CHAPTER 2

SETS AND FUNCTIONS

CONTENTS

Sets are the most basic of discrete structures and also the most general. Several of the discrete structures we will study are built out of sets.

We have been using sets on an informal basis when we talked about universes of discourse. We will now define what a set is and start working with sets more formally.

2.1 Sets

2.1.1 Definitions and Notation

2.1.2 Cardinality

2.1.3 Power Set

2.1.4 Cartesian Products

2.1.1 Definitions and notation

Definition. A ***set*** is an unordered collection of elements.

Examples.

- **{1, 2, 3}** is the set containing "1" and "2" and "3."
- **{1, 1, 2, 3, 3}** = {1, 2, 3} since repetition is ir**re**levant.
- **{1, 2, 3}** = {3, 2, 1} since sets are unordered.
- **{0, 1, 2, 3, …}** is a way we denote an infinite set (in this case, the natural numbers).
- $\varnothing$ **= {}** is the empty set, or the set containing no element.

Note: $\varnothing \neq \{\varnothing\}$

Some Important Sets

N = *natural numbers* = {0,1,2,3….}

Z = *integers* = {…,-3,-2,-1,0,1,2,3,…}

$\mathbf{Z}^+$ = *positive integers* = {1,2,3,…..}

R = set of *real numbers*

$\mathbf{R}^+$ = set of *positive real numbers*

C = set of *complex numbers*.

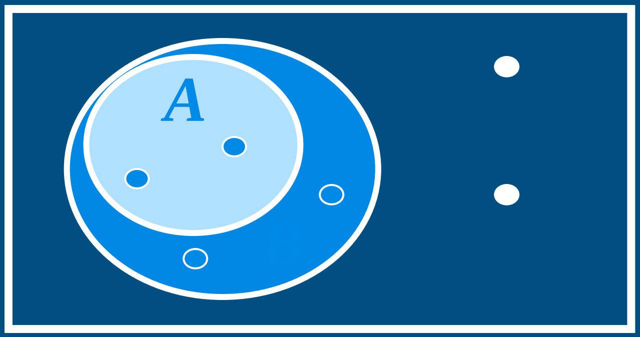
Q = set of rational numbers

- $x \in S$ means "x is an element of set S."
- $x \notin S$ means "x is not an element of set S."
- $A \subseteq B$ means "A is a subset of B."

or, "B contains A."

or, "every element of A is also in B."

or, $\forall x\ ((x \in A) \rightarrow (x \in B))$.



Venn Diagram

Ex 7/125. For each of the following sets, determine whether 2 is an element of that set.

a) $\{x \in \mathbf{R} \mid x \text{ is an integer greater than } 1\}$

b) $\{x \in \mathbf{R} \mid x \text{ is the square of an integer}\}$

c) $\{2,\{2\}\}$

d) $\{\{2\},\{\{2\}\}\}$

e) $\{\{2\},\{2,\{2\}\}\}$

f) $\{\{\{2\}\}\}$

- $\boldsymbol{A \subseteq B}$ means "A is a subset of B."
- $\boldsymbol{A \supseteq B}$ means "A is a superset of B."
- $\boldsymbol{A = B}$ if and only if A and B have exactly the same elements

iff, $A \subseteq B$ and $B \subseteq A$

iff, $A \subseteq B$ and $A \supseteq B$

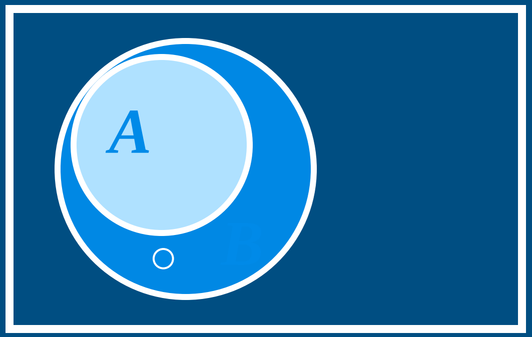
iff, $\forall x\,((x \in A) \leftrightarrow (x \in B))$.

So to show equality of sets A and B, show:

$$A \subseteq B \;\; and \;\; B \subseteq A$$

$A \subset B$ means "A is a proper subset of B."

- $A \subseteq B$, and $A \neq B$.
- $\forall x\,((x \in A) \rightarrow (x \in B)) \wedge \exists x\,((x \in B) \wedge (x \notin A))$



Quick examples.

- $\{1,2,3\} \subseteq \{1,2,3,4,5\}$
- $\{1,2,3\} \subset \{1,2,3,4,5\}$

Is $\emptyset \subseteq \{1,2,3\}$?

Yes! $\forall x\ (x \in \emptyset) \rightarrow (x \in \{1,2,3\})$ holds, because $(x \in \emptyset)$ is false.

Is $\emptyset \in \{1,2,3\}$? **No!**

Is $\emptyset \subseteq \{\emptyset,1,2,3\}$? **Yes!**

Is $\emptyset \in \{\emptyset,1,2,3\}$? **Yes!**

Quiz time:

Is $\{x\} \subseteq \{x\}$? Yes

Is $\{x\} \in \{x,\{x\}\}$? Yes

Is $\{x\} \subseteq \{x,\{x\}\}$? Yes

Is $\{x\} \in \{x\}$? No

Ex 9 /125. Determine whether each of these statements is true or false.

a) $0 \in \emptyset$

b) $\emptyset \in \{0\}$

c) $\{0\} \subset \emptyset$

d) $\emptyset \subset \{0\}$

e) $\{0\} \in \{0\}$

f) $\{0\} \subset \{0\}$

g) $\{\emptyset\} \subseteq \{\emptyset\}$

Ways to define sets

- Explicitly: {An, Binh, Dung, Duong}
- Implicitly: {1,2,3,…}, or {2,3,5,7,11,13,17,…}
- Set builder: $\{ x : x \text{ is prime} \}$, $\{ x \mid x \text{ is odd} \}$.
- In general $\{ x : P(x)\}$, where $P(x)$ is some predicate.

We read
"the set of all x such that $P(x)$"

In general $\{ x : P(x)\}$, where $P(x)$ is some predicate

Example. Let $D(x,y)$ denote the predicate "x is divisible by y"

And $P(x)$ denote the predicate

$$\forall y \, ((y > 1) \wedge (y < x)) \rightarrow \neg D(x,y)$$

Then

$$\{ x : \forall y \, ((y > 1) \wedge (y < x)) \rightarrow \neg D(x,y) \}.$$

is precisely the set of all primes

Ex 1/125. List the members of these sets.

a) $\{x \mid x \text{ is a real number such that } x^2 = 1\}$

b) $\{x \mid x \text{ is a positive integer less than } 12\}$

c) $\{x \mid x \text{ is the square of an integer and } x < 100\}$

d) $\{x \mid x \text{ is an integer such that } x^2 = 2\}$

2.1.2 Cardinality

If S is finite, then the ***cardinality*** of S, denoted by $|S|$, is the number of distinct elements in S.

If $S = \{1,2,3\}$ $|S| = 3$.

If $S = \{3,3,3,3,3\}$ $|S| = 1$.

If $S = \varnothing$ $|S| = 0$.

If $S = \{ \varnothing, \{\varnothing\}, \{\varnothing,\{\varnothing\}\} \}$ $|S| = 3$.

If $S = \{0,1,2,3,\ldots\}$, $|S|$ is infinite. (more on this later)

Ex 19/125. What is the cardinality of each of these sets?

a) $\{a\}$

b) $\{\{a\}\}$

c) $\{a, \{a\}\}$

d) $\{a, \{a\}, \{a, \{a\}\}\}$

2.1.3 Power sets

If S is a set, then the ***power set*** of S is

$$P(S) = 2^S = \{ A : A \subseteq S \}.$$

We say, "$P(S)$ is the set of all subsets of S."

If $S = \{a\}$ $\quad 2^S = \{\varnothing, \{a\}\}.$

If $S = \{a,b\}$ $\quad 2^S = \{\varnothing, \{a\}, \{b\}, \{a,b\}\}.$

If $S = \varnothing$ $\quad 2^S = \{\varnothing\}.$

If $S = \{\varnothing,\{\varnothing\}\}$ $\quad 2^S = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing,\{\varnothing\}\}\}.$

Fact: if S is finite, $|2^S| = 2^{|S|}$. (if $|S| = n$, $|2^S| = 2^n$)

2.1.4 Cartesian Product

The ***Cartesian Product*** of two sets A and B is:

$$A \times B = \{ (a, b) : a \in A \wedge b \in B\}$$

If A = {1,2,3}, and B = {x,y}, then

We'll use these special sets soon!

