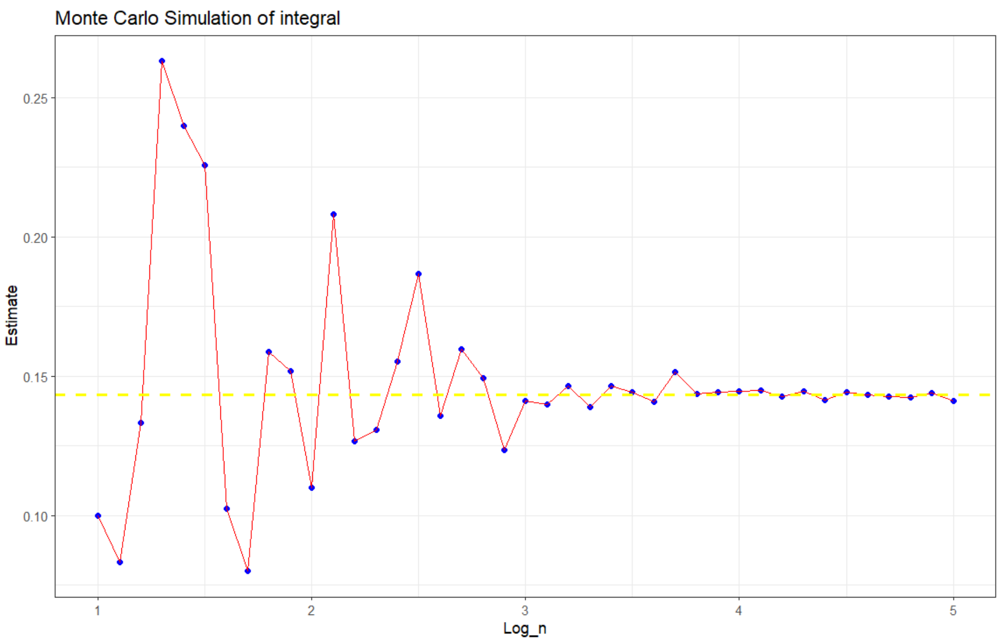

Figure 1: Markov simulation of integral. Note horizontal line equals 0.1434 calculated by https://www.integral-calculator.com/

4 Jill Waiting Time

a

We will define 6pm to be 0 and 7pm to be 1. The pdf of a single friend arriving is

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwiise} \end{cases} \tag{30}$$

Yielding a cdf of:

$$F(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 \leq x \leq 1 \\ 1 & x \geq 1 \end{cases} \tag{31}$$

If we simply choose T as the time that the last friend arrives at, the CDF $F_a(t)$ of everyone arriving at that time or earlier is:

$$F_a(T) = P(X_1 \leq t \cap X_2 \leq t \cap X_3 \leq t) \tag{32}$$

where $T = \max\{X_1, X_2, X_3\}$. However since the events are independent of each other, we can multiply the probabilities:

$$\begin{aligned} F_a(T) &= P(X_1 \leq t)P(X_2 \leq t)P(X_1 \leq t) & (33) \\ &= F(t)F(t)F(t) & (34) \\ &= t^3 & (35) \end{aligned}$$

Taking the derivative and setting the supports to yield the pdf of all friends arriving:

$$f_a(t) = \begin{cases} 3t^2 & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{36}$$

