Chapter 5
Relations

Discrete Structures for Computing on August 31, 2021

Huynh Tuong Nguyen, Tran Tuan Anh, Nguyen An Khuong, Le Hong Trang
Faculty of Computer Science and Engineering
University of Technology - VNUHCM
trtanh@hcmut.edu.vn - htnguyen@hcmut.edu.vn

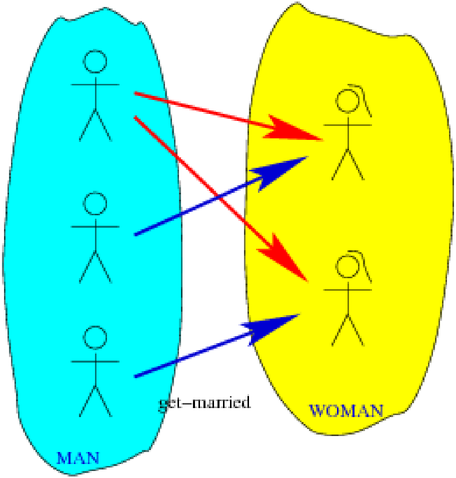
Contents

Course outcomes

	Course learning outcomes
L.O.1	Understanding of logic and discrete structures
	L.O.1.1 – Describe definition of propositional and predicate logic L.O.1.2 – Define basic discrete structures: set, mapping, graphs
L.O.2	Represent and model practical problems with discrete structures
	L.O.2.1 – Logically describe some problems arising in Computing L.O.2.2 – Use proving methods: direct, contrapositive, induction L.O.2.3 – Explain problem modeling using discrete structures
L.O.3	Understanding of basic probability and random variables
	L.O.3.1 – Define basic probability theory L.O.3.2 – Explain discrete random variables
L.O.4	Compute quantities of discrete structures and probabilities
	L.O.4.1 – Operate (compute/ optimize) on discrete structures L.O.4.2 – Compute probabilities of various events, conditional ones, Bayes theorem

Huynh Tuong Nguyen, Tran Tuan Anh, Nguyen An Khuong, Le Hong Trang

Introduction



Function?

Relation

Definition

Let A and B be sets. A **binary relation** (*quan hệ hai ngôi*) from a set A to a set B is a set

$$R \subseteq A \times B$$

- Notations:

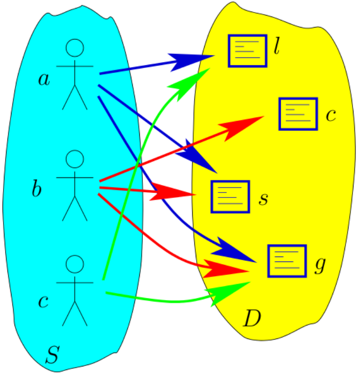
  $(a, b) \in R \longleftrightarrow aRb$

- **n-ary relations?**

Example

Example

Let $A = \{a, b, c\}$ be the set of students, $B = \{l, c, s, d\}$ be the set of the available optional courses. We can have relation R that consists of pairs (a, b), where a is a student enrolled in course b.



$$R = \{(a,l),(a,s),(a,g),(b,c),(b,s),(b,g),(c,l),(c,g)\}$$

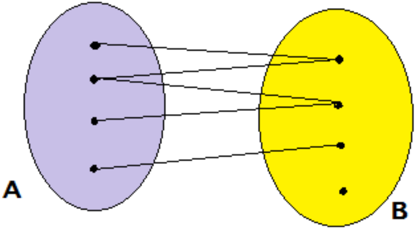
R	l	c	s	g
a	x		x	x
b		x	x	x
c	x			x

Functions as Relations

- **Is a function a relation?**
- **Yes!**
- $f:\ A \to B$
  $R = \{(a, b) \mid b = f(a)\}$

Functions as Relations

- **Is a relation a function?**
- **No**



- Relations are a generalization of functions

Relations on a Set

Huynh Tuong Nguyen, Tran Tuan Anh, Nguyen An Khuong, Le Hong Trang

Definition

A relation on the set A is a relation from A to A.

Example

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$ (a *là ước số của* b)?

