

Homework 1

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See <https://github.com/nguyencquan/Markov2026> for code used in simulations.

1 Power Law Sampling

Assume we have some distribution

$$\begin{cases} Cx^{-\gamma} & x \geq x_0, \\ 0, & x < x_0 \end{cases} \quad (1)$$

a) Where $x_0 > 0$ and $\gamma > 3$

A valid pdf must have a total probability of 1, which is calculated below:

$$1 = \int_{-\infty}^{\infty} Cx^{-\gamma} dx = C \int_{x_0}^{\infty} x^{-\gamma} dx = \frac{C}{x^{\gamma-1}(1-\gamma)}|_{x_0}^{\infty} = \frac{C}{x_0^{\gamma-1}(\gamma-1)} \quad (2)$$

$$C = x_0^{\gamma-1}(\gamma-1) \quad (3)$$

b) First calculating the cumulative distribution we take a step from equation 2 except take the integral from x_0 to x instead.

$$1 = \int_{-\infty}^{\infty} Cx^{-\gamma} dx = C \int_{x_0}^x x^{-\gamma} dx = \frac{C}{x^{\gamma-1}(1-\gamma)}|_{x_0}^x = \frac{C}{x_0^{\gamma-1}(\gamma-1)} + \frac{C}{x^{\gamma-1}(1-\gamma)} \quad (4)$$

Substituting in C

$$F(x) = \begin{cases} 1 - \frac{x_0^{\gamma-1}}{x^{\gamma-1}}, & x \geq x_0 \\ 0, & x < x_0 \end{cases} \quad (5)$$

To calculate the inverse, note we can do root math since $x_0 > 0$

$$p = 1 - \frac{x_0^{\gamma-1}}{x^{\gamma-1}} \quad (6)$$

$$\frac{x_0^{\gamma-1}}{x^{\gamma-1}} = 1 - p \quad (7)$$

$$\frac{x_0}{x} = (1-p)^{\frac{1}{\gamma-1}} \quad (8)$$

$$x = \frac{x_0}{(1-p)^{\frac{1}{\gamma-1}}} \quad (9)$$

c) Also we can note that we can use $U(0, 1)$ without $1 - p$ to speed it up since their distribution is exactly the same.

1. Sample from a normal distribution $U(0, 1)$

2. substitue value into p for $\frac{x_0}{p^{\frac{1}{\gamma-1}}}$

d)

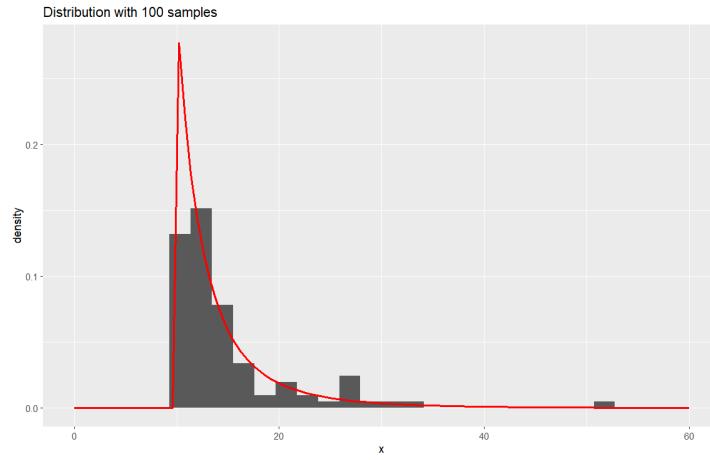


Figure 1: Graph of pdf and random sampling of 100 points, given $\gamma = 4$ and $x_0 = 10$