b

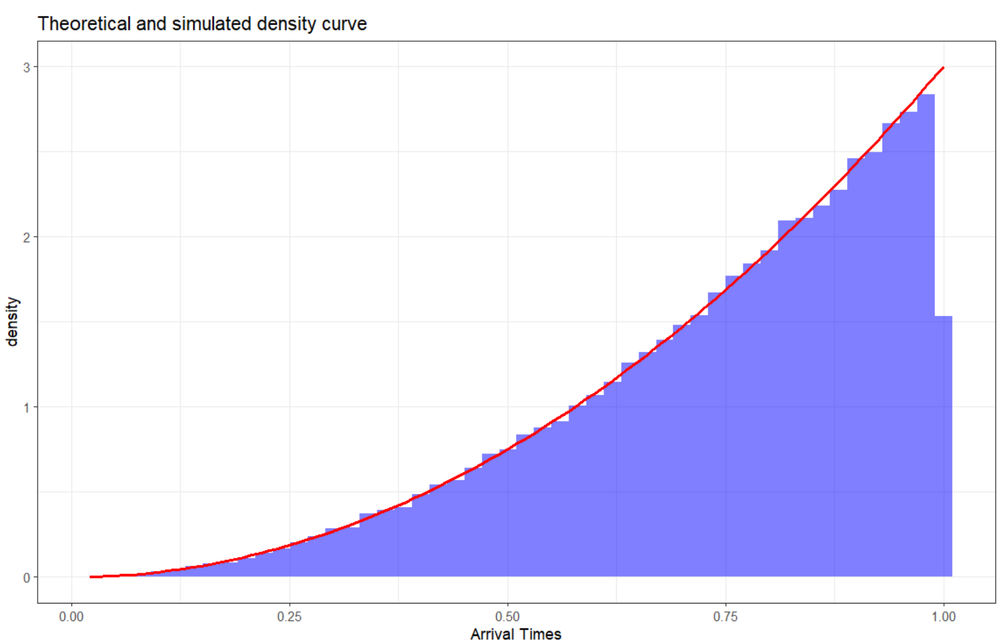

Figure 2: Simulated arrival times by sampling three uniform distribution and choosing the largest value 10^5 times. Red line represents theoretical pdf of $3x^2$

5 Moment Generating Function

a

To approach this we can start with the moment generating function of Y given some N. Note we can multiply the moment generating function as each event is independent of each other.

$$\Phi_{Y|N}(s) = \Pi_{i=1}^{N}\Phi_X(s) = (\Phi_X(s))^N \tag{37}$$

However by using the tower property and equation 38.

$$\begin{align}
(\Phi_X(s))^N &= E[e^{sY}|N] \tag{38}\\
\Phi_Y(s) = E[e^{sY}] &= E[E[e^{sY}|N]] \tag{39}\\
&= E[(\Phi_X(s))^N] \tag{40}\\
&= \sum_{N=0}^{\infty}(\Phi_X(s))^N \frac{\lambda^N e^{-\lambda}}{N!} \tag{41}\\
&= e^{-\lambda}\sum_{N=0}^{\infty}\frac{(\Phi_X(s)\lambda)^N}{N!} \tag{42}\\
&= e^{-\lambda}e^{\Phi_X(s)\lambda} \tag{43}\\
&= e^{\lambda(\Phi_X(s)-1)} \tag{44}
\end{align}$$

b

Calculate the first moment for expected value, note $\Phi_X(0) = E[e^{0X}] = E[1] = 1$

$$\begin{align}
E[Y] &= \frac{d}{ds}\Phi_Y(0) \tag{45}\\
&= \lambda\Phi'_X(s)e^{\lambda(\Phi_X(s)-1)}|_{s=0} \tag{46}\\
&= \lambda E[X] \tag{47}
\end{align}$$

As a lemma, the second moment at 0 is calculated below

$$\begin{align}
E[Y^2] &= \frac{d^2}{ds^2}\Phi_Y(0) \tag{48}\\
&= \frac{d}{ds}\lambda\Phi'_X(s)e^{\lambda(\Phi_X(s)-1)}|_{s=0} \tag{49}\\
&= \lambda\Phi''_X(s)e^{\lambda(\Phi_X(s)-1)} + (\lambda\Phi'_X(s))^2e^{\lambda(\Phi_X(s)-1)}|_{s=0} \tag{50}\\
&= \lambda E[X^2] + (\lambda E[X])^2 \tag{51}
\end{align}$$

To calculate the variance of Y:

$$\begin{align}
Var[Y] &= E[Y^2] - (E[Y])^2 \tag{52}\\
&= \lambda E[X^2] + (\lambda E[X])^2 - (\lambda E[X])^2 \tag{53}\\
&= \lambda E[X^2] \tag{54}
\end{align}$$

6 Finding PDFs

By attempting to setup the equation in the form of $F_Z(z) = P(Z \leq z) = P(Y/X \leq z) = P(Y \leq Xz)$, solve for the CDF of Y.

$$F_y(y) = \int_{-\infty}^{y} f_y(y)dy = \int_{0}^{y} \lambda_y e^{-\lambda_y y} dy \tag{55}$$
$$= -e^{-\lambda_y y}|_0^y \tag{56}$$
$$= 1 - e^{-\lambda_y y} \tag{57}$$

Since the probability of Y depends on the probability of X. The law of total probability can be used as X can be split into partitions that span the entire sample space. The bounds are those where the pdf of x is not 0.

