Chapter 2
Logics (cont.)

Discrete Structures for Computing on September 3, 2021

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Course outcomes

	Course learning outcomes
L.O.1	Understanding of logic and discrete structures
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L.O.2	Represent and model practical problems with discrete structures
	L.O.2.1 – Logically describe some problems arising in Computing L.O.2.2 – Use proving methods: direct, contrapositive, induction L.O.2.3 – Explain problem modeling using discrete structures
L.O.3	Understanding of basic probability and random variables
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L.O.4	Compute quantities of discrete structures and probabilities
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Limits of Propositional Logic

- $x > 3$
- All square numbers are not prime numbers. 100 is a square number. Therefore 100 is not a prime number.

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Predicates

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Definition

A predicate (*vị từ*) is a statement containing one or more variables. If values are assigned to all the variables in a predicate, the resulting statement is a proposition (*mệnh đề*).

- $x > 3 \rightarrow P(x)$
- $5 > 3 \rightarrow P(5)$
- A predicate with n variables $P(x_1, x_2, ..., x_n)$

Example:

- $x > 3$ (predicate)
- $5 > 3$ (proposition)
- $2 > 3$ (proposition)

Truth value

- $x > 3$ is true or false?
- $5 > 3$
- For every number $x, x > 3$ holds
- There is a number x such that $x > 3$

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Quantifiers

- $\forall$: Universal – *Với mọi*
  - $\forall x P(x) = P(x)$ is T for all x
- $\exists$: Existential – *Tồn tại*
  - $\exists x P(x) =$ There exists an element x such that $P(x)$ is T
- We need a domain of discourse for variable

Example

Let $P(x)$ be the statement "$x < 2$". What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real number?

- $P(3) = 3 < 2$ is false
- $\Rightarrow \forall x P(x)$ is false

- 3 is a counterexample (*phản ví dụ*) of $\forall x P(x)$

Example

What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all real number?

Example

Express the statement "Some student in this class comes from Central Vietnam."

Solution 1

- $M(x) = x$ comes from Central Vietnam
- Domain for x is the students in the class
- $\exists x M(x)$

Solution 2

- Domain for x is all people
- ...

Negation of Quantifiers

Statement	Negation	Equivalent form
$\forall x P(x)$	$\neg(\forall x P(x))$	$\exists x \neg P(x)$
$\exists x P(x)$	$\neg(\exists x P(x))$	$\forall x \neg P(x)$

Example

- All CSE students study Discrete Math 1
- Let C(x) denote "x is a CSE student"
- Let S(x) denote "x studies Discrete Math 1"
- $\forall x : C(x) \to S(x)$
- $\exists x : \neg(C(x) \to S(x)) \equiv \exists x : C(x) \wedge \neg S(x)$
- There is a CSE student who does not study Discrete Math 1.

Another Example

Example

Translate these:

- All lions are fierce.
- Some lions do not drink coffee.
- Some fierce creatures do not drink coffee.

Solution

Let $P(x)$, $Q(x)$ and $R(x)$ be the statements "x is a lion", "x is fierce" and "x drinks coffee", respectively.

- $\forall x(P(x) \to Q(x))$.
- $\exists x(P(x) \wedge \neg R(x))$.
- $\exists x(Q(x) \wedge \neg R(x))$.

The Order of Quantifiers

- The order of quantifiers is important, unless all the quantifiers are universal quantifiers or all are existential quantifiers
- Read from left to right, apply from inner to outer

Example

$\forall x\ \forall y\ (x + y = y + x)$
T for all $x, y \in \mathbb{R}$

Example

$\forall x\ \exists y\ (x + y = 0)$ is T,
while
$\exists y\ \forall x\ (x + y = 0)$ is F

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Translating Nested Quantifiers

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Example

$\forall x\ (C(x) \vee \exists y\ (C(y) \wedge F(x,y))\ )$

Provided that:

- $C(x)$: x has a computer,
- $F(x,y)$: x and y are friends,
- $x, y \in$ all students in your school.

