CHAPTER 3

COUNTING

3. Counting

3.1 The Basics of Counting

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3.1 The Basics of Counting

3.1.1 Basic Counting Principles

3.1.2 Inclusion-Exclusion Principle.

3.1.3 Tree Diagrams

3.1.1 Basic Counting Principles

Counting problems are of the following kind:

- "How many different 8-letter passwords are there?"
- "How many possible ways are there to pick 11 soccer players out of a 20-player team?"
- Most importantly, counting is the basis for computing probabilities of discrete events.
- ("What is the probability of winning the lottery?")

Basic Counting Principles

The sum rule:

If a first task can be done in ***m*** ways and a second task in ***n*** ways, and if these two tasks cannot be done at the same time, then there are ***m* + *n*** ways to do either task.

Example. The department will award a free computer to either a CS student or a CS professor. How many different choices are there, if there are 530 students and 15 professors?

Solution. There are 530 + 15 = 545 choices.

Generalized sum rule:

If we have tasks T_1, T_2, …, T_m that can be done in n_1, n_2, …, n_m ways, respectively, and no two of these tasks can be done at the same time, then there are

$$n_1 + n_2 + \ldots + n_m$$

ways to do one of these tasks.

The product rule:

Suppose that a procedure can be broken down into two successive tasks. If there are n_1 ways to do the first task and n_2 ways to do the second task after the first task has been done, then there are $\boldsymbol{n_1 \times n_2}$ ways to do the procedure.

Generalized product rule:

If we have a procedure consisting of sequential tasks T_1, T_2, …, T_m that can be done in n_1, n_2, …, n_m ways, respectively, then there are

$$\boldsymbol{n_1 \times n_2 \times \ldots \times n_m}$$

ways to carry out the procedure.

Example. How many different license plates are there that contain exactly three English letters?

Solution:

There are 26 possibilities to pick the first letter, 26 possibilities for the second one, and 26 for the last one.

So there are $26 \cdot 26 \cdot 26 =$ **17576** different license plates.

Example. How many bit strings are there of length eight?

Example. How many bit strings of length eight both begin and end with a 1?

Example. How many functions are there from an m-element set A to an n-element set B?

Solution:

Let $A = \{a_1, a_2, \ldots, a_m\}$. Then a function $f : A \to B$ is completely determined by the selection of the images $f(a_1), f(a_2), \ldots, f(a_m)$.

There are n choices (among the elements of B) for each of these images

So there are $\underbrace{n \cdot n \cdot \ldots \cdot n}_{m} = \boldsymbol{n^m}$ different functions.

Example. How many one-to-one functions are there from an m-element set A to an n-element set B?

Solution: A function $f: A \to B$ as above is one-to-one iff the images $f(a_1), f(a_2), \ldots, f(a_m)$ are different. There are:

- ✓ n choices for $f(a_1)$ (among the elements of B)
- ✓ $n - 1$ choices for $f(a_2)$ (among the elements of B except $f(a_1)$)
- ✓ ………………………………
- ✓ $n - m + 1$ choices for $f(a_m)$ (among the elements of B except $f(a_1), f(a_2), \ldots, f(a_{m-1})$)

So there are $n\,(n - 1) \ldots (n - m + 1)$ one-to-one functions

Example. Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Solution: Let P be the total number of possible passwords, and let P_6, P_7, and P_8 denote the number of possible passwords of length 6, 7, and 8, respectively. By the sum rule,

$$P = P_6 + P_7 + P_8.$$

We will now find P_6, P_7, and P_8.

To find P_6 it is easier to find the number of strings of uppercase letters and digits that are six characters long, including those with no digits, and subtract from this the number of strings with no digits.

By the product rule, the number of strings of six characters is 36^6, and the number of strings with no digits is 26^6. Hence,

$$P_6 = 36^6 - 26^6 = 1{,}867{,}866{,}560.$$

Similarly, we have

$$P_7 = 36^7 - 26^7 = 70{,}332{,}353{,}920$$

$$P_8 = 36^8 - 26^8 = 2{,}612{,}282{,}842{,}880.$$

Consequently,

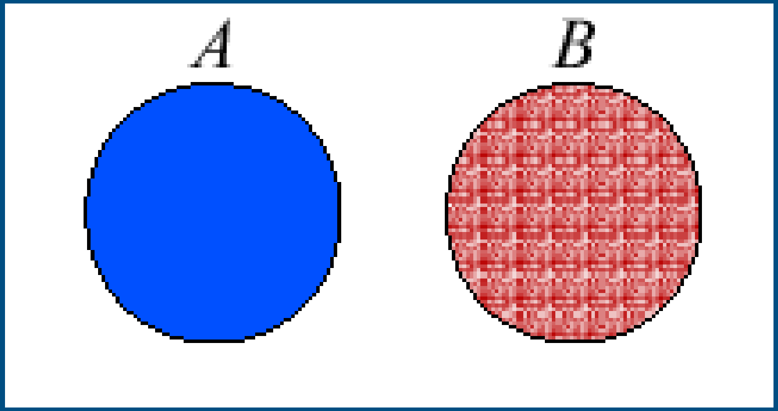
$$P = P_6 + P_7 + P_8 = 2{,}684{,}483{,}063{,}360.$$

Basic Counting Principles

The sum and product rules can also be phrased in terms of ***Set Theory***.

Sum rule: Let $A_1, A_2, \ldots, A_m$ be disjoint sets. Then the number of ways to choose any element from one of these sets is

$$|A_1 \cup A_2 \cup \ldots \cup A_m| = |A_1| + |A_2| + \ldots + |A_m|.$$



Recall that The ***Cartesian Product*** of two sets A and B is:

$$A \times B = \{ (a, b) : a \in A \wedge b \in B\}$$

If $|A| = m$ and $|B| = n$, then $|A \times B| = m\,n$

Product rule: Let $A_1, A_2, \ldots, A_m$ be finite sets. Then the number of ways to choose one element from each set in the order $A_1, A_2, \ldots, A_m$ is

$$|A_1 \times A_2 \times \ldots \times A_m| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_m|.$$

Ex 21. How many positive integers between 50 and 100

a) are divisible by 7? Which integers are these?

b) are divisible by 11? Which integers are these?

c) are divisible by both 7 and 11? Which integers are these?

Ex 25. How many strings of three decimal digits

a) do not contain the same digit three times?

b) begin with an odd digit?

c) have exactly two digits that are 4s?

3.1.2 Inclusion-Exclusion

Example. How many bit strings of length 8 either start with a 1 or end with 00?

Task 1: Construct a string of length 8 that starts with a 1

- There is one way to pick the first bit (1),
- two ways to pick the second bit (0 or 1),
- two ways to pick the third bit (0 or 1),

…………………………

- two ways to pick the eighth bit (0 or 1).

Product rule: Task 1 can be done in $1 \cdot 2^7 = 128$ ways.

Task 2: Construct a string of length 8 that ends with 00.

- There are two ways to pick the first bit (0 or 1),
- two ways to pick the second bit (0 or 1),

……………………………

- two ways to pick the sixth bit (0 or 1),
- one way to pick the seventh bit (0), and
- one way to pick the eighth bit (0).

Product rule: Task 2 can be done in $2^6 = 64$ ways.

- Since there are 128 ways to do Task 1 and 64 ways to do Task 2, does this mean that there are 192 bit strings either starting with 1 or ending with 00 ?
- **No**, because here Task 1 and Task 2 can be done at the same time.
- When we carry out Task 1 and create strings starting with 1, some of these strings end with 00.
- Therefore, we sometimes do Tasks 1 and 2 at the same time, so the sum rule does not apply.

- If we want to use the sum rule in such a case, we have to subtract the cases when Tasks 1 and 2 are done at the same time.
- How many cases are there, that is, how many strings start with 1 and end with 00?
  - There is one way to pick the first bit (1),
  - two ways for the second, …, sixth bit (0 or 1),
  - one way for the seventh, eighth bit (0).
- Product rule: In $2^5 = 32$ cases, Tasks 1 and 2 are **carried out at the same time**.

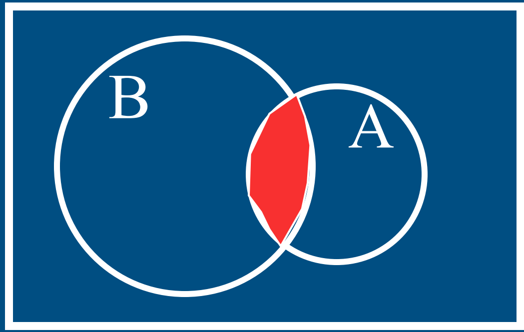
- Since there are 128 ways to complete Task 1 and 64 ways to complete Task 2, and in 32 of these cases Tasks 1 and 2 are completed at the same time, there are:

  $128 + 64 - 32 = 160$ ways to do either task.