Solution:

$$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$$

R	1	2	3	4
1	x	x	x	x
2		x		x
3			x	
4				x

Properties of Relations

Reflexive (*phản xạ*)	$xRx, \forall x \in A$
Symmetric (*đối xứng*)	$xRy \rightarrow yRx, \forall x, y \in A$
Antisymmetric (*phản đối xứng*)	$(xRy \wedge yRx) \rightarrow x = y, \forall x, y \in A$
Transitive (*bắc cầu*)	$(xRy \wedge yRz) \rightarrow xRz, \forall x, y, z \in A$

Example

Example

Consider the following relations on $\{1, 2, 3, 4\}$:

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$,
$R_2 = \{(1,1), (1,2), (2,1)\}$,
$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$,
$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$,
$R_5 = \{(3,4)\}$

Solution:

- Reflexive: R_3
- Symmetric: R_2, R_3
- Antisymmetric: R_4, R_5
- Transitive: R_4, R_5

Example

Huynh Tuong Nguyen, Tran Tuan Anh, Nguyen An Khuong, Le Hong Trang

Example

What is the properties of the **divides** (*ước số*) relation on the set of positive integers?

Solution:

- $\forall a \in \mathbb{Z}^+, a \mid a$: reflexive
- $1 \mid 2$, but $2 \nmid 1$: not symmetric
- $\exists a, b \in \mathbb{Z}^+, (a \mid b) \wedge (b \mid a) \rightarrow a = b$: antisymmetric
- $a \mid b \Rightarrow \exists k \in \mathbb{Z}^+, b = ak; b \mid c \Rightarrow \exists l \in \mathbb{Z}^+, c = bl$. Hence, $c = a(kl) \Rightarrow a \mid c$: transitive

Example

Example

What are the properties of these relations on the set of integers:

$$R_1 = \{(a, b) \mid a \leq b\}$$

$$R_2 = \{(a, b) \mid a > b\}$$

$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$$

Combining Relations

Because relations from A to B are **subsets** of $A \times B$, two relations from A to B can be combined in any way two sets can be combined.

Example

Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. List the combinations of relations $R_1 = \{(1, 1), (2, 2), (3, 3)\}$ and $R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$.

Solution: $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$ and $R_2 - R_1$.

Example

Let A and B be the set of all students and the set of all courses at school, respectively. Suppose $R_1 = \{(a, b) \mid a$ has taken the course $b\}$ and $R_2 = \{(a, b) \mid a$ requires course b to graduate$\}$. What are the relations $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 \oplus R_2$, $R_1 - R_2$, $R_2 - R_1$?

Composition of Relations

Definition

Let R be **relations** from A to B and S be from B to C. Then the **composite** (*hợp thành*) of S and R is

$$S \circ R = \{(a, c) \in A \times C \mid \exists b \in B \ (aRb \wedge bSc)\}$$

Example

$R = \{(0, 0), (0, 3), (1, 2), (0, 1)\}$
$S = \{(0, 0), (1, 0), (2, 1), (3, 1)\}$
$S \circ R = \{(0, 0), (0, 1), (1, 1)\}$

Power of Relations

Huynh Tuong Nguyen, Tran Tuan Anh, Nguyen An Khuong, Le Hong Trang

Definition

Let R be a relation on the set A. The **powers** (*lũy thừa*) $R^n, n = 1, 2, 3, \ldots$ are defined recursively by

$$R^1 = R \qquad \text{and} \qquad R^{n+1} = R^n \circ R.$$

Example

Let $R = \{(1,1), (2,1), (3,2), (4,3)\}$. Find the powers $R^n, n = 2, 3, 4, \ldots$.

Solution:
$R^2 = \{(1,1), (2,1), (3,1), (4,2)\}$
$R^3 = \{(1,1), (2,1), (3,1), (4,1)\}$
$R^4 = \{(1,1), (2,1), (3,1), (4,1)\}$
$\ldots$

Representing Relations Using Matrices

Definition

Suppose R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$, R can be represented by the **matrix** $\mathbf{M}_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if}(a_i, b_j) \in R \\ 0 & \text{if}(a_i, b_j) \notin R \end{cases}$$

Example

R is relation from $A = \{1, 2, 3\}$ to $B = \{1, 2\}$. Let $R = \{(2, 1), (3, 1), (3, 2)\}$, the matrix for R is

$$\mathbf{M}_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Determine whether the relation has certain properties (reflexive, symmetric, antisymmetric,...)