Answer

For every student x in your school, x has a computer or there is a student y such that y has a computer and x and y are friends.

Translating Nested Quantifiers

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Example

$\exists x \forall y \forall z \quad (((F(x,y) \wedge F(x,z) \wedge (y \neq z)) \rightarrow \neg F(y,z)))$

Provided that:

- $F(x,y)$: x, y are friends
- $x, y, z \in$ all students in your school.

Answer

There is a student x, so that for every student y, every student z not the same as y, if x and y are friends, and x and z are friends, then y and z are not friends.

Translating into Logical Expressions

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Example

1. "There is a student in the class has visited Hanoi".
2. "Every student in the class has visited Nha Trang or Vung Tau".

Answer

Assume:
$C(x)$: x has visited Hanoi
$D(x)$: x has visited Nha Trang
$E(x)$: x has visited Vung Tau
We have:

1. $\exists x C(x)$
2. $\forall x (D(x) \vee E(x))$

Translating into Logical Expressions

Example

If a person is a woman and a parent, then this person is mother of someone.

Solution

We define:

- $W(x)$: x is woman
- $P(x)$: x is a parent
- $M(x, y)$: x is mother of y

We have: $\forall x((W(x) \wedge P(x)) \rightarrow \exists y M(x, y))$

Example

"Every people has only one best friend."
Assume:

- $B(x, y)$: y is the best friend of x

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Translating into Logical Expressions

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Example

"Every people has only one best friend."
Assume:

- $B(x, y)$: y is the best friend of x

Solution

$\forall x \exists y \forall z (B(x, y) \wedge ((y \neq z) \rightarrow \neg B(x, z)))$

Inference

Example

- If I have a girlfriend, I will take her to go shopping.
- Whenever I and my girlfriend go shopping and that day is a special day, I will surely buy her some expensive gift.
- If I buy my girlfriend expensive gifts, I will eat noodles for a week.
- Today is March 8.
- March 8 is such a special day.
- Therefore, if I have a girlfriend,...
- I will eat noodles for a week.

Propositional Rules of Inferences

Rule of Inference	Name
p $p \to q$ $\therefore q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	Hypothetical syllogism (*Tam đoạn luận giả định*)
$p \vee q$ $\neg p$ $\therefore q$	Disjunctive syllogism (*Tam đoạn luận tuyển*)

Propositional Rules of Inferences

Rule of Inference	Name
$\dfrac{p}{\therefore p \vee q}$	Addition (*Quy tắc cộng*)
$\dfrac{p \wedge q}{\therefore p}$	Simplification (*Rút gọn*)
$\dfrac{\begin{array}{c} p \\ q \end{array}}{\therefore p \wedge q}$	Conjunction (*Kết hợp*)
$\dfrac{\begin{array}{l} p \vee q \\ \neg p \vee r \end{array}}{\therefore q \vee r}$	Resolution (*Phân giải*)

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Example

If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have a barbecue tomorrow.

Solution

- p: It is raining today
- q: We will not have a barbecue today
- r: We will have barbecue tomorrow

$$\begin{array}{l} p \to q \\ \underline{q \to r} \\ \therefore p \to r \end{array}$$

Hypothetical syllogism

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Example

- It is not sunny this afternoon ($\neg p$) and it is colder than yesterday (q)
- We will go swimming (r) only if it is sunny
- If we do not go swimming, then we will take a canoe trip (s)
- If we take a canoe trip, then we will be home by sunset (t)
- We will be home by sunset (t)

1. $\neg p \wedge q$ — Hypothesis
2. $\neg p$ — Simplification using (1)
3. $r \rightarrow p$ — Hypothesis
4. $\neg r$ — Modus tollens using (2) and (3)
5. $\neg r \rightarrow s$ — Hypothesis
6. s — Modus ponens using (4) and (5)
7. $s \rightarrow t$ — Hypothesis
8. t — Modus ponens using (6) and (7)

Fallacies

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Definition

Fallacies (*ngụy biện*) resemble rules of inference but are based on contingencies rather than tautologies.