$A \times B$ = {(1,x), (2,x), (3,x), (1, y), (2, y), (3, y)}

Note. $(a, b) = (c, d)$ iff $a = c$, and $b = d$

A,B finite $\rightarrow |A \times B| = |A||B|$

The ***Cartesian Product*** *of n sets* $A_1, A_2, \ldots, A_n$ is:

$$A_1 \times A_2 \times \ldots \times A_n =$$
$$= \{(a_1, a_2, \ldots, a_n) : a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}$$

Note. $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ iff
$$a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n$$

We'll use these special sets soon!

Ex 27/125. Let $A = \{a, b, c, d\}$ and $B = \{y, z\}$. Find

a) A × B.

b) B × A.

Ex 32/125. Let $A = \{a, b, c\}$, $B = \{x, y\}$, and $C = \{0, 1\}$. Find

a) A × B × C.

d) C× B × B.

2.2 Set Operations

2.2.1 Introduction

2.2.2 Sets Identities

2.2.3 Generalized Set Operations

2.2.4 Computer Representation of Sets

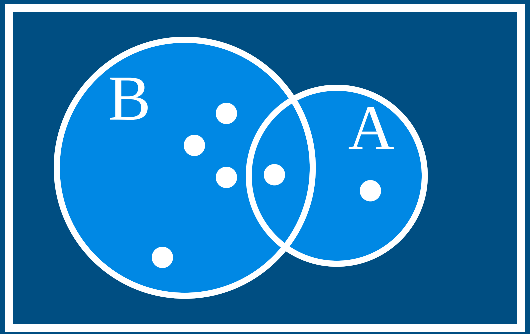
2.2.1 Introduction

The ***union*** of two sets A and B is:

$$\boldsymbol{A} \cup \boldsymbol{B} = \{\, x : x \in A \vee x \in B\}$$

If A = {1,2,3}, and B = {2, 4}, then

$A \cup B$ = {1,2,3,4}

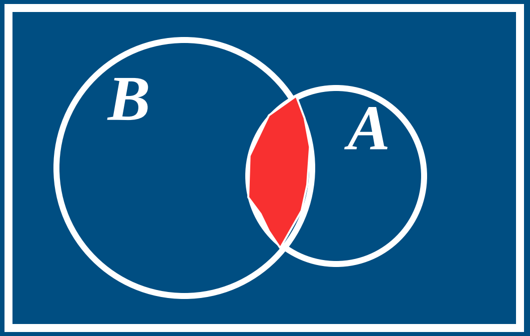


The ***intersection*** of two sets A and B is:

$$\mathbf{A} \cap \mathbf{B} = \{\, x : x \in A \wedge x \in B \}$$

If A = {1, 2, 3}, and B = {2, 4}, then

$A \cap B$ = {2}

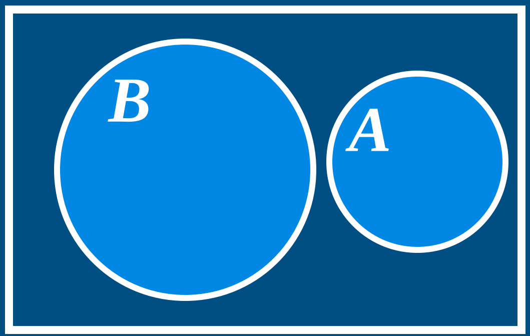


The *intersection* of two sets A and B is:

$$A \cap B = \{ x : x \in A \wedge x \in B \}$$

If A = {x : x is a US president}, and
B = {x : x is in this room}, then

$A \cap B$ = {x : x is a US president in this room} = $\varnothing$



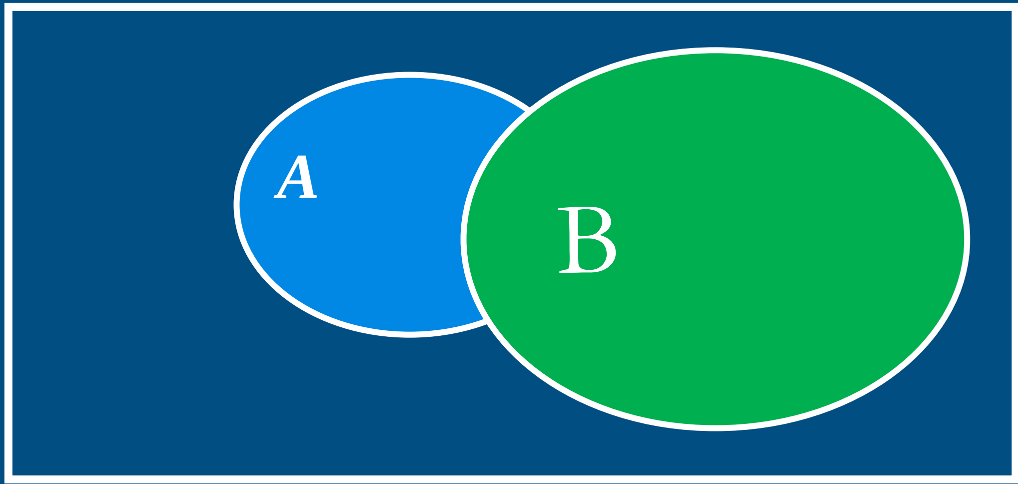
Sets whose intersection is empty are called *disjoint sets*

The ***difference*** of two sets A and B is:

$$A - B = \{x : x \in A \wedge x \notin B\}$$

If $A = \{1,2,3,4,5,6\}$, and
$B = \{1,2,4,7\}$, then

$A - B = \{3,5,6\}$

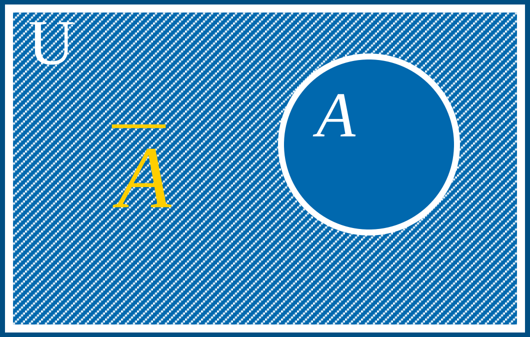


The ***complement*** of a set A is:

$$\overline{A} = \{x : x \notin A\}$$

If $A = \{x : x$ is not shaded$\}$, then

$$\bar{A} = \{x : x \text{ is shaded}\}$$



$\overline{\varnothing} = U$
and
$\overline{U} = \varnothing$

Example. $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ $A = \{1, 2, 3, 4, 5\}$,

$B = \{4, 5, 6, 7, 8\}$

1. $A \cup B$

   $\{1, 2, 3, 4, 5, 6, 7, 8\}$

2. $A \cap B$

   $\{4, 5\}$

3. $\bar{A}$

   $\{0, 6, 7, 8, 9, 10\}$

4. $\bar{B}$

   $\{0, 1, 2, 3, 9, 10\}$

5. $A - B$

   $\{1, 2, 3\}$

6. $B - A$

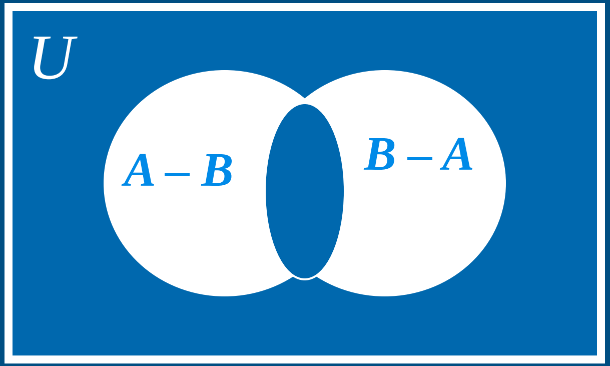
   $\{6, 7, 8\}$

The ***symmetric difference***, $A \oplus B$, is:

$$\boldsymbol{A \oplus B} = \{\, x : (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\}$$

$$= (A - B) \cup (B - A)$$

$$= \{\, x : x \in A \oplus x \in B\}$$



Ex 26. Draw the Venn diagrams for each of these combinations of the sets A, B, and C.

a) $A \cap (B \cup C)$

b) $A \cap B \cap C$

c) $(A - B) \cup (A - C) \cup (B - C)$

Ex 4. Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$. Find

a) $A \cup B$.

b) $A \cap B$.

