Use these instructions for the remainder of the exam.

Your J-number

$$JN \quad := \quad (J_k \in 0..9 | k \in 1..8) \quad = \quad \underline{00298436}$$

Hence

$$J_1 = 0 \quad J_2 = 0 \quad J_3 = 2 \quad J_4 = 9 \quad J_5 = 8 \quad J_6 = 4 \quad J_7 = 3 \quad J_8 = 6$$

R Write **the elements of** every numerical set or multiset appearing in this exam **in increasing order.**

D $$\mathcal{U} \quad := \quad 0..9 \quad = \quad \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

D $$SJ \quad := \quad \left\{J_k \in 0..9 \middle| k \in 1..8\right\} \quad = \{0, 2, 3, 4, 6, 8, 9\}$$

$$\nu(SJ) \quad = \quad 7$$

D $$SJN \quad := \left((SJ)_k \in 0..9 \middle| k \in 1..(\nu(SJ))\right) = \quad (0, 2, 3, 4, 6, 8, 9)$$

D $$\forall l \in 1..(\nu(SJ)) \quad n_l \quad := \quad \nu\left(\left\{k \in 1..8 \middle| J_k = (SJ)_l\right\}\right)$$

D $$MSJ \quad := \quad \left\{(SJ)_m : n_m \middle| m \in 1..(\nu(SJ))\right\}$$

$$= \quad \{0:2, 2:1, 3:1, 4:1, 6:1, 8:1, 9:1\}$$

D $$\nu(MSJ) \quad = \quad \Sigma_{m \in 1..(\nu(SJ))} n_m \quad = \quad 8$$

D $s := mnm_{k \in 1..8}\{J_k | J_k \neq 0\} = 2$ ✓

D $b := mxm_{k \in 1..8}\{J_k | k \in 1..8\} = 9$ ✓

D Define Y := Yes N := No Pf := Proof W := Witness

D $\forall x \in \mathbb{Z}$ $Evn(x) :\Leftrightarrow x$ is even.

D $\forall x \in \mathbb{Z}$ $Odd(x) :\Leftrightarrow x$ is odd.

D $\forall x \in \mathbb{N}$ $Prm(x) :\Leftrightarrow x$ is prime.

R **Note that in a set an element may occur exactly once**. Therefore, there may not be any repeated elements in any set in the above. **So, if you do not have the correct sets, all your answers will be wrong.**

R If answers to different instances of some question on the exam contradict each other you will get a score of zero for every instance.

R **You will lose points for making any extraneous mark on the exam.**

R You may only submit complete answers to questions marked explicitly on the exam. You will lose points for unfinished work.

R Separate pieces of work for any part of any problem recorded on a blank page intended for the purpose must be indexed by the index of the problem and also by numbers starting at 0 that record the order in which the pieces were completed; such parts must be separated from each other by straight lines parallel to the edges of the paper drawn with a ruler. You will lose points for otherwise.

R Any sloppiness, untidiness, and any departure from proper format (as indicated in class) will lead to a score of 0.

i)

$$\nu(MSJ) = 8, \quad \nu(SJ) = 7$$

$$P(\nu(MSJ), 0)$$
$$= P(8,0)$$
$$= \left(P(n,k)\right)\left\langle {n \leftarrow 8 \atop k \leftarrow 0} \right\rangle$$
$$= \left(\frac{n!}{(n-k)!}\right)\langle \text{〃} \rangle$$
$$= \frac{8!}{(8-0)!}$$
$$= \frac{\cancel{8!}}{\cancel{8!}}$$
$$= \frac{1}{1}$$
$$= 1$$

$$P(\nu(SJ), 0)$$
$$= P(7,0)$$
$$= \left(P(n,k)\right)\left\langle {n \leftarrow 7 \atop k \leftarrow 0} \right\rangle$$
$$= \left(\frac{n!}{(n-k)!}\right)\langle \text{〃} \rangle$$
$$= \frac{7!}{(7-0)!}$$
$$= \frac{\cancel{7!}}{\cancel{7!}}$$
$$= \frac{1}{1}$$
$$= 1$$

$$P\left(P\left(\nu(MSJ), 0\right), P\left(\nu(SJ), 0\right)\right)$$
$$= P(1,1)$$
$$= \left(P(n,1)\right)\langle n \leftarrow 1 \rangle$$
$$= (n)\langle n \leftarrow 1 \rangle$$
$$= 1$$

6 (1a) Compute, $\forall n \in \mathbb{N}$, $\forall k \in 0..n$ the following, and express the result as a polynomial in n. **Show computations or substitutions into appropriate formulas exactly as shown in class on the facing page and write the answers on this page.**

2 (i) $$P\left(P\left(v\left(MSJ\right),0\right),P\left(v\left(SJ\right),0\right)\right)$$