Example

If you do correctly every questions in mid-term exam, you will get 10 grade. You got 10 grade.

Therefore, you did correctly every questions in mid-term exam.

Is $[(p \to q) \wedge q] \to p$ a tautology?

Rules of Inference for Quantified Statements

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Rule of Inference	Name
$\dfrac{\forall x P(x)}{\therefore P(c)}$	Universal instantiation (*Cụ thể hóa phổ quát*)
$\dfrac{P(c) \text{for an arbitrary } c}{\therefore \forall x P(x)}$	Universal generalization (*Tổng quát hóa phổ quát*)
$\dfrac{\exists x P(x)}{\therefore P(c) \text{for some element } c}$	Existential instantiation (*Cụ thể hóa tồn tại*)
$\dfrac{P(c) \text{for some element } c}{\therefore \exists x P(x)}$	Existential generalization (*Tổng quát hóa tồn tại*)

Example

- A student in this class has not gone to class
- Everyone in this class passed the first exam
- Someone who passed the first exam has not gone to class

Hint

- $C(x)$: x is in this class
- $B(x)$: x has gone to class
- $P(x)$: x passed the first exam
- Premises???

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1. $\exists x(C(x) \land \neg B(x))$	Premise
2. $C(a) \land \neg B(a)$	Existential instantiation from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \to P(x))$	Premise
5. $C(a) \to P(a)$	Universal instantiation from (4)
6. $P(a)$	Modus ponens from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \land \neg B(a)$	Conjunction from (6) and (7)
9. $\exists x(P(x) \land \neg B(x))$	Existential generalization from (8)

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Given the predicate $p(x) :'' x^2 - 3x + 2 = 0''$. What is the truth value (chân trị) of the following propositions:

a) $p(0)$

b) $p(1)$

c) $p(2)$

d) $\exists x, p(x)$

e) $\forall x, p(x)$

Let $x, y \in \mathrm{Z}^+$, and the predicate: $p(x, y)$: "x is a divisor of y"
Determine the truth value of the following propositions:

a) $p(2, 3)$
b) $p(2, 6)$
c) $\forall y, p(1, y)$
d) $\forall x, p(x, x)$
e) $\forall x \exists y, p(x, y)$
f) $\exists y \forall x, p(x, y)$
g) $\forall x \forall y, (p(x, y) \wedge p(y, x)) \rightarrow (x = y)$
h) $\forall x \forall y \forall z (p(x, y) \wedge p(y, z)) \rightarrow (p(x, z))$

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Provided that:

- $F(x, y)$: x is father of y,
- $M(x, y)$: x is mother of y,
- $S(x, y)$: x is sister of y,
- $B(x, y)$: x is brother of y,
- $H(x, y)$: x is spouse (wife/husband) of y,
- $O(x, y)$: x is elder than y.

Express each of these statements using predicates:

a) 'He (a person) has an elder sister and younger brother'.

b) 'All of her brothers are younger than her'.

c) 'Thuyen has only one husband' (Thuyen is a private name).

d) 'One of his sisters is younger than him'.

e) 'Everyone has grandfather, grandmother, maternal grandfather, maternal grandmother'.

f) 'A father of a person cannot be a mother of other ones'.

Solutions:

a) 'He (a person) has an elder sister and younger brother'.
$\exists x \exists y (S(x,m) \land O(x,m) \land B(y,m) \land \neg O(y,m))$.

b) 'All of her brothers are younger than her'.
$\forall x (B(x,m) \to \neg O(x,m))$.

c) 'Thuyen has only one husband' (Thuyen is a private name).
$\exists x \forall y\ H(x, \text{Thuyen}) \land H(y, \text{Thuyen}) \to (x = y)$
or $\exists x \forall y\ H(x, \text{Thuyen}) \land (x \neq y) \to \neg H(y, \text{Thuyen})$.