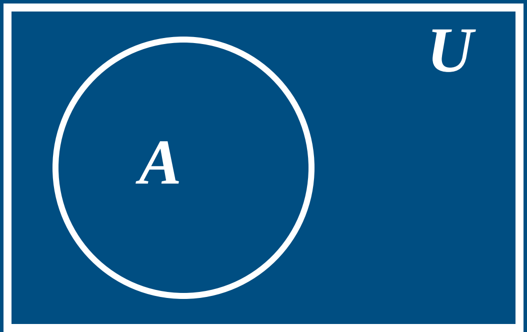
c) $A - B$.

d) $B - A$.

2.2.2 Set Identities

- **Identity** $A \cap U = A$

  $A \cup \varnothing = A$



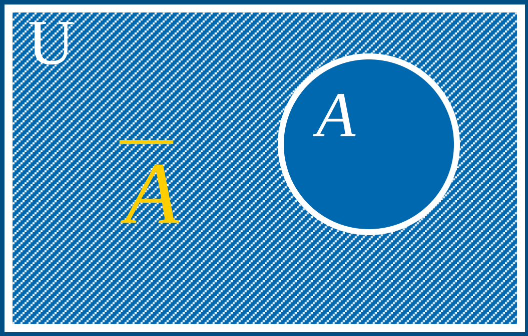
- **Domination** $A \cup U = U$

  $A \cap \varnothing = \varnothing$

- **Idempotent** $A \cup A = A$

  $A \cap A = A$

- *Excluded Middle* $$A \cup \overline{A} = U$$



- *Uniqueness* $$A \cap \overline{A} = \varnothing$$

- *Double complement* $$\overline{\overline{A}} = A$$

- **Commutativity**

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

- **Associativity**

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

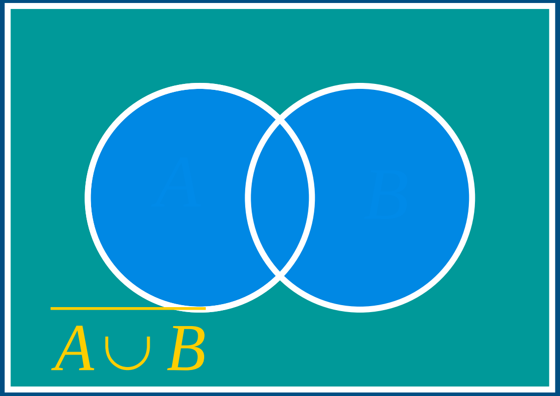
- **Distributivity**

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

- DeMorgan's I

- DeMorgan's II $\overline{A \cap B} = \overline{A} \cup \overline{B}$



4 Ways to prove identities

- Show that $A \subseteq B$ and that $A \supseteq B$.

New & important

- Use a membership table.

Like truth tables

- Use previously proven identities.

Like $\equiv$

- Use logical equivalences to prove equivalent set definitions.

Not hard, a little tedious

Prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$

$$x \in \overline{A \cup B} \equiv x \notin A \cup B \equiv \neg(x \in A \vee x \in B)$$

$$\equiv (\neg x \in A) \wedge (\neg x \in B) \equiv x \notin A \wedge x \notin B$$

$$\equiv x \in \overline{A} \wedge x \in \overline{B} \equiv x \in \overline{A} \cap \overline{B}$$

Prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$

using a membership table.

0 : x is not in the specified set

1 : otherwise

Haven't we seen this before?

A	B	$\overline{A}$	$\overline{B}$	$\overline{A} \cap \overline{B}$	$A \cup B$	$\overline{A \cup B}$
1	1	0	0	0	1	0
1	0	0	1	0	1	0
0	1	1	0	0	1	0
0	0	1	1	1	0	1

Prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$

using logically equivalent set definitions

$$\overline{(A \cup \mathrm{B})} = \{x : \neg(x \in A \vee x \in B)\}$$

$$= \{x : \neg(x \in A) \wedge \neg(x \in B)\}$$

$$= \{x : (x \in \overline{A}) \ \wedge \ (x \in \overline{B})\}$$

$$= \overline{A} \cap \overline{B}$$

Prove that $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$

using known identities

$$\overline{A \cup (B \cap C)} = \overline{A} \cap (\overline{B \cap C})$$
$$= \overline{A} \cap (\overline{B} \cup \overline{C})$$
$$= (\overline{B} \cup \overline{C}) \cap \overline{A}$$
$$= (\overline{C} \cup \overline{B}) \cap \overline{A}$$

Ex 19. Show that if A, B and C are sets, then

- (a) $A \cap (B \backslash A) = \emptyset$
- (b) $A \backslash (A \backslash B) = A \cap B$
- (c) $A \backslash B = A \backslash (A \cap B) = (A \cup B) \backslash B$
- (d) $(A \backslash B) \cup (A \backslash C) = A \backslash (B \cap C)$
- (e) $(A \backslash B) \cup (B \backslash A) = (A \cup B) \backslash (A \cap B)$

2.2.3 Generalized Set Operations

Generalized Union

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

$$= \{x : x \in A_1 \vee x \in A_2 \vee \ldots \vee x \in A_n\}$$

Example. Let $U = \mathbf{N}$, and define:

$$A_i = \{i, i+1, i+2, \ldots\}$$

Then

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{1,2,3,\ldots\}$$

Generalized Intersection

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

$$= \{x : x \in A_1 \wedge x \in A_2 \wedge \ldots \wedge x \in A_n\}$$

Example. Let $U = \mathbf{N}$, and define:

$$A_i = \{i,\ i+1,\ i+2,\ \ldots\}$$

Then

$$\bigcap_{i=1}^{n} A_i = \{n, n+1, n+2, \ldots\}$$

Example. Let $A = \{0, 2, 4, 6, 8\}$, $B = \{0, 1, 2, 3, 4\}$, and $C = \{0, 3, 6, 9\}$. What are $A \cup B \cup C$ and $A \cap B \cap C$?

2.2.4 Computer Representation of Sets

Let $U = \{x_1, x_2, \ldots, x_n\}$, and choose an arbitrary order of the elements of U, say

$$x_1, x_2, \ldots, x_n$$

Let $A \subseteq U$. Then the ***bit string representation*** of A is the bit string of length n : $a_1\, a_2 \ldots a_n$ such that $a_i = 1$ if $x_i \in A$, and 0 otherwise.

Example. If $U = \{x_1, x_2, \ldots, x_6\}$, and $A = \{x_1, x_3, x_5, x_6\}$, then the bit string representation of A is

(101011)

Sets as bit strings

Ex. If U = $\{x_1, x_2, \ldots, x_6\}$, $A = \{x_1, x_3, x_5, x_6\}$, and $B = \{x_2, x_3, x_6\}$.

Then we have a quick way of finding the bit string corresponding to of $A \cup B$ and $A \cap B$.

A	1	0	1	0	1	1
B	0	1	1	0	0	1
$A \cup B$	1	1	1	0	1	1
$A \cap B$	0	0	1	0	0	1

Bit-wise OR

Bit-wise AND

Ex 57. Show how bitwise operations on bit strings can be used to find these combinations of $A = \{a, b, c, d, e\}$, $B = \{b, c, d, g, p, t, v\}$, $C = \{c, e, i, o, u, x, y, z\}$, and $D = \{d, e, h, i, n, o, t, u, x, y\}$.

a) $A \cup B$

b) $A \cap B$

c) $(A \cup D) \cap (B \cup C)$

d) $A \cup B \cup C \cup D$

2.3 Functions

2.3.1 Introduction

2.3.2 One-to-One and Onto Functions.

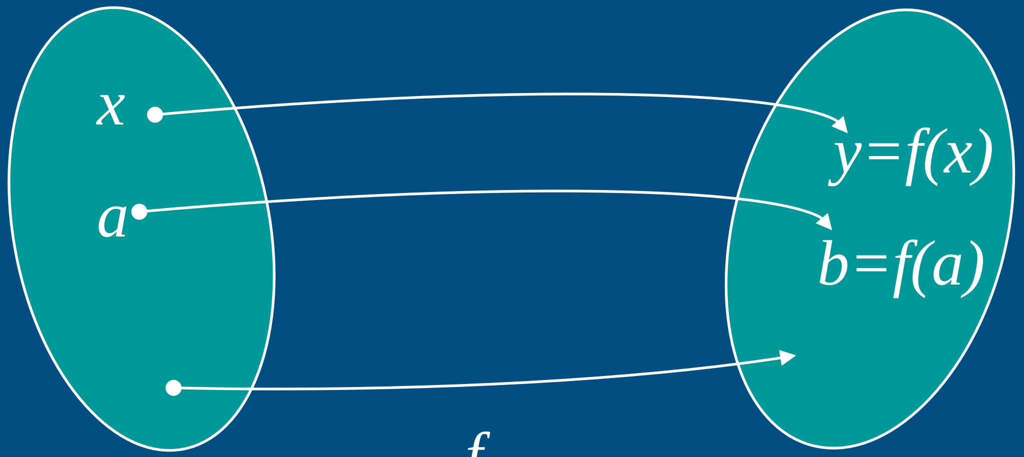
2.3.3 Inverse Functions and Composition of Functions