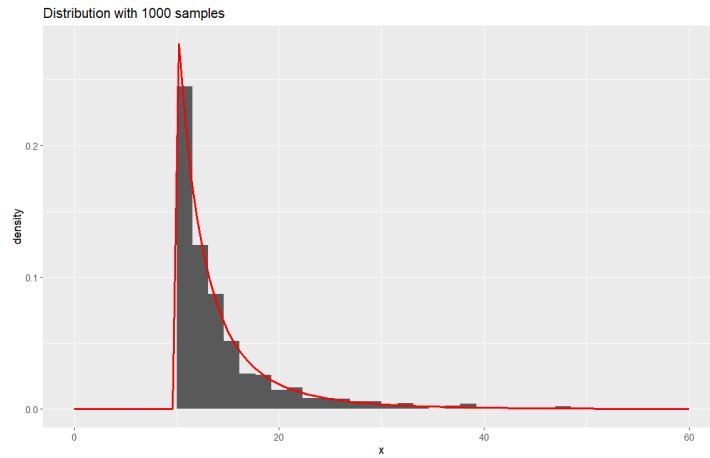


Figure 2: Graph of pdf and random sampling of 1000 points, given $\gamma = 4$ and $x_0 = 10$

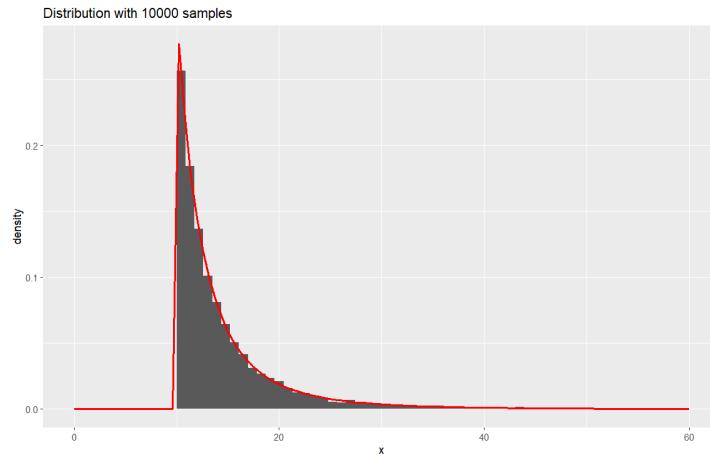


Figure 3: Graph of pdf and random sampling of 10000 points, given $\gamma = 4$ and $x_0 = 10$

2 Gamma Sampling

Suppose we have $X \sim \text{Gamma}(2, 1)$ with pdf

$$f(x) = \begin{cases} xe^{-x}, & x > 0 \\ 0, & x \leq 0 \end{cases} \quad (10)$$

a) To calculate the CDF we take the integral of the pdf from 0 to x. Using integration by part with $u = x$ and $dv = e^{-x}$

$$\int_{-\infty}^x f(x)dx = \int_0^x xe^{-x}dx = -xe^{-x} + \int_0^x e^{-x}dx \quad (11)$$

$$= -e^{-x}(x+1)|_0^x \quad (12)$$

$$= 1 - e^{-x}(x+1) \quad (13)$$

Hence we have

$$F(x) = \begin{cases} 1 - e^{-x}(x+1), & x > 0 \\ 0, & x \leq 0 \end{cases} \quad (14)$$

b I will be using Newtons method to find the inverse of the CDF. The method will guess a root at one and runs 5 times then repeated until the root is positive. There is a sanity check where if the root stays negative an error occurs. Fortunately the code works. The solution to the CDF will be calcualted by solving the root of $F(x) - u = 0$. The function took 5.1287 seconds to run.

c To solve for the optimal value for c, I will use the inequality:

$$C \frac{1}{2} e^{-\frac{x}{2}} \geq xe^{-x} \quad (15)$$

$$C \geq 2xe^{-\frac{x}{2}} = h(x) \quad (16)$$

$$(17)$$

To solve for the critical point of the equation, take the derivative and solve for the roots

$$h'(x) = 2e^{-\frac{x}{2}} - xe^{-\frac{x}{2}} \quad (18)$$

$$= e^{-\frac{x}{2}}(2-x) = 0 \quad (19)$$

$$(20)$$

Since the derivative is a strictly decreasing function and is positive when $x < 2$, the critical value is the point where $h(x)$ is the absolute maximum yielding

$$C \geq h(2) = 4/e \quad (21)$$

Since efficiency is determined by $1/C$ the smallest possible value of C to maximize efficiency is $4/e$.

