Homework 1

Quan Nguyen

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See https://github.com/nguyencquan/Markov2026 for code used in simulations.

1 Power Law Sampling

Assume we have some distribution

$$\begin{cases} Cx^{-\gamma} & x \geq x_0, \\ 0, & x < x_0 \end{cases} \tag{1}$$

a) Where $x_0 > 0$ and $\gamma > 3$
A valid pdf must have a total probability of 1, which is calculated below:

$$1 = \int_{-\infty}^{\infty} Cx^{-\gamma} dx = C \int_{x_0}^{\infty} x^{-\gamma} dx = \frac{C}{x^{\gamma-1}(1-\gamma)}|_{x_0}^{\infty} = \frac{C}{x_0^{\gamma-1}(\gamma-1)} \tag{2}$$

$$C = x_0^{\gamma-1}(\gamma-1) \tag{3}$$

b) First calculating the cummulative distribution we take a step from equation 2 except take the integral from x_0 to x instead.

$$1 = \int_{-\infty}^{\infty} Cx^{-\gamma} dx = C \int_{x_0}^{x} x^{-\gamma} dx = \frac{C}{x^{\gamma-1}(1-\gamma)}|_{x_0}^{x} = \frac{C}{x_0^{\gamma-1}(\gamma-1)} + \frac{C}{x^{\gamma-1}(1-\gamma)} \tag{4}$$

Substituting in C

$$F(x) = \begin{cases} 1 - \frac{x_0^{\gamma-1}}{x^{\gamma-1}}, & x \geq x_0 \\ 0, & x < x_0 \end{cases} \tag{5}$$

To calculate the inverse, note we can do root math since $x_0 > 0$

$$p = 1 - \frac{x_0^{\gamma-1}}{x^{\gamma-1}} \tag{6}$$

$$\frac{x_0^{\gamma-1}}{x^{\gamma-1}} = 1 - p \tag{7}$$

$$\frac{x_0}{x} = (1-p)^{\frac{1}{\gamma-1}} \tag{8}$$

$$x = \frac{x_0}{(1-p)^{\frac{1}{\gamma-1}}} \tag{9}$$

c) Also we can note that we can use $U(0,1)$ without $1-p$ to speed it up since their distribution is exactly the same.
1. Sample from a normal distribution $U(0,1)$
2. substitue value into p for $\frac{x_0}{p^{\frac{1}{\gamma-1}}}$

d)

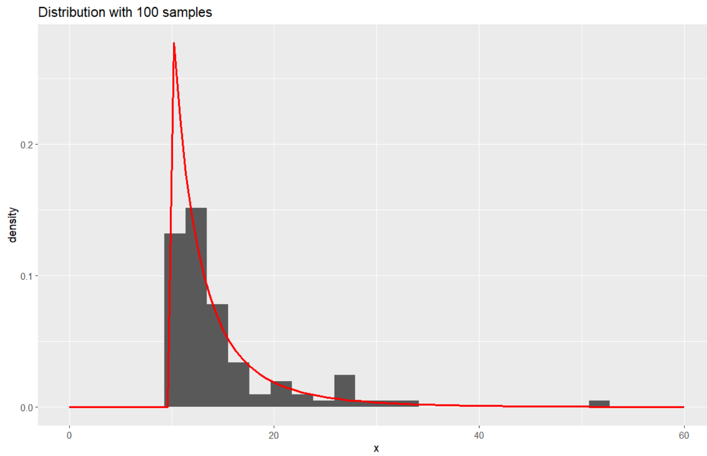

Figure 1: Graph of pdf and random sampling of 100 points, given $\gamma = 4$ and $x_0 = 10$