- In set theory, this corresponds to sets *A* and *B* that are not disjoint. Then we have:

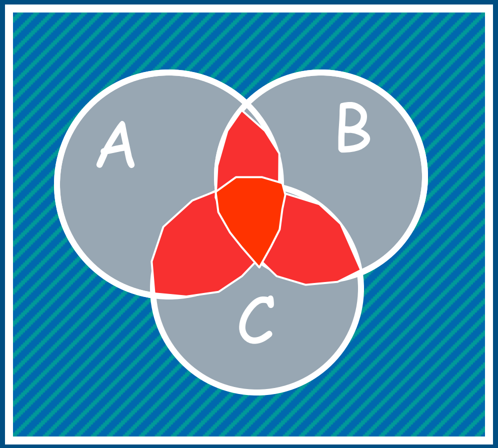
$$|A \cup B| = |A| + |B| - |A \cap B|$$

This is called the **Principle of Inclusion-Exclusion**



- **The Principle of Inclusion-Exclusion** for three sets:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



Ex. How many bit strings of length seven either begin with two 0s or end with three 1s?

Ex. Use the principle of inclusion–exclusion to find the number of positive integers less than 1,000,000 that are not divisible by either 4 or by 6.

Ex 22/396. How many positive integers less than 1000

a) are divisible by 7?

b) are divisible by 7 but not by 11?

c) are divisible by both 7 and 11?

d) are divisible by either 7 or 11?

e) are divisible by exactly one of 7 and 11?

f) are divisible by neither 7 nor 11?

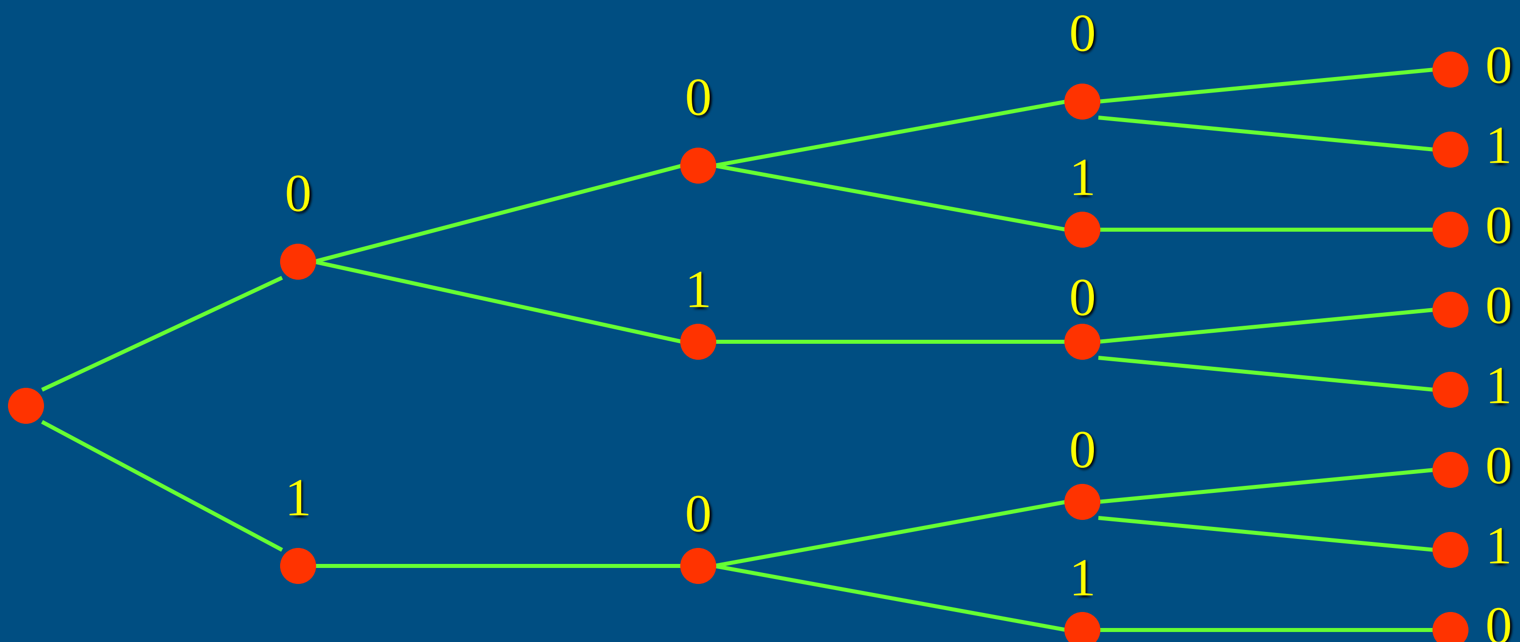
g) have distinct digits?

h) have distinct digits and are even?

3.1.3 Tree Diagrams

How many bit strings of length four do not have two consecutive 1s?

Task 1	Task 2	Task 3	Task 4
(1st bit)	(2nd bit)	(3rd bit)	(4th bit)



There are 8 strings.

Ex 64. Use a tree diagram to find the number of bit strings of length four with no three consecutive 0s.

3.2 The Pigeonhole Principle

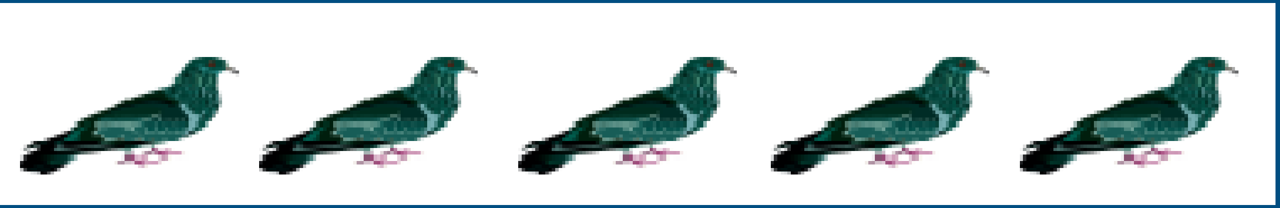
3.2.1 The Pigeonhole Principle

3.2.2 The Generalized Pigeonhole Principle

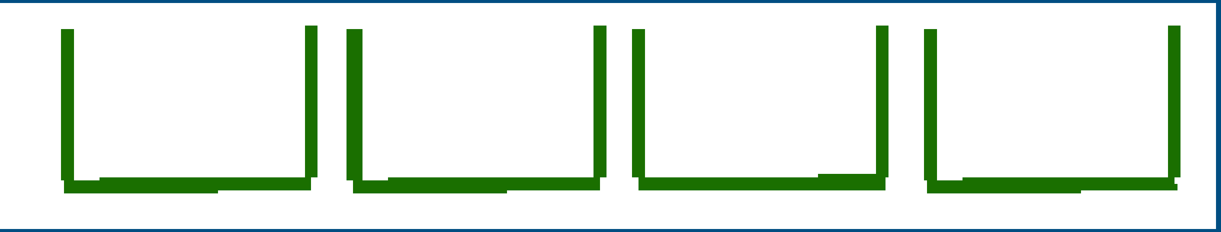
3.2.3 Applications of the Pigeonhole Principle

