CHAPTER 4

RELATIONS

4. Relations

4.1 Relations and Their Properties

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4.3 Equivalent Relations, Congruences, Arithmetic Operations on Zn

4.4 Partial Orderings, Hasse Diagram

4.1 Relations and Their Properties

4.1.1 Definition

4.1.2 Properties of Relations

4.1.3 Combining Relations

4.1.1 Definitions

Definition. Let A and B be sets. A ***binary relation*** from A to B is a subset R of $A \times B$.

A relation from a set A to itself is called a ***relation*** on A.

Example: Let A = {a, b, c} and B = {1, 2, 3}. Then

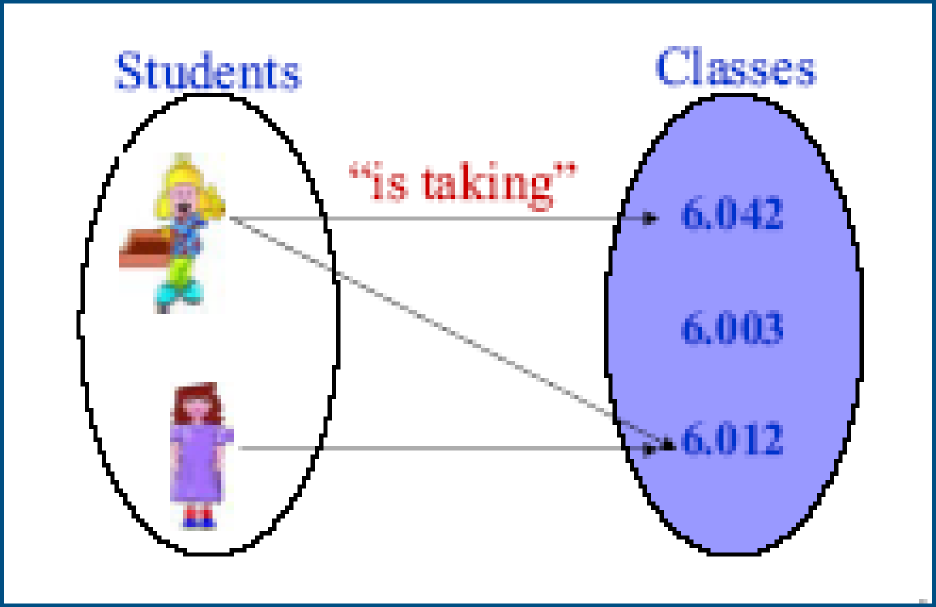
$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3), (c, 1), (c, 2), (c, 3)\}$.

So, in particular, $R=\{(a, 3), (c, 1), (c, 2), (c, 3)\}$ is a binary relation from A to B. Since $(a, 3) \in R$, we say that

"a is related to 3" or "a R 3"

Example. A = students; B = courses.

$R = \{(a, b) \mid \text{student } a \text{ is enrolled in class } b\}$



Example: Let A = {1, 2, 3, 4} and define

$$R = \{(a, b) \mid a \text{ divides } b\}.$$

Then

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}.$$

Example. List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$, where $(a, b) \in R$ if and only if

a) $a = b$.

b) $a + b = 4$.

c) $a > b$.

d) $\gcd(a, b) = 1$.

Example. Consider these relations on the set of integers:

$$R_1 = \{(a, b) \mid a \leq b\},$$

$$R_2 = \{(a, b) \mid a > b\},$$

$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\},$$

$$R_4 = \{(a, b) \mid a = b\},$$

$$R_5 = \{(a, b) \mid a = b + 1\},$$

$$R_6 = \{(a, b) \mid a + b \leq 3\}.$$

Which of these relations contain each of the pairs (1, 1), (1, 2), (2, 1), (1, −1), and (2, 2)?

Example: If A is a finite set with $|A| = n$, how many different relations are there on A?

Solution. Since a relation on A is simply a subset of $A \times A$, then we are really asking "how many subsets are there of $A \times A$" or $|P(A \times A)|$?

We know that $|A \times A| = n^2$ so $|P(A \times A)| = 2^{|A \times A|} = 2^{n^2}$.

4.1.2 Properties of Relations

Definition. A relation R on a set A is ***reflexive*** if:

$$(a, a) \in R \text{ for all } a \in A$$

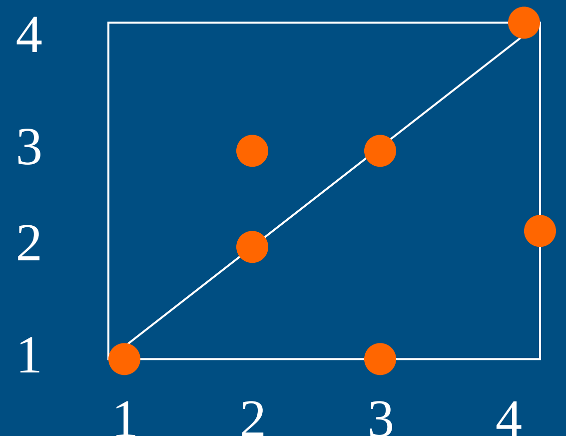
Example. On the set $A = \{1, 2, 3, 4\}$, the relation:

- $R_1 = \{(1,1), (1,2), (2,1), (2, 2), (3, 4), (4, 1), (4, 4)\}$ is not reflexive since $(3, 3) \notin R_1$
- $R_2 = \{(1,1), (1,2), (1,4), (2, 2), (3, 3), (4, 1), (4, 4)\}$ is reflexive since $(1,1), (2, 2), (3, 3), (4, 4) \in R_2$

- The relation $\leq$ on $\boldsymbol{Z}$ is reflexive since $a \leq a$ for all $a \in \boldsymbol{Z}$
- The relation $>$ on $\boldsymbol{Z}$ is not reflexive since $1 \not> 1$
- The relation " | " ("divides") on $\boldsymbol{Z}^+$ is reflexive since any integer a divides itself

Note. A relation R on a set A is reflexive iff it contains the diagonal of $A \times A$:

$$\Delta = \{(a, a); a \in A\}$$



Definition. A relation R on a set A is ***symmetric*** if:

$$\forall a \in A\ \forall b \in A,\ (a\ R\ b) \rightarrow (b\ R\ a)$$

The relation R is said to be ***antisymmetric*** if:

$$\forall a \in A\ \forall b \in A,\ (a\ R\ b) \wedge (b\ R\ a) \rightarrow (a = b)$$

Example.

- The relation $R_1 = \{(1,1), (1,2), (2,1)\}$ on the set $A = \{1, 2, 3, 4\}$ is symmetric
- The relation $\leq$ on **Z** is not symmetric.

  However it is antisymmetric since

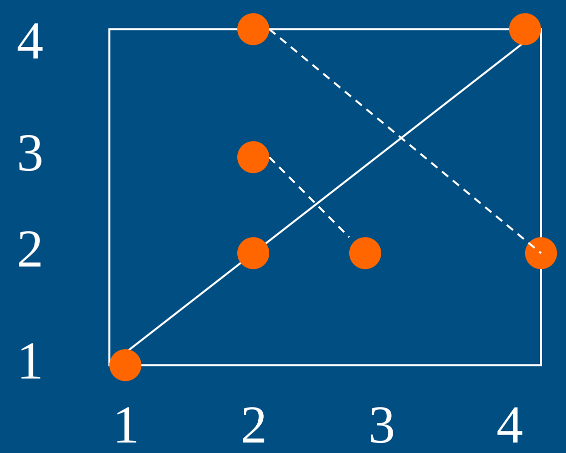
$$(a \leq b) \wedge (b \leq a) \rightarrow (a = b)$$

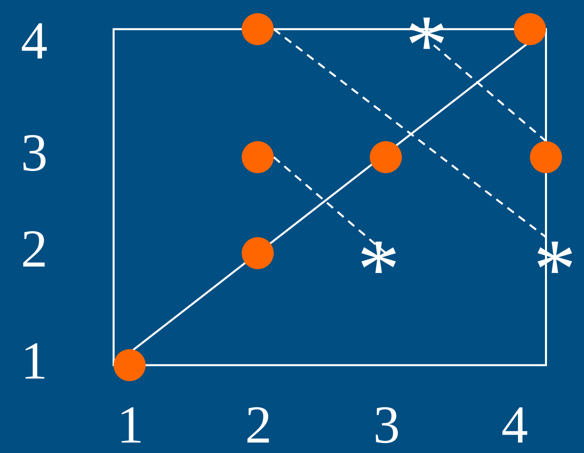
- The relation " | " ("divides") on Z^+ is not symmetric. However it is antisymmetric

$$(a \mid b) \wedge (b \mid a) \rightarrow (a = b)$$

Note. A relation R on a set A is symmetric iff it is self symmetric with respect to the diagonal Δ of $A \times A$.