2.3.4 The Graphs of Functions

2.3.5 Some Important Functions

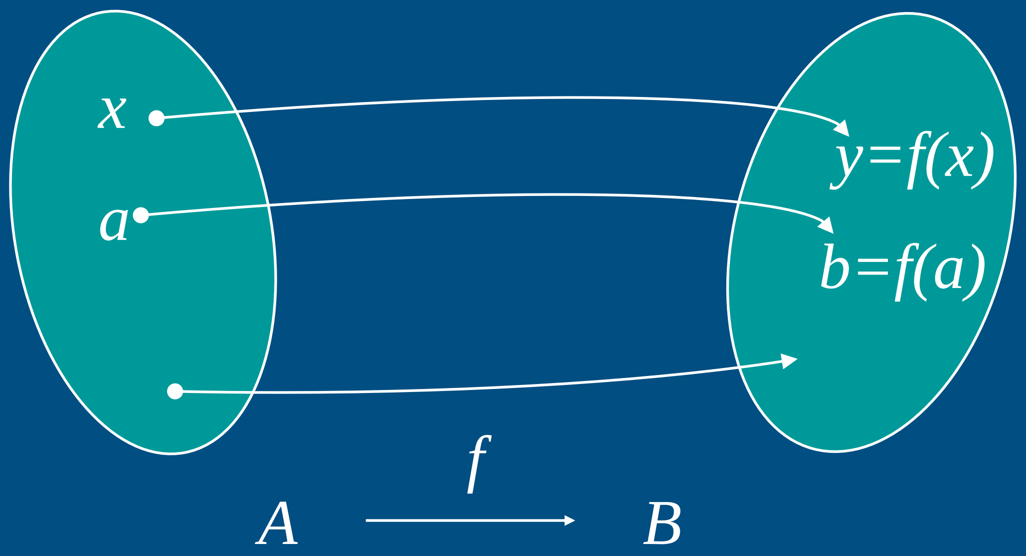
2.3.1 Introduction

Definition. A ***function*** *f* is a rule that assigns to each element x in a set A exactly one element $y=f(x)$ in a set B



$$A \xrightarrow{f} B$$

- A is the ***domain***, B is the ***codomain*** of f.

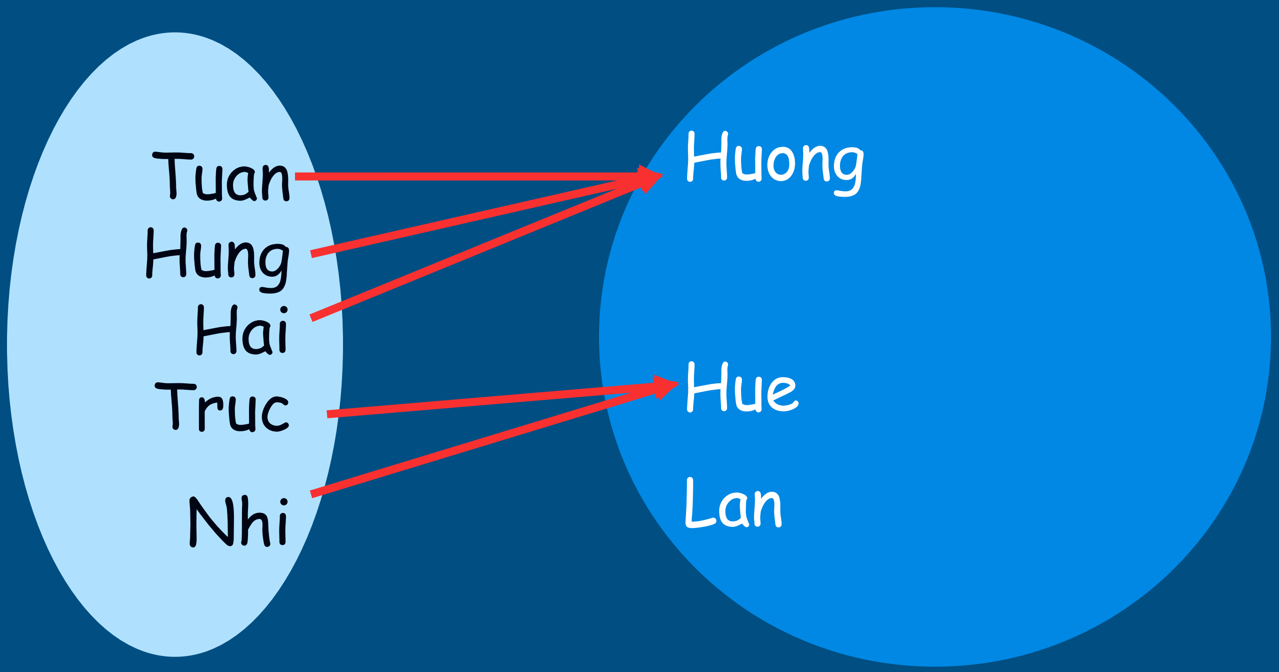


- $b = f(a)$ is the ***image*** of a and a is a ***preimage*** of b.
- The ***range*** of f is the set $\{f(a), a \in A\}$

Example.

A = {Tuan, Hung, Hai, Nhi, Truc}
B = {Huong, Hue, Lan}

Let $f: A \to B$ be defined as $f(a)$ = mother(a).



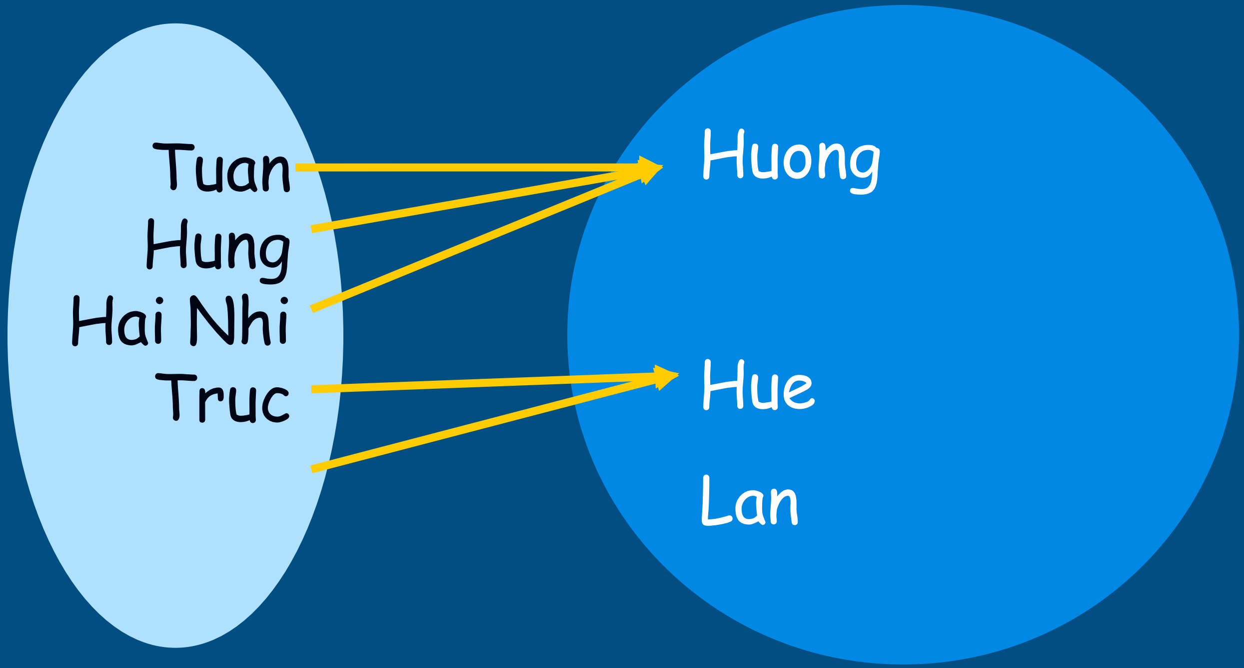
Ex 3/152. Determine whether f is a function from the set of all bit strings to the set of integers if

a) $f(S)$ is the position of a 0 bit in S.

b) $f(S)$ is the number of 1 bits in S.

c) $f(S)$ is the smallest integer i such that the ith bit of S is 1 and $f(S) = 0$ when S is the empty string, the string with no bits.