3.2.1 The Pigeonhole Principle

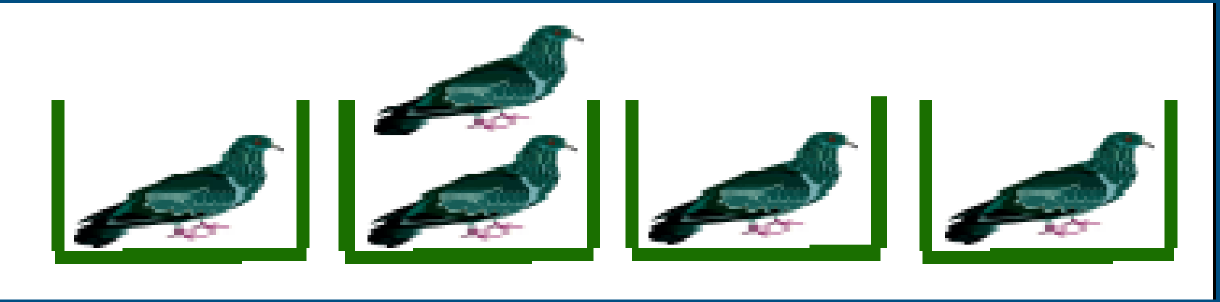
If there are more pigeons



than pigeonholes



then one hole must contain two or more pigeons



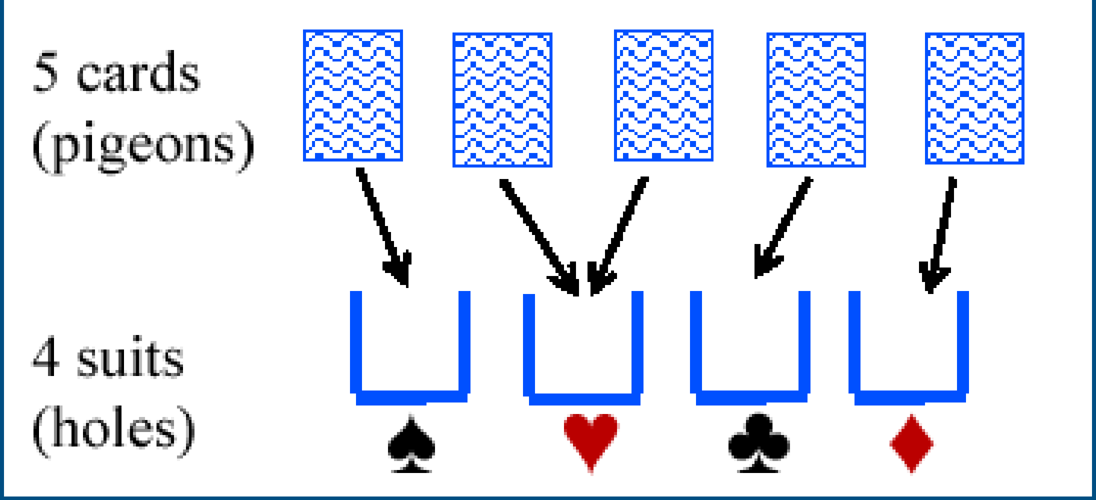
The pigeonhole principle: If $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Example 1: If there are 11 players in a soccer team that wins 12-0, there must be at least one player in the team who scored at least twice.

Example 2: If you have 6 classes from Monday to Friday, there must be at least one day on which you have at least two classes.

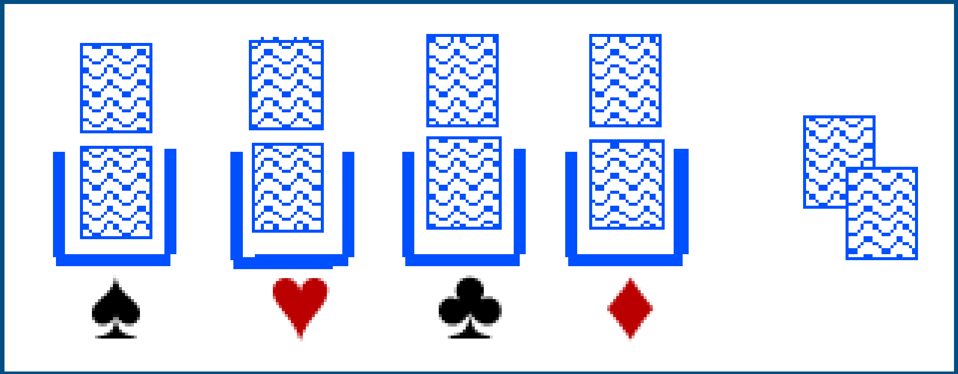
The pigeonhole principle: If $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Example 3: We can pick a set of five cards to guarantee that two cards are of the same suit



3.2.2 The Generalized Pigeonhole Principle

If we draw 10 cards, how many cards of a same suit are guaranteed?



If there are at most 2 cards for each suit, then the number of cards is ≤ 8

Thus some suit must contain at least $\lceil 10/4 \rceil = 3$ cards

The generalized pigeonhole principle: If N objects are placed into k boxes, then there is some box containing at least $\lceil N/k \rceil$ of the objects.

Example 1: In a 60-student class, at least 12 students will get the same letter grade (*A*, *B*, *C*, *D*, or *F*).

Example 2: In a 61-student class, at least 13 students will get the same letter grade.

Example 3: Assume you have a drawer containing a random distribution of a dozen brown socks and a dozen black socks. It is dark, so how many socks do you have to pick to be sure that among them there is a matching pair?

Solution: There are two types of socks, so if you pick at least 3 socks, there must be either at least two brown socks or at least two black socks.

Generalized pigeonhole principle: $\lceil 3/2 \rceil = 2$.

3.2.3 Applications of the Pigeonhole Principle

Example. How many cards must be selected from a standard deck of 52 cards to ensure that we get at least 3 cards of the same suit?

Solution. Since there are 4 suits, if we only select 8 cards then it is possible that we get 2 cards of each suit. So 8 is not enough to guarantee at least 3 cards of the same suit.

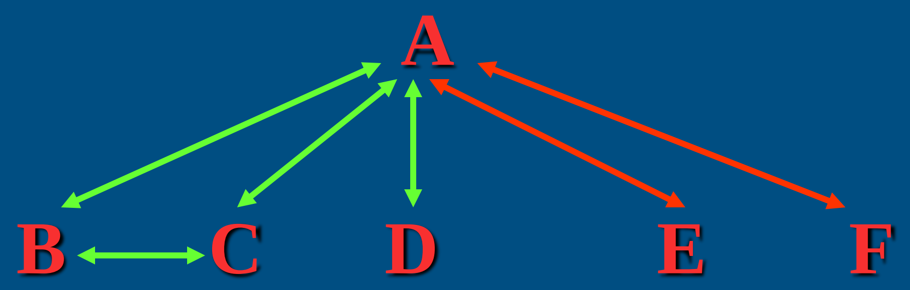
However, if we select 9 cards then the Pigeonhole Principle tells us that we will get at least $\lceil 9/4 \rceil = 3$ cards of the same suit.

So 9 is the least we can select to guarantee at least 3 cards of the same suit

Example Assume in a group of 6 people, any pair consists of either 2 friends or 2 enemies. Then there are either 3 mutual friends or 3 mutual enemies.

Solution: Some person **A** either has at least $\lceil 5/2 \rceil = 3$ friends or 3 enemies.

- Assume that he has 3 friends **B, C, D**.



  - If **B, C, D** are mutual enemies: OK
  - Otherwise 2 of them are friends, say **B, C**.
  - Thus **A, B, C** form a group of 3 mutual friends.
- The case that **A** has 3 enemies is treated similarly

Example. A computer network consists of six computers. Each computer is directly connected to at least one of the other computers. Show that there are at least two computers in the network that are directly connected to the same number of other computers.

Solution. Each computer can be connected to 1, 2, 3, 4, or 5 other computers.

Since there are 6 computers, there are more computers than possible numbers of connections, by Dirichlet's principle at least two computers that have the same number of connections.

Example. How many numbers must be selected from the set {1; 3; 5; 7; 9; 11; 13; 15} to guarantee that at least one pair of these numbers add up to 16?

Solution. The answer is 5. Divide the above numbers into the following 4 groups: {1; 15}, {3; 13}, {5; 11}, {7; 9}.

If we choose 5 numbers out of 4 groups, then by Dirichlet's principle we'll have at least 2 numbers in the same group, and their sum will be equal to 16.

It is not sufficient to choose 4 numbers. For example, we could choose 1, 3, 5, 7, and the sum of no 2 of these is equal to 16.

3.3 Permutations and Combinations.

3.3.1 Permutations

3.3.2 Combinations.

3.3.3 Binomial Coefficients.

3.3.4 Permutations with repetition

3.3.5 Combinations with repetition

3.3.1 Permutations

Example. In how many ways can we select two students from a group of four students to stand in line for a picture?

In how many ways can we arrange all four of these students in a line for a picture?

Example: Let $S = \{1, 2, 3\}$.

- The arrangement 3, 1, 2 is a ***permutation*** of S.
- The arrangement 3, 2 is a ***2-permutation*** of S.

The number of r-permutations of a set with n distinct elements is denoted by $P(n, r)$.

- there are n choices for the first element,
- $(n - 1)$ for the second one,
- $(n - 2)$ for the third one…

So $\quad \mathbf{P(n, r) = n \cdot (n - 1) \cdot (n - 2) \cdot \ldots \cdot (n - r + 1).}$

Example. Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

Answer: 7.6.5.4.3.2.1=5040

Definition. The *factorial* of a non-negative integer n, denoted by **n!**, is the product of all positive integers less than or equal to n, **n! = n.(n-1).(n-2)…2.1**

Example:

- $P(8, 3) = 8 \cdot 7 \cdot 6 = 336$

$$= \frac{8.7.6.5.4.3.2.1}{5.4.3.2.1} = \frac{8!}{5!}$$

General formula:

$$P(n, r) = \frac{n!}{(n-r)!}$$

In particular $\boldsymbol{P(r, r) = r\,!}$

Ex 9. How many possibilities are there for the win, place, and show (first, second, and third) positions in a horse race with 12 horses if all orders of finish are possible?

Solution. We have to choose 3 from 12, with necessary of order so the result is P(12,3) = 12.11.10 = 1320

Ex 10. There are six different candidates for governor of a state. In how many different orders can the names of the candidates be printed on a ballot?

Example. How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?

Solution: Because the letters *ABC* must occur as a block, we can find the answer by finding the number of permutations of six objects, namely, the block *ABC* and the individual letters *D*, *E*, *F*, *G*, and *H*.

Because these six objects can occur in any order, there are $6! = 720$ permutations of the letters *ABCDEFGH* in which *ABC* occurs as a block.