The relation R is antisymmetric iff the only self symmetric parts lie on the diagonal Δ of $A \times A$.





Definition. A relation R on a set A is ***transitive*** if:

$$\forall a \in A\ \forall b \in A\ \forall c \in A,\ (a\ R\ b) \wedge (b\ R\ c) \rightarrow (a\ R\ c)$$

Example.

- The relation $R = \{(1,1), (1,2), (2,1), (2, 2), (1, 3), (2, 3)\}$ on the set $A = \{1, 2, 3, 4\}$ is transitive

- The relations $\leq$ and "$|$" on $\boldsymbol{Z}$ are transitive

$$(a \leq b) \wedge (b \leq c) \rightarrow (a \leq c)$$

$$(a \mid b) \wedge (b \mid c) \rightarrow (a \mid c)$$

Ex 3. For each of these relations on the set {1, 2, 3, 4}, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive.

a) {(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)}

b) {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}

c) {(2, 4), (4, 2)}

d) {(1, 2), (2, 3), (3, 4)}

e) {(1, 1), (2, 2), (3, 3), (4, 4)}

f) {(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)}

Ex 4. Determine whether the relation *R* on the set of all people is reflexive, symmetric, antisymmetric, and/or transitive, where $(a, b) \in R$ if and only if

a) *a* is taller than *b*.

b) *a* and *b* were born on the same day.

c) *a* has the same first name as *b*.

d) *a* and *b* have a common grandparent.

4.1.3 Combining Relations

Let R_1 and R_2 be relations from A to B, then the set operations

$$R_1 \cup R_2,\ R_1 \cap R_2,\ R_1 - R_2,\ R_2 - R_1,\quad R_1 \oplus R_2$$

are defined as usual.

Example.

- ✓ $A = \{\text{students}\}, \quad B : \{\text{courses}\}$
- ✓ $R_1 = \{(a, b)$, student a takes the course $b\}$
- ✓ $R_2 = \{(a, b)$, student a requires course b to graduate$\}$

- ❖ $R_1 \cap R_2 = \{(a, b)$, student a takes course b and requires course b to graduate$\}$
- ❖ $R_1 - R_2 = \{(a, b)$, student a takes course b but doesn't need b to graduate (b is elective course)$\}$

Example. Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The relations $R_1 = \{(1, 1), (2, 2), (3, 3)\}$ and $R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$ can be combined to obtain

- $R_1 \cup R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (3, 3)\}$,
- $R_1 \cap R_2 = \{(1, 1)\}$,
- $R_1 - R_2 = \{(2, 2), (3, 3)\}$,
- $R_2 - R_1 = \{(1, 2), (1, 3), (1, 4)\}$.

Ex 30. Let $R_1 = \{(1, 2), (2, 3), (3, 4)\}$ and $R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (3, 4)\}$ be relations from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$. Find

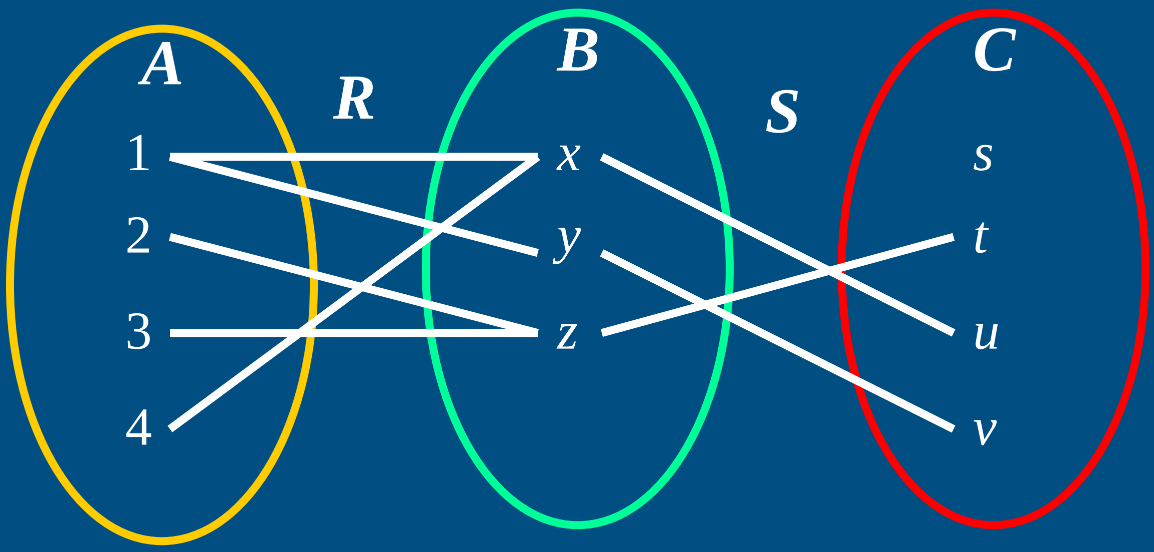
a) $R_1 \cup R_2$. b) $R_1 \cap R_2$.

c) $R_1 - R_2$. d) $R_2 - R_1$.

Definition. Let R be a relation from A to B and S a relation from B to C. Their ***composite*** is defined as :

$$S \circ R = \{(a, c) \in A \times C,\ \exists\, b \in B: (a, b) \in R \wedge (b, c) \in S\}$$

Example.



$$S \circ R = \{(1,u),(1,v),(2,t),(3,t),(4,u)\}$$

Example.

Let $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$, $C = \{0, 1, 2\}$

$$R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$$

$$S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$$

Then the composite relation is

$$S \circ R = \{(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)\}$$

Example. Let A = {all people} and $R = \{(a, b) \in A \times A,$ a is a parent of $b\}$. Then the composite relation is

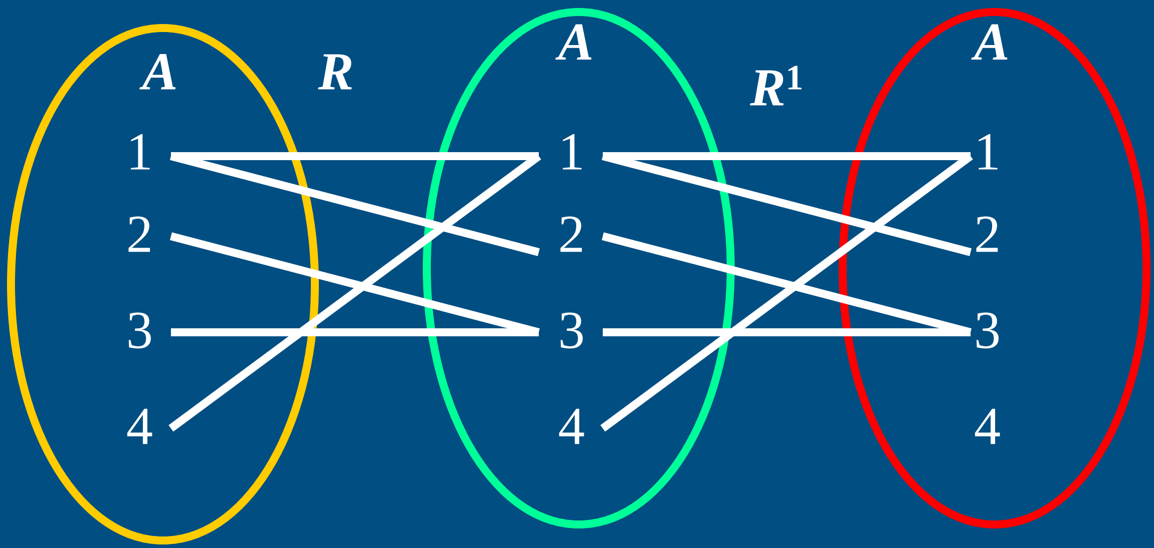
$$R \circ R = \{(a, c) \in A \times A,\ a \text{ is a grandparent of } c\}$$