For any set $S \subseteq A$, ***image(S)*** $= \{f(x) : x \in S\} = f(S)$



- image({Tuan, Hung}) = {Huong}
- image(A) = {Huong, Hue}

What about the range?

Some say it means codomain, others say, image.

Ex 6/152. Find the domain and range of these functions.

a) the function that assigns to each pair of positive integers the first integer of the pair

b) the function that assigns to each positive integer its largest decimal digit

c) the function that assigns to a bit string the number of ones minus the number of zeros in the string

d) the function that assigns to each positive integer the largest integer not exceeding the square root of the integer

e) the function that assigns to a bit string the longest string of ones in the string

Algebra of functions: Let f and g be functions with domains A and B. Then the functions $f+g$, $f-g$, fg and f/g are defined as follows:

$$(f+g)(x) = f(x) + g(x) \qquad \text{domain} = A \cap B$$

$$(f-g)(x) = f(x) - g(x) \qquad \text{domain} = A \cap B$$

$$(fg)(x) = f(x)g(x) \qquad \text{domain} = A \cap B$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad \text{domain} = \{x \in A \cap B \mid g(x) \neq 0\}$$

Example. Let $f(x) = x^2$, $g(x) = x - x^2$ be functions from **R** to **R**, find $f + g$ and fg.

Solution. We have

$$(f + g)(x) = x^2 + (x - x^2) = x$$

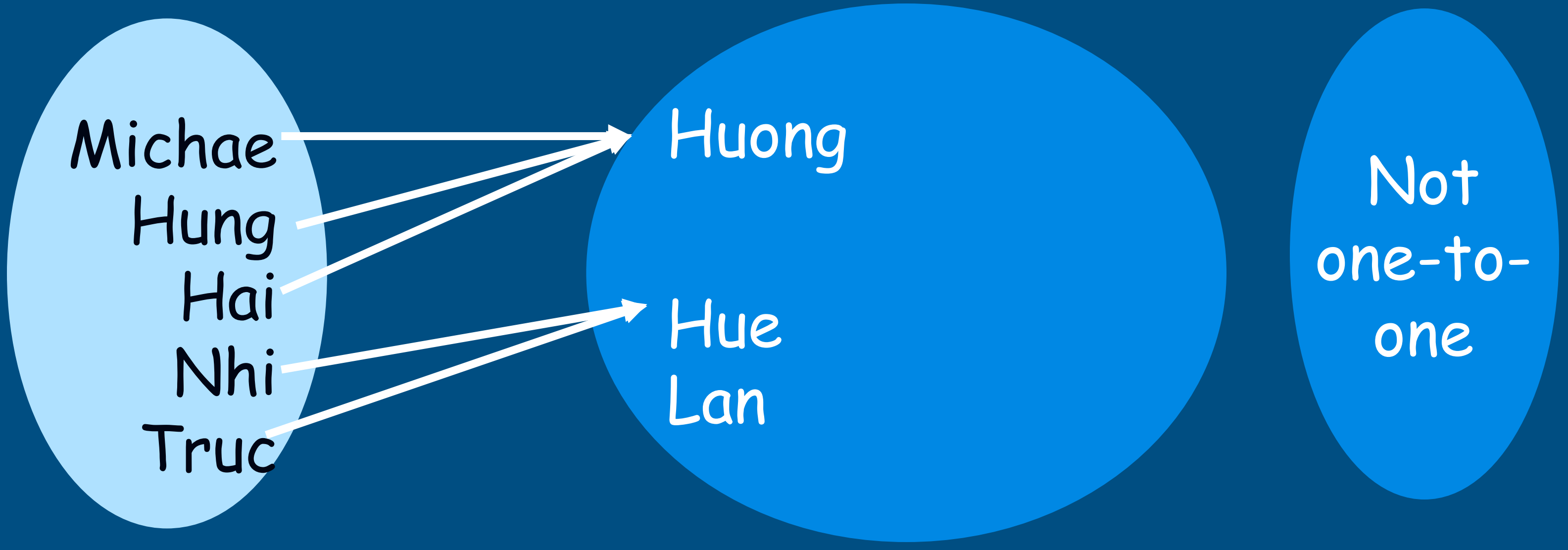
and

$$(fg)(x) = x^2 (x - x^2) = x^3 - x^4$$

2.3.2 One-to-One and Onto Functions

Definition. A function $f: A \rightarrow B$ is ***one-to-one*** (***injective***, ***an injection***) if

$$\forall x, \forall y, \quad f(x) = f(y) \rightarrow x = y$$



Every $b \in B$ has at most 1 preimage.

Ex 10/152. Determine whether each of these functions from $\{a, b, c, d\}$ to itself is one-to-one.

a) $f(a) = b, f(b) = a, f(c) = c, f(d) = d$

b) $f(a) = b, f(b) = b, f(c) = d, f(d) = c$

c) $f(a) = d, f(b) = b, f(c) = c, f(d) = d$

Ex 12/152. Determine whether each of these functions from **Z** to **Z** is one-to-one.

a) $f(n) = n - 1$ **b)** $f(n) = n^2 + 1$

c) $f(n) = n^3$ **d)** $f(n) = n^4$-n

Remark. A function f: A $\rightarrow$ B is ***one-to-one*** iff

$$\forall x, \forall y, \quad x \neq y \rightarrow f(\mathrm{x}) \neq f(y)$$

Recall that

- A function f is ***strictly increasing*** on an interval $I \subseteq \mathbf{R}$ if

$$\forall x, \forall y, x < y \rightarrow f(\mathrm{x}) < f(y)$$

- f is strictly ***decreasing*** on I if

$$\forall x, \forall y, \; x < y \rightarrow f(\mathrm{x}) > f(y)$$

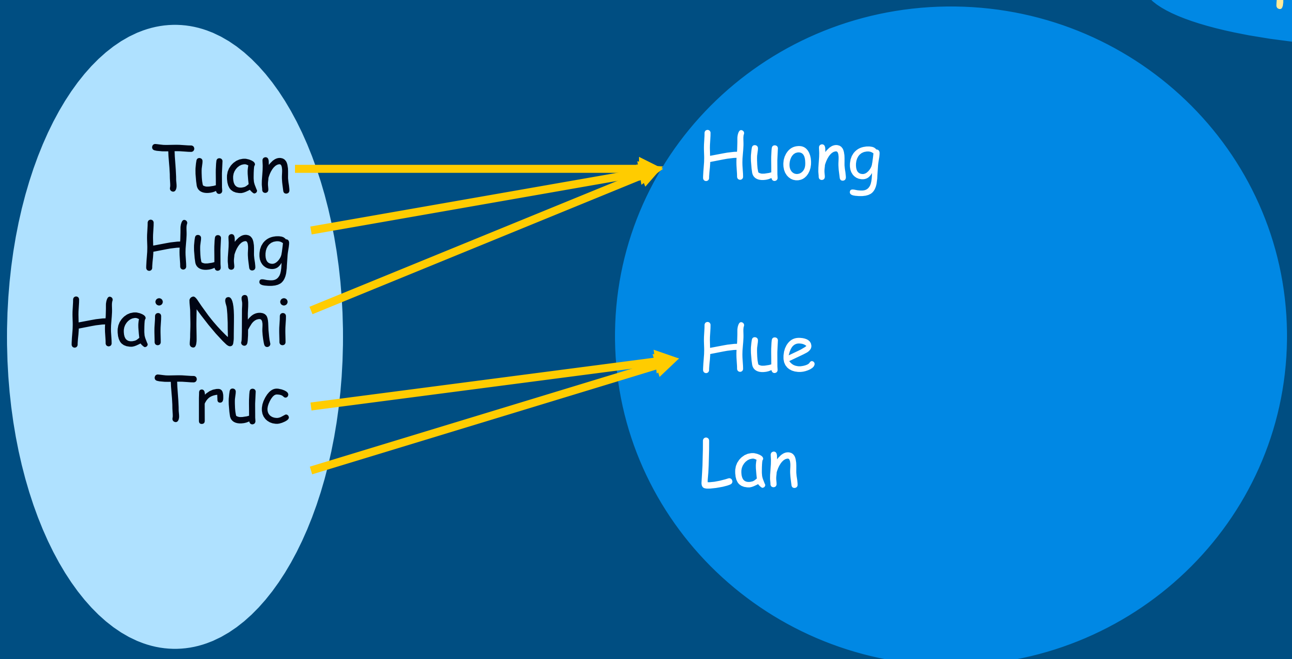
- It is clear that a strictly increasing or strictly decreasing function is one-to-one.