Ex 21. How many permutations of the letters *ABCDEFG* contain

- **a)** the string *BCD*?
- **b)** the string *CFGA*?
- **c)** the strings *BA* and *GF*?
- **d)** the strings *ABC* and *DE*?
- **e)** the strings *ABC* and *CDE*?
- **f)** the strings *CBA* and *BED*?

Ex 25. One hundred tickets, numbered 1, 2, 3, . . . , 100, are sold to 100 different people for a drawing. Four different prizes are awarded, including a grand prize (a trip to Tahiti). How many ways are there to award the prizes if

a) there are no restrictions?

b) the person holding ticket 47 wins the grand prize?

c) the person holding ticket 47 wins one of the prizes?

d) the person holding ticket 47 does not win a prize?

e) the people holding tickets 19 and 47 both win prizes?

f) the people holding tickets 19, 47, and 73 all win prizes?

g) the people holding tickets 19, 47, 73, and 97 all win prizes?

h) none of the people holding tickets 19, 47, 73, and 97 wins a prize?

i) the grand prize winner is a person holding ticket 19, 47, 73, or 97?

j) the people holding tickets 19 and 47 win prizes, but the people holding tickets 73 and 97 do not win prizes?

3.3.2 Combinations

Definition. An ***r-combination*** of elements of a set is an unordered selection of r elements from the set.

Thus, an r-combination is simply a subset of the set with r elements.

The number of r-combinations of a set with n distinct elements is denoted by $\boldsymbol{C(n, r)}$.

Example: Let $S = \{1, 2, 3, 4\}$. Then $\{1, 3, 4\}$ is a 3-combination from S.

Example: $C(4, 2) = 6$, since, for example, the 2-combinations of a set $\{1, 2, 3, 4\}$ are:

$\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}$.

- How can we calculate $C(n, r)$?
- Consider that we can obtain the r-permutation of an n-element set A in two steps:
  - **First,** we form all the r-combinations of the set (there are $C(n, r)$ such r-combinations).
  - **Then,** for each r-combination B of A we select an arbitrary ordering $(b_1, b_2, \ldots, b_r)$ of B.

  This is precisely an r-permutation of B (there are $P(r, r)$ such permutations).
- Therefore, we have:

$$P(n, r) = C(n, r) \cdot P(r, r)$$

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{\frac{n!}{(n-r)!}}{r!}$$

$$= \frac{n!}{r!(n-r)!} = \frac{n(n-1)\cdots(n-r+1)}{r!}$$

- **Now we can answer our initial question:** how many ways are there to pick a set of 3 people from a group of 6 (disregarding the order of picking)?

$$C(6,3) = \frac{6\cdot 5\cdot 4}{3\cdot 2\cdot 1} = \frac{120}{6} = 20$$

Thus there are 20 different ways to pick a set of 3 people

Corollary: Let n and r be nonnegative integers with $r \leq n$. Then

$$C(n, r) = C(n, n - r).$$

- Note that "picking a group of r people from a group of n people" **is the same as** "splitting a group of n people into a group of r people and another group of $(n - r)$ people".

Example: A soccer club has 2 goal keepers, 7 defenders, 6 midfielders and 3 strikers. For today's match, the coach wants to adopt the 4 – 4 – 2 formation. How many possible ways to pick a first team?

Solution: To pick a first team, the coach needs to pick:

- one goal keeper out of two: 2 ways
- 4 defenders out of seven:

$$C(7,4) = C(7,3) = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$$

- 4 midfielders out of six:

$$C(6,4) = C(6,2) = \frac{6 \cdot 5}{2 \cdot 1} = 15$$

- 2 strikers out of three:

$$C(3,2) = C(3,1) = \frac{3}{1} = 3$$

By the Product Rule, the number of possible ways to pick a first team is

$$2 \times 35 \times 15 \times 3 = 3150$$

Example. How many bit strings of length n contain exactly r 1s?

Solution: The positions of r 1s in a bit string of length n form an r-combination of the set $\{1, 2, 3, \ldots, n\}$. Hence, there are $C(n, r)$ bit strings of length n that contain exactly r 1s.

Ex 11. How many bit strings of length 10 contain

a) exactly four 1s?

b) at most four 1s?

c) at least four 1s?

d) an equal number of 0s and 1s?

Ex 34. Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have more women than men?

Ex 35. How many bit strings contain exactly eight 0s and ten 1s if every 0 must be immediately followed by a 1?

3.3.3 Binomial Coefficients

C(n, r) is also called a ***binomial coefficient*** and denoted by

$$C(n,r) = \binom{n}{r}$$

- Recall that a ***binomial expression*** is the sum of two terms, such as $(x + y)$. Consider $(x + y)^2 = (x + y)(x + y)$.

- When expanding such an expression, we have to form all possible products of a term in the first factor and a term in the second factor:

$$(x + y)^2 = x\,x + x\,y + y\,x + y\,y$$

- Then we can sum identical terms:

$$\mathbf{(x + y)^2 = x^2 + 2\,x\,y + y^2}$$

Now expanding $(x + y)^3 = (x + y)(x + y)(x + y)$ we have

$$(x + y)^3 = xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$$
$$= x^3 + 3x^2y + 3xy^2 + y^3$$

Notice that

- There is only one term x^3, because there is only one possibility: choose x from all three factors: C(3, 3) = 1.
- There is three times the term x^2y, because we have to choose x in two out of the three factors: C(3, 2) = 3.
- Similarly, there is three times the term xy^2 (C(3, 1) = 3) and only one term y^3 (C(3, 0) = 1).

Binomial Theorem. Let x and y be variables, and n a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

Example.

$$(x+y)^4 = \sum_{j=0}^{n} \binom{4}{j} x^{4-j} y^j$$

$$= \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4$$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Example. What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x + y)^{25}$?

Solution: From the binomial theorem it follows that this coefficient is

$$\binom{25}{13} = \frac{25!}{13!\,12!} = 5{,}200{,}300.$$

Ex. What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j \qquad (*)$$

By putting $x = y = 1$, we obtain the following

Corollary. Let n be a nonnegative integer. Then

$$\sum_{j=0}^{n} \binom{n}{j} = 2^n$$

Note. The left hand side is the sum of the numbers of subsets of an n element set consisting of 0, 1, …, n elements.

Therefore we **found** again the number of subsets of an n-element set is 2^n

Question 1. Find the expansion of $(x + 2y)^4$.

Question 2. What is the coefficient of x^2y^{10} in the expansion of

$$(x + y)^{12}?$$

Question 3. What is the coefficient of x^3y^7 in the expansion of

$$(2x - 3y)^{10}?$$

Pascal's Triangle

With the help of Pascal's triangle, the Binomial Formula can considerably simplify the process of expanding powers of binomial expressions

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
\ldots & \ldots & \ldots & \ldots & & \ldots & \ldots & & \ldots
\end{array}
$$

In Pascal's triangle, each number is:

- ✓ the sum of the numbers to its upper left and upper right

3.3.4 Permutations with Repetition

Example. How many strings of length r can be formed from the English alphabet?

Solution. There are 26 choices for each character, so by the Product Rule there are

$$26 \cdot 26 \cdot \ldots \cdot 26 = 26^r \text{ strings.}$$

Theorem. The number of ***r-permutation*** of a set of n objects with repetition allowed is n^r

Ex 1. In how many different ways can five elements be selected in order from a set with three elements when repetition is allowed?

Ex 8. How many different ways are there to choose a dozen donuts from the 21 varieties at a donut shop?

3.3.5 Combinations with Repetition

Example. How many ways are there to select 4 pieces of fruits from a bowl containing plenty of apples, oranges and pears?

Solution. There are 15 ways:

- 4aps, 3aps+1or, 3aps+1pear, 2aps+2ors, 2aps+1or+1pear
- 4ors, 3ors+1ap, 3ors+1pear, 2ors+2prs, 2ors+1ap+1pear
- 4prs, 3prs+1ap, 3prs+1or, 2prs+2aps, 2prs+1ap+1or

Each selection is called a *4-combinations* from a three-element set with repetition allowed

Theorem. The number of ***r-combinations*** from a set of n elements with repetition allowed is

$$C(n + r - 1, r)$$

The number of ways to select 4 pieces of fruits from a bowl containing apples, oranges and pears is:

$$C(3 + 4 - 1, 4) = C(6, 4) = C(6, 2) = (6 \cdot 5)/2 = 15$$

Ex. How many different combinations of pennies, nickels, dimes, quarters, and half dollars can a piggy bank contain if it has 20 coins in it?

Example. How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where x_1, x_2, x_3 are nonnegative integers?

Solution. Each solution is a way to select 11 items from 3-element set so that there are x_1, x_2, x_3 items of each types.

Thus the number of solutions is:

$$C(3 + 11 - 1, 11) = C(13, 11) = C(13, 2) = 78$$

Ex 14. How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 17,$$

where x_1, x_2, x_3, and x_4 are nonnegative integers?