Definition. Let R be a relation on A. The ***powers*** R^n are defined recursively by:

$$R^1 = R \qquad \text{and} \qquad R^{n+1} = R^n \circ R$$

Definition. Let R be a relation on A. The ***powers*** R^n are defined recursively by:

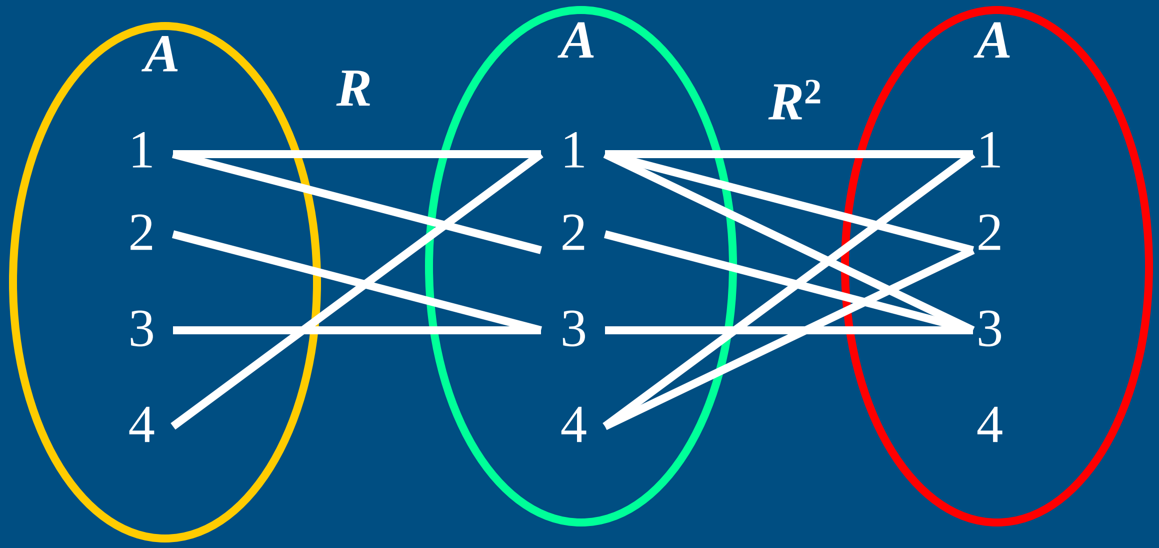
$$R^1 = R \qquad \text{and} \qquad R^{n+1} = R^n \circ R$$



$$R^2 = R^1 \circ R = \{(1,1),(1,2),(1,3),(2,3),(3,3),(4,1),(4,2)\}$$

Definition. Let R be a relation on A. The ***powers*** R^n are defined recursively by:

$$R^1 = R \qquad \text{and} \qquad R^{n+1} = R^n \circ R$$



$R^4 =$

$R^5 =$

$R^6 =$

…

$R^3 = R^2 \circ R =$
$\{(1,1),(1,2),(1,3),(2,3),(3,3),(4,1),(4,2),(4,3)\}$

Ex 56. Let R be the relation on the set {1, 2, 3, 4, 5} containing the ordered pairs (1, 1), (1,3), (2, 3), (3, 4), (3, 5), (4, 2), (5, 2), and (5,4).

Find R^2 R^3 R^4 R^5.

Theorem. The relation R on A is transitive iff

$$R^n \subseteq R \qquad \text{for } n = 1, 2, 3, \ldots$$

Ex 3. For each of these relations on the set {1, 2, 3, 4}, decide whether it is transitive.

a) {(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)}

b) {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}

c) {(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)}

4.2 Representing Relations

4.2.1 Introduction

4.2.2 Matrices

4.2.3 Representing Relations

4.2.4 Bit Operations on Representing Matrices

4.2.1 Introduction

Let R be a relation from $A = \{1,2,3,4\}$ to $B = \{u,v,w\}$:

$$R = \{(1,u), (1,v), (2,w), (3,w), (4,u)\}.$$

Then we can represent R as:

	u	v	w
1	1	1	0
2	0	0	1
3	0	0	1
4	1	0	0

This is a 4×3-matrix whose entries indicate membership in R

- Now assume that **A** and **B** are ***zero-one matrices***, then we can view them as bit strings.
- Therefore it makes sense to talk about bitwise operations on such matrices

Definition. Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be zero-one $m \times n$ matrices. The ***joint*** $\mathbf{A} \vee \mathbf{B} = [c_{ij}]$ and the ***meet*** $\mathbf{A} \wedge \mathbf{B} = [d_{ij}]$ are $m \times n$ matrices defined by

$$c_{ij} = a_{ij} \vee b_{ij} \text{ for } 1 \leq i \leq m \text{ , } 1 \leq j \leq n$$
$$d_{ij} = a_{ij} \wedge b_{ij} \text{ for } 1 \leq i \leq m \text{ , } 1 \leq j \leq n$$

$$\mathbf{A} \vee \mathbf{B} = [c_{ij}] \text{ and } \mathbf{A} \wedge \mathbf{B} = [d_{ij}]$$

$$c_{ij} = a_{ij} \vee b_{ij} \text{ for } 1 \le i \le m,\ 1 \le j \le n$$

$$d_{ij} = a_{ij} \wedge b_{ij} \text{ for } 1 \le i \le m,\ 1 \le j \le n$$

Example.

Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

Then $\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

and $\mathbf{A} \wedge \mathbf{B} = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Definition. Let $\mathbf{A} = [a_{ij}]$ be an $m \times n$ zero-one matrix and $\mathbf{B} = [b_{jk}]$ be an $n \times p$ zero-one matrix.
Then the ***Boolean product*** $\mathbf{A} \odot \mathbf{B} = [c_{ik}]$ is the $m \times p$ zero-one matrix defined by

$$c_{ik} = (a_{i1} \wedge b_{1k}) \vee (a_{i2} \wedge b_{2k}) \vee \ldots \vee (a_{in} \wedge b_{nk})$$

$$\text{for } 1 \le i \le m\ ,\ 1 \le k \le p$$

Note. The $(i, k)^{\text{th}}$ entry of $\mathbf{A} \odot \mathbf{B}$ is obtained by:

- ✓ carrying out the AND operator to every element of the i^{th} row of the matrix **A** to the corresponding element of the k^{th} column of **B**,
- ✓ then take the OR operator on n obtained results

$$c_{ik} = (a_{i1} \wedge b_{1k}) \vee (a_{i2} \wedge b_{2k}) \vee \ldots \vee (a_{in} \wedge b_{nk})$$

$$1 \le i \le m\,,\ 1 \le k \le p$$

Example. Let $\mathbf{A} = \begin{bmatrix} & \\ & \\ 1 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & & 0 \\ 0 & & 1 \end{bmatrix}$

$$c_{31} = (a_{31} \wedge b_{11}) \vee (a_{32} \wedge b_{21}) = (1 \wedge 1) \vee (0 \wedge 0) = 1$$

The other entries of $\mathbf{A} \odot \mathbf{B}$ are obtained in the same way:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Definition. Let R be a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$, then the ***representing matrix*** of R is the $m \times n$ zero-one matrix $\mathbf{M}_R = [m_{ij}]$ defined by

$$m_{ij} = \begin{cases} 0 & \text{if } (a_i, b_j) \notin R \\ 1 & \text{if } (a_i, b_j) \in R \end{cases}$$

Example. Let R be the relation from $A = \{1, 2, 3\}$ to $B = \{1, 2\}$ such that $a\,R\,b$ if $a > b$.

