CHAPTER 1

Logic and Proofs

CONTENT

1.1 Logic

Logic is used to give a **precise meaning** to statements made in a mathematical system. We will study the rules and components of logical statements to gain a deeper understanding of reasoning in mathematics.

1.1.1 Propositions

1.1.2 Fundamental Logical Operators

1.1.3 Implications

1.1.4 Precedence of Logical Operators

1.1.5 Logic and Bit Operations

1.1.1 Propositions

A ***proposition*** is a declarative statement that is either TRUE or FALSE but not both.

not both!

Examples.

Washington, D.C. is the capital of the US — TRUE

$1 + 1 = 2$ — TRUE

$2 + 2 = 3$ — FALSE

- We use letters to denote propositions such as p, q, r, and s.
- We denote the truth value of a proposition as T (true) or F (false).

A ***proposition*** is a declarative statement that is either TRUE or FALSE.

not both!

These are NOT propositions:

Study hard!! (Imperative sentence)

Do you like Discrete Structures? (Question sentence)

3 + 2

$x + y = z$

Exercise. Which of these sentences are propositions? What are the truth values of those that are propositions?

a) Paris is the capital of France.

b) The sun rises in the west.

c) $2 + 3 = 5$.

d) $x + 7 = 10$.

e) $x^2 + 1 > 0$.

f) Answer this question.

g) Do not pass go.

h) What time is it?

1.1.2 Fundamental Logical Operators

Suppose p is a proposition. The ***negation*** of p is written $\neg p$ and has meaning:

"It is not the case that p."

$\neg p$ is read "**NOT** p"

Example 1: Let p be "Today is Friday".

Then ¬p is "Today is not Friday.

Example 2 : Let q be " 2 > 1".

Then ¬p is "2 ≤ 1".

Example. What is the negation of each of these propositions?

a) An has a smartphone.

b) There is no pollution in HCM City.

c) $2 + 1 = 3$.

d) The summer in Vietnam is hot and sunny.

e) Duc and Nam are brothers.

f) There are 13 items in a baker’s dozen.

g) Vy sent more than 100 text messages every day.

Definition. A ***truth table*** displays the relationship between the truth values of propositions.

They are useful as a visual display for the workings of a logical operator such as ¬, and can also be used to determine the truth value of a compound proposition based on the truth values of its component propositions

Truth table for negation:

p	$\neg p$
T	F
F	T

Conjunction: $p \wedge q$ corresponds to English "and."

Proposition $p \wedge q$ is true when p and q are both true.

Example. Let p be "Today is Tuesday" and q be "Classes are cancelled".

Then $p \wedge q$ is the proposition "Today is Tuesday and classes are cancelled".

Truth table for conjunction:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example. Determine whether these statements are true or false.

a) $3 > 4$ and Paris is the capital of France

b) 2 is an even and prime number

c) An is drinking water and singing a song

d) $3^2 > 9$ and the sun rises in the west

Disjunction: $\boldsymbol{p \vee q}$ corresponds to English "OR"

Proposition $\boldsymbol{p \vee q}$ is true when p or q (or both) is true.

Example. The students major in CS **or** Business can take this course.

Truth table for disjunction:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example. Determine whether these statements are true or false.

a) $3 > 4$ or Paris is the capital of France

b) 4 is an even number or 5 is prime number

d) $\pi > 4$ or the sun rises in the west

Exclusive Or: $p \oplus q$ corresponds to English "either…or… (but not both)"

Proposition $p \oplus q$ is true when p or q (but not both) is true.

Truth table for exclusive or:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

1.1.3 Implications

Implication (Conditional Statements) : $p \rightarrow q$ corresponds to English "*if…then…*"

Proposition $p \rightarrow q$ is false when p is true and q is false; it is true otherwise

Example. If it is sunny, then I will go to the beach.

If 2 = 1 then Descartes and Pascal are one person.

Truth table for implication:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example. Determine whether each of these conditional statements is true or false.

a) If $1 + 1 = 2$, then $2 + 2 = 5$.

b) If $1 + 1 = 3$, then $2 + 2 = 4$.

c) If $1 + 1 = 3$, then $2 + 2 = 5$.

d) If monkeys can fly, then $1 + 1 = 3$.

e) If $1 + 1 = 3$, then dogs can fly.

f) If $1 + 1 = 2$, then dogs can fly.

g) If $2 + 2 = 4$, then $1 + 2 = 3$.

There are a number of ways to indicate or express an implication in a mathematical statement:

$$p \rightarrow q$$

"if p, then q"

"if p, q"

"p is sufficient for q"

"q if p"

"q when p"

"p implies q"

"p only if q"

"q whenever p"

"q is necessary for p"

"q follows from p"

Ex. Let p and q be the propositions

p :You drive over 65 miles per hour.

q :You get a speeding ticket.

Write these propositions using p and q and logical connectives (including negations).

a) You do not drive over 65 miles per hour.

b) You drive over 65 miles per hour, but you do not get a speeding ticket.

c) You will get a speeding ticket if you drive over 65 miles per hour.

d) If you do not drive over 65 miles per hour, then you will not get a speeding ticket.

e) Driving over 65 miles per hour is sufficient for getting a speeding ticket.

f) You get a speeding ticket, but you do not drive over 65 miles per hour.

g) Whenever you get a speeding ticket, you are driving over 65 miles per hour.

The implication can be reduced to the other fundamental logical operators.

Example. Consider the truth tables of $p \to q$ and $\neg p \vee q$

p	q	$p \to q$	$\neg p$	$\neg p \vee q$
T	T	**T**	F	**T**
T	F	**F**	F	**F**
F	T	**T**	T	**T**
F	F	**T**	T	**T**

They are the same

Biconditional: $p \leftrightarrow q$ corresponds to English "… ***if and only if***"

$p \leftrightarrow q$ is true when p and q have the same truth value; it is false otherwise.