Ex 15. How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21,$$

where x_i , $i = 1, 2, 3, 4, 5$, is a nonnegative integer such that

a) $x_1 \geq 1$?

b) $x_i \geq 2$ for $i = 1, 2, 3, 4, 5$?

c) $0 \leq x_1 \leq 10$?

d) $0 \leq x_1 \leq 3$, $1 \leq x_2 < 4$, and $x_3 \geq 15$?

3.4 Recurrence Relations

3.4.1 Recurrence Relations

3.4.2 Modeling with Recurrence Relations

3.4.3 Linear Homogeneous Recurrence Relations with Constant Coefficients

3.4.4 Linear Non-Homogeneous Recurrence Relations with Constant Coefficients

3.4.1 Recurrence Relations

Example. How many bit strings of length n do not contain two consecutive bit 0?

- Let $\boldsymbol{a_n}$ be the number of such strings, and $x_1x_2 \ldots x_n$ be such a string:
- if $x_n = 1$ then the string is $x_1x_2 \ldots x_{n-1}1$, where $x_1x_2 \ldots x_{n-1}$ is an arbitrary string of length $n - 1$ without two consecutive bit 0: there are $\boldsymbol{a_{n-1}}$ such strings
- if $x_n = 0$ then the string is $x_1x_2 \ldots x_{n-2}10$, where $x_1x_2 \ldots x_{n-2}$ is an arbitrary string of length $n - 2$ without two consecutive bit 0: there are $\boldsymbol{a_{n-2}}$ such strings

Thus $\boldsymbol{a_n = a_{n-1} + a_{n-2}}$

- This is an example of a ***recurrence relation***

Definition. A ***recurrence relation*** for the sequence $\{a_n\}$ is an equations that express a_n in terms of the previous terms $a_0, a_1, \ldots, a_{n-1}$

The sequence $\{a_n\}$ is called a ***solution*** of the recurrence relation

Example: Determine whether $a_n = 3n$ is a solution of

$a_n = 2a_{n-1} - a_{n-2}$

Solution: $\boldsymbol{2a_{n-1} - a_{n-2}} = 2[3(n-1)] - 3(n-2) = 3n = \boldsymbol{a_n}$.

Therefore $a_n = 3n$ is a solution of $a_n = 2a_{n-1} - a_{n-2}\boldsymbol{a_n}$.

Example: Determine whether $a_n = 5$ is a solution of

$$a_n = 2a_{n-1} - a_{n-2}$$

Solution: $2a_{n-1} - a_{n-2} = 2(5) - 5 = 5 = a_n$.
Therefore $a_n = 5$ is a solution of $a_n = 2a_{n-1} - a_{n-2}$

- Thus a recurrence relation may have more than one solutions.
- The relation in the last example will have a unique solution satisfying the ***initial conditions***

$$a_0 = 3 \text{ and } a_1 = 5.$$

- Given, indeed these initial conditions we can prove by induction that a_2 , a_3 , …, a_n , … are uniquely determined.

Ex 7. a) Find a recurrence relation for the number of bit strings of length n that contain a pair of consecutive 0s.

b) What are the initial conditions?

c) How many bit strings of length seven contain two consecutive 0s?

3.4.2 Modeling with Recurrence Relations

Example: A sum of P_0 is deposited in the saving account with the interest rate of r % *compounded* annually. How much will be in the account after n years?

Solution: Let P_n be the amount after n years, then P_n is obtained from P_{n-1} and the interest yielded from that amount in one year:

$$P_n = P_{n-1} + r\,P_{n-1} = (1+r)\,P_{n-1}$$

Solution: Let P_n be the amount after n years, then :

$$\boldsymbol{P_n = P_{n-1} + r\,P_{n-1} = (1+r)\,P_{n-1}}$$

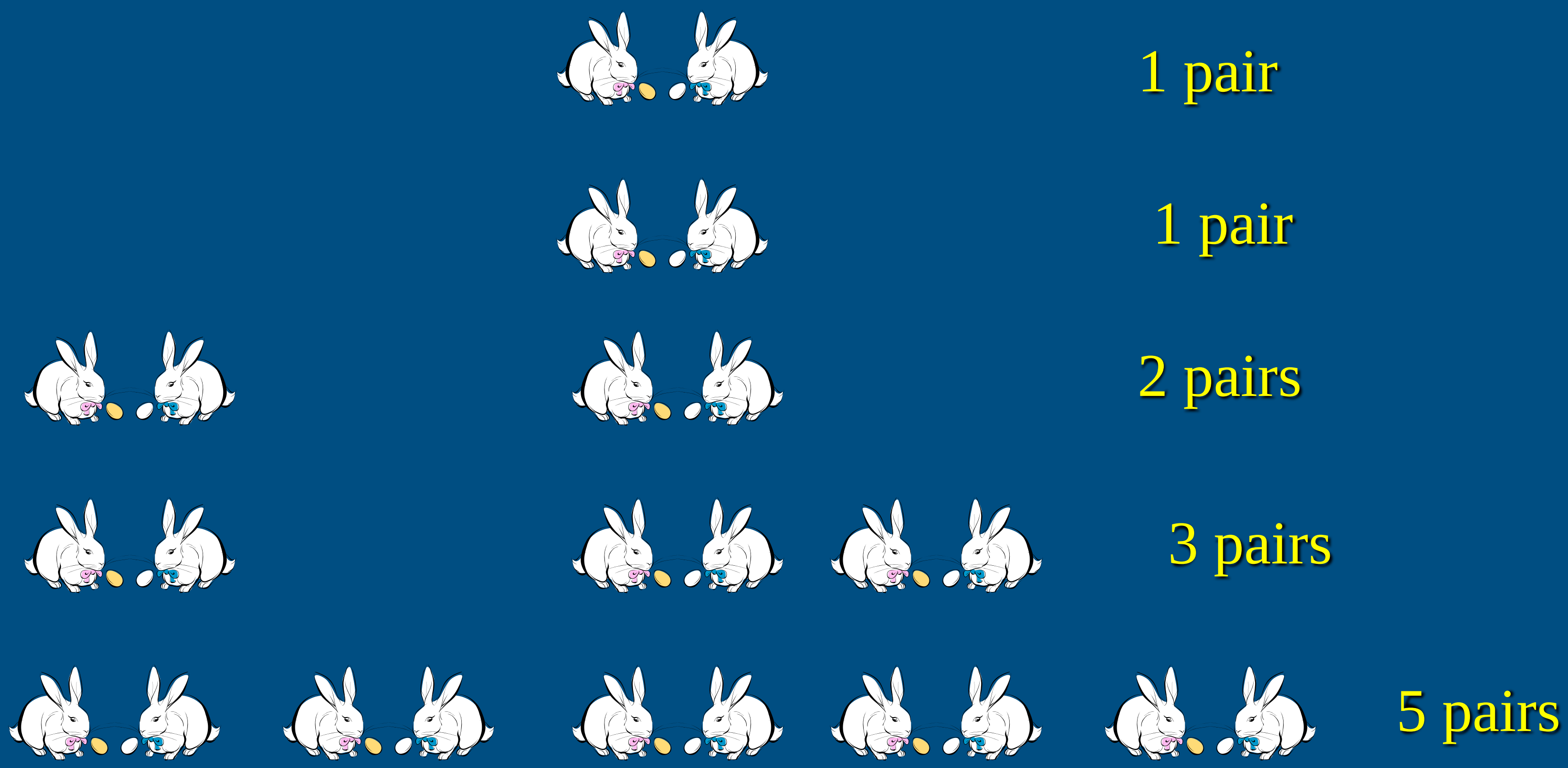
This is a ***geometric progression***.

The solution of this relation is obtained as follows:

$$P_n = (1+r)\,P_{n-1} = (1+r)^2\,P_{n-2} = \ldots = (1+r)^n\,P_0$$

$$\boldsymbol{P_n = (1+r)^n\,P_0}$$

Example: A young pair of rabbits (1 male + 1 fem) is placed on an island. A 2 month old – pair will produce another pair. How many pairs of rabbits after *n* months?