Then the representing matrix of R is

	1	2
1	0	0
2	1	0
3	1	1

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

Example. Let R be the relation from $A = \{a_1, a_2, a_3\}$ to $B = \{b_1, b_2, b_3, b_4, b_5\}$ represented by the matrix

$$\begin{array}{c} \\ \mathbf{M}_R = \end{array} \begin{array}{c} \begin{array}{ccccc} b_1 & b_2 & b_3 & b_4 & b_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array}$$

Then R consists of the pairs:

$\{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}$

Ex 2. Represent each of these relations on {1, 2, 3, 4} with a matrix.

a) {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}

b) {(1, 1), (1, 4), (2, 2), (3, 3), (4, 1)}

c) {(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3)}

d) {(2, 4), (3, 1), (3, 2), (3, 4)}

4. List the ordered pairs in the relations on $\{1, 2, 3, 4\}$ corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order).

a) $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

- Let R be a relation on a set A, then the matrix $\mathbf{M}_R$ that represents R is a ***square matrix***
- R is ***reflexive*** if and only if all ***diagonal entries*** of $\mathbf{M}_R$ are equal to 1: $m_{ii} = 1$ for all i

	u	v	w
u	1	1	0
v	0	1	1
w	0	0	1

- R is symmetric if and only if $\mathbf{M}_R$ is ***symmetric***

$$m_{ij} = m_{ji} \qquad \text{for all } i, j$$

	u	v	w
u	1	0	1
v	0	0	1
w	1	1	0

- R is ***antisymmetric*** if and only if $\mathbf{M}_R$ satisfies:

$$m_{ij} = 0 \text{ or } m_{ji} = 0 \quad \text{if } i \neq j$$

	u	v	w
u	1	0	1
v	0	0	0
w	0	1	1

Ex 9. How many nonzero entries does the matrix representing the relation R on $A = \{1, 2, 3, \ldots, 100\}$ consisting of the first 100 positive integers have if R is

a) $\{(a, b) \mid a > b\}$?

b) $\{(a, b) \mid a = b\}$?

c) $\{(a, b) \mid a = b + 1\}$?

d) $\{(a, b) \mid a = 1\}$?

e) $\{(a, b) \mid ab = 1\}$?

4.2.4 Bit-Operations on Representing Matrices

- Since the representing matrices are ***zero-one matrices*** we can carry out the matrix operations on these matrices
- Let R_1 and R_2 be relations, then we have the following identities on representing matrices

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2}$$

$$\mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2}$$

Example. Let R_1 and R_2 be relations on $A = \{u, v, w\}$

R_1	u	v	w
u	1	0	1
v	0	0	1
w	1	1	0

R_2	u	v	w
u	1	1	0
v	0	1	1
w	0	0	1

Then $R_1 \cap R_2$ is represented by:

$$\mathbf{M}_{R_1 \cap R_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2}$$

Theorem. The representing matrix $\mathbf{M}_{S \circ R}$ of the relation $S \circ R$ is precisely the matrix $\mathbf{M}_R \odot \mathbf{M}_S = [t_{ik}]$, where

$$t_{ik} = (r_{i1} \wedge s_{1k}) \vee (r_{i2} \wedge s_{2k}) \vee \ldots \vee (r_{in} \wedge s_{nk})$$

$$\text{for } 1 \leq i \leq m\,,\ 1 \leq k \leq n$$

Example. Find the representing matrix of the relation R^2, where R is the relation with representing matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Solution. The representing matrix for R^2 is

$$\mathbf{M}_R \odot \mathbf{M}_R = \mathbf{M}_{R^2}$$

Hence

$$\mathbf{M}_{R^2} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

14. Let R_1 and R_2 be relations on a set A represented by the matrices

$$\mathbf{M}_{R_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Find the matrices that represent

a) $R_1 \cup R_2$. **b)** $R_1 \cap R_2$.

c) $R_2 \circ R_1$. **d)** $R_1 \circ R_1$.

15. Let R be the relation represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Find the matrices that represent

a) R^2. **b)** R^3. **c)** R^4.

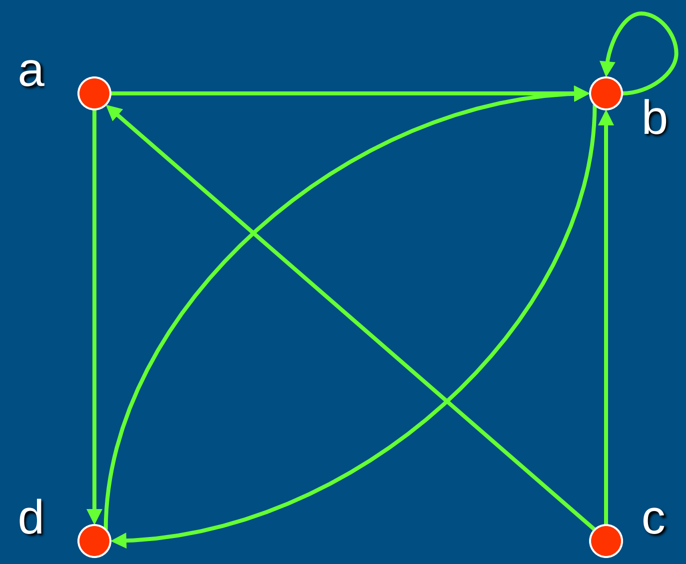
4.2.5 Representing Relations Using Digraphs

Definition: A ***directed graph***, or ***digraph***, consists of a set V of ***vertices*** (or ***nodes***) together with a set E of ordered pairs of elements of V called ***edges*** (or ***arcs***).

The vertex a is called the ***initial vertex*** of the edge (a, b), and the vertex b is called the ***terminal vertex*** of this edge.

We can use arrows to display graphs.

Example. Display the digraph with $V = \{a, b, c, d\}$, $E = \{(a, b), (a, d), (b, b), (b, d), (c, a), (c, b), (d, b)\}$.



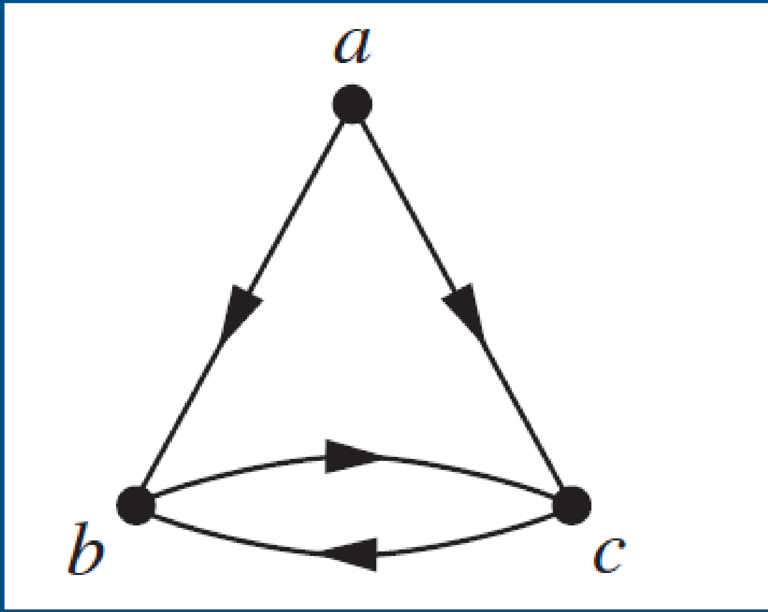
An edge of the form (b, b) is called a **loop**.