Example. It is cloudy if and only if it is raining.

You can take the flight if and only if you hold a ticket.

Truth table for biconditional:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Ex. Determine whether these biconditionals are true or false.

a) $2 + 2 = 4$ if and only if $1 + 1 = 2$.

b) $1 + 1 = 2$ if and only if $2 + 3 = 4$.

c) $1 + 1 = 3$ if and only if monkeys can fly.

d) $0 > 1$ if and only if $2 > 1$.

There are a number of ways to indicate or express an biconditional statement:

$$\mathbf{p \leftrightarrow q}$$

“*p* is necessary and sufficient for *q*”

“if *p* then *q*, and conversely”

“*p* iff *q*.”

These propositions are **logically equivalent.**

Definition. If p and q are propositions, then we say that p is ***equivalent*** to q if their truth tables are the same.

If p is equivalent to q, then we write $p \equiv q$.

- Thus $p \rightarrow q \equiv \neg p \vee q$
- We will, in the following, substitute any proposition by an equivalent proposition as many times as we like

Another equivalent proposition of $p \to q$ is its ***Contrapositive*** $\neg q \to \neg p$

- We can verify easily that the ***contrapositives*** propositions $p \to q$ and $\neg q \to \neg p$ have the same the truth tables

Example. "If it is noon, then I am hungry."

"If I am not hungry, then it is not noon."

- Note that $p \to q$ is not equivalent to its ***converse*** $q \to p$

Example. "If it is noon, then I am hungry."

is **NOT** equivalent to

"If I am hungry, then it is noon."

- Neither is it equivalent to its ***inverses***: $\neg p \to \neg q$

Example. "If it is noon, then I am hungry."

is **NOT** equivalent to

"If it is not noon, then I am not hungry.

- However the ***converses*** $q \to p$ and the ***inverses*** $\neg p \to \neg q$ are equivalent to each other because $\neg p \to \neg q$ is precisely the ***contrapositive*** of $q \to p$

Example. "If I am hungry, then it is noon"

is ***equivalent*** to

"If it is not noon, then I am not hungry"

Example. What are the contrapositive, the converse, and the inverse of the conditional statement

"The home team wins whenever it is raining?"

Solution. The statement can be rewritten as

"If it is raining, then the home team wins."

Contrapositive:

"If the home team does not win, then it is not raining."

Converse:

"If the home team wins, then it is raining."

Inverse is

"If it is not raining, then the home team does not win."

1.1.4 Precedence of Logical Operators

- In a compound proposition with several logical operators, we have to use parentheses ( ) to specify their order of execution.
- However to reduce the number of parentheses used we adopted the following ***precedence convention***.

Ex. $p \wedge q \vee r$ means $(p \wedge q) \vee r$

rather than $p \wedge (q \vee r)$

$p \vee q \rightarrow r$ means $(p \vee q) \rightarrow r$

Operator	*Precedence*
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Example. Construct a truth table for the compound propositions $p \lor q \to r$

p	q	r	$p \lor q$	$p \lor q \to r$
F	F	F	F	T
F	F	T	F	T
F	T	F	T	F
F	T	T	T	T
T	F	F	T	F
T	F	T	T	T
T	T	F	T	F
T	T	T	T	T

Exercise. Construct a truth table for the compound propositions $(p \oplus q) \land \neg r \to p$

1.1.5 Bit Operations

- We can use a bit to represent a truth value: bit 1 for ***true*** and bit 0 for ***false***.
- A ***Boolean variable*** has value either true or false, and can be represented by a bit.
- By replacing true by 1 and false by 0 in the truth tables of logical operators, we obtain the corresponding tables for ***bit operations***.

Example. $1 \wedge 1 = 1$ $\quad 1 \vee 0 = 1$ $\quad 1 \oplus 1 = 0$ $\quad \neg 1 = 0$

$1 \wedge 0 = 0$ $\quad 0 \vee 0 = 0$ $\quad 1 \oplus 0 = 1$ $\quad \neg 0 = 1$

The operators $\neg$, $\wedge$, $\vee$ and $\oplus$ are also denoted by *NOT*, *AND*, *OR* and *XOR*

Definition. A ***bit string*** is a sequence of 0 or 1 bits. The ***length*** of a bit string is the number of bits in the string.

- The ***bitwise*** *AND*, *OR* and *XOR* of two strings of the same length is the string whose bits are the *AND*, *OR* and *XOR* of the corresponding bits of the two strings

Example. The bitwise *AND*, *OR* and *XOR* of

	01 1001 0110	
and	11 0001 1101	are:
AND :	01 0001 0100	
OR :	11 1001 1111	
XOR :	10 1000 1011	

Ex 44p16. Evaluate each of these expressions.

a) $1\,1000 \wedge (0\,1011 \vee 1\,1011)$

b) $(0\,1111 \wedge 1\,0101) \vee 0\,1000$

c) $(0\,1010 \oplus 1\,1011) \oplus 0\,1000$

d) $(1\,1011 \vee 0\,1010) \wedge (1\,0001 \vee 1\,1011)$

1.2 Propositional Equivalences

1.2.1 Logical Equivalence

1.2.2 The Laws of Logic

1.2.1 Logical Equivalence

Recall that

> The propositions p and q are logically equivalent if they have the same truth tables. We also write $\boldsymbol{p \equiv q}$.

A ***tautology*** (denoted by **T**) is a compound proposition that is always true.

A ***contradiction*** (denoted by **F**) is a compound proposition that is always false.