Solution: The number of pairs after the first few months are: **1, 1, 2, 3, 5, …**

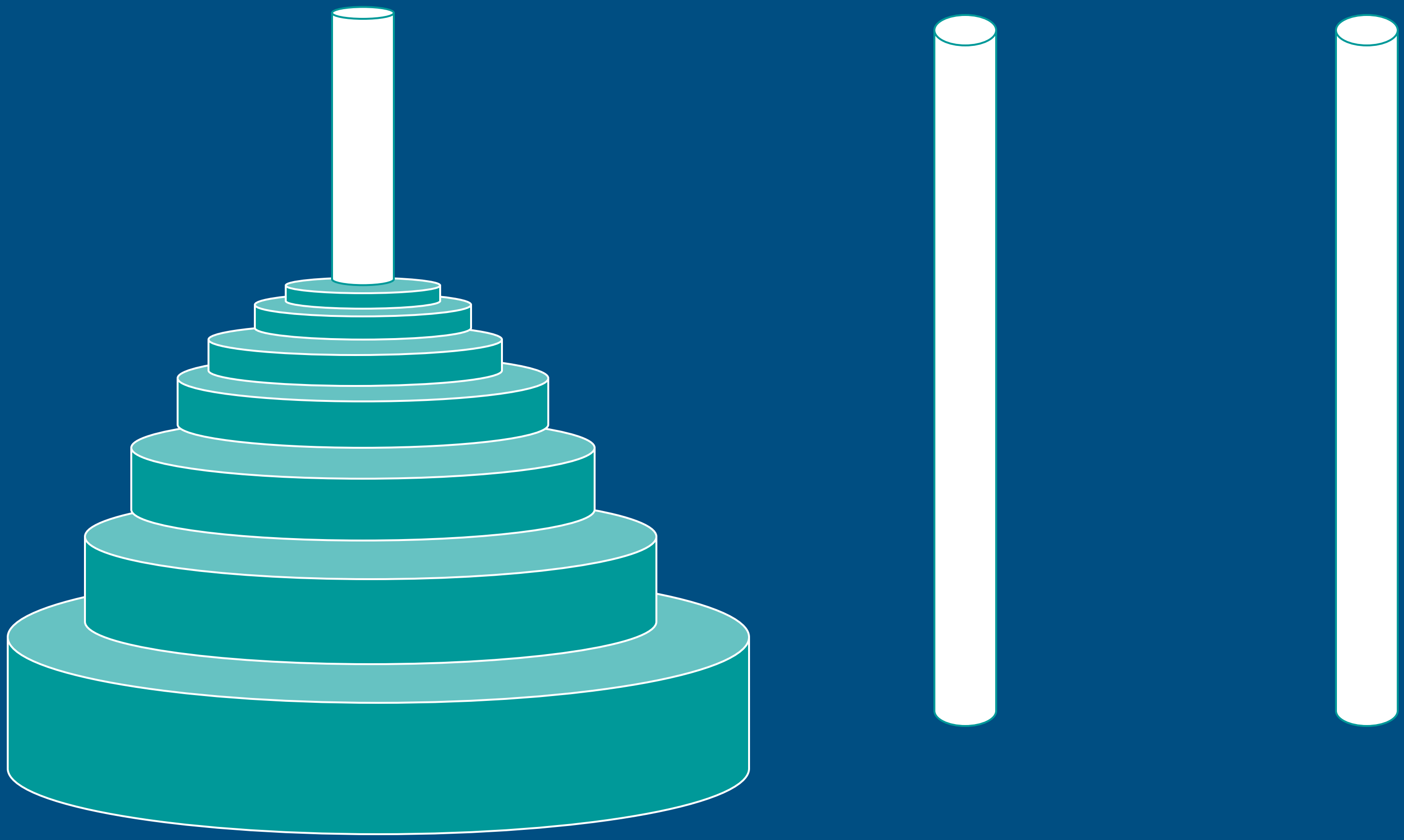
Solution: Let f_n be the number of pairs of rabbits after n months, the above results suggest the following relation:

$$f_n = f_{n-1} + f_{n-2}$$

In fact the right hand side is precisely the number of pairs in the previous month (f_{n-1}) plus the number of newborn pairs given by the pairs of at least two month old (f_{n-2})

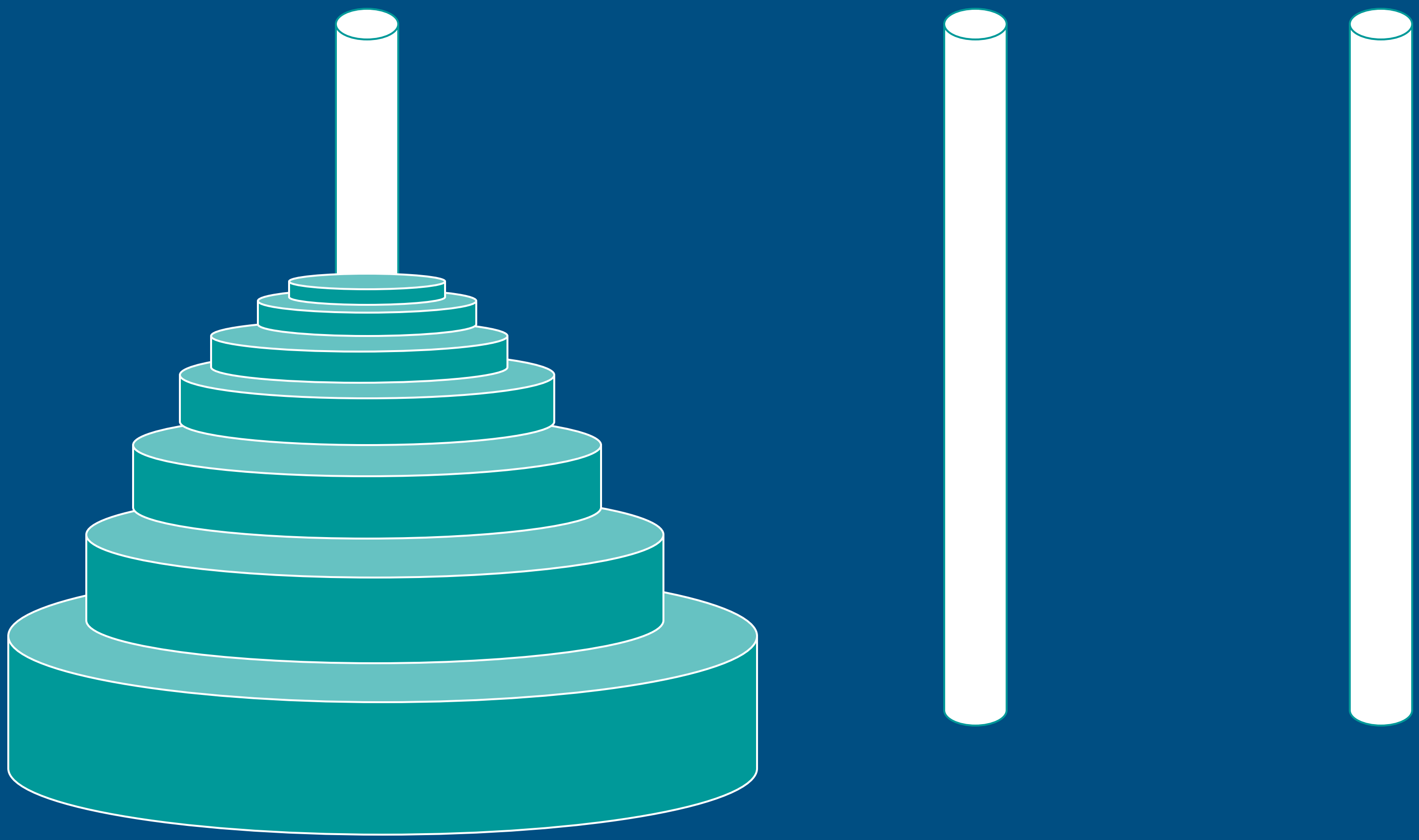
- With the initial conditions $f_1 = 1$ and $f_2 = 1$ the solution is unique
- The numbers f_n with the above initial conditions are also called the ***Fibonacci numbers***.

Example: The tower of Hanoi puzzle consists of three pegs mounted on a board and disks with different sizes.