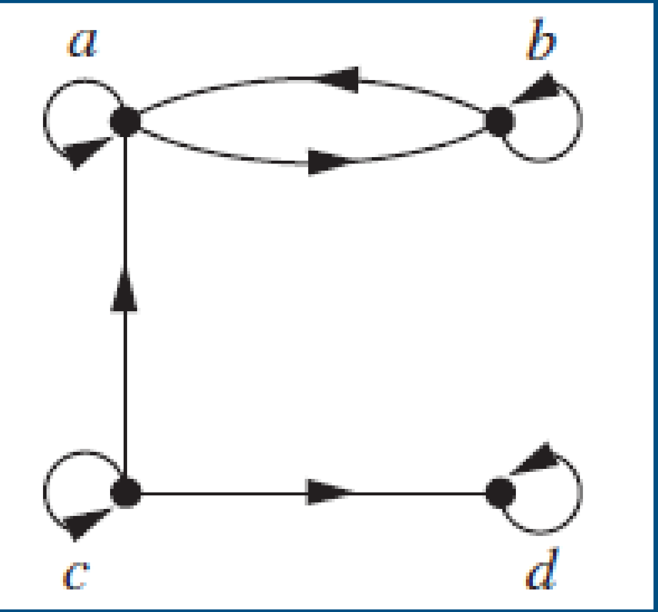
Obviously, we can represent any relation R on a set A by the digraph with A as its vertices and all pairs $(a, b) \in R$ as its edges.

Vice versa, any digraph with vertices V and edges E can be represented by a relation on V containing all the pairs in E.

This ***one-to-one correspondence*** between relations and digraphs means that any statement about relations also applies to digraphs, and vice versa.

Example. List the ordered pairs in the relations represented by the directed graphs





4.3 Equivalence Relations

Definition. A relation R on a set A is called an ***equivalence relation*** if it is reflexive, symmetric, and transitive.

Example. Let A be any set and define $R = \{(a, b) \mid a = b\}$. That is, every element of A is related to itself only.

R is an equivalence relation. R is clearly reflexive, symmetric and transitive. R is called the trivial equivalence on A.

Ex. Which of these relations on {0, 1, 2, 3} are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) {(0, 0), (1, 1), (2, 2), (3, 3)}

b) {(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)}

c) {(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)}

d) {(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2),(3, 3)}

e) {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)}

Example. Let A be the set of real numbers and define

$$R = \{(a, b) \mid a - b \text{ is an integer}\}.$$

R is an equivalence relation

Solution. R is reflexive since $a - a = 0$ for all $a \in A$ and 0 is an integer.

R is symmetric since if $a - b$ is an integer, then $-(a - b) = b - a$ is also an integer.

R is transitive since if $a - b$ is an integer and $b - c$ is an integer, then $a - c = (a - b) + (b - c)$ is also an integer as the sum of two integers.

Example. Let $m \in Z^+$. Then $R = \{(a, b) \mid a \equiv b \pmod{m}\}$ is an equivalence relation over the set of integers.

Proof. Let $m \in Z+$. Let $a \in Z$. Then $(a, a) \in R$ since $m \mid (a - a)$. So R is reflexive.

Now let $a, b \in Z$ such that $(a, b) \in R$. That is $a \equiv b \pmod{m}$ which means that $m \mid (a - b)$. So $a - b = km$ for some integer k. Then $b - a = -(a - b) = -km = (-k)m$. So $m \mid (b - a)$. So $b \equiv a \pmod{m}$. So $(b, a) \in R$. Hence R is symmetric.

Now let $a, b, c \in Z$ such that $(a, b), (b, c) \in R$. That is $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$ which means that $m \mid (a - b)$ and $m \mid (b - c)$. Then $m \mid ((a - b) + (b - c))$. That is, $m \mid (a - c)$. So $a \equiv c \pmod{m}$. So $(a, c) \in R$. Hence R is transitive. So R is an equivalence relation.♣

Example. Let $A = Z$ and $R = \{(a, b) \mid a \text{ divides } b\}$.

R is not an equivalence relation because it is not symmetric.

For example, 1 divides 2 but 2 does not divide 1.

Definition. Let R be an equivalence relation over a set A. The set of all elements that are related to an element x of A is called the ***equivalence class*** of x.

The equivalence class of x with respect to R is denoted as $[x]_R$. When only one relation is under consideration, we will use just $[x]$ to denote the equivalence class of a with respect to R.

Remark: $\mathbf{[x]_R = \{a \in A \mid (a, x) \in R\}}$.

Ex. Let R ={(1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4,4)} be a relation on {1,2,3,4}

a) Show that R is an equivalence relation.

b) Write down the equivalence class [1] ,[4]

Ex. Let A = {1; 2; 3; 5; 6; 10; 11; 12} and R be a relation defined on A by

$$R = \{(a; b) \mid a - b \text{ is divisible by } 4\}:$$

a) Show that R is an equivalence relation.

b) Write down the equivalence class [2], [6]

Example. We have seen that $R = \{(a, b) \mid a \equiv b \pmod{3}\}$ is an equivalence relation over the set of integers. What is [1], [2], [4]?

$$[1] = \{s \in Z \mid s \equiv 1 \pmod{3}\} = \{\ldots, -5, -2, 1, 4, 7, \ldots\}$$
$$= \{3k + 1 \mid k \in Z\}$$

$$[2] = \{s \in Z \mid s \equiv 2 \pmod{3}\} = \{\ldots, -4, -1, 2, 5, 8, \ldots\}$$
$$= \{3k + 2 \mid k \in Z\}$$

$$[4] = \{s \in Z \mid s \equiv 4 \pmod{3}\} = \{\ldots, -2, 1, 4, 7, 10, \ldots\}$$
$$= \{3k + 4 \mid k \in Z\}$$
$$= \{3k + 1 \mid k \in Z\} = [1].$$

We call 1 and 4 representatives for the equivalence class [1] or [4].

4.4 Partial Orderings

4.4.1 Introduction

4.4.2 Lexicographic Order

4.4.3 Hasse Diagrams

4.4.4 Maximal and Minimal Elements

4.4.5 Upper Bounds and Lower Bounds

4.4.1 Introduction

Example. Let R be the relation on the real numbers:

$$a\ R\ b \text{ if and only if } a \le b$$

Quiz time:

- Is R reflexive? Yes
- Is R transitive? Yes
- Is R symmetric? No
- Is R antisymmetric? Yes

4.4.1 Introduction

Definition. A relation R on a set A is a ***partial order*** if it is ***reflexive, antisymmetric*** and ***transitive***.

We often denote a partial order by $\prec$

The pair $(A, \prec)$ is called a ***partially ordered set*** or a ***poset***

Reflexive: $a \prec a$

Antisymmetric: $(a \prec b) \land (b \prec a) \to (a = b)$

Transitive: $(a \prec b) \land (b \prec c) \to (a \prec c)$

Ex. Which of these relations on {0, 1, 2, 3} are partial orderings? Determine the properties of a partial ordering that the others lack.

a) {(0, 0), (1, 1), (2, 2), (3, 3)}

b) {(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)}

c) {(0, 0), (1, 1), (1, 2), (2, 2), (3, 3)}

d) {(0, 0), (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)}

e) {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)}

Ex 3. Is (S,R) a poset if S is the set of all people in the world and $(a, b) \in R$, where a and b are people, if

a) a is taller than b?

b) a is not taller than b?

c) $a = b$ or a is an ancestor of b?

d) a and b have a common friend?

Definition. A relation R on a set A is a ***partial order*** if it is reflexive, antisymmetric and transitive.

Example. The divisibility relation " | "on the set of positive integers is a partial ordering, i.e. $(\mathbf{Z}^+, |)$ is a poset

Reflexive?

Yes, $x \mid x$ since $x = 1 \cdot x$

Transitive?

Yes?

$a \mid b$ means $b = ka$, $b \mid c$ means $c = jb$.
Then $c = j(ka) = jka$: $a \mid c$

Example. The divisibility relation " | "on the set of positive integers is a partial ordering, i.e. $(\mathbf{Z}^+, |)$ is a poset

Antisymmetric?

Yes?

$a \mid b$ means $b = ka$, $b \mid a$ means $a = jb$.

Then $a = jka$

It follows that $j = k = 1$, i.e. $a = b$

Example. Is $(\mathbf{Z}, |)$ a poset?

Not a poset.

Antisymmetric?

No

$3|-3$, and $-3|3$,

but $3 \neq -3$.

Ex. Is $(2^S, \subseteq)$, where 2^S the set of all subsets of S, a poset?

Yes, A poset.

Reflexive?

Yes, $A \subseteq A, \forall A \in 2^S$

Transitive?