$$P(Y \leq Xz) = \int_{0}^{\infty} P(y \leq zx|x) f_x(x)dx \tag{58}$$
$$= \int_{0}^{\infty} [1 - e^{-\lambda_y xz}][\lambda_x e^{-\lambda_x x}]dx \tag{59}$$
$$= \lambda_x \int_{0}^{\infty} e^{-\lambda_x x} - e^{-x(\lambda_x + \lambda_y z)} dx \tag{60}$$
$$= -e^{-\lambda_x x} + \frac{\lambda_x e^{-x(\lambda_x + \lambda_y z)}}{\lambda_x + \lambda_y z}|_0^{\infty} \tag{61}$$
$$= 1 - \frac{\lambda_x}{\lambda_x + \lambda_y z} \tag{62}$$
$$= \frac{\lambda_y z}{\lambda_x + \lambda_y z} \tag{63}$$

The bounds of z are determined from $0 \leq \frac{Y}{X} \leq z$ as Y and X are positive.

$$F_Z(z) = \begin{cases} \frac{\lambda_y z}{\lambda_x + \lambda_y z} & z \geq 0 \\ 0 & z < 0 \end{cases} \tag{64}$$

To calculate the pdf take the derivative with respect to z.

$$\frac{dF_Z(z)}{dz} = \frac{\lambda_y(\lambda_x + \lambda_y z) - \lambda_y^2 z}{(\lambda_x + \lambda_y z)^2} \tag{65}$$
$$= \frac{\lambda_y \lambda_x}{(\lambda_x + \lambda_y z)^2} \tag{66}$$

This results in a pdf of

$$f_z(z) = \begin{cases} \frac{\lambda_y \lambda_x}{(\lambda_x + \lambda_y z)^2} & z \geq 0 \\ 0 & z < 0 \end{cases} \tag{67}$$

7 Central Limit Theorem - Convergence

$$Y_n = \frac{\sum_{i=1}^{n} X_i - n}{\sqrt{n}} \tag{68}$$

a

To calculate the mean of Y_n we take the expected value. Note $E[X_i] = \lambda = 1$

$$\begin{aligned}
\mu_{Y_n} = E[Y_n] &= E\left[\frac{\sum_{i=1}^{n} X_i - n}{\sqrt{n}}\right] &(69)\\
&= \frac{1}{\sqrt{n}} E\left[\sum_{i=1}^{n} X_i - n\right] &(70)\\
&= \frac{1}{\sqrt{n}} \left(E\left[\sum_{i=1}^{n} X_i\right] - n\right) &(71)\\
&= \frac{1}{\sqrt{n}} \left(\sum_{i=1}^{n} E\left[X_i\right] - n\right) &(72)\\
&= \frac{n\lambda - n^2}{\sqrt{n}} &(73)\\
&= 0 &(74)
\end{aligned}$$

To calculate the variance we use $Var[A + B] = Var[A] + Var[B]2Covar[A, B]$. However, the covariance of independent random variables is 0. Keep in mind $Var[X_i] = \lambda$. Hence:

$$\begin{aligned}
Var[Y_n] &= Var\left[\frac{\sum_{i=1}^{n} X_i - n}{\sqrt{n}}\right] &(75)\\
&= \frac{Var\left[\sum_{i=1}^{n} X_i\right]}{n} &(76)\\
&= \frac{\sum_{i=1}^{n} Var\left[X_i\right]}{n} &(77)\\
&= \frac{\sum_{i=1}^{n} \lambda}{n} &(78)\\
&= \frac{n\lambda}{n} &(79)\\
&= 1 &(80)
\end{aligned}$$

b

We will setup the moment generating function:

$$\begin{aligned}
\Phi_{Y_n}(s) &= E\left[\exp(s\frac{\sum_{i=1}^{n} X_i - n}{\sqrt{n}})\right] &(81)\\
&= \exp(\frac{-sn}{\sqrt{n}})E\left[\exp(\frac{s}{\sqrt{n}}\sum_{i=1}^{n} X_i)\right] &(82)\\
&= \exp(-s\sqrt{n})E\left[\prod_{i=1}^{n} \exp(\frac{s}{\sqrt{n}} X_i)\right] &(83)\\
& &(84)
\end{aligned}$$

However since each event X_i is independent of each other we can rewrite this as

$$\exp(-s\sqrt{n})E\left[\prod_{i=1}^{n} \exp(\frac{s}{\sqrt{n}} X_i)\right] = \exp(-s\sqrt{n})\prod_{i=1}^{n} E\left[\exp(\frac{s}{\sqrt{n}} X_i)\right] \tag{85}$$