d) 'One of his sisters is younger than him'.
$\exists x \forall y (S(x,m) \land \neg O(x,m) \land S(y,m) \land (x \neq y) \to O(y,m))$.

e) 'Everyone has grandfather, grandmother, maternal grandfather, maternal grandmother'. $\forall x \exists y \exists z \exists y_1 \exists y_2 \exists z_1 \exists z_2$
$(F(y,x) \land M(z,x) \land F(y_1,y) \land M(y_2,y) \land F(z_1,z) \land M(z_2,z))$.

f) 'A father of a person cannot be a mother of other ones'.
$\exists x \exists y \forall z (F(x,y) \to \neg M(x,z))$.

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Translating the following nested quantifiers:

a) $B(c,m) \wedge (O(c,m) \vee O(m,c))$.

b) $B(c,m) \wedge F(a,m) \to O(a,c) \wedge F(a,c)$.

c) $\forall x \forall y (S(x,m) \wedge B(c,y) \to x = y)$.

d) $\exists x((S(x,m) \vee H(c,x)) \vee \exists x(H(x,m) \wedge O(x,m)))$.

e) $\forall x \forall y (S(x,m) \wedge S(y,m) \to O(x,y) \vee O(y,x))$

Given a predicate $N(x)$ "x has been to Da Lat" with the domain is the all students in Mathematics class. Translate the following predicates into English

a) $\exists x N(x)$
b) $\forall x N(x)$
c) $\neg \exists x N(x)$
d) $\exists x \neg N(x)$
e) $\neg \forall x N(x)$
f) $\forall x \neg N(x)$

a) There is a student in this class has been to Da Lat.
b) All students in Math class have been to Da Lat.
c) There is no exists a student in Math class has gone to Da Lat.
d) There is a student in this class has never gone to Da Lat.
e) Not all students in Math class have ever been to Da Lat.
f) All students in Math class have never been to Da Lat.

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Given the predicate $N(x)$ "x studies more than 5 hours in class every weekday" with the domain is the all students in Mathematics class. Express the following predicates:

a) $\exists x N(x)$

b) $\forall x N(x)$

c) $\exists x \neg N(x)$

d) $\forall x \neg N(x)$

What is the propositional formula for the following pseudo code:

```
for (i = 0; i<numObjects; i++) {
  Object x = Objects(i);
  if isMushroom(x)
    if isPoisonous(x) && isPurple(x)
    return false;
}
return true;
```

- There are no mushrooms that are poisonous and purple.
- $\forall x Mushroom(x) \rightarrow \neg(Poisonous(x) \wedge Purple(x))$

What is the propositional formula for the following pseudo code:

```
for (i=0; i<numObjects; i++) {
   Object x = Objects(i);
   if isMushroom(x) && isPoisonous(x) && isPurple(x)
      return true;
}
return false;
```

- There is a mushroom that is purple and poisonous.
- $\exists x Mushroom(x) \wedge Poisonous(x) \wedge Purple(x)$

Giving the following pseudo code:

```
//— Look for first match
for (x=0; x<numKids; x++)
   if isParent(Peter, kids[x])
      match1Found = true;

//— Now look for a second match
for (y=0; (y<numKids)&&(y!=x); y++)
   if isParent(Peter, kids[y])
      match2Found = true;

return match1Found && match2Found;
```

Knowing that: kids array has 3 elements: { Alice, Bob, Charles } and Peter only have 1 child Alice.
What is the propositional formula for "Peter has at least 2 children".
$\exists x \exists y (ParentOf(Peter, x) \land ParentOf(Peter, y) \land \neg(x = y))$

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Let P(x) be "x can speak Russian" and Q(x) be "x can use Java". Formalize the following:
Giving the space is ll students in your university.

a) There is a student in your university that can speak Russian and can use Java.
b) There is a student in your university that can speak Russian but can't use Java.
c) Every student in your university can speak Russian or can use Java.
d) None of the student in your university can speak Russian or can use Java.