Huynh Tuong Nguyen, Tran Tuan Anh, Nguyen An Khuong, Le Hong Trang

Representing Relations Using Digraphs

Huynh Tuong Nguyen, Tran Tuan Anh, Nguyen An Khuong, Le Hong Trang

Definition

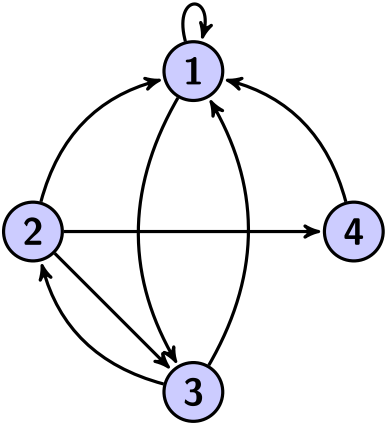
Suppose R is a relation in $A = \{a_1, a_2, \ldots, a_m\}$, R can be represented by the **digraph** (*đồ thị có hướng*) $G = (V, E)$, where

$$V = A$$
$$(a_i, a_j) \in E \text{ if } (a_i, a_j) \in R$$

Example

Given a relation on $A = \{1, 2, 3, 4\}$,
$R = \{(1,1), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (4,1)\}$
Draw corresponding digraph.

Resulting digraph



Closure

Definition

The **closure** (*bao đóng*) of relation R with respect to **property** P is the relation S that

i. **contains** R

ii. **has** property P

iii. is **contained in any** relation satisfying **(i) and (ii)**.

S is the "smallest" relation satisfying (i) & (ii)

Huynh Tuong Nguyen, Tran Tuan Anh, Nguyen An Khuong, Le Hong Trang

Reflexive Closure

Huynh Tuong Nguyen, Tran Tuan Anh, Nguyen An Khuong, Le Hong Trang

Example

Let $R = \{(a,b),(a,c),(b,d),(d,c)\}$

The **reflexive closure** of R

$$\{(a,b),(a,c),(b,d),(d,c),(a,a),(b,b),(c,c),(d,d)\}$$

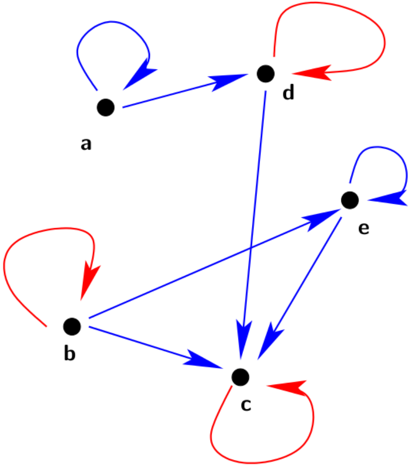
$$R \cup \Delta$$

where

$$\Delta = \{(a,a) \mid a \in A\}$$

diagonal relation (*quan hệ đường chéo*).

Reflexive Closure



Huynh Tuong Nguyen, Tran Tuan Anh, Nguyen An Khuong, Le Hong Trang

Symmetric Closure

Huynh Tuong Nguyen, Tran Tuan Anh, Nguyen An Khuong, Le Hong Trang

Example

Let $R = \{(a,b),(a,c),(b,d),(c,a),(d,e)\}$

The **symmetric closure** of R

$$\{(a,b),(a,c),(b,d),(c,a),(d,e),(b,a),(d,b),(e,d)\}$$

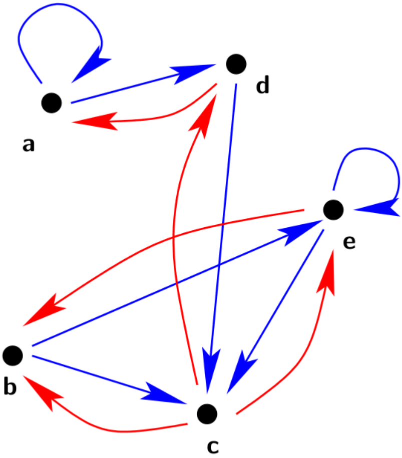
$$R \cup R^{-1}$$

where

$$R^{-1} = \{(b,a) \mid (a,b) \in R\}$$

inverse relation (*quan hệ ngược*).