$A \subseteq B, B \subseteq C$. Does that mean $A \subseteq C$?

Yes

Antisymmetric?

$A \subseteq B, B \subseteq A$. Does that mean $A = B$?

Yes

Definition. Let $(S, \prec)$ be a poset. Let a and b are elements of S. If $a \prec b$ then we say that b is an ***upper bound*** of a and a is a ***lower bound*** of b.

The elements a and b are said to be ***comparable*** if either a is a lower bound of b or b is a lower bound of a.

Otherwise, they are said to be ***incomparable***.

A poset $(S, \prec)$ such that every two elements are comparable is called a ***totally ordered set***

We also say that $\prec$ is a ***total order*** or a ***linear order*** on S

Example. The relation "$\leq$ " on the set of positive integers is a total order.

The elements *a* and *b* are ***comparable*** if either *a* is a lower bound of *b* or *b* is a lower bound of *a*.

Otherwise, they are ***incomparable***.

A poset $(S, \prec)$ such that every two elements are comparable is called a ***totally ordered set***

We also say that $\prec$ is a ***total order*** or a ***linear order*** on *S*

Example. The divisibility relation " | "on the set of positive integers is not a total order, since the elements 5 and 7 are not comparable

4.4.2 Lexicographic Order

Ex. A straight forward partial order on bit strings of length n, is defined as:

$$a_1a_2\ldots a_n \le b_1b_2\ldots b_n$$

if and only if $a_i \le b_i$, $\forall\, i$.

With respect to this order, 0110 and 1000 are "incomparable" …

We can't tell which is "bigger."

For many applications in computer, it is convenient to have a total order on bit strings, or more generally on strings of characters:

This is the lexicographic order

Let $(A, \leq)$ and $(B, \leq')$ be two totally ordered sets. We define a partial order $\prec$ on $A \times B$ as follows:

$$(a_1, b_1) \prec (a_2, b_2) \text{ if and only if}$$
$$a_1 < a_2 \text{ or } (a_1 = a_2 \text{ and } b_1 <' b_2)$$

Now we can verify that this is a total order on $A \times B$ called the ***lexicographic order***

Note that if A and B are well ordered by $\leq$ and $\leq'$ respectively, then $A \times B$ is also well ordered by $\prec$

Note also that this definition can be extended to the cartesian product of a finite number of totally ordered sets

Recall that if Σ is a finite set called an alphabet, then the set of strings on Σ, denoted by Σ^* is defined by:

- $\lambda \in \Sigma^*$, where λ denotes the null or empty string.
- If $x \in \Sigma$, and $w \in \Sigma^*$, then $wx \in \Sigma^*$, where wx is the concatenation of string w with symbol x.

Example. Let $\Sigma = \{a, b, c\}$. Then

$$\Sigma^* = \{\lambda, a, b, c, aa, ab, ac, ba, bb, bc, ca, cb, cc, aaa, aab, \ldots\}$$

Now assume that $\leq$ is a total order on Σ, then we can define a total order $\prec$ on Σ^* as follows.

Let $s = a_1 a_2 \ldots a_m$ and $t = b_1 b_2 \ldots b_n$ be two strings in Σ^*

Then $s \prec t$ if and only if

- either $a_i = b_i$ for $1 \leq i \leq m$ so that

$$t = a_1 a_2 \ldots a_m b_{m+1} b_{m+2} \ldots b_n$$

- or there exists $k < m$ such that
  - ✓ $a_i = b_i$ for $1 \leq i \leq k$ and
  - ✓ $a_{k+1} < b_{k+1}$ so that

$$s = a_1 a_2 \ldots a_k a_{k+1} a_{k+2} \ldots a_m$$
$$t = a_1 a_2 \ldots a_k b_{k+1} b_{k+2} \ldots b_n$$

- We can prove again that $\prec$ is a total order on the set Σ^* called the ***lexicographic order*** on Σ^*

Example. If Σ is the English alphabet with the usual order on the characters: a < b < … < z, then the lexicographic order is precisely the order of the words in a dictionary

For example

✓ $discreet \prec discrete$

$$\begin{array}{cccccccc} d & i & s & c & r & e & e & t \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ d & i & s & c & r & e & t & e \end{array}$$

$$e \precneqq t$$

✓ $discreet \prec discreetness$

$$\begin{array}{ccccccccccccc} d & i & s & c & r & e & e & t & & & & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & & & \\ d & i & s & c & r & e & e & t & n & e & s & s \end{array}$$

$\prec$ is a total order called the ***lexicographic order*** on **Σ***

Example. If $\Sigma = \{0, 1\}$ with the usual order $0 < 1$, then **Σ*** is the set of all bit strings.

We have

✓ $0110 \prec 10$

✓ $0110 \prec 01100$

Ex 17. Find the lexicographic ordering of these n-tuples:

a) $(1, 1, 2), (1, 2, 1)$ **b)** $(0, 1, 2, 3), (0, 1, 3, 2)$

c) $(1, 0, 1, 0, 1), (0, 1, 1, 1, 0)$

Ex 18. Find the lexicographic ordering of these strings of lowercase
English letters:

a) *quack*, *quick*, *quicksilver*, *quicksand*, *quacking*

b) *open*, *opener*, *opera*, *operand*, *opened*

c) *zoo*, *zero*, *zoom*, *zoology*, *zoological*

Ex 19. Find the lexicographic ordering of the bit strings 0, 01, 11, 001, 010, 011, 0001, and 0101 based on the ordering $0 < 1$.

4.4.3 Hasse Diagrams

A poset can be represented visually using a special kind of graphs called the ***Hasse diagram***

To define the Hasse diagram we need the concept of direct upper bound.

Definition. An element b in a poset $(S, \prec)$ is said to be a ***direct upper bound*** of an element a in S if

b is an upper bound of a, and there is no upper bound c of a such that

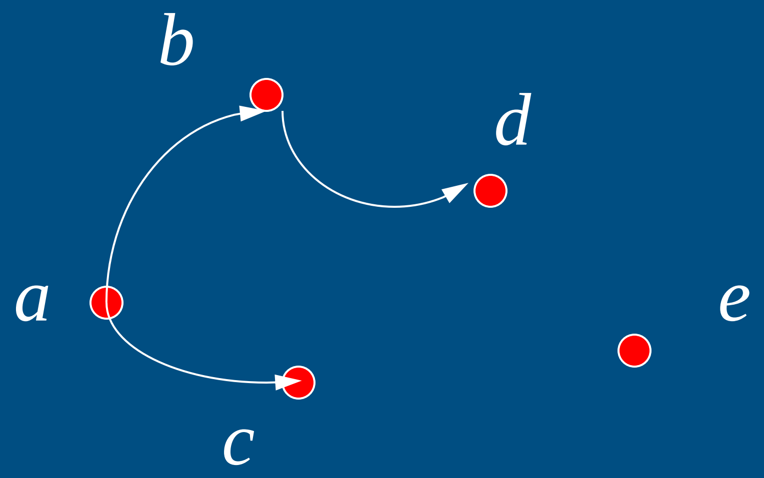
$$a \prec c \prec b, \quad a \neq c \neq b$$

4.4.3 Hasse Diagrams

- Now the ***Hasse diagram*** of a finite poset $(S, \prec)$ is the graph:

✓whose vertices are points in the plane in one-to-one correspondence with S,

✓two vertices a, b are joined by an arc directed from a to b if b is a direct upper bound of a



$a \prec b \prec d, \quad a \prec c$

4.4.3 Hasse Diagrams

Ex. The Hasse diagram of the poset ({1,2,3,4}, ≤) can be drawn as

Note. We did not draw up arrows for the arcs by adopting the convention that arcs are always directed upward

Ex 22. Draw the Hasse diagram for divisibility on the set

a) {1, 2, 3, 4, 5, 6}. **b)** {3, 5, 7, 11, 13, 16, 17}.

c) {2, 3, 5, 10, 11, 15, 25}. **d)** {1, 3, 9, 27, 81, 243}.