a) $\exists x(P(x) \wedge Q(x))$
b) $\exists x(P(x) \wedge \neg Q(x))$
c) $\forall x(P(x) \vee Q(x))$
d) $\forall x \neg(P(x) \wedge Q(x))$

Let L(x,y) be "x love y", where the space of x and y is the set of all people in the world. Use logical quantifier to express the following:

a) Everybody loves Jerry.
b) Everybody loves someone.
c) There is a person who everybody loves.
d) Nobody loves everybody.
e) There is someone Lydica doesn't love.
f) There is someone nobody loves.
g) There is exact one person everybody loves.
h) There are exact two person Lynn loves.
i) Everybody loves themselves.
j) There is a person who love nobody but himself.

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Giving the following:
.   -P(x): "x is a math problem".
.   -Q(x): "x is hard" (based on a well-defined standard).
.   -R(x): "x is easy" (based on a well-defined standard - same as above).
.   -S(x): "x is not solvable".
Translate the following propositional formulas to natural English

a) $\forall x(P(x) \rightarrow (Q(x) \iff \neg R(x)))$
b) $\exists x(S(x) \wedge \neg P(x))$

There are many ways to translate a formula to natural language and the following is one of them

a) If x is a math problem, to say x is hard is the same as saying x is not easy.
b) There is unsolvable non-math problem.

Translate the following propositional formulas to natural English where:
$F(p)$ is "Printer p is broken",
$B(p)$ is "Printer p is currently printing another document",
$L(j)$ is "Printing job j is lost",
and $Q(j)$ is "Printing job j is in queue."

a) $\exists p(F(p) \wedge B(p)) \rightarrow \exists j L(j)$
b) $\forall p B(p) \rightarrow \exists j Q(j)$
c) $\exists j(Q(j) \wedge L(j)) \rightarrow \exists p F(p)$
d) $(\forall p B(p) \wedge \forall j Q(j)) \rightarrow \exists j L(j)$

Formalize the following sentences:

a) Nobody is perfect.
b) not everyone is perfect.
c) All your friends are perfect.
d) At least one of your friend is perfect.
e) Everybody is your friend and they are perfect.
f) Not everybody is your friend or there is somebody not perfect.

Giving: C(x): x is perfect.
D(x): x is your friend.
E(x): x is someone else.

Giving the following Predicate:
- P(x): Program x satisfies ABET standard.
- Q(x,y): Program x has the same educational goal as program y.
- R(x): Educational outcome from program x is verifiable.

Which of the following formalize this sentence : "Every program that has the same educational goal as a ABET satisfied program and verifiable Educational outcome also satisfies ABET standard"

A. $\forall x(P(x) \land \neg Q(x)) \to \exists x(R(x))$

B. $\forall x(\exists y(Q(x,y) \land P(y) \land R(x)) \to P(x))$

C. $\forall x(\exists y(Q(x,y) \land P(y) \land R(x)) \to P(x) \lor R(x))$

D. $\forall x(\forall y(Q(x,y) \land P(y) \lor R(x)) \to P(x))$

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Let:

- $P(x, y)$: x is parent of y.
- $M(x)$: x is male .

Given:
$F(v, w) = M(v) \wedge \exists x \exists y (P(x, y) \wedge P(x, v) \wedge (y \neq v) \wedge P(y, w))$,
then $F(v, w)$ means:

A) v is brother of w

B) v is cousin of w

C) v is uncle of w

D) v is grand father of w

Formalize the following sentences using predicate logic:

a) When a hard drive has less than 30GB free space, a warning will be issued to all the users.
b) Do not back up the files if anyone is logging in the system.
c) YouTube's videos will be buffered if there are at least 8MB memory and 56kb/s line rate.
d) Few computer student is good at programming.
e) No computer student is not hard working.
f) Not all computer students are smart.
g) All the Pompeians are either loyal to or hate Caesar.
h) Everyone is loyal to someone.
i) People only want to assassinate the dictator whom they are not loyal to.