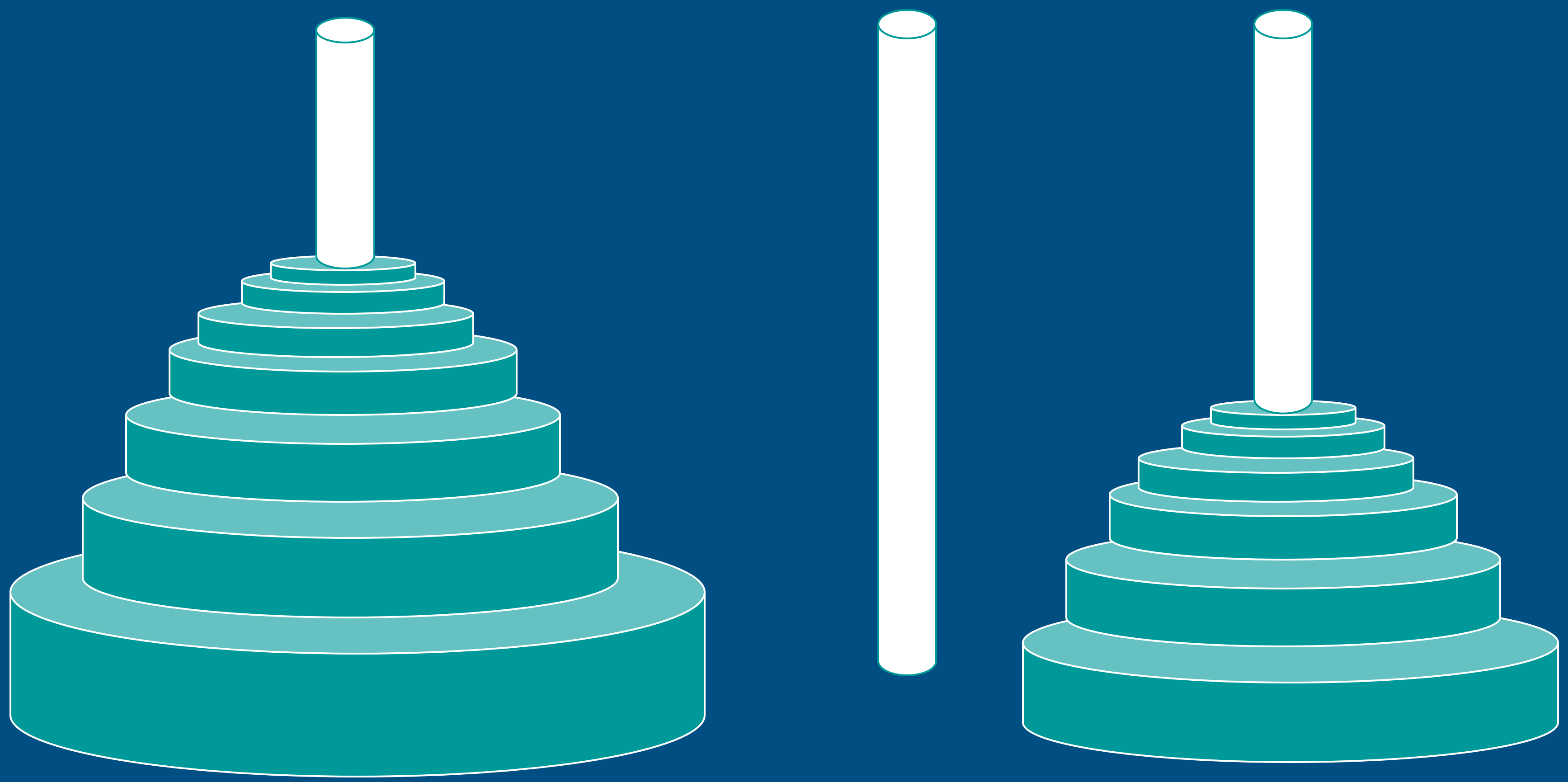


How can we move the disks to the 2nd peg, following the rule: larger disks are never placed on top of smaller ones?

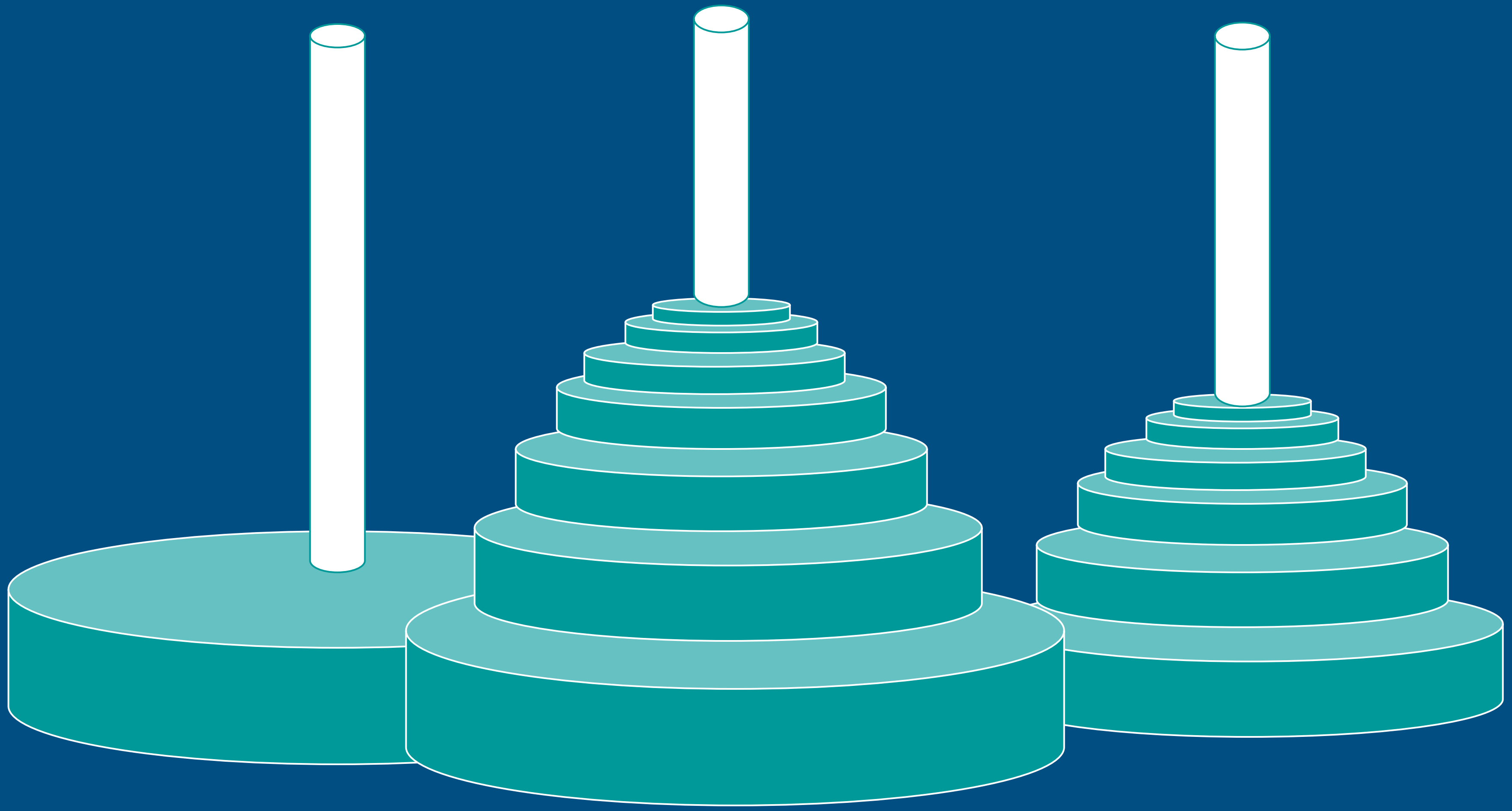
How can we move the disks to the 2nd peg, one in a time, following the rule: larger disks are never placed on top of smaller ones?



- Let H_n be the minimum number of moves to complete the puzzle. First we must move the top $(n - 1)$ disks to the 3rd peg, using at least H_{n-1} moves



- We need one more move to take the largest disk to peg 2
- Then carry $(n - 1)$ smaller disks from 3rd peg to the 2nd peg, using at least H_{n-1} moves .



3.4.2 Modeling with Recurrence Relations

- First, move the top $(n - 1)$ disks to the 3^{rd} peg, using at least H_{n-1} moves

- one more move to take the largest disk to peg 2
- carry $(n - 1)$ smaller disks from 3^{rd} peg to the 2^{nd} peg, using at least H_{n-1} moves .

Thus

$$H_n = 2H_{n-1} + 1$$

$$H_n = 2\,H_{n-1} + 1$$

- To solve this recurrence relation, we write

$$H_n + 1 = 2\,H_{n-1} + 2 = 2(H_{n-1} + 1)$$

- This is a geometric progression, so the solution is:

$$H_n + 1 = C\,2^n$$

Since $H_1 = 1$, we have $C = 1$ and

$$\boldsymbol{H_n = 2^n - 1}$$

E.g. $H_{64} = 18{,}446{,}744{,}073{,}709{,}551{,}615$:

It takes 500 billion years to solve the puzzle !!

3.4.3 Linear Homogeneous Recurrence Relations with Constant Coefficients

Definition. A ***linear homogeneous recurrence relation of degree k with constant coefficients*** is a relation of the form

(1) $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

where $c_1, c_2, \ldots, c_k$ are real numbers and $c_k \neq 0$

- Note that there is a unique solution satisfying the *initial conditions*

$$a_0 = C_0,\ a_1 = C_1, \ldots, a_{k-1} = C_{k-1}$$

Example. The recurrence relations $a_n = a_{n-1} + a_{n-2}{}^2$ is not linear, while the relation $H_n = 2H_{n-1} + 1$ is not homogeneous.

Ex1. Determine which of these are linear homogeneous recurrence relations with constant coefficients. Also, find the degree of those that are.

a) $a_n = 3a_{n-1} + 4a_{n-2} + 5a_{n-3}$

b) $a_n = 2na_{n-1} + a_{n-2}$

c) $a_n = a_{n-1} + a_{n-4}$

d) $a_n = a_{n-1} + 2$

e) $a_n = a^2{}_{n-1} + a_{n-2}$

f) $a_n = a_{n-2}$

g) $a_n = a_{n-1} + n$

(1) $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

Now we look for a solution of (1) in the form $a_n = r^n$.