Ex 23. Draw the Hasse diagram for divisibility on the set

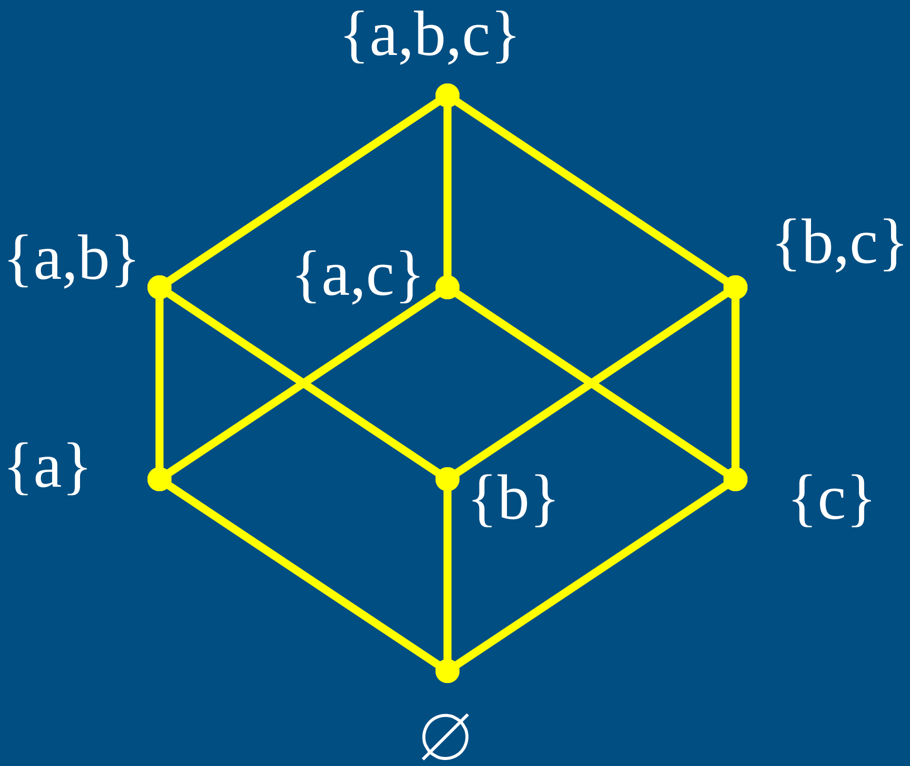
a) {1, 2, 3, 4, 5, 6, 7, 8}. **b)** {1, 2, 3, 5, 7, 11, 13}.

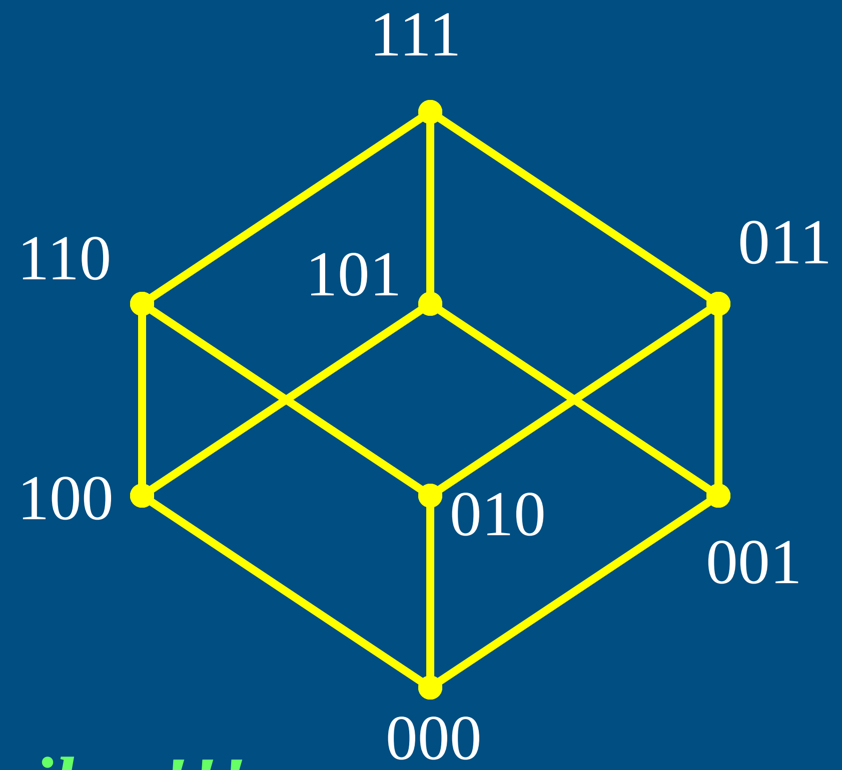
c) {1, 2, 3, 6, 12, 24, 36, 48}.

d) {1, 2, 4, 8, 16, 32, 64}.

Example. The Hasse diagram of P({a,b,c})

and the Hasse diagram of the set of bit strings of length 3 with natural bitwise order



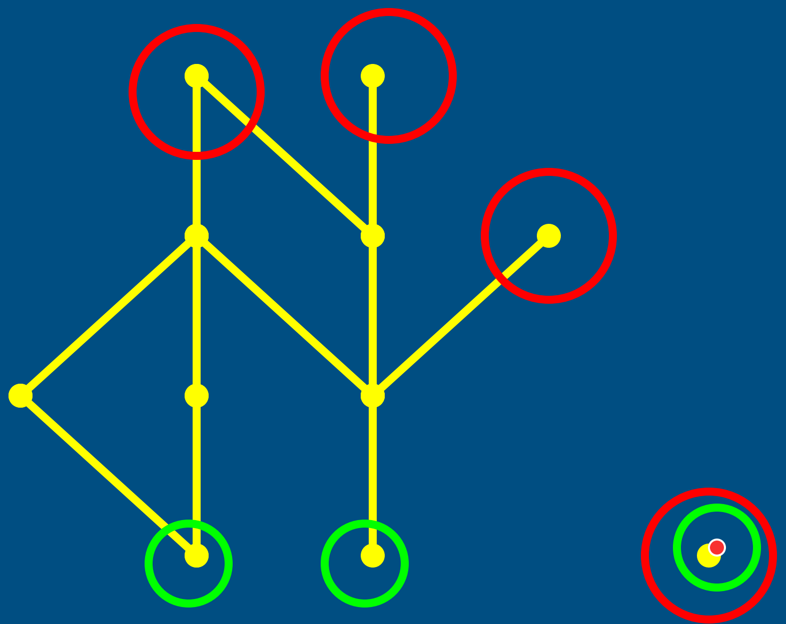


They look similar !!!

4.4.4 Maximal & Minimal Elements

Consider this poset:

- ✓ Each Red is ***maximal***: there is no proper upper bound
- ✓ Each Green is ***minimal***: there is no proper lower bound
- ✓ There is no arc starting from a maximal element
- ✓ There is no arc ending at a minimal element

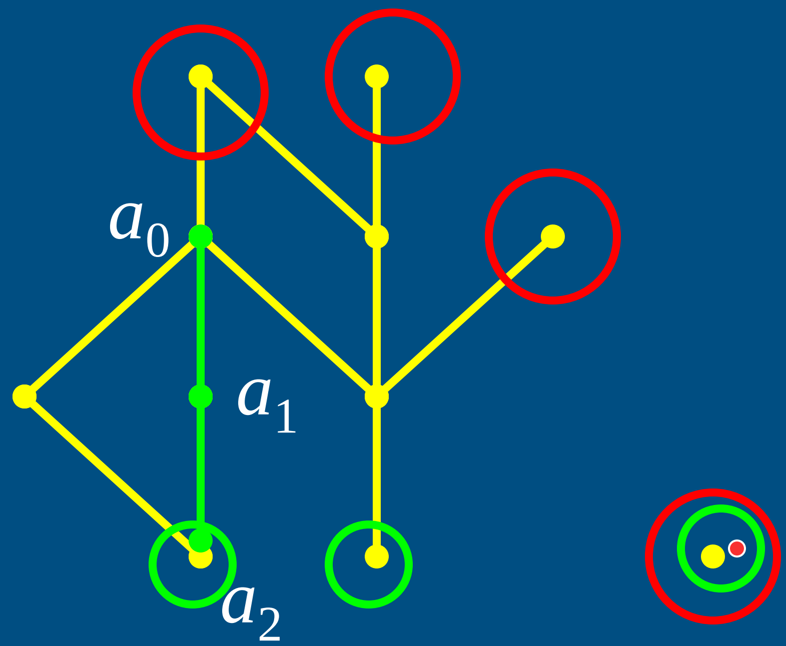


Note. In a finite poset S, maximal and minimal elements always exist.