Thus p and q are logical equivalent if and only if $p \leftrightarrow q$ is a tautology

1.2.2 The Laws of Logic

Following are the Laws of Logic which can be expressed via logical equivalences

- *Identity* $p \wedge \mathbf{T} \equiv p$

  $p \vee \mathbf{F} \equiv p$

p	$\mathbf{T}$	$p \wedge \mathbf{T}$
T	T	T
F	T	F

- *Domination* $p \vee \mathbf{T} \equiv \mathbf{T}$

  $p \wedge \mathbf{F} \equiv \mathbf{F}$

- *Idempotent* $p \vee p \equiv p$

  $p \wedge p \equiv p$

- *Double negation* $\neg(\neg p) \equiv p$

- *Negation* $p \vee \neg p \equiv \mathrm{T}$

  $p \wedge \neg p \equiv \mathrm{F}$

- *Commutativity*

$$p \vee q \equiv q \vee p$$

$$p \wedge q \equiv q \wedge p$$

- *Associativity*

$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$

$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$

- *Distributivity*

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

- *DeMorgan's*

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

- *Absorption*

$$p \wedge (p \vee q) \equiv p$$

$$p \vee (p \wedge q) \equiv p$$

Proofs of some famous Equivalences

- ***Distributivity*** $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

DeMorgan's

$$\neg(p \lor q) \equiv \neg p \land \neg q$$

p	q	$\neg p$	$\neg q$	$\neg p \land \neg q$	$p \lor q$	$\neg(p \lor q)$
T	T	F	F	F	T	F
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	F	T	T	T	F	T

DeMorgan's

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$\neg(\neg p \vee \neg q) \equiv \neg\neg p \wedge \neg\neg q$ DeMorgan's for $\neg p$ and $\neg q$

$\equiv p \wedge q$ Double negation

Therefore

$\neg(p \wedge q) \equiv \neg\neg(\neg p \vee \neg q)$

$\equiv \neg p \vee \neg q$ Double negation

- Show that $[p \wedge (p \rightarrow q)] \rightarrow q$ is a tautology.
- We apply the Laws of Logic in each steps.

$[p \wedge (p \rightarrow q)] \rightarrow q$	
$\equiv [p \wedge (\neg p \vee q)] \rightarrow q$	substitution for $\rightarrow$
$\equiv [(p \wedge \neg p) \vee (p \wedge q)] \rightarrow q$	distributivity
$\equiv [\mathbf{F} \vee (p \wedge q)] \rightarrow q$	negation
$\equiv (p \wedge q) \rightarrow q$	identity
$\equiv \neg(p \wedge q) \vee q$	substitution for $\rightarrow$
$\equiv (\neg p \vee \neg q) \vee q$	DeMorgan's
$\equiv \neg p \vee (\neg q \vee q)$	associativity
$\equiv \neg p \vee \mathbf{T}$	negation
$\equiv \mathbf{T}$	domination

The following logical equivalences involving implications can be checked using the Laws of Logic

- $p \to q \equiv \neg\, q \to \neg\, p$
- $\boldsymbol{\neg\, (p \to q) \equiv p \wedge \neg\, q}$
- $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
- $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
- $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
- $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

- We also have logical equivalences involving ***Biconditionals***

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$

$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$

$$\neg (p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

Exercise. Show that $(\neg p \to r) \wedge (q \to r)$ and $(p \to q) \to r$ are logically equivalent.

Exercise. Show that $(p \to q) \wedge [\neg q \wedge (q \to r)]$ and $\neg q \wedge \neg p$ are logically equivalent.

1.3 Predicates and Quantifiers

1.3.1 Predicates

1.3.2 Quantifiers

1.3.3 Negations

1.3.4 Biding Variables

1.3.1 Predicates

- We will now discuss the area of predicate logic. Predicate logic builds on propositional logic by introducing quantifiers to handle special types of "variables" in a mathematical statement.
- With quantifiers and predicates, we can express a great deal more than we could using propositional logic.

Example. Which statements are propositions:

- Tuan loves ice cream. YES
- *X* loves ice cream. NO
- Everyone loves ice cream. YES
- Someone loves ice cream. YES

Example. Which statements are propositions:

$3 + 2 = 5$

YES

$X + 2 = 5$

NO

$X + 2 = 5$ for any choice of X in $\{1, 2, 3\}$

YES

$X + 2 = 5$ for some X in $\{1, 2, 3\}$

YES

Proposition, YES or NO?

$12 > 4$ YES

$X > 4$ NO

$X > 4$ for any choice of X in $\{3, 4, 5\}$ YES

$X > 4$ for some X in $\{1, 2, 3\}$ YES

The efficiency of Predicates.

An eats pizza at least once a week.

Binh eats pizza at least once a week.

Ha eats pizza at least once a week.

Minh eats pizza at least once a week.

Thu eats pizza at least once a week.

Huy eats pizza at least once a week.

Viet eats pizza at least once a week.

An eats pizza at least once a week.

……………………………………………

……………………………………………

……………………………………………

Define:

$P(x)$ = "x eats pizza at least once a week."

Universe of Discourse - x is a student in Discrete Math class

- Note that $P(x)$ is not a proposition, $P(\text{Thinh})$ is.

The efficiency of Predicates.

Definition. A predicate, or propositional function, is a function defined on a set *U* and returns a proposition as its value.

The set *U* is called the ***universe of discourse***.

