Exercises for chapter 2
Predicate Logic & Proof

1 Introduction

In the exercise below, we will familiarize ourselves with knowledge related to proving methods (direct, contradiction, contrapositive) and inductive method.
Students should review the theory of chapter 2 before doing the exercises.

2 Sample exercises

Exercise 1.

Prove that "$\forall n \in \mathbb{Z}^+, n \geq 1,\ 10^{n+1} + 11^{2n-1} \vdots 111$".
Solution. *We can prove this statement by using Mathematical induction*

a) *Basic Step, $P(1)$ is true because $10^2 + 11^1 = 111$.*

b) *Inductive step:*

Assume that the statement is true for $n = k$, that means $10^{k+1} + 11^{2k-1} \vdots 111$. On the other words, there exists a integer number x such that $10^{k+1} + 11^{2k-1} = 111.x$.

*We need to prove the above statement is true for $n = k + 1$, that means $10^{k+2} + 11^{2k+1} \vdots 111$.
Expands the left expression, we have:*
$10^{k+2} + 11^{2k+1} = (10^{k+2} - 11^2.10^{k+1}) + (11^2.10^{k+1} + 11^{2k+1}) = 10^{k+1}(10 - 11^2) + 11^2(10^{k+1} + 11^{2k-1}) = 10^{k+1}(-111) + 121(10^{k+1} + 11^{2k-1}) \vdots 111$.

Therefore, $10^{k+2} + 11^{2k+1} \vdots 111$; It means this statement is tru for all integer number $n \geq 1$ due to Mathematical induction.

□

Exercise 2.

Prove by inductive method that the sum of sequence $1 + 3 + 5 + 7 + ... + 2n - 1$ is a perfect square number, $\forall n \geq 1$.
Solution. *Let $S_n = 1 + 3 + 5 + 7 + ... + 2n - 1$.
so that $S_{n+1} = S_n + (2n + 1)$.*

a) *$P(1)$ is true, because 1 itself is a perfect square number.*

b) *Assume S_n is a perfect square number with $n \geq 1$, it means, there is exists an integer number x satisfies $S_n = x^2$.
We need to prove the statement "S_{n+1} is also a perfect square number" is true.
We have $S_{n+1} = S_n + (2n + 1) = x^2 + 2n + 1$.
Select (set) $x = n$, we can get $S_{n+1} = (x + 1)^2$.
Eventually, we have proved the statement by using inductive method.*

□

3 Practice exercises

Exercise 3.
Let $P(x)$ denote the statement "$x \leq 4$". What are these truth values?

a) $P(0)$

b) $P(4)$

c) $P(6)$

Exercise 4.
Let $Q(x)$ be the statement "$x + 1 > 2x$". If the domain consists of all integers, what are these truth values?

a) $Q(0)$

b) $Q(-1)$

c) $Q(1)$

d) $\exists x Q(x)$

e) $\forall x Q(x)$

f) $\exists x \neg Q(x)$

g) $\forall x \neg Q(x)$

Exercise 5.
Let $P(x)$ be the statement "x spends more than five hours every weekday in class," where the domain for x consists of all students. Express each of these quantifications in English.

a) $\exists x P(x)$

b) $\forall x P(x)$

c) $\exists x \neg P(x)$

d) $\forall x \neg P(x)$

Exercise 6.
Translate these statements into English, where $C(x)$ is "x is a comedian" and $F(x)$ is "x is funny" and the domain consists of all people.

a) $\forall x(C(x) \rightarrow F(x))$

b) $\forall x(C(x) \wedge F(x))$

c) $\exists x(C(x) \rightarrow F(x))$

d) $\exists x(C(x) \wedge F(x))$

Exercise 7.
Let $P(x)$ be the statement "x can speak English" and let $Q(x)$ be the statement "x knows the computer language C++." Express each of these sentences in terms of $P(x), Q(x)$, quantifiers, and logical connectives. The domain for quantifiers consists of all students at your school.

a) There is a student at your school who can speak English and who knows Java.

b) There is a student at your school who can speak English but who doesn't know Java.

c) Every student at your school either can speak English or knows Java.

d) No student at your school can speak English or knows Java.

Exercise 8.
Let $Q(x, y)$ be the statement "x has sent an e-mail message to y," where the domain for both x and y consists of all students in your class. Express each of these quantifications in English.

a) $\exists x \exists y Q(x, y)$

b) $\exists x \forall y Q(x, y)$

c) $\forall x \exists y Q(x, y)$

d) $\exists y \forall x Q(x, y)$

e) $\forall y \exists x Q(x, y)$

f) $\forall x \forall y Q(x, y)$

Exercise 9.
Let $L(x, y)$ be the statement "x loves y," where the domain for both x and y consists of all people in the world. Use quantifiers to express each of these statements.

a) Everybody loves Jerry.

b) Everybody loves somebody.

c) There is somebody whom everybody loves.

d) There is somebody whom Lydia does not love.

e) There is somebody whom no one loves.

f) There is exactly one person whom everybody loves.