Onto Functions

Definition. A function $f: A \to B$ is ***onto*** (***surjective***, a ***surjection***) if $\forall b \in B, \exists a \in A, \ f(a) = b$

Every $b \in B$ has at least 1 preimage.

Not onto



Example. Is the function $f(x) = x^2$ from **Z** to **Z** onto?

Solution. The function f is not onto since there is no x in **Z** such that $x^2 = -1$

Example. Is the function $f(x) = x + 1$ from **Z** to **Z** onto?

Solution. The function f is onto since for every y in **Z,** there is an element x in **Z** such that $x + 1 = y$ (by taking $x = y - 1$)

Ex 14/152 . Determine whether $f : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ is onto if

a) $f(m, n) = 2m - n$.

b) $f(m, n) = m^2 - n^2$.

c) $f(m, n) = m + n + 1$.

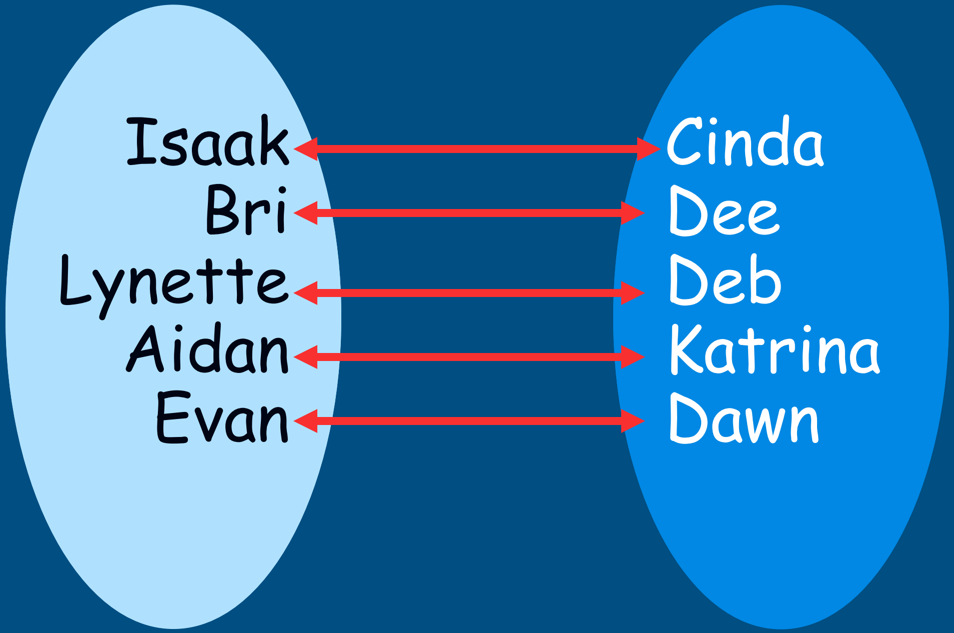
d) $f(m, n) = |m| - |n|$.

e) $f(m, n) = m^2 - 4$.

Bijection

Definition. A function f: $A \to B$ is bijective if it is one-to-one and onto. We also say that f is a ***bijection***

Every b ∈ B has exactly 1 preimage.



Ex 22./ 152. Determine whether each of these functions is a bijection from **R** to **R**.

a) $f(x) = -3x + 4$

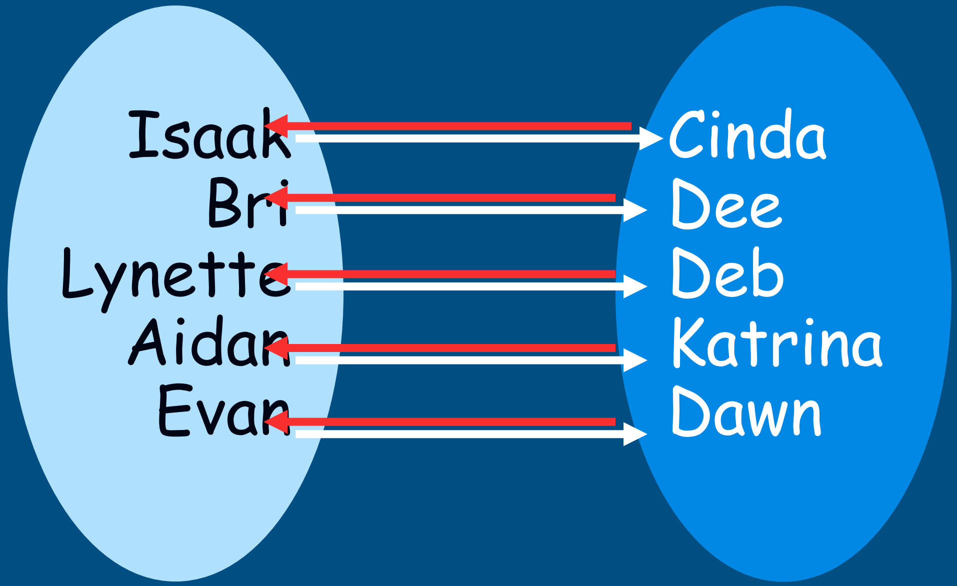
b) $f(x) = -3x^2 + 7$

c) $f(x) = (x + 1)/(x + 2)$

d) $f(x) = x^5 + 1$

2.3.3 Inverse Functions and Compositions of Functions

Definition. Let $f : A \to B$ be a ***bijection***. Then the inverse function of f, denoted by f^{-1} is the function that assigns each element b in B the unique element a in A such that $f(a) = b$. Thus $f^{-1}(b) = a$.



f^{-1}(Cinda) = Isaak, f^{-1}(Dee) = *Bri*, …, f^{-1}(Dawn) = Evan

Example. Is the function $f(x) = x^2$ from **Z** to **Z** invertible? (i.e. the inverse function exists)

Solution. The function f is not onto. Therefore it is not a bijection, and hence not invertible

Example. Is the function $f(x) = x + 1$ from **Z** to **Z** invertible?

Solution. The function f is a bijection so it is invertible.

Example. Is the function $f(x) = x + 1$ from **Z** to **Z** invertible? What is its inverse?

Solution. The function f is a bijection so it is invertible. To find the inverse, let y be any element in **Z**, we find the element x in **Z** such that $y = f(x) = x + 1$.

Solving this equation we obtain $x = y - 1$.
Hence $f^{-1}(y) = y - 1$.

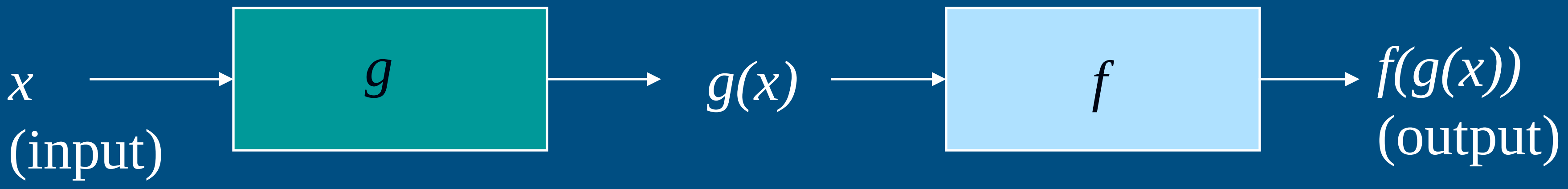
We also write $f^{-1}(x) = x - 1$.

Ex 69/152. Find the inverse function of $f(x) = x^3 + 1$.