- We often denote a predicate by ***P*(*x*)**

- Note that *P*(*x*) is not a proposition, but *P*(*a*) where *a* is some fixed element of *U* is a proposition with well determined truth value

Predicates

Example. Let $Q(x,y)$ = "$x > y$"

Which statements are propositions:

$Q(x,y)$ NO

$Q(3,4)$ YES

$Q(x,9)$ NO

$Q(x,y)$ is a predicates in two free variables x and y in **R** **(real numbers)**

1.3.2 Quantifiers
The universal quantifier

Let $P(x)$ be a predicate on some universe of discourse U.

- One way to obtain a proposition from $P(x)$ is to substitute x by a fixed element of U.
- Another way to obtain a proposition from $P(x)$ is to use the universal quantifier.

Consider the statement:

"$P(x)$ is true for all x in the universe of discourse."

- We write it **$\forall x\, P(x)$** and say "for all x, $P(x)$ "

- The symbol $\forall$ is the ***universal quantifier***.

The universal quantifier

Let $P(x)$ be a predicate on some universe of discourse U.

Consider the statement

"$P(x)$ is true for all x in the universe of discourse."

We write it $\forall x\, P(x)$, and say "for all x, $P(x)$ "

The proposition $\forall x\, P(x)$ is called the ***universal quantification*** of the predicate $P(x)$. It is

- TRUE if $P(a)$ is true when we substitute x by any element a in U
- FALSE if there is an element a in U for which $P(a)$ is false.

Examples.

- Let $P(x)$ be the predicate $x + 1 > x$, where the universe of discourse are the real numbers.

Then $\forall x\, P(x)$ is a TRUE proposition because for all real number x , $x + 1$ is always greater than x

- Let $Q(x)$ be the predicate $x < 1$, where the universe of discourse are the real numbers.

Then $\forall x\, Q(x)$ is a FALSE proposition because we can find a real number, say 3 such that $3 < 1$ is a FALSE proposition.

In the special case that the universe of discourse, U, is finite, say $U = \{x_1, x_2, x_3, \ldots, x_n\}$. Then

$$\forall x\, P(x)$$

corresponds to the proposition:

$$P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$$

We can write a program to loop through the elements in the universe and check each for truthfulness. If all are true, then the proposition is true. Otherwise it is false!

The existential quantifier

Yet another way of changing a predicate into a proposition.

Let $P(x)$ is a predicate on some universe of discourse.

The ***existential quantification*** of $P(x)$ is the proposition:

> "There exists an element x in the universe of discourse such that $P(x)$ is true."

We write it $\exists x\, P(x)$, and say "for some x, $P(x)$"

$\exists$ is called the ***existential quantifier***

- $\exists x\, P(x)$ is FALSE if $P(x)$ is false for every single x.
- $\exists x\, P(x)$ is TRUE if there is an a for which $P(\mathrm{a})$ is true.

Examples.

- Let $P(x)$ be the predicate $x > 3$, where the universe of discourse are the real numbers.

Then $\exists\, x\, P(x)$ is a TRUE proposition because we can find a real number, say 4 such that $4 > 3$ is a TRUE proposition

- Let $Q(x)$ be the predicate $x = x + 1$, where the universe of discourse are the real numbers.

Then $\exists\, x\, Q(x)$ is a FALSE proposition because for all real number x , x and $x + 1$ are distinct real numbers.

In the special case that the universe of discourse, U, is finite, say $U = \{x_1, x_2, x_3, \ldots, x_n\}$. Then

$$\exists x\, P(x)$$

corresponds to the proposition:

$$P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$$

We can write a program to loop through the elements in the universe and check each for truthfulness.

The program will stop when we find an element x_i such that $P(x_i)$ is true. In this case the Proposition is true.

Otherwise, it is false

Ex 11/53 Let $P(x)$ be the statement "$x = x^2$." If the domain consists of the integers, what are these truth values?

a) $P(0)$ **b)** $P(1)$ **c)** $P(2)$

d) $P(-1)$ **e)** $\exists x P(x)$ **f)** $\forall x P(x)$

Ex 13/53. Determine the truth value of each of these statements if the domain consists of all integers.

a) $\forall n(n + 1 > n)$ b) $\exists n(2n = 3n)$

c) $\exists n(n = -n)$ d) $\forall n(3n \leq 4n)$

Predicates - not so boring examples

Suppose the universe of discourse is all creatures, and define the following:

$L(x)$ = "x is a lion."

$F(x)$ = "x is fierce."

$C(x)$ = "x drinks coffee."

All lions are fierce. $\forall x\, (L(x) \rightarrow F(x))$

Some lions don't drink coffee. $\exists x\, (L(x) \wedge \neg C(x))$

Some fierce creatures don't drink coffee.

$\exists x\, (F(x) \wedge \neg C(x))$

Ex 7/53 Translate these statements into English, where

C(x) is "x is a comedian" and

F(x) is "x is funny"

and the domain consists of all people.

a) $\forall x(C(x) \rightarrow F(x))$

b) $\forall x(C(x) \land F(x))$

c) $\exists x(C(x) \rightarrow F(x))$

d) $\exists x(C(x) \land F(x))$

a) $\forall x(C(x) \rightarrow F(x))$

All comedians are funny

b) $\forall x(C(x) \land F(x))$

Every person is funny and a comedian.

c) $\exists x(C(x) \rightarrow F(x))$

There exists a person such that if this person is a comedian then this person is funny

d) $\exists x(C(x) \land F(x))$

There is a person who is a comedian and funny

Ex 9 /53 Let

P(x) = “x can speak Russian” and let

Q(x) =“x knows the computer language C++.”

Express each of these sentences in terms of P(x), Q(x), quantifiers, and logical connectives. The domain for quantifiers consists of all students at your school.

a) There is a student at your school who can speak Russian and who knows C++.

b) There is a student at your school who can speak Russian but who doesn’t know C++.

c) Every student at your school either can speak Russian or knows C++.

d) No student at your school can speak Russian or knows C++.

a) There is a student at your school who can speak Russian and who knows C++.

