Homework 4

Quan Nguyen

February 2026

Homework code located at: https://github.com/nguyencquan/Markov2026

1 Gambler's Ruin with Retirement

a) Find the probability that they retire before losing all their money.
Let a_i be the probability retiring before going broke, with i money then the recurrence:

$$a_i = pa_{i+1} + qa_{i-1} + s$$

Solving the homogenous equation with the guess $a_i = r^i$ we get

$$r^i = pr^{i+1} + qr^{i-1}$$

$$r = pr^2 + q$$

$$0 = pr^2 - r + q$$

This is a quadratic equation with the solution:

$$r = \frac{1 \pm \sqrt{1-4pq}}{2p}$$

Resulting in the general equation:

$$a_i = c_1(\frac{1+\sqrt{1-4pq}}{2p})^i + c_2(\frac{1-\sqrt{1-4pq}}{2p})^i$$

To solve for the particular solution guess a constant solution being A:

$$A = pA + qA + s$$

$$A(1-p-q) = s$$

$$\implies A = \frac{s}{1-p-q} = 1$$

Hence we get the full solution:

$$a_i = c_1(\frac{1+\sqrt{1-4pq}}{2p})^i + c_2(\frac{1-\sqrt{1-4pq}}{2p})^i + 1$$

Dealing with boundary conditions, $a_0 = 0$ and $\lim_{k\to\infty} a_k = 1$(since if you have an infinite amount of money, the probability of never quitting is 0)

$$a_0 = 0 = c_1 + c_2 + 1$$

For the infinity boundary condition, we need to look to see if there are any terms, that would not converge to 0. Suppose the characteristic equation $f(r) = pr^2 - r + q$ then $f(0) = q > 0$ and $f(1) = p - 1 + q < 0$ as $p + q + s = 1$ and $p, q, s > 0$. Since $p > 0$ this equation is concave up so the equation is decreasing between $r = 0$ and $r = 1$, the first critical value must occur between $r = 0$ and $r = 1$ while the other one is greater then $r = 1$ hence we choose the smaller critical value being $\frac{1-\sqrt{1-4pq}}{2p}$. Factoring these two boundary conditions we get:

$$a_i = -(\frac{1-\sqrt{1-4pq}}{2p})^i + 1$$

c)
Running the simulation, I got 9.79092 being the expected payout.

2 Greedy Management

a) We can model each cycle being a binomial distribution if we treat each machine breaking as a Bernouli distribution hence the entire cycle is a binomial distribution $Bin(k, n, p)$. Each row being being n or number of machines X_i and each column being k or number of machine at X_{i+1}. We will define $p = 9/10$ or the probability that a machine does not break.
Note the rows and columns are numbered from 0 to 5 and indicates the amount of functional machines.

$$p_{ij} = \begin{cases} 1, & i = 0, j = 5 \\ \binom{i}{j}(9/10)^j(1/10)^{i-j}, & 0 < i \leq 5 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ Bin(0,1,9/10) & Bin(1,1,9/10) & 0 & 0 & 0 & 0 \\ Bin(0,2,9/10) & Bin(1,2,9/10) & Bin(2,2,9/10) & 0 & 0 & 0 \\ Bin(0,3,9/10) & Bin(1,3,9/10) & Bin(2,3,9/10) & Bin(3,3,9/10) & 0 & 0 \\ Bin(0,4,9/10) & Bin(1,4,9/10) & Bin(2,4,9/10) & Bin(3,4,9/10) & Bin(4,4,9/10) & 0 \\ Bin(0,5,9/10) & Bin(1,5,9/10) & Bin(2,5,9/10) & Bin(3,5,9/10) & Bin(4,5,9/10) & Bin(5,5,9/10) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ .1 & 0.90000 & 0 & 0 & 0 & 0 \\ .01 & .18000 & .8100 & 0 & 0 & 0 \\ .001 & .02700 & .2430 & .7290 & 0 & 0 \\ .0001 & .00360 & .0486 & .2916 & .65610 & 0 \\ .00001 & .00045 & .0081 & .0729 & .32805 & .59049 \end{bmatrix}$$

b)
To see when the machine will fail, we will do one state conditioning for when it reaches state 0. The modified transition probability matrix is p'

$$e_i = 1 + \sum_j e_j p_{ij}$$

We will also say $e_0 = 0$ as this is our end state, and solve using code by rearanging the bottom to:

$$(I - p')e = [1, 1, 1, 1, 1]^T$$

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0.90000 & 0 & 0 & 0 & 0 \\ .18000 & .8100 & 0 & 0 & 0 \\ .02700 & .2430 & .7290 & 0 & 0 \\ .00360 & .0486 & .2916 & .65610 & 0 \\ .00045 & .0081 & .0729 & .32805 & .59049 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{bmatrix}$$

Based on the code result, it will break in approximately 22.17162 weeks.
c)
We are essentially finding a stationary distribution. If it is stationary then

$$\vec{\pi} = \vec{\pi} p$$

Furthermore we will say that $\pi_0 = 1$
Resulting in the following equation

$$[\pi_1, \pi_2, \pi_3, \pi_4, \pi_5] = [\pi_1, \pi_2, \pi_3, \pi_4, \pi_5] \begin{bmatrix} 0.90000 & 0 & 0 & 0 & 0 \\ .18000 & .8100 & 0 & 0 & 0 \\ .02700 & .2430 & .7290 & 0 & 0 \\ .00360 & .0486 & .2916 & .65610 & 0 \\ .00045 & .0081 & .0729 & .32805 & .59049 \end{bmatrix} + [0, 0, 0, 0, 1]$$