- ✓ In fact, we can start from any element $a_0 \in S$.

  If a_0 was not minimal, then there exists $a_1 \prec a_0$, and so on until a minimal element is found.

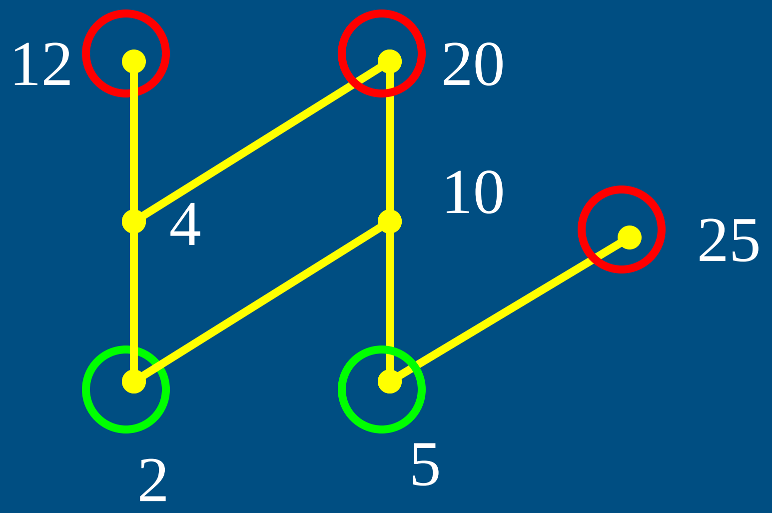
- ✓ The maximal elements are found in a similar way.



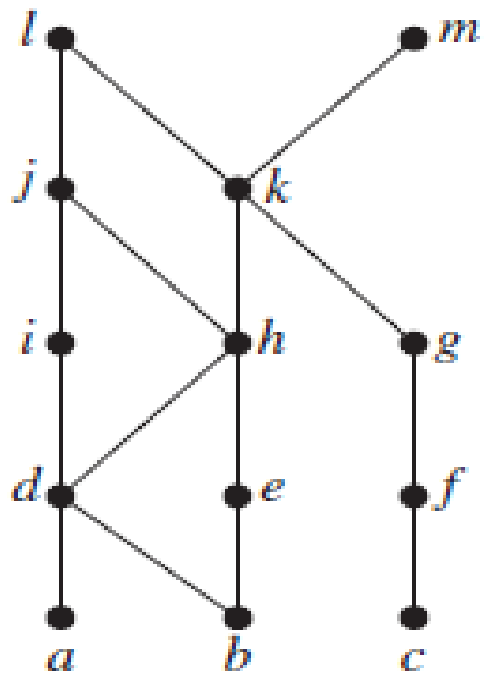
Example. What are the maximal and minimal elements of the poset ({2, 4, 5, 10, 12, 20, 25}, |) ?

Solution. From the Hasse diagram, we see that 12, 20, 25 are maximal elements and 2, 5 are minimal elements

Thus the maximal and minimal elements of a poset are not necessarily unique



32. Answer these questions for the partial order represented by this Hasse diagram.



a) Find the maximal elements.

b) Find the minimal elements.

c) Is there a greatest element?

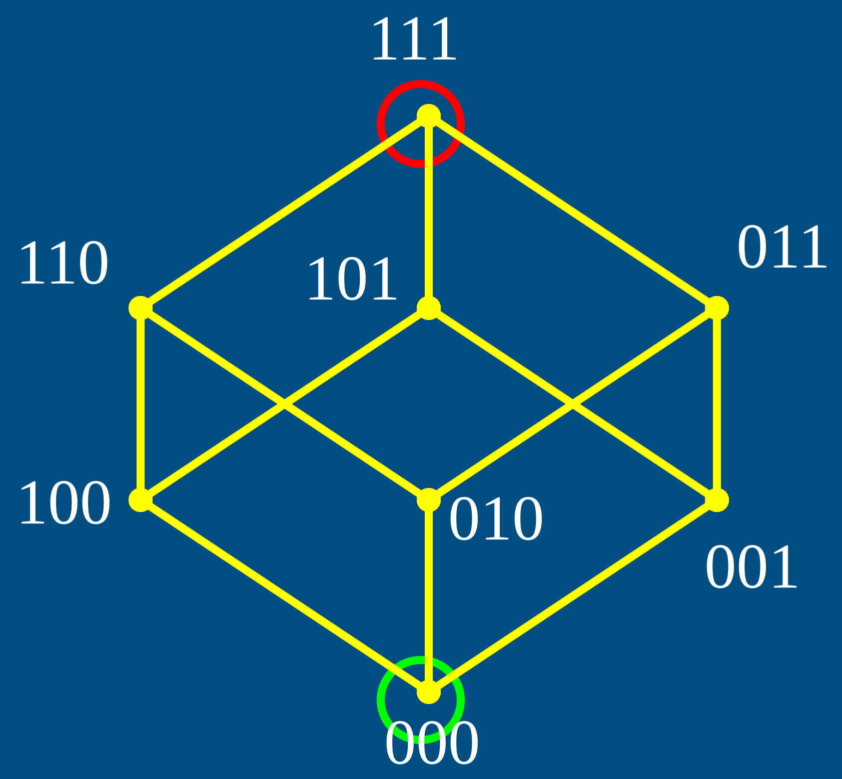
Example. What are the maximal and minimal elements of the poset consisting of bit strings of length 3?

Solution. From the Hasse diagram, we see that 111 is the unique maximal element and 000 is the unique minimal element

111 is also the ***greatest element*** and
000 is the ***least element***
in the sense:

$$000 \prec abc \prec 111$$

for all string *abc*



Ex 34. Answer these questions for the poset $(\{2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72\}, |)$.

a) Find the maximal elements.

b) Find the minimal elements.

c) Is there a greatest element?

d) Is there a least element?

e) Find all upper bounds of $\{2, 9\}$.

f) Find the least upper bound of $\{2, 9\}$, if it exists.

g) Find all lower bounds of $\{60, 72\}$.

h) Find the greatest lower bound of $\{60, 72\}$, if it exists.

In fact we have

Theorem. In a finite poset, if the maximal element is unique, then it is the greatest element .

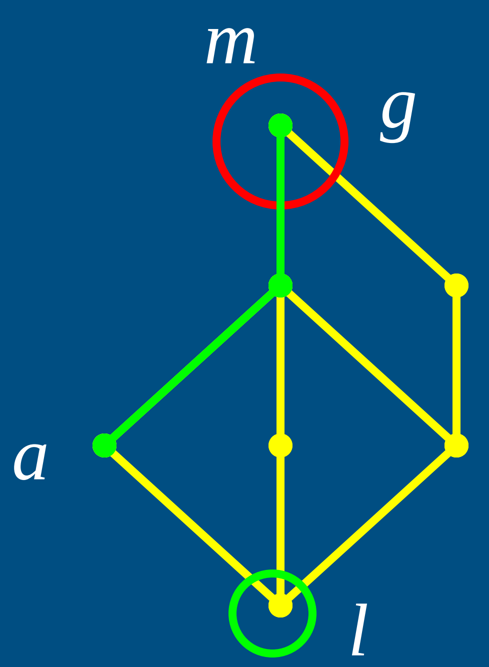
Similarly for the least element.

Proof. Let g be the unique maximal element.

Let a be an arbitrary element, then there is a maximal element m such that

$$a \prec m$$

Since g is unique we must have $m = g$, i.e. $a \prec g$



Therefore g is the greatest element.

Similar proof for the existence of the least element l