$\exists x\ (P(x) \wedge Q(x))$

b) There is a student at your school who can speak Russian but who doesn't know C++.

$\exists x\ (P(x) \wedge \neg Q(x))$

c) Every student at your school either can speak Russian or knows C++.

$\forall x\ (P(x) \vee Q(x))$

d) No student at your school can speak Russian or knows C++.

$\neg(\exists x\ (P(x) \vee Q(x)))$

Ex 25/54. Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

a) No one is perfect.

b) Not everyone is perfect.

c) All your friends are perfect.

d) At least one of your friends is perfect.

a) No one is perfect.

$\neg\exists x\,(P(x))$

b) Not everyone is perfect.

$\neg\forall x(P(x))$

c) All your friends are perfect.

$\forall x\,(F(x) \rightarrow P(x))$

d) At least one of your friends is perfect.

$\exists x\,(F(x) \wedge P(x))$

1.3.3 Negations

Another:

B(x) = "x is a hummingbird."

L(x) = "x is a large bird."

H(x) = "x lives on honey."

R(x) = "x is richly colored."

$\forall x\,(B(x) \rightarrow R(x))$

All hummingbirds are richly colored.

No large birds live on honey.

$\neg\exists x\,(L(x) \wedge H(x))$

Birds that do not live on honey are dully colored.

$\forall x\,(\neg H(x) \rightarrow \neg R(x))$

No large birds live on honey. $\neg\exists x\ (\mathrm{L}(x) \wedge \mathrm{H}(x))$

$\forall x\ P(x)$ means "$P(x)$ is true for every x."

What about $\neg\forall x\ P(x)$?

Not["$P(x)$ is true for every x."]

"There is an x for which $P(x)$ is not true."

$$\exists x\ \neg P(x)$$

So, $\neg\forall x\ P(x)$ is the same as $\exists x\ \neg P(x)$.

Predicates - quantifier negation

No large birds live on honey.

$\neg\exists x\ (\mathrm{L}(x) \wedge \mathrm{H}(x))$

$\exists x\ P(x)$ means "$P(x)$ is true for some x."

What about $\neg\exists x\ P(x)$?

Not["$P(x)$ is true for some x."]

"$P(x)$ is not true for all x."

$\forall x\ \neg P(x)$

So, $\neg\exists x\ P(x)$ is the same as $\forall x\ \neg P(x)$.

No large birds live on honey. $\neg\exists x\ (\mathrm{L}(x) \wedge \mathrm{H}(x))$

So, $\neg\forall x\ P(x)$ is the same as $\exists x\ \neg P(x)$.

and, $\neg\exists x\ P(x)$ is the same as $\forall x\ \neg P(x)$.

General rule: to negate a quantification,

- move negation ($\neg$) to the right,
- change the quantifier from $\forall$ to $\exists$, and from $\exists$ to $\forall$.

No large birds live on honey.

$\neg\exists x\ (L(x) \wedge H(x)) \equiv \forall x\ \neg(L(x) \wedge H(x))$ **Negation rule**

$\equiv \forall x\ (\neg L(x) \vee \neg H(x))$ **DeMorgan's**

$\equiv \forall x\ (L(x) \rightarrow \neg H(x))$ **Subst for $\rightarrow$**

Large birds do not live on honey.

Ex 33/54. Express each of these statements using quantifiers. Then form the negation of the statement, so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the phrase “It is not the case that.”)

a) Some old dogs can learn new tricks.

b) No rabbit knows calculus.

c) Every bird can fly.

d) There is no dog that can talk.

e) There is no one in this class who knows French and Russian.

a) Some old dogs can learn new tricks.

No dog can learn new tricks

b) No rabbit knows calculus.

Some rabbits know calculus

c) Every bird can fly.

Some birds can not fly

d) There is no dog that can talk.

There exists a dog that can talk

e) There is no one in this class who knows French and Russian.

Someone in this class knows French and Russian

1.3.4 Biding Variables

A variable is ***bound*** if it is known or quantified. Otherwise, it is ***free***.

Examples.

- $P(x)$ — x is free
- $P(5)$ — x is bound to 5
- $\forall x\, P(x)$ — x is bound by quantifier

Reminder: in a proposition, all variables must be bound.

Predicates - multiple quantifiers

To bind many variables, use many quantifiers.

Example. $P(x,y) = $ "$x > y$"

- $\forall x\, P(x,y)$ — NOT a proposition
- $\forall x \forall y\, P(x,y)$ — FALSE proposition
- $\forall x \exists y\, P(x,y)$ — TRUE proposition
- $\forall x\, P(x, 3)$ — FALSE proposition

Predicates - the meaning of multiple quantifiers

- "$\forall x \forall y\, P(x,y)$" means $P(x,y)$ is true for every possible combination of x and y.
- "$\exists x \exists y\, P(x,y)$" means $P(x,y)$ is true for some choice of x and y (together).
- "$\forall x \exists y\, P(x,y)$" means for every x we can find a (possibly different) y so that $P(x,y)$ is true.
- "$\exists x \forall y\, P(x,y)$" means there is (at least one) particular x for which $P(x,y)$ is always true.

Notice: quantifier order is not interchangeable!

Example. $N(x,y)$ = "x is sitting next to y"

$\forall x \forall y\, N(x,y)$ - everyone is sitting next to everyone else.

False

$\exists x \exists y\, N(x,y)$ - there are two people sitting next to each other.

True?

$\forall x \exists y\, N(x,y)$ - every person is sitting next to somebody.

True?