Rewrite this equation as:

$$\pi' = \pi' p' + [0, 0, 0, 0, c]$$
$$\pi'^T = p'^T \pi'^T + [0, 0, 0, 0, 1]^T$$
$$(I - p'^T)\pi'^T = [0, 0, 0, 0, 1]^T$$

Solving this equation using R where $c = 1$ arbitrarily, we get:

$$\pi' = [9.491222, 4.745705, 3.163356, 2.329396, 2.441943]$$

Subbing back in the 1 for $\pi_0 = 1$ we get

$$\pi = c[1, 9.491222, 4.745705, 3.163356, 2.329396, 2.441943]$$

Note there is a c since we need to normalize the data so the sum equals 1, so we define c as 1 divided by the sum of all the terms in the matrix above resulting in:

$$\pi = [0.04315624, 0.40960542, 0.20480677, 0.13651855, 0.10052797, 0.10538506]$$

Hence at a given time as time approaches infinity there is a .41 chance that there will be one machine working.

3 Convergence to the stationary distribution

a)

$$[q_n(1)(1-a) + q_n(2)b, q_n(1)(a) + q_n(2)(1-b)] = [q_n(1), q_n(2)]p$$

$$\implies q_{n+1}(1) = q_n(1)(1-a) + q_n(2)b$$

$$\implies q_{n+1}(2) = q_n(1)(a) + q_n(2)(1-b)$$

b)
Since we are working with the stationary distribution:
Also some simplification was done from line 8 to line 9 since we are working with a stationary distribution Since: $\vec{\pi} = \vec{\pi}p$

$$\begin{aligned}
[\pi_1 + x_{n+1}, \pi_2 + y_{n+1}] &= [\pi_1 + x_n, \pi_2 + y_n]p && (1)\\
&= [(\pi_1 + x_n)(1-a) + (\pi_2 + y_n)b, (\pi_1 + x_n)a + (\pi_2 + y_n)(1-b)] && (2)\\
&= [x_n(1-a) + y_nb + [(1-a)\pi_1 + b\pi_2], x_na + y_n(1-b) + [a\pi_1 + (1-b)\pi_2]] && (3)\\
&= [x_n(1-a) + y_nb + \pi_1, x_na + y_n(1-b) + \pi_2] && (4)\\
\implies [x_{n+1}, y_{n+1}] &= [(1-a)x_n + by_n, ax_n + (1-b)y_n] && (5)
\end{aligned}$$

c)
Using some properties of stationary distributions:

$$q_n(1) + q_n(2) = \pi_1 + \pi_2 + x_n + y_n = 1$$

This implies that $x_n = -y_n$ as $q_n(1) + q_n(2) = \pi_1 + \pi_2 = 1$ since the probability of being in 1 or 2 at n is 1 and the the sum of all stationary distribution is 1 hence we get the equations:

$$[x_{n+1}, y_{n+1}] = [(1-a-b)x_n, (1-b-a)y_n]$$

Which can be rewritten as

$$x_{n+1} = (1-a-b)^n x_0$$

$$y_{n+1} = (1-b-a)^n y_0$$

We also know that $-2 < -a-b < 0$. Hence $-1 < 1-a-b < 1$ which means $(1-a-b)^n$ approaches 0 exponentially implying x_n and y_n approaches 0 exponentially over each iteration.

4 Ring

a)
We know that $b+c=a+d+c=1$ Hence we can also say $b=a+d=1-c$, but more importantly $1-d=a+c$
b)
Since the even nodes behave the same and the odd nodes also behave the same, we can describe the behavior with π_{even} and π_{odd}. Note we will be doing substitution so everything is in terms of d. Resulting in the equation:

$$[\pi_e, \pi_o] = [\pi_e, \pi_o]\begin{bmatrix} d & 1-d \\ 1 & 0 \end{bmatrix} = [d\pi_e + \pi_o, (1-d)\pi_e]$$

Now let us let $\pi_o = c$ then

$$\begin{aligned} \pi_e &= d\pi_e + c \\ \pi_o &= c \\ \implies [\pi_e, \pi_o] &= c[\frac{1}{1-d}, 1] \end{aligned}$$

Since the total probability has to add up to 1 $c = \frac{1}{\frac{1-d+1}{1-d}} = \frac{1-d}{2-d}$
Resulting in the final equation:

$$[\pi_e, \pi_o] = [\frac{1}{2-d}, \frac{1-d}{2-d}]$$

If we want to calculate the probability of each number then:

$$\pi_i = \begin{cases} \frac{1}{5(2-d)}, & \text{i is even} \\ \frac{1-d}{5(2-d)}, & \text{i is odd} \end{cases}$$

5 Matrix equations for hitting time.

a)
If we want hitting time, we will add 1 as long as it is not in the final state using the equation:

$$q_i = 1 + \sum_j q_j p_{ij}$$

However

$$q_N = 0$$

Define $\hat{p}$ as p except we remove the $N'th$ row and column and $\vec{q}$ as the column vector $[q_1, q_2, q_3, ..., q_{N-1}]^T$
Then we get the matrix equation:

$$\vec{q} = 1_{N-1} + \hat{p}\vec{q}$$

b)

$$\begin{aligned} (I-\hat{p})q &= 1_{N-1} \\ q &= (I-\hat{p})^{-1}1_{N-1} \end{aligned}$$