Exercise 10.
Let $M(x, y)$ be "x has sent y an e-mail message" and $T(x, y)$ be "x has telephoned y". where the domain consists of all students in your class. Use quantifiers to express each of these statements

a) Chou has never sent an e-mail message to Koko.

b) Arlene has never sent an e-mail message to or telephoned Sarah.

c) Jose has never received an e-mail message from Deborah.

d) Every student in your class has sent an e-mail message to Ken.

e) No one in your class has telephoned Nina.

f) Everyone in your class has either telephoned Avi or sent him an e-mail message.

Exercise 11.
Let $C(x)$ be the statement "x has a cat," let $D(x)$ be the statement "x has a dog," and let $F(x)$ be the statement "x has a ferret." Express each of these statements in terms of $C(x), D(x), F(x)$, quantifiers, and logical connectives. Let the domain consist of all students in your class.

a) A student in your class has a cat, a dog, and a ferret.

b) All students in your class have a cat, a dog, or a ferret.

c) Some student in your class has a cat and a ferret, but not a dog.

d) No student in your class has a cat, a dog, and a ferret.

e) For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has this animal as a pet.

Exercise 12.
Express each of these system specifications using predicates, quantifiers, and logical connectives.
$A(x)$: User x has access to an electronic mailbox.
$A(x, y)$: Group member x can access resource y.
$S(x, y)$: System/Router x is in state y.
$T(x)$: The throughput is at least x kbps.
$M(x, y)$: Resource x is in mode y.

a) Every user has access to an electronic mailbox.

b) The system mailbox can be accessed by everyone in the group if the file system is locked.

c) The firewall is in a diagnostic state only if the proxy server is in a diagnostic state.

d) At least one router is functioning normally if the throughput is between 100 kbps and 500 kbps and the proxy server is not in diagnostic mode.

Exercise 13.
What rule of inference is used in each of these arguments?

a) Alice is a mathematics major. Therefore, Alice is either a mathematics major or a computer science major.

b) Jerry is a mathematics major and a computer science major. Therefore, Jerry is a mathematics major.

c) If it is rainy, then the pool will be closed. It is rainy. Therefor, the pool is closed.

d) If it snows today, then university will close. The university is not closed today. Therefore, it did not snow today.

e) If I go swimming, then I will stay in the sun too long. If I stay in the sun too long, then I will sunburn. Therefore, if I go swimming, then I will sunburn.

Exercise 14.
What is wrong with this argument? Let $H(x)$ be "x is happy." Given the premise $\exists x H(x)$, we conclude that $H(\text{Lola})$. Therefore, Lola is happy.
Exercise 15.
Use rules of inference to show that if $\forall x(P(x) \vee Q(x)), \forall x(\neg Q(x) \vee S(x)), \forall x(R(x) \rightarrow \neg S(x))$ and $\exists x \neg P(x)$ are true, then $\exists x \neg R(x)$ is true.
Exercise 16.
Use a direct proof to show that the sum of two odd integers is even.
Exercise 17.
Use a direct proof to show that the product of two odd numbers is odd.
Exercise 18.
Use a direct proof to show that every odd integer is the difference of two squares.
Exercise 19.
Proof that if $n + m$ and $n + p$ are even integers, where m, n, p are integers, then $m + p$ is even. What kind of proof did you use?
Exercise 20.
Prove that the sum of two rational numbers is rational.

Exercise 21.
Use a proof by contradiction to prove that the sum of an irrational number and a rational number is irrational.
Exercise 22.
Prove that if x is irrational, then $1/x$ is irrational.
Exercise 23.
Use a proof by contraposition to show that if $x + y \geq 2$, where x and y are real numbers, then $x \geq 1$ or $y \geq 1$.
Exercise 24.
Show that if n is an integer and $n^3 + 2015$ is odd, then n is even using

a) a proof by contraposition.

b) a proof by contradiction.

Exercise 25.
Prove that if n is an integer and $3n + 2$ is even, then n is even using

a) a proof by contraposition.

b) a proof by contradiction.

Exercise 26.
Prove that if n is a positive integer, then n is odd if and only if $5n + 6$ is odd.
Exercise 27.
Show that these statements about the integer x are equivalent: (i) $3x + 2$ is even, (ii) $x + 5$ is odd, (iii) x^2 is even.
Exercise 28.
Prove that if n is an integer, these four statements are equivalent: (i) n is even, (ii) $n + 1$ is odd, (iii) $3n + 1$ is odd, (iv) $3n$ is even.
Exercise 29.
Prove by induction that $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$
Exercise 30.
Prove that $2^n > 2n$ for every positive integer $n > 2$.
Exercise 31.
Prove that $3^{2n-1} + 1$ is divisible by 4 for all $n \geq 1$
Exercise 32.
Prove that $6^n - 1$ is divisible by 5 for all $n \geq 1$.
Exercise 33.
Prove that $n! > 2^n$ for all $n \geq 1$.
Exercise 34.
Let the Fibonacci sequence be defined by $F_0 = 0, F_1 = 1, F_{n+2} = F_n + F_{n+1}$ for $n \geq 0$. Prove that F_{3n} is even for $n \geq 1$.
Exercise 35.
Let the "Tribonacci sequence" be defined by $T_1 = T_2 = T_3 = 1$ and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 4$. Prove that $T_n < 2^n$ for all $n \geq 1$.