$\exists x \forall y\, N(x,y)$ - a particular person is sitting next to everyone else.

False

Ex 9/65. Let L(x, y) = “x loves y,” where the domain for both x and y consists of all people in the world.

Use quantifiers to express each of these statements.

a) Everybody loves Jerry.

b) Everybody loves somebody.

c) There is somebody whom everybody loves.

d) Nobody loves everybody.

e) There is somebody whom Lydia does not love.

f) There is somebody whom no one loves.

g) There is exactly one person whom everybody loves.

h) There are exactly two people whom Lynn loves.

i) Everyone loves himself or herself.

j) There is someone who loves no one besides himself or herself.

1.4 Rules of Inference

1.4.1 Definitions

1.2.2 Rules of Inference

1.4.3 Fallacies

1.4.4 Rules of Inference and Quantifiers

1.4.5

1.4.1 Definition

An argument in propositional logic is a sequence of propositions. All but the final proposition in the argument are called premises and the final proposition is called the conclusion.

- ✓"if you have a current password, then you can log onto the network"
- ✓"you have a current password"

premises

Therefore,

- "you can log onto the network"

conclusion

1.4.2. Rules of Inference

An <u>argument form</u> in propositional logic is a sequence of compound propositions involving propositional variables.

"if $\boxed{\text{you have a current password,}}$ → **p**
then $\boxed{\text{you can log onto the network"}}$ → q

✓ "you have a current password"

Therefore,

"you can log onto the network"

$$\begin{array}{l} p \to q \\ \underline{p \qquad\quad} \\ \therefore\ q \end{array}$$

Valid arguments in propositional logic

An <u>argument</u> is <u>valid</u> if the truth of all its premises implies that the conclusion is <u>true</u>.

If we have an argument with premises p_1, p_2, p_3, …………,p_n and conclusion q

This argument is valid when
$(p_1 \wedge p_2 \wedge p_3 \wedge \ldots\ldots\ldots\ldots \wedge p_n) \rightarrow q$ is a <u>tautology</u>

We can use a truth table to show that an argument form is valid. By showing that whenever the premises are true, the conclusion must also be true.

$$\begin{array}{c} p \to q \\ \underline{\quad p \quad} \\ \therefore\ q \end{array}$$

p	q	p→q	(p→q) ∧ p	[p∧(p→q)] →q
F	F	T	F	T
F	T	T	F	T
T	F	F	F	T
T	T	T	T	T

We can use a truth table to show that an argument form is valid. By showing that whenever the premises are true, the conclusion must also be true.

If an argument form involves 10 different propositional variables, to use truth table, $2^{10}=1024$ rows are needed. This is a tedious (long and boring) approach.

Instead of using a truth table to show that an argument form is valid, we can use Rules of inference.

The tautology $(p \wedge (p \rightarrow q)) \rightarrow q$ is the basis of the rule of inference called Modus Ponens (or law of detachment- mode that affirms)

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

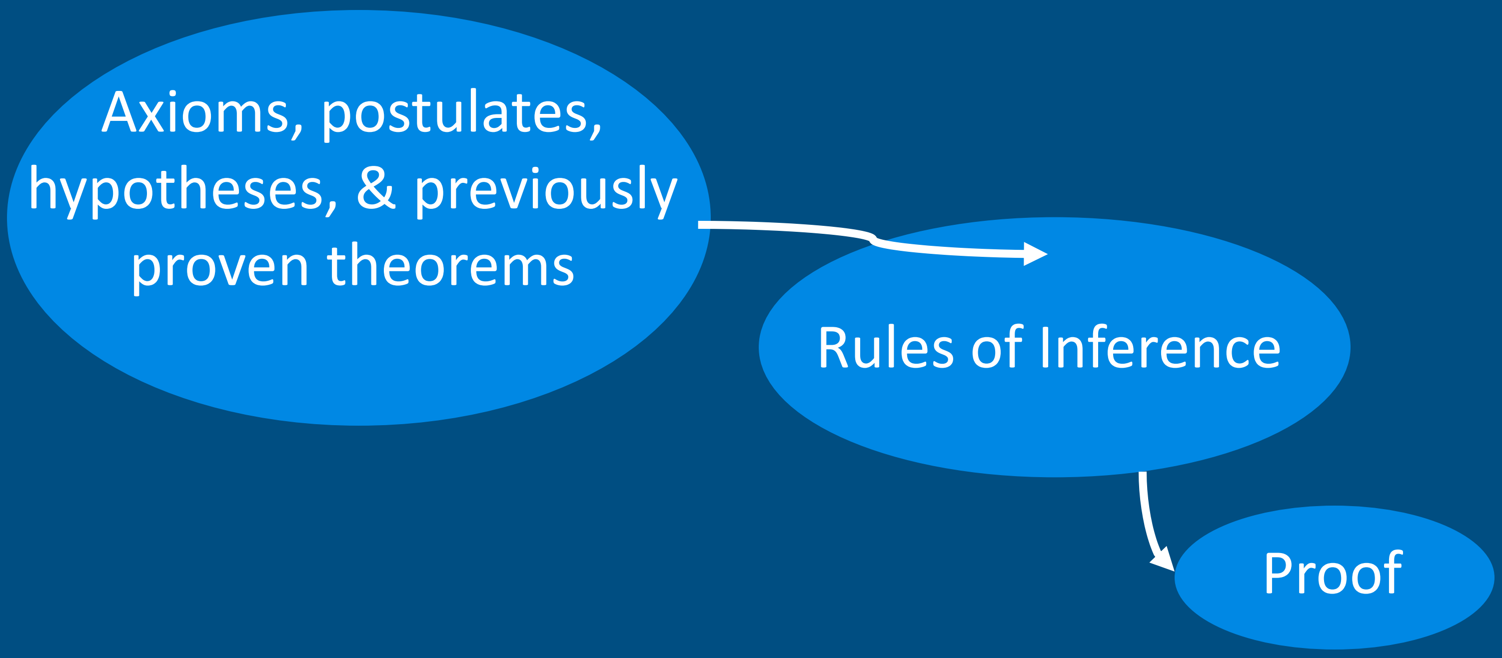
The hypotheses are written in a column, followed by horizontal bar, followed by the $\therefore$ and the conclusion.