Calculating one of the expected values for some i:

$$E\left[\exp(\frac{s}{\sqrt{n}}X_i)\right] = \sum_{x_i=0}^{\infty} \exp(\frac{s}{\sqrt{n}}x_i)\frac{\lambda^{x_i}e^{-\lambda}}{x_i!} \tag{86}$$

$$= \sum_{x_i=0}^{\infty} \frac{e^{\frac{sx_i}{\sqrt{n}}}}{ex_i!} \tag{87}$$

$$= e^{-1}\sum_{x_i=0}^{\infty} \frac{\left(e^{\frac{s}{\sqrt{n}}}\right)^{x_i}}{x_i!} \tag{88}$$

$$= e^{-1}\exp(e^{\frac{s}{\sqrt{n}}}) \tag{89}$$

$$= \exp(e^{\frac{s}{\sqrt{n}}} - 1) \tag{90}$$

Substituting everything back into the original equation for the moment:

$$\Phi_{Y_n}(s) = \exp(-s\sqrt{n})\prod_{i=1}^{n}\exp(e^{\frac{s}{\sqrt{n}}} - 1) = \exp(-s\sqrt{n})\exp(n(e^{\frac{s}{\sqrt{n}}} - 1)) \tag{91}$$

$$= \exp(ne^{\frac{s}{\sqrt{n}}} - n - s\sqrt{n}) \tag{92}$$

c

To prove that the limit of $\Phi_{Y_n}(s) \to e^{s^2/2}$ as $n \to \infty$, we will be expanding $\ln(\Phi_{y_n}(s))$ in powers of $s/\sqrt{n}$ using the taylor series specifically for $e^{\frac{s}{\sqrt{n}}}$

$$\ln(\Phi_{y_n}(s)) = ne^{\frac{s}{\sqrt{n}}} - n - s\sqrt{n} \tag{93}$$

$$= n\left(1 + s/\sqrt{n} + \frac{(s/\sqrt{n})^2}{2!} + \frac{(s/\sqrt{n})^3}{3!} + \frac{(s/\sqrt{n})^4}{4!} + ...\right) - n - s\sqrt{n} \tag{94}$$

$$= n\left(\frac{(s/\sqrt{n})^2}{2!} + \frac{(s/\sqrt{n})^3}{3!} + \frac{(s/\sqrt{n})^4}{4!} + ...\right) \tag{95}$$

$$= \frac{s^2}{2!} + (\frac{s^3}{n^{\frac{1}{2}}3!} + \frac{s^4}{n4!} + ...) \tag{96}$$

$$= \frac{s^2}{2!} + \sum_{i=3}^{\infty}\frac{s^i}{n^{\frac{i-2}{2}}i!} \tag{97}$$

Note since $\lim_{n\to\infty} n^{-a} = 0$ for any $a > 0$

$$lim_{n\to\infty}ln(\Phi_{Y_n}(s)) = \frac{s^2}{2!} + \sum_{i=3}^{\infty} 0 \tag{98}$$

$$= \frac{s^2}{2!} \tag{99}$$

$$\implies lim_{n\to\infty}\Phi_{Y_n}(s) = e^{\frac{s^2}{2}} \tag{100}$$

Note that this is the same moment generating function as the standard gaussian distribution.

d

Expanding the polynomial yields:

$$\Phi_{Y_n}(s) = 1 + \frac{s^2}{2!} + \frac{s^3}{3!\sqrt{n}} + Os^4 \tag{101}$$

By looking at the coefficient

$$E\left[Y_n\right] = 0 \tag{102}$$

$$E\left[Y_n^2\right] = 1 \tag{103}$$

$$E\left[Y_n^3\right] = \frac{1}{\sqrt{n}} \tag{104}$$

$$\tag{105}$$

Now to calculate skewdness:

$$
\begin{aligned}
\frac{E\left[(Y_n-\mu)^3\right]}{\sigma^3} &= \frac{E\left[Y_n^3-3Y_n^2\mu+3Y_n\mu^2-\mu^3\right]}{\sigma^3} && (106)\\
&= \frac{E\left[Y_n^3\right]-3\mu E\left[Y_n^2\right]+3\mu^2 E\left[Y_n\right]-\mu^3}{\sigma^3} && (107)\\
&= \frac{\frac{1}{\sqrt{n}}-3\mu-\mu^3}{\sigma^3} && (108)\\
& && (109)
\end{aligned}
$$

Substituting in $\mu = 0$ and $\sigma = 1$ from (a)

$$S(n) = \frac{1}{\sqrt{n}} \tag{110}$$

Note this is a monotone decreasing function, hence as n increases, the skewdness of the normal curve derived from the mean value theorem decreases over time to 0, since the limit of $S(n)$ as n approaches infinity is 0.