$= 1$ ✓

2 (ii) $$P\left(C\left(v\left(MSJ\right),0\right),C\left(v\left(SJ\right),0\right)\right)$$

$= 1$ ✓

2 (iii) $$C\left(C\left(v\left(MSJ\right),v\left(SJ\right)\right),C\left(v\left(MSJ\right),v\left(SJ\right)\right)\right)$$

$= 1$ ✓

⑥

(ii) $\nu(MSJ) = 8, \quad \nu(SJ) = 7$

$$C(\nu(MSJ), 0)$$

$$= C(8,0)$$

$$= C(n,k)\left\langle \begin{matrix} n \leftarrow 8 \\ k \leftarrow 0 \end{matrix} \right\rangle$$

$$= \left(\frac{n!}{k!((n-k)!)}\right)\left\langle \begin{matrix} n \leftarrow 8 \\ k \leftarrow 0 \end{matrix} \right\rangle$$

$$= \frac{8!}{0!((8-0)!)}$$

$$= \frac{\cancel{8!}}{1(\cancel{8!})}$$

$$= \frac{1}{1}$$

$$= 1$$

$$C(\nu(SJ), 0)$$

$$= C(7,0)$$

$$= C(n,k)\left\langle \begin{matrix} n \leftarrow 7 \\ k \leftarrow 0 \end{matrix} \right\rangle$$

$$= \left(\frac{n!}{k!(n-k)!}\right)\left\langle \begin{matrix} n \leftarrow 7 \\ k \leftarrow 0 \end{matrix} \right\rangle$$

$$= \frac{7!}{0!((7-0)!)}$$

$$= \frac{\cancel{7!}}{1\,\cancel{(7!)}}$$

$$= \frac{1}{1}$$

$$= 1$$

$$P\left(C(\nu(MSJ), 0), C(\nu(SJ), 0)\right)$$

$$= P(1, 1)$$

$$= (P(n, 1))\langle n \leftarrow 1 \rangle$$

$$= (n)\langle n \leftarrow 1 \rangle$$

$$= 1$$

(iii)

$\nu(MSJ) = 8, \quad \nu(SJ) = 7$

$$C\left(\nu(MSJ), \nu(SJ)\right)$$

$$= C(8,7)$$

$$= C(n,k) \left\langle \begin{matrix} n \leftarrow 8 \\ k \leftarrow 7 \end{matrix} \right\rangle$$

$$= \left(\frac{n!}{k!((n-k)!)}\right) \left\langle \,''\, \right\rangle$$

$$= \frac{8!}{7!((8-7)!)}$$

$$= \frac{8 \cdot \cancel{7!}}{\cancel{7!}(1!)}$$

$$= \frac{8}{1}$$

$$= 8$$

$$C\left(C\left(\nu(MSJ), \nu(SJ)\right), C\left(\nu(MSJ), \nu(SJ)\right)\right)$$

$$= C(8,8)$$

$$= C(n,k) \left\langle \begin{matrix} n \leftarrow 8 \\ k \leftarrow 8 \end{matrix} \right\rangle$$

$$= \left(\frac{n!}{k!((n-k)!)}\right) \left\langle \,''\, \right\rangle$$

$$= \frac{\cancel{8!}}{\cancel{8!}((8-8)!)}$$

$$= \frac{1}{1(0!)}$$

$$= \frac{1}{1}$$

$$= 1$$

6 (1b) **Provide complete reasoning by making a systematic, exhaustive list in the form of a table,** to count the following:

3 (i) D $$F := \left\{(x, y) \in \left(SJ \right) \times \left(SJ \right) \middle| Prm(x + y)\right\}$$

compute: $v\left(F \right) = \_\_\_$

3 (ii) D $$G := \left\{(x, y) \in \left(SJ \right) \times \left(SJ \right) \middle| Prm(xy)\right\}$$

compute: $v\left(G \right) = \_\_\_$

0

(i) ⓪

$$\varphi(0) = (9(0))(9(0)+1)$$

$$= 0(0+1)$$

$$= 0$$

(ii) ①

$$\varphi(n+1) = (9(n+1))(9(n+1)+1)$$

$$= (9n+9)(9n+9+1)$$

$$= (9n+9)(9n+10)$$

$$= 81n^2 + 171n + 90$$

5 (1c) Carry out the following instructions:

$$\text{D} \quad e \quad := \left(mnm_{k \in 1..8}\{J_k \in 0..9 | Evn(J_k)\} \right) + 1 = (0) + 1 = 1$$

$$\text{D} \quad b \quad := \left(mxm_{k \in 1..8}\{J_k \in 0..9 | J_k\} \right) \cancel{+1} = 9 = 9$$

$$1\ (0) \quad be = (9)(1) = 9$$

Find recursive definition(s) for the functions defined below, indicating the base case(s) and recursive step(s) explicitly.