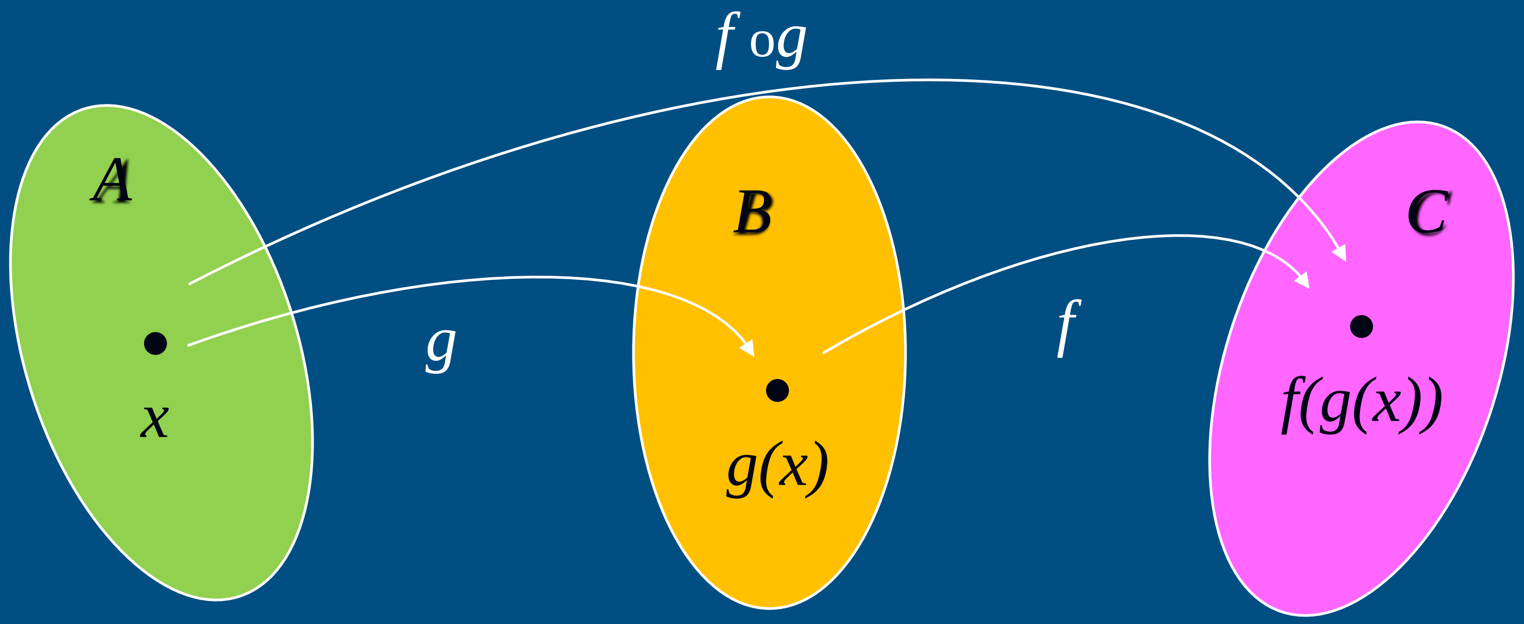
Definition. The ***composition*** of a function $g: A \to B$ and a function $f : B \to C$ is the function $f \circ g : A \to C$ defined by

$$f \circ g(x) = f(g(x))$$

Note. The domain of $f \circ g$ is also the domain of g, and the codomain of $f \circ g$ is also the codomain of f.



Arrow diagram for $f \circ g$



Example. Let $f(x) = x^2$ and $g(x) = x - 3$ are functions from **R** to **R**.

Find the compositions $f \circ g$ and $g \circ f$

Solution.

- $(f \circ g)(x) = f(g(x)) = f(x - 3) = (x - 3)^2$
- $(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 - 3$

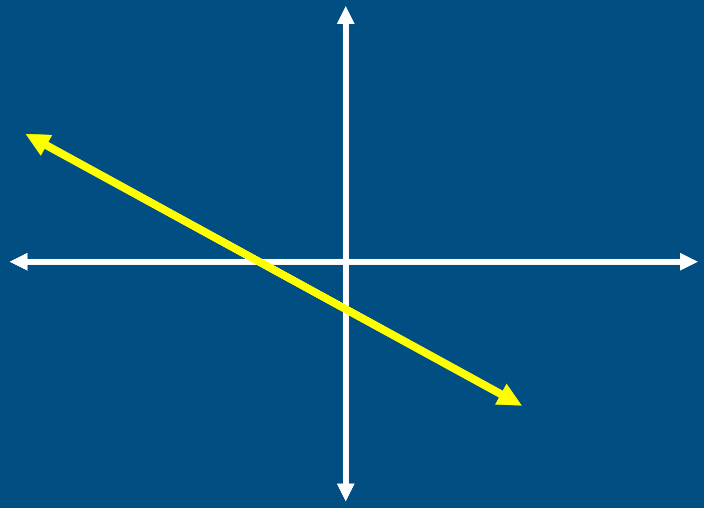
This shows that in general: $f \circ g \neq g \circ f$

Ex 38 /152. Let $f(x) = 2x + a$ and $g(x) = bx + 1$, where *a and b* are constants. Find a, b such that $f \circ g = g \circ f$.

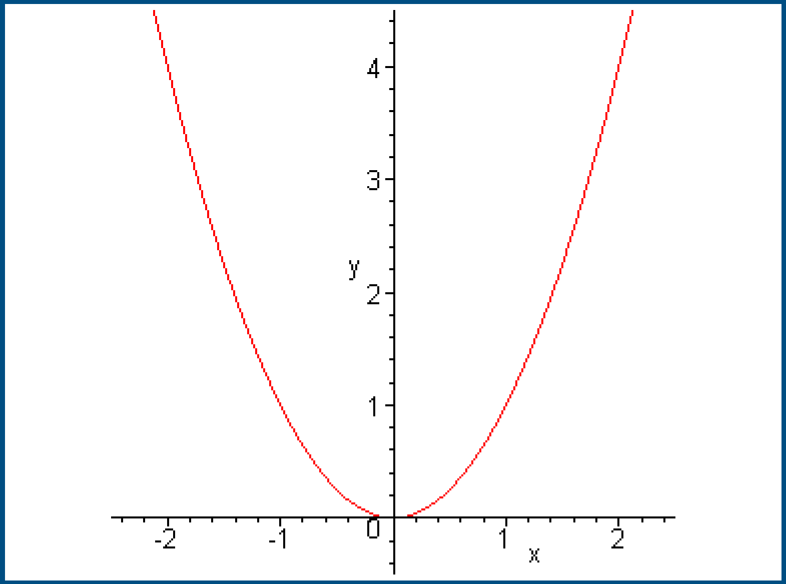
2.3.4 The Graph of a Function

Definition. Let $f : A \to B$ be a function. Then the ***graph*** of f, is the set of ordered pair (a, b) with a in A and b $=f(a)$.

Example. The graph of the function $f : \mathbf{R} \to \mathbf{R}$ such that $f(x) = - x / 2 - 25$



Example. The graph of the function $f : \mathbf{R} \to \mathbf{R}$ such that $f(x)=x^2$



Ex 62/ 152. Draw the graph of the function $f(n) = 1 - n^2$ from $\mathbf{Z}$ to $\mathbf{Z}$.

2.3.5 Some Important Functions

Ceiling

$f(x) = \lceil x \rceil$ the least integer y so that $x \le y$.

Ex: $\lceil 1.2 \rceil = 2; \lceil -1.2 \rceil = -1; \lceil 1 \rceil = 1$

Floor

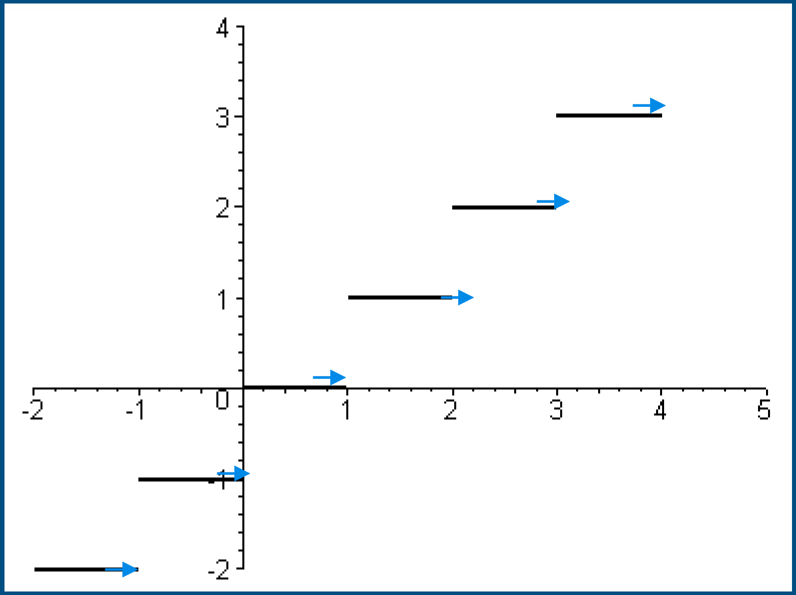
$f(x) = \lfloor x \rfloor$ the greatest integer y so that $y \le x$.