Terminology

- A **theorem** is a statement that can be shown to be true.
- A **lemma** is a simple theorem used as an intermediate result in the proof of another theorem.
- A **corollary** is a proposition that follows directly from a theorem that has been proved.
- A **conjecture** is a statement whose truth value is unknown. Once it is proven, it becomes a theorem.

A ***theorem*** is a statement that can be shown to be true.

A ***proof*** is the means of doing so.



Modus Ponens

I am Luyen.
If I am Luyen, then I am a lecturer of Mathematics.

$\therefore$ I am a lecturer of Mathematics.

$$
\begin{array}{l}
p \\
p \rightarrow q \\
\hline
\therefore q
\end{array}
$$

tautology:

$$(p \wedge (p \rightarrow q)) \rightarrow q$$

Inference Rule:

Modus Ponens

Modus Tollens

I am not a great actor.
If I am Thanh, then I am a great actor.
$\therefore$ I am not Thanh!

$$\begin{array}{l} \neg q \\ p \rightarrow q \\ \hline \therefore \neg p \end{array}$$

tautology:

$(\neg q \land (p \rightarrow q)) \rightarrow \neg p$

Inference Rule:

Modus Tollens

Addition

I am not a great actor.
$\therefore$ I am not a great actor or I am tall.

$$\frac{p}{\therefore\ p \lor q}$$

tautology:

$p \rightarrow (p \lor q)$

Inference Rule:

Addition

Conjunction

- I not a great actor.
- I am a lecturer

$\therefore$ I am not a great actor and I am a lecturer.

p

q

$\therefore\ p \land q$

tautology:

$p \land q \rightarrow (p \land q)$

Inference Rule:

Conjuction

Simplification

I am not a great actor and you are sleepy.
$\therefore$ you are sleepy.

$$\frac{p \wedge q}{\therefore\ p}$$

tautology:

$(p \wedge q) \rightarrow p$

Inference Rule:

Simplification

Disjunctive Syllogism

I am BM major or CS major.
I am not a BM major.
$\therefore$ I am a CS major!

$$\begin{array}{l} p \lor q \\ \neg q \\ \hline \therefore p \end{array}$$

tautology:

$((p \lor q) \land \neg q) \to p$

Inference Rule:

Disjunctive Syllogism

Hypothetical Syllogism

If you are a CS major, then you must pass Math 2215.

If you passed Math 2215, then you are good in logic.

$\therefore$ If you are a CS major, then you are good in logic.

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore\ p \rightarrow r \end{array}$$

tautology:

$$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$$

Inference Rule:

Hypothetical Syllogism

Rule of Inference	Tautology	Name
p $\therefore\ p \vee q$	$p \rightarrow p \vee q$	Addition
$p \wedge q$ $\therefore\ p$	$p \wedge q \rightarrow p$	Simplification
p, q $\therefore\ p \wedge q$	$(p) \wedge (q) \rightarrow p \wedge q$	Conjunction
$p, p \rightarrow q$ $\therefore\ q$	$p \wedge (p \rightarrow q) \rightarrow q$	Modus Ponens
$\neg q, p \rightarrow q$ $\therefore\ \neg p$	$\neg q \wedge (p \rightarrow q) \rightarrow \neg p$	Modus Tollens
$p \rightarrow q, q \rightarrow r$ $\therefore\ p \rightarrow r$	$(p \rightarrow q) \wedge (q \rightarrow r)$ $\rightarrow (p \rightarrow r)$	Hypothetical Syllogism
$p \vee q, \neg p$ $\therefore\ q$	$(p \vee q) \wedge \neg p \rightarrow q$	Disjunctive Syllogism

Ex 3/78. Which rule of inference is used in each argument below?

a) Alice is a Math major. Therefore, Alice is either a Math major or a CS major.
b) Jerry is a Math major and a CS major. Therefore, Jerry is a Maths major.
c) If it is rainy, then the pool will be closed. It is rainy. Therefore, the pool is closed.
d) If it snows today, the university will close. The university is not closed today. Therefore, it did not snow today.
e) If I go swimming, then I will stay in the sun too long. If I stay in the sun too long, then I will sunburn. Therefore, if I go swimming, then I will sunburn.
f) I go swimming or eat an ice cream. I did not go swimming. Therefore, I eat an ice cream.

Using Rules of Inference to Build Arguments

A formal proof of a conclusion C, given premises p_1, $p_2, \ldots, p_n$ consists of a sequence of steps, each of which applies some inference rule to premises or to previously-proven statements (as hypotheses) to yield a new true statement (the conclusion).

- A proof demonstrates that if the premises are true, then the conclusion is true (i.e., valid argument).

Example.

Show that the hypotheses

"It is not sunny and it is cold (q)."
"we will swim (r) only if it is sunny (p),
"If we do not swim, then we will canoe (s)."
"If we canoe, then we will be home early (t)."