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \ldots + c_k r^{n-k}$$

Thus r must be a solution of the ***characteristic equation***

$$\boldsymbol{r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_k = 0}$$

We say also that r is a ***characteristic root***

Theorem. Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then all solutions of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ have the form

$$\boldsymbol{a_n = \alpha_1 r_1^n + \alpha_2 r_2^n}$$

where α_1 and α_2 are constants

Example. Find the solution of $a_n = a_{n-1} + 2a_{n-2}$ satisfying the initial conditions $a_0 = 2$ and $a_1 = 7$

Solution. The characteristic equation $r^2 - r - 2 = 0$ has two distinct roots $r_1 = 2$ and $r_2 = -1$.

Hence the general solution has the form $a_n = \alpha_1 2^n + \alpha_2(-1)^n$, where α_1 and α_2 are constants.
This solution satisfies the initial conditions if

$$a_0 = 2 = \alpha_1 + \alpha_2$$
$$a_1 = 7 = 2\alpha_1 - \alpha_2$$

Solving this system we get $\alpha_1 = 3$ and $\alpha_2 = -1$.
Therefore

$$a_n = 3 \cdot 2^n - (-1)^n$$

Example. Find an explicit formula for the Fibonacci numbers: $f_n = f_{n-1} + f_{n-2}$, with $f_0 = 0$ and $f_1 = 1$

Solution. The characteristic equation $r^2 - r - 1 = 0$ has two distinct roots

$$r_1 = \frac{1+\sqrt{5}}{2}, \quad r_2 = \frac{1-\sqrt{5}}{2}$$

Hence the general solution has the form $f_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, where α_1 and α_2 are constants.

This solution satisfies the initial conditions if

$$f_0 = \alpha_1 + \alpha_2 = 0$$

$$f_1 = \alpha_1 \frac{1+\sqrt{5}}{2} + \alpha_2 \frac{1-\sqrt{5}}{2} = 1$$

$$f_0 = \alpha_1 + \alpha_2 = 0$$

$$f_1 = \alpha_1 \frac{1+\sqrt{5}}{2} + \alpha_2 \frac{1-\sqrt{5}}{2} = 1$$

Solving this system we get

$$\alpha_1 = \frac{1}{\sqrt{5}}, \quad \alpha_2 = -\frac{1}{\sqrt{5}}$$

Therefore

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Theorem. Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has a double root r_0. Then all solutions of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ have the form

$$\boldsymbol{a_n = \alpha_1 r_0^{\,n} + \alpha_2\, n\, r_0^{\,n}}$$

where α_1 and α_2 are constants

Example. Find the solution of $a_n = 6a_{n-1} - 9a_{n-2}$ satisfying the initial conditions

$$a_0 = 1 \text{ and } a_1 = 6$$

The double root of $r^2 - 6r + 9 = 0$ is $r_0 = 3$.
Hence the general solution has the form $a_n = \alpha_1 3^n + \alpha_2 n\, 3^n$, where α_1 and α_2 are constants.

This solution satisfies the initial conditions if

$$a_0 = 1 = \alpha_1$$
$$a_1 = 6 = 3\alpha_1 + 3\alpha_2$$

Solving this system we get $\alpha_1 = 1$ and $\alpha_2 = 1$.
Therefore $\mathbf{a_n = 3^n + n\, 3^n}$

Ex 3. Solve these recurrence relations together with the initial conditions given.

a) $a_n = 2a_{n-1}$ for $n \geq 1$, $a_0 = 3$

b) $a_n = a_{n-1}$ for $n \geq 1$, $a_0 = 2$

c) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$

d) $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 6$, $a_1 = 8$

e) $a_n = -4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 1$

f) $a_n = 4a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 4$

g) $a_n = a_{n-2}/4$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$

Theorem. Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r_k - c_1 r_{k-1} - \cdots - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \ldots$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

Example. Find the solution to the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with the initial conditions $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$.

Solution: The characteristic polynomial of this recurrence relation is

$$r^3 - 6r^2 + 11r - 6.$$

The characteristic roots are $r = 1$, $r = 2$, and $r = 3$, because

$$r^3 - 6r^2 + 11r - 6 = (r - 1)(r - 2)(r - 3).$$

Hence, the solutions to this recurrence relation are of the form

$$a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot 3^n.$$

To find the constants α_1, α_2, and α_3, use the initial conditions. This gives

$$a_0 = 2 = \alpha_1 + \alpha_2 + \alpha_3,$$

$$a_1 = 5 = \alpha_1 + \alpha_2 \cdot 2 + \alpha_3 \cdot 3,$$

$$a_2 = 15 = \alpha_1 + \alpha_2 \cdot 4 + \alpha_3 \cdot 9.$$

When these three simultaneous equations are solved for α_1, α_2, and α_3, we find that $\alpha_1 = 1$, $\alpha_2 = -1$, and $\alpha_3 = 2$.

Hence, the unique solution to this recurrence relation and the given initial conditions is the sequence $\{a_n\}$ with

$$a_n = 1 - 2^n + 2 \cdot 3^n.$$

Ex 12. Find the solution to $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n = 3, 4, 5, \ldots$, with $a_0 = 3$, $a_1 = 6$, and $a_2 = 0$.

3.4.4 Linear Non-Homogeneous Recurrence Relations with Constant Coefficients

Definition. A ***linear non-homogeneous recurrence relation with constant coefficients*** is a relation of the form

(5) $$\boldsymbol{a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n)}$$

where $c_1, c_2, \ldots, c_k$ are real numbers and $F(n) \neq 0$ is a function depending only on n. The relation

(1) $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

is called the ***associated homogeneous recurrence relation***

Example. $a_n = a_{n-1} + 2^n$ is a linear non-homogeneous recurrence relation with constant coefficients. The associated linear homogeneous relation is $a_n = a_{n-1}$

Theorem. If $\{a_n^{(p)}\}$ is a particular solution of the linear non-homogeneous recurrence relation

(5) $\boldsymbol{a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n)}$

and $\{a_n^{(h)}\}$ is a general solution of the associated linear homogeneous recurrence relation (1),

then $\{a_n^{(p)} + a_n^{(h)}\}$ is the general solution of (5)

Example. Find all solutions of

$$a_n = 5a_{n-1} - 6a_{n-2} + 4^n$$

Solution. The associated linear homogeneous relation is

$$a_n = 5a_{n-1} - 6a_{n-2}$$

It has two characteristic roots 3 and 2, and hence its general solution is $a_n^{(h)} = \alpha_1 3^n + \alpha_2 2^n$

$$a_n = 5a_{n-1} - 6a_{n-2} + 4^n$$

Since $F(n) = 4^n$, we look for a particular solution of the form

$$a_n^{(p)} = C \cdot 4^n$$

We have

$$C \cdot 4^n = 5\,C \cdot 4^{n-1} - 6\,C \cdot 4^{n-2} + 4^n$$

$$16\,C = 20\,C - 6\,C + 16$$

$$C = 8$$

$$a_n^{(p)} = 8 \cdot 4^n$$

Hence the general solution is

$$a_n = \alpha_1 3^n + \alpha_2 2^n + 8 \cdot 4^n$$

Note. If $F(n) = s^n P(n)$, where $P(n)$ is a polynomial of degree t, then we look for a particular solution of the form $s^n Q(n)$, where:

- $Q(n)$ is a polynomial of degree t if s is not a characteristic root
- $Q(n) = n^m Q_1(n)$ where $Q_1(n)$ is a polynomial of degree t if s is a characteristic root of multiplicity m

Example. Find all solutions of

$$a_n = 6a_{n-1} - 9a_{n-2} + n\,3^n$$

Solution. The associated linear homogeneous relation is

$$a_n = 6a_{n-1} - 9a_{n-2}$$

Solution. The associated linear homogeneous relation $a_n = 6a_{n-1} - 9a_{n-2}$ has a double characteristic root $r = 3$.

Hence we look for a particular solution of the form

$$a_n^{(p)} = n^2 \, Q(n) \, 3^n$$

where $Q(n)$ is a polynomial of degree 1 : $Q(n) = (an + b)$

$$n^2(an + b)3^n = 6(n-1)^2(a(n-1) + b)\, 3^{n-1} - 9(n-2)^2(a(n-2) + b)\, 3^{n-2} + n\, 3^n$$

$$6an - n - 6a + 2b = 0$$

Solving this we get $a = 1/6$ and $b = 1/2$

$$a_n^{(p)} = n^2 \frac{n+3}{6} 3^n$$

Ex 23. Consider the nonhomogeneous linear recurrence relation $a_n = 3a_{n-1} + 2^n$.

a) Show that $a_n = -2^{n+1}$ is a solution of this recurrence relation.

b) Find all solutions of this recurrence relation.

c) Find the solution with $a_0 = 1$.

Ex. Consider the nonhomogeneous linear recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 2n+1$. Find the solution with $a_0 = 1$, $a_1 = 3$