Symmetric Closure



Transitive Closure

Huynh Tuong Nguyen, Tran Tuan Anh, Nguyen An Khuong, Le Hong Trang

Example

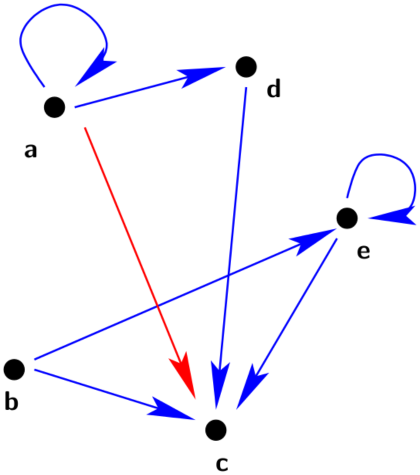
Let R $= \{(a,b),(a,c),(b,d),(d,e)\}$

The **transitive closure** of R

$$\{(a,b),(a,c),(b,d),(d,e),(a,d),(b,e),(a,e)\}$$

$$\cup_{n=1}^{\infty} R^n$$

Transitive Closure



Equivalence Relations

Definition

A relation on a set A is called an **equivalence relation** (*quan hệ tương đương*) if it is reflexive, symmetric and transitive.

Example (1)

The relation $R = \{(a,b) | a \text{ and } b \text{ are in the same provinces}\}$ is an equivalence relation. a is equivalent to b and vice versa, denoted $a \sim b$.

Example (2)

$$R = \{(a,b) \mid a = b \vee a = -b\}$$

R is an equivalence relation.

Example (3)

$$R = \{(x,y) \mid |x-y| < 1\}$$

Is R an equivalence relation?

Example

Example (Congruence Modulo m - Đồng dư modulo m)

Let m be a positive integer with $m > 1$. Show that the relation

$$R = \{(a, b) \mid a \equiv b \pmod{m}\}$$

is an equivalence relation on the set of integers.

Equivalence Classes

Definition

Let R be an **equivalence relation** on the set A. The set of all elements that are related to an element a of A is called the **equivalence class** (*lớp tương đương*) of a, denoted by

$$[a]_R = \{s \mid (a, s) \in R\}$$

Example

The equivalence class of "Thủ Đức" for the equivalence relation "in the same provinces" is { "Thủ Đức", "Gò Vấp", "Bình Thạnh", "Quận 10",...}

Example

Example

What are the equivalence classes of 0, 1, 2, 3 for congruence modulo 4?

Solution:

$$[0]_4 = \{..., -8, -4, 0, 4, 8, ...\}$$
$$[1]_4 = \{..., -7, -3, 1, 5, 9, ...\}$$
$$[2]_4 = \{..., -6, -2, 2, 6, 10, ...\}$$
$$[3]_4 = \{..., -5, -1, 3, 7, 11, ...\}$$

Equivalence Relations and Partitions

Theorem

Let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

i aRb

ii $[a] = [b]$

iii $[a] \cap [b] \neq \emptyset$

Example 1

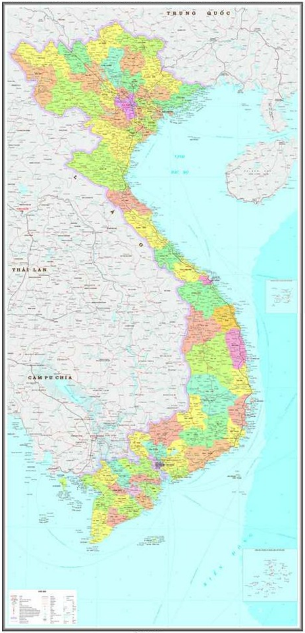
Example

Suppose that $S = \{1, 2, 3, 4, 5, 6\}$. The collection of sets $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$, and $A_3 = \{6\}$ forms a partition of S, because these sets are disjoint and their union is S

The equivalence classes of an equivalence relation R on a set S form a **partition** of S.

Every partition of a set can be used to form an equivalence relation.

Example 2



Example

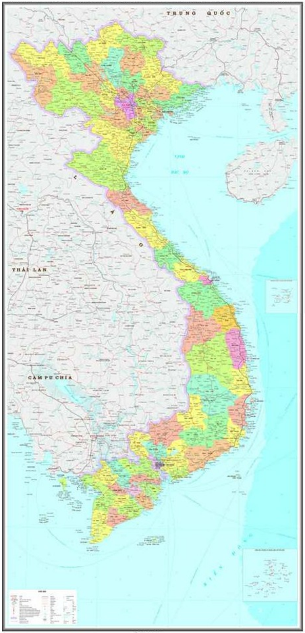
Divides set of all cities and towns in Vietnam into set of 64 provinces. We know that:

- there are no provinces with no cities or towns
- no city is in more than one province
- every city is accounted for

Definition

A **partition** of a Vietnam is a collection of non-overlapping non-empty subsets of Vietnam (provinces) that, together, make up all of Vietnam.