Also note that the given distribution $g(x)$ is aa pdf for an exponential distribution with $\lambda = 1/2$ Using an acceptace rejection algorithm, it took 3.1873 seconds to complete.

d After running the simulation code, the experiment was completed in 0.0872 seconds.

f The fastest algorithm is summing the exponential distributions, then it is the acceptance-rejection criterion, and the slowest is using the root solver. Note the root solver doesn't even give very accurate results and takes even more time if I want it to be more accurate.

g

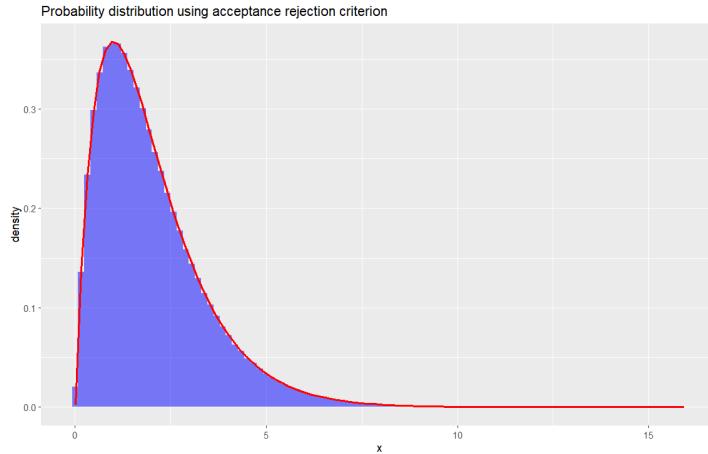


Figure 4: Graph for the pdf of xe^{-x} using acceptance rejection criterion.

3 Mixture Sampling to Polynomial Sampling

a To show that these two distributions are the same, we will be calculating the CDF of Y. Note the main proof follows the law of total probability:

$$Y(y) = P(Y \leq y) = \sum_{j=1}^N P(X_J \leq y | J = j)P(J = j) \quad (22)$$

$$= \sum_{j=1}^N P(X_j \leq y)P(J = j) \quad (23)$$

$$= \sum_{j=1}^N P(X_j \leq y)a_j \quad (24)$$

$$= \sum_{j=1}^N \int_0^y f_j(x)dx a_j \quad (25)$$

$$= \sum_{j=1}^N \int_0^y f_j(x)a_j dx \quad (26)$$

$$= \int_0^y \sum_{j=1}^N a_j f_j(x) dx \quad (27)$$

$$= \int_0^y f(x) dx = F(y) \quad (28)$$

b To solve for the CDF, I will be taking the integral of f_j from 0 to x.

$$\int_0^x f_j(x) dx = \int_0^x (j+1)x^j dx = x^{j+1}|_0^x = x^{j+1} \quad (29)$$

To sample from this distribution I will use inverse sampling. The inverse of the CDF is $x = u^{\frac{1}{j+1}}$. Hence the pseudo code will be.

1. Generate a random value $u \sim U(0, 1)$

2. return $u^{\frac{1}{j+1}}$

c I will be finding a way to setup $f(z)$ Using the sum that is given above. Since the highest power of $f(z)$ is 5, $N = 5$ resulting in the following sum:

$$\sum_{j=1}^5 a_j f_j(z) = a_1 2z + a_2 3z^2 + a_3 4z^3 + a_4 5z^4 + a_5 6z^5 \quad (30)$$

Furthermore $a_1 = \frac{1}{2}$, $a_5 = \frac{1}{2}$ while all other a_j equal 0. Using the fact from question a due to the law of total probability, we can do conditional sampling.

1. Sample $j \sim U(0, 1)$

2. if $j \geq .5$ $j = 1$ else $j = 5$

3. generate $u \sim U(0, 1)$

4. return $u^{\frac{1}{j+1}}$