Chapter 6
Counting

Discrete Structures for Computing on August 31, 2021

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Contents

Course outcomes

	Course learning outcomes
L.O.1	Understanding of logic and discrete structures
	L.O.1.1 – Describe definition of propositional and predicate logic L.O.1.2 – Define basic discrete structures: set, mapping, graphs
L.O.2	Represent and model practical problems with discrete structures
	L.O.2.1 – Logically describe some problems arising in Computing L.O.2.2 – Use proving methods: direct, contrapositive, induction L.O.2.3 – Explain problem modeling using discrete structures
L.O.3	Understanding of basic probability and random variables
	L.O.3.1 – Define basic probability theory L.O.3.2 – Explain discrete random variables
L.O.4	Compute quantities of discrete structures and probabilities
	L.O.4.1 – Operate (compute/ optimize) on discrete structures L.O.4.2 – Compute probabilities of various events, conditional ones, Bayes theorem

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Introduction

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Example

- In games: playing card, gambling, dices,...
- How many allowable passwords on a computer system?
- How many ways to choose a starting line-up for a football match?

- **Combinatorics** (*tổ hợp*) is the study of arrangements of objects
- Counting of objects with certain properties is an important part of combinatorics

Applications of Combinatorics

- Number theory
- Probability
- Statistics
- Computer science
- Game theory
- Information theory
- ...

Problems

- Number of passwords a hacker should try if he wants to use brute force attack
- Number of possible outcomes in experiments
- Number of operations used by an algorithm

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Product Rule

Example

There are 32 routers in a computer center. Each router has 24 ports. How many different ports in the center?

Solution

There are two tasks to choose a port:

1. *picking a router*
2. *picking a port on this router*

Because there are 32 ways to choose the router and 24 ways to choose the port no matter which router has been selected, the number of ports are 32 $\times$ 24 $=$ 768 ports.

Definition (Product Rule (Luật nhân))

Suppose that a procedure can be broken down into a **sequence of two tasks**. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 \times n_2$ ways to do the procedure.

Can be extended to T_1, T_2, ..., T_m tasks in sequence.

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More examples

Example (1)

Two new students arrive at the dorm and there are 12 rooms available. How many ways are there to assign different rooms to two students?

Example (2)

How many different bit strings of length seven are there?

Example (3)

How many one-to-one functions are there from a set with m elements to one with n elements?

Sum Rule

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Example

A student can choose a project from one of three fields: Information system (32 projects), Software Engineering (12 projects) and Computer Science (15 projects). How many ways are there for a student to choose?

Solution: 32 + 12 + 15

Definition (Sum Rule (Luật cộng))

If a task can be done **either** in one of n_1 ways or in one of n_2 ways, there none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Can be extended to n_1, n_2, ..., n_m disjoint ways.

Using Both Rules

Example

In a computer language, the name of a variable is a string of one or two alphanumeric characters, where uppercase and lowercase letters are not distinguished. Moreover, a variable name must begin with a letter and must be **different from** the five strings of two characters that are reserved for programming use. How many different variables names are there in this language?

Solution

Let V equal to the number of different variable names.
Let V_1 be the number of these that are one character long, V_2 be the number of these that are two characters long. Then, by sum rule, $V = V_1 + V_2$.
Note that $V_1 = 26$, because it must be a letter. Moreover, there are $26 \cdot 36$ strings of length two that begin with a letter and end with an alphanumeric character. However, five of these are excluded, so $V_2 = 26 \cdot 36 - 5 = 931$. Hence $V = V_1 + V_2 = 957$ different names for variables in this language.

Inclusion-Exclusion

Example

How many bit strings of length eight either start with a 1 bit **or** end with the two bits 00?

Solution

- *Bit string of length eight that begins with a 1 is* $2^7 = 128$ *ways*
- *Bit string of length eight that ends with 00 is* $2^6 = 64$ *ways*
- *Bit string of length eight that begins with 1 and ends with 00:* $2^5 = 32$ *ways*

Number of satisfied bit strings are $2^7 + 2^6 - 2^5 = 160$ *ways.*

Inclusion-Exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Inclusion-Exclusion

$|A \cup B \cup C| = ???$

Example

In a certain survey of a group of students, 87 students indicated they liked Arsenal, 91 indicated that they liked Chelsea and 91 indicated that they liked MU. Of the students surveyed, 9 liked only Arsenal, 10 liked only Chelsea, 12 liked only MU and 40 liked all three clubs. How many of the student surveyed liked both MU and Chelsea but not Arsenal?