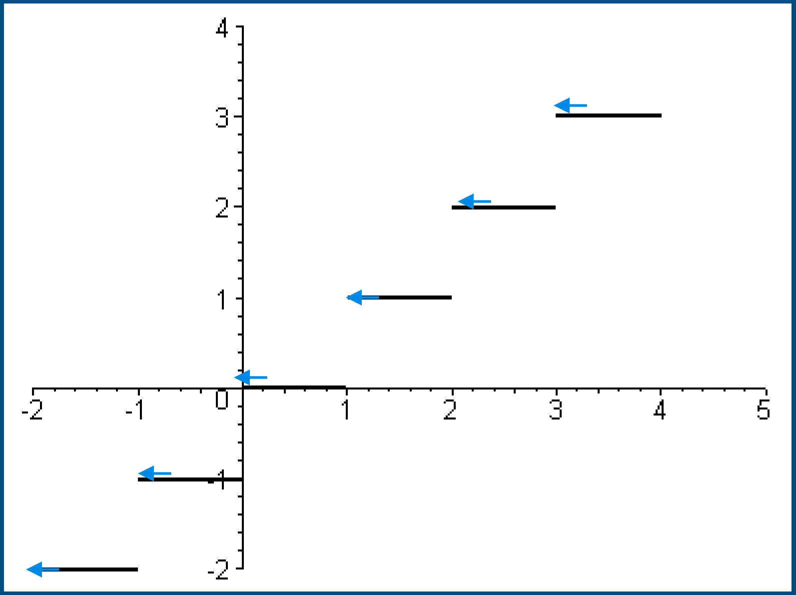
Ex: $\lfloor 1.8 \rfloor = 1; \lfloor -1.8 \rfloor = -2; \lfloor -5 \rfloor = -5$

0

Quiz. What is $\lceil -1.2 + \lfloor 1.1 \rfloor \rceil$?

The graph of the floor function





The graph of the ceiling function

30. Let $S = \{-1, 0, 2, 4, 7\}$. Find $f(S)$ if

a) $f(x) = 1$.

b) $f(x) = 2x + 1$.

c) $f(x) = \lceil x/5 \rceil$.

d) $f(x) = \lfloor (x^2 + 1)/3 \rfloor$.

64. Draw the graph of the function $f(x) = \lfloor x/2 \rfloor$ from **R** to **R**.

65. Draw the graph of the function $f(x) = \lfloor x \rfloor + \lfloor x/2 \rfloor$ from **R** to **R**.

2.4 Mathematical Induction

2.4.1 Proofs by Mathematical Induction

2.4.2 Inductive Definitions

2.4.1 Proofs by Mathematical Induction

Prove that if a set S has $|S| = n$, then $|P(S)| = 2^n$

Proof. Let $R(n)$ be the predicate:

"If $|S| = n$, then $|P(S)| = 2^n$"

- **Base case** (n=0): $S = \emptyset$, $P(S) = \{\emptyset\}$ and $|P(S)| = 1 = 2^0$

- **Assume** $R(k)$: "If $|S| = k$, then $|P(S)| = 2^k$"

Inductive hypothesis

Prove that $R(k+1)$. It means

if $|S'| = k+1$, then $|P(S')| = 2^{k+1}$

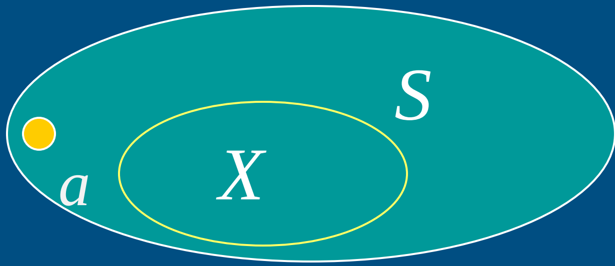
$S' = S \cup \{a\}$ for some $S \subset S'$ with $|S| = k$, and $a \in S'$, $a \notin S$

Mathematical Induction

$S' = S \cup \{a\}$, $S \subset S'$ with $|S| = k$, and $a \in S'$, $a \notin S$

Partition the power set of S' into the sets containing a and those not.

Let X be an arbitrary element of $P(S')$,
Then either $a \in X$ or $a \notin X$:



$$\mathbf{P(S') = \{X : a \in X\} \cup \{X : a \notin X\}}$$

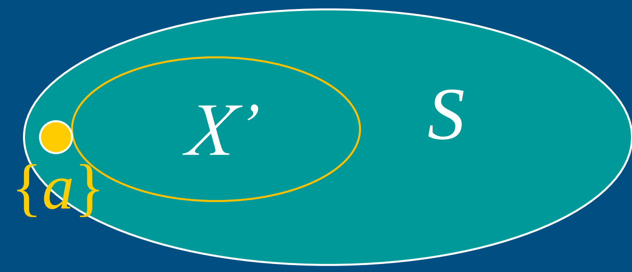
If $a \notin X$, then X is a subset of S so that

$$\{X : a \notin X\} = P(S)$$

If $a \in X$ then we can write: $X = \{a\} \cup X'$ with $a \notin X'$

So $|\{X : a \in X\}| = |\{X' : a \notin X'\}|$

$= |\mathrm{P}(S)|$



Hence $|\mathrm{P}(S')| = |\{X : a \in X\}| + |\mathrm{P}(S)|$

$= 2\,|\mathrm{P}(S)|$

$= 2 \cdot 2^k = 2^{k+1}$

Therefore $R(n)$ is true for all n

Why does it work?

Theorem. Let $P(n)$ be a predicate with the universe of discourse $\mathbf{N}$. Assume that

- $P(0)$ is true
- $\forall k, P(k) \rightarrow P(k+1)$

Then $\forall n, P(n)$ is true

PRINCIPLE OF MATHEMATICAL INDUCTION

To prove that *P(n)* is true for all positive integers $n \geq N_o$, where *P(n)* is a propositional function, we complete two steps:

- Basis Step: We verify that P(N_o) is true.

- Inductive Step: We show that the conditional statement P(k) → P(k + 1) is true for all positive integers *k*.

Example. Show that if n is a positive integer, then

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Solution. Let

$$P(n) = "1 + 2 + \cdots + n = \frac{n(n+1)}{2}"$$

Basis Step: P(1) is true, because $1 = \frac{1(1+1)}{2}$

Inductive Step: For the inductive hypothesis we assume that P(k) holds for an arbitrary positive integer k. That is, we assume that

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

Under this assumption, it must be shown that P(k + 1) is true, namely, that

$$1 + 2 + \cdots + k + (k + 1) = \frac{(k + 1)(k + 2)}{2}$$

is also true.

When we add k + 1 to both sides of the equation in P(k), we obtain

$$1 + 2 + \cdots + k + (\boldsymbol{k} + \mathbf{1}) = \frac{k(k+1)}{2} + (\boldsymbol{k} + \mathbf{1})$$

$$= \frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

This last equation shows that P(k + 1) is true under the assumption that P(k) is true. This completes the inductive step.

Example. Show that if n is a positive integer, then

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

Ex 31./ 330. Prove that 2 divides $n^2 + n$ whenever n is a positive integer.

Ex 10/330. a) Find a formula for

$$\frac{1}{1\,.\,2} + \frac{1}{2\,.\,3} + \frac{1}{3\,.\,4} + \cdots + \frac{1}{n\,.\,(n+1)}$$

by examining the values of this expression for small values of n.

b) Prove the formula you conjectured in part (a).

2.4.2 Inductive Definitions

We completely understand the function $f(n) = n\,!$, right?

As a reminder, here's the definition:

$$n\,! = n \cdot (n-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1, \qquad n \geq 1$$

But equivalently, we could define it like this:

$$n! = \begin{cases} n \cdot (n-1)!, & \text{if } n > 1 \\ 1, & \text{if } n = 1 \end{cases}$$

Recursive Case

Base Case

Inductive (Recursive) Definition

Another VERY common example:

Fibonacci Numbers

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ f(n-1) + f(n-2) & \text{if } n > 1 \end{cases}$$

Base Cases

Recursive Case

Is there a non-recursive definition for the Fibonacci Numbers?

$$f(n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Ex 1/357. Find f (1), f (2), f (3), and f (4) if f (n) is defined recursively by f (0) = 1 and for n = 0, 1, 2, . . .

a) $f(n + 1) = f(n) + 2$.

b) $f(n + 1) = 3f(n)$.

c) $f(n + 1) = 2f(n)$.

d) $f(n + 1) = f(n)^2 + f(n) + 1$.