Relation in a Partition



- We divided based on relation

$$R = \{(a, b) | a \text{ and } b \text{ are in the same provinces}$$

- "Thủ Đức" is related (equivalent) to "Gò Vấp"
- "Đà Lạt" is not related (not equivalent) to "Long Xuyên"

Partial Order Relations

- Order words such that x comes before y in the dictionary
- Schedule projects such that x must be completed before y
- Order set of integers, where $x < y$

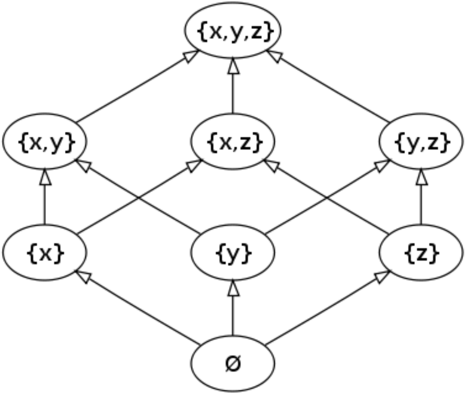
Definition

A relation R on a set S is called a **partial ordering** (*có thứ tự bộ phận*) if it is reflexive, antisymmetric and transitive. A set S together with a partial ordering R is called a partially ordered set, or **poset** (*tập có thứ tự bộ phận*), and is denoted by (S, R) or $(S, \preccurlyeq)$.

Example

- $(\mathbb{Z}, \geq)$ is a poset
- Let S a set, $(P(S), \subseteq)$ is a poset

Example



Totally Order Relations

Example

In the poset $(\mathbb{Z}^+, |)$, 3 and 9 are comparable (*so sánh được*), because $3 \mid 9$, but 5 and 7 are not, because $5 \nmid 7$ and $7 \nmid 5$.

$\rightarrow$ That's why we call it **partially** ordering.

Definition

If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a **totally ordered** (*có thứ tự toàn phần*). A totally ordered set is also called a **chain** (*dây xích*).

Example

The poset $(\mathbb{Z}, \leq)$ is totally ordered.

Maximal & Minimal Elements

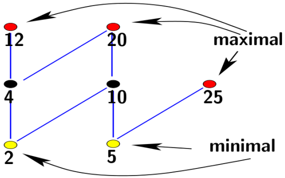
Huynh Tuong Nguyen, Tran Tuan Anh, Nguyen An Khuong, Le Hong Trang

Definition

- a is **maximal** (*cực đại*) in the poset $(S, \preccurlyeq)$ if there is no $b \in S$ such that $a \prec b$.
- a is **minimal** (*cực tiểu*) in the poset $(S, \preccurlyeq)$ if there is no $b \in S$ such that $b \prec a$.

Example

Which elements of the poset $(\{2, 4, 5, 10, 12, 20, 25\}, |)$ are minimal and maximal?



Greatest Element& Least Element

Definition

- a is the **greatest element** (*lớn nhất*) of the poset $(S, \preccurlyeq)$ if $b \preccurlyeq a$ for all $b \in S$.
- a is the **least element** (*nhỏ nhất*) of the poset $(S, \preccurlyeq)$ if $a \preccurlyeq b$ for all $b \in S$.

The greatest and least element are unique if it exists.

Example

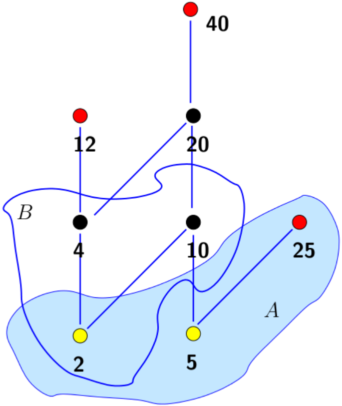
Let S be a set. In the poset $(P(S), \subseteq)$, the least element is $\emptyset$ and the greatest element is S.

Upper Bound & Lower Bound

Definition

Let $A \subseteq (S, \preccurlyeq)$.

- If u is an element of S such that $a \preccurlyeq u$ for all elements $a \in A$, then u is called an **upper bound** (*cận trên*) of A.
- If l is an element of S such that $l \preccurlyeq a$ for all elements $a \in A$, then l is called a **lower bound** (*cận dưới*) of A.



Example

- Subset A does not have upper bound and lower bound.
- The upper bound of B are $20, 40$ and the lower bound is 2.