Pigeonhole Principle

Examples

Example (1)

Among any group of 367 people, there must be at least two with the **same birthday**.

Because there are only 366 possible birthdays.

Example (2)

In any group of 27 English words, there must be at least two that begin with the **same letter**.

Because there are 26 letters in the English alphabet.

Exercise

Example

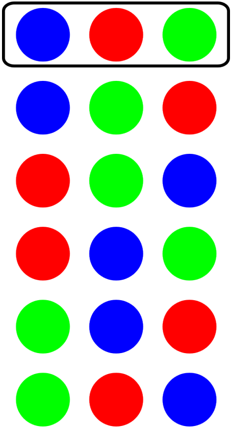
Prove that if seven distinct numbers are selected from $\{1, 2, \ldots, 11\}$, then some two of these numbers sum to 12.

Solution

1. **Pigeons**: *seven numbers from* $\{1, 2, \ldots, 11\}$
2. **Pigeonholes**: *corresponding to six sets,* $\{1, 11\}$, $\{2, 10\}$, $\{3, 9\}$, $\{4, 8\}$, $\{5, 7\}$, $\{6\}$
3. **Assigning rule**: *selected number gets placed into the pigeonhole corresponding to the set that contains it.*
4. **Apply the pigeon hole**: *seven numbers are selected and placed in six pigeonholes, some pigeonhole contains two numbers.*

Examples – Permutations

How many ways can we arrange three students to stand in line for a picture?



Number of choices: $6 = 3!$

Permutations

Definition

A **permutation** (*hoán vị*) of a set of distinct objects is an ordered arrangement of these objects.

An ordered arrangement of r elements of a set is called an r-**permutation** (*hoán vị chập* r).

$$P(n, r) = \frac{n!}{(n-r)!}$$

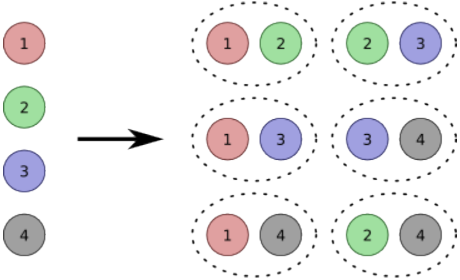
Example

How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

$$P(100, 3) = 100 \cdot 99 \cdot 98 = 970,200$$

Examples – Combinations

How many ways to choose two students from a group of four to offer scholarship?



Number of choices: 6

Combinations

Definition (Combinations)

An r-**combination** (*tổ hợp chập* r) of elements of a set is an unordered selection of r elements from the set. Thus, an r-combination is simply a subset of the set with r elements.

$$C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example

How many ways are there to select eleven players from a 22-member football team to start up?

$$C(22,11) = \frac{22!}{11!11!} = 705432$$

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Exercises – Permutations with Repetition

1. Suppose that a salesman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the salesman use when visiting these cities?
2. Suppose that there are 9 faculty members in CS department and 11 in CE department. How many ways are there to select a defend committee if the committee is to consist of three faculty members from the CS and four from the CE department?

Permutations with Repetition

Theorem

The number of r-permutations of a set of n objects with repetition allowed is n^r.

Example

How many strings of length r can be formed from the English alphabet?

By product rule, we see that there are 26^r strings of length r.

Example

Question: How many ways we can choose 3 students from the faculties of Computer Science, Electrical Engineering and Mechanical Engineering?

CCC	CEM
CCE	EEE
CCM	EEM
CEE	EMM
CMM	MMM

Example

CCC	CEM	$\star\star\star\mid\mid$	$\star\mid\star\mid\star$
CCE	EEE	$\star\star\mid\star\mid$	$\mid\star\star\star\mid$
CCM	EEM	$\star\star\mid\mid\star$	$\mid\star\star\mid\star$
CEE	EMM	$\star\mid\star\star\mid$	**???**
CMM	MMM	$\star\mid\mid\star\star$	$\mid\mid\star\star\star$

How many ways to put $\star$ and $\mid$???

Combinations with Repetition

Theorem

There are $C(n+r-1, r)$ r-combinations from a set with n elements when repetition of elements is allowed.

Example

How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where x_1, x_2, and x_3 are nonnegative integers?

Examples

Question: How many permutations are there of **MISSISSIPPI**?

MISSISSIPPI $\equiv$ **MISSISSIPPI**

Permutations with Indistinguishable Objects

Theorem

The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, . . . , and n_k indistinguishable objects of type k, is

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

Example

How many permutations are there of MISSISSIPPI?