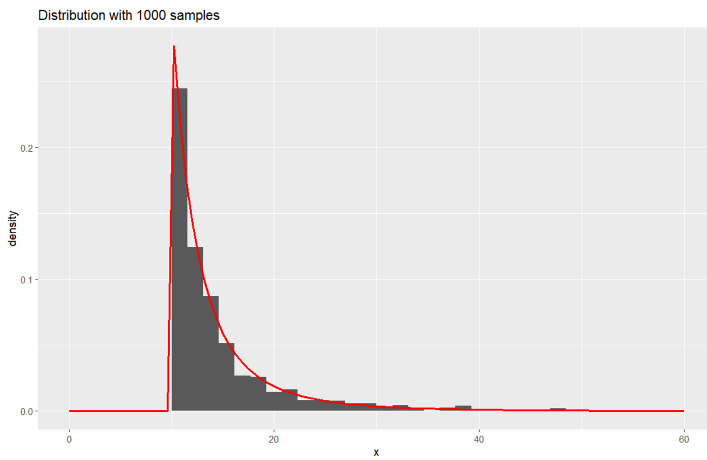

Figure 2: Graph of pdf and random sampling of 1000 points, given $\gamma = 4$ and $x_0 = 10$

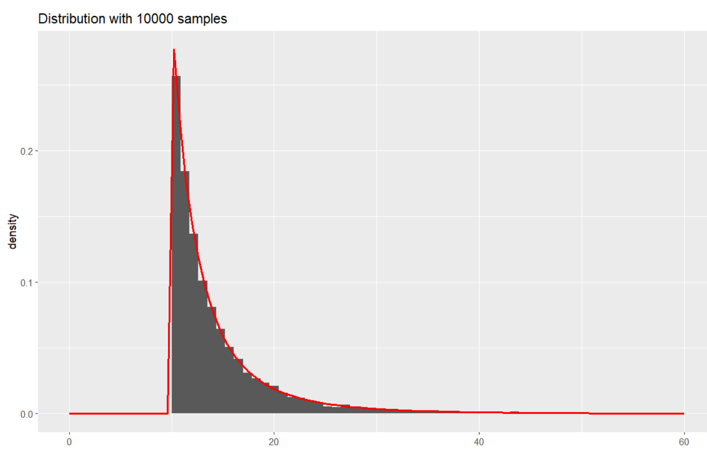

Figure 3: Graph of pdf and random sampling of 10000 points, given $\gamma = 4$ and $x_0 = 10$

2 Gamma Sampling

Suppose we have $X \sim Gamma(2, 1)$ with pdf

$$f(x) = \begin{cases} xe^{-x}, & x > 0 \\ 0, & x \leq 0 \end{cases} \tag{10}$$

a) To calculate the CDF we take the integral of the pdf from 0 to x. Using integration by part with $u = x$ and $dv = e^{-x}$

$$\int_{-\infty}^{x} f(x)dx = \int_{0}^{x} xe^{-x}dx = -xe^{-x} + \int_{0}^{x} e^{-x}dx \tag{11}$$
$$= -e^{-x}(x+1)|_0^x \tag{12}$$
$$= 1 - e^{-x}(x+1) \tag{13}$$

Hence we have

$$F(x) = \begin{cases} 1 - e^{-x}(x+1), & x > 0 \\ 0, & x \leq 0 \end{cases} \tag{14}$$

b I will be using Newtons method to find the inverse of the CDF. The method will guess a root at one and runs 5 times then repeated until the root is positive. There is a sanity check where if the root stays negative an error occurs. Fortunately the code works. The solution to the CDF will be calcualted by solving the root of $F(x) - u = 0$. The function took 5.1287 seconds to run.

c To solve for the optimal value for c, I will use the inequality:

$$C\frac{1}{2}e^{-\frac{x}{2}} \geq xe^{-x} \tag{15}$$
$$C \geq 2xe^{\frac{-x}{2}} = h(x) \tag{16}$$
$$\tag{17}$$

To solve for the critical point of the equation, take the derivative and solve for the roots

$$h'(x) = 2e^{\frac{-x}{2}} - xe^{\frac{-x}{2}} \tag{18}$$
$$= e^{\frac{-x}{2}}(2-x) = 0 \tag{19}$$
$$\tag{20}$$

Since the derivative is a strictly decreasing function and is positive when $x < 2$, the critical value is the point where $h(x)$ is the absolute maximum yielding

$$C \geq h(2) = 4/e \tag{21}$$

Since efficiency is determined by $1/C$ the smallest possible value of C to maximize efficiency is $4/e$.
Also note that the given distribution $g(x)$ is aa pdf for an exponential distribution with $\lambda = 1/2$ Using an acceptace rejection algorithm, it took 3.1873 seconds to complete.