Leads to the conclusion
"We will be home early" t

$\neg p \wedge q$, $r \to p$, $\neg r \to s$, $s \to t$

We can use the truth table

Premises $\neg p \wedge q$, $r \to p$, $\neg r \to s$, $s \to t$

Conclusion t

Step	Reason
1. $\neg p \wedge q$	Hypothesis
2. $\neg p$	Simplification using (1)
3. $r \to p$	Hypothesis
4. $\neg r$	Modus Tollens using (2) (3)
5. $\neg r \to s$	Hypothesis
6. s	Modus ponens using (4) (5)
7. $s \to t$	Hypothesis
8. t	Modus ponens using (6) (7)

$$\begin{array}{c} \neg p \wedge q \\ r \to p \\ \neg r \to s \\ s \to t \\ \hline \therefore t \end{array}$$

A little proof…

Here's what you know:

Minh is a BM major or a CS major.

If Minh does not like discrete math, he is not a CS major.

If Minh likes discrete math, he is smart.

Minh is not a BM major.

Can you conclude Minh is smart?

$$
\begin{array}{l}
B \lor C \\
\neg D \to \neg C \\
D \to S \\
\hline
\therefore \quad \neg B
\end{array}
$$

1. $B \vee C$
2. $\neg D \rightarrow \neg C$
3. $D \rightarrow S$
4. $\neg B$

5. C	Disjunctive Syllogism (1,4)
6. $C \rightarrow D$	Contrapositive of 2
7. $C \rightarrow S$	Hypothetical Syllogism (6,3)
8. S	Modus Ponens (5,7)

Minh is smart!

1.4.3 Fallacies

Rules of inference, appropriately applied give ***valid*** arguments.

Mistakes in applying rules of inference are called ***fallacies***.

If I am Descartes, then I am a mathematician

I am a mathematician!

$\therefore$ I am Descartes

Affirming the conclusion.

I' m Luyen

Fallacies:

$((p \to q) \land q) \to p$

Not at tautology.

If you don’t give me \$10, I bite your ear.

I bite your ear!

$\therefore$ You didn’t give me \$10.

Affirming the conclusion.

Fallacies:

$((p \to q) \land q) \to p$

Not at tautology.

If it rains then it is cloudy.

It does not rain.

$\therefore$ It is not cloudy

Denying the hypothesis.

Fallacies:

$((p \to q) \land \neg p) \to \neg q$

Not at tautology.

If it is a bicycle, then it has 2 wheels.

It is not a bicycle.

$\therefore$ It doesn't have 2 wheels.

Denying the hypothesis.

Motor cycle

Fallacies:

$((p \to q) \wedge \neg p) \to \neg q$

Not at tautology.

1.4.4. Rules of Inference for Quantified Statements

$$\frac{\forall x\, P(x)}{\therefore\ P(c) \text{ if } c \in U}$$

Universal instantiation

$$\frac{P(c) \text{ for an arbitrary } c \in U}{\therefore\ \forall x\, P(x)}$$

Universal generalization

$$\frac{\exists x\, P(x)}{\therefore\ P(c) \text{ for some element } c \in U}$$

Existential instantiation

$$\frac{P(c) \text{ for some element } c \in U}{\therefore\ \exists x\, P(x)}$$

Existential generalization

Rules of Inference for Quantified Statements

Example.

Every US student is a genius.
An is an US student.
Therefore, An is a genius.

U(x): “x is an US student.”
G(x): “x is a genius.”

Rules of Inference for Quantified Statements

- The following steps are used in the argument:
- Step 1: $\forall x\,(U(x) \to G(x))$ Hypothesis
- Step 2: $U(An) \to G(An)$ Univ. instantiation using Step 1

Step 3: $U(An)$ Hypothesis

Step 4: $G(An)$ Modus ponens using Steps 2 & 3

$$\frac{\forall x\, P(x)}{\therefore P(c) \text{ if } c \in U}$$

Universal instantiation

Example

Anyone performs well is either intelligent or a good actor.

If someone is intelligent, then he/she can count from 1 to 10.

Gary performs well.

Gary can only count from 1 to 3.

Therefore, not everyone is both intelligent and a good actor

P(x): x performs well

I(x): x is intelligent

A(x): x is a good actor

C(x): x can count from 1 to 10

Hypotheses:

Anyone performs well is either intelligent or a good actor.

$$\forall x\ (P(x) \rightarrow I(x) \lor A(x))$$

If someone is intelligent, then he/she can count from 1 to 10.

$$\forall x\ (I(x) \rightarrow C(x)\ )$$

Gary performs well.

$$P(G)$$

Gary can only count from 1 to 3.

$$\neg C(G)$$

Conclusion: not everyone is both intelligent and a good actor

$$\neg \forall x(I(x) \land A(x))$$

The following steps are used in the argument:

Step 1: $\forall x\,(P(x) \rightarrow I(x) \vee A(x))$	Hypothesis
Step 2: $P(G) \rightarrow I(G) \vee A(G)$	Univ. Inst. Step 1
Step 3: $P(G)$	Hypothesis
Step 4: $I(G) \vee A(G)$	Modus ponens Steps 2 & 3
Step 5: $\forall x\,(I(x) \rightarrow C(x))$	Hypothesis
Step 6: $I(G) \rightarrow C(G)$	Univ. inst. Step5
Step 7: $\neg C(G)$	Hypothesis
Step 8: $\neg I(G)$	Modus tollens Steps 6 & 7
Step 9: $\neg I(G) \vee \neg A(G)$	Addition Step 8
Step 10: $\neg(I(G) \wedge A(G))$	Equivalence Step 9
Step 11: $\exists x \neg(I(x) \wedge A(x))$	Exist. general. Step 10
Step 12: $\neg\forall x\,(I(x) \wedge A(x))$	Equivalence Step 11

Conclusion: $\neg\forall x\,(I(x) \wedge A(x))$, not everyone is both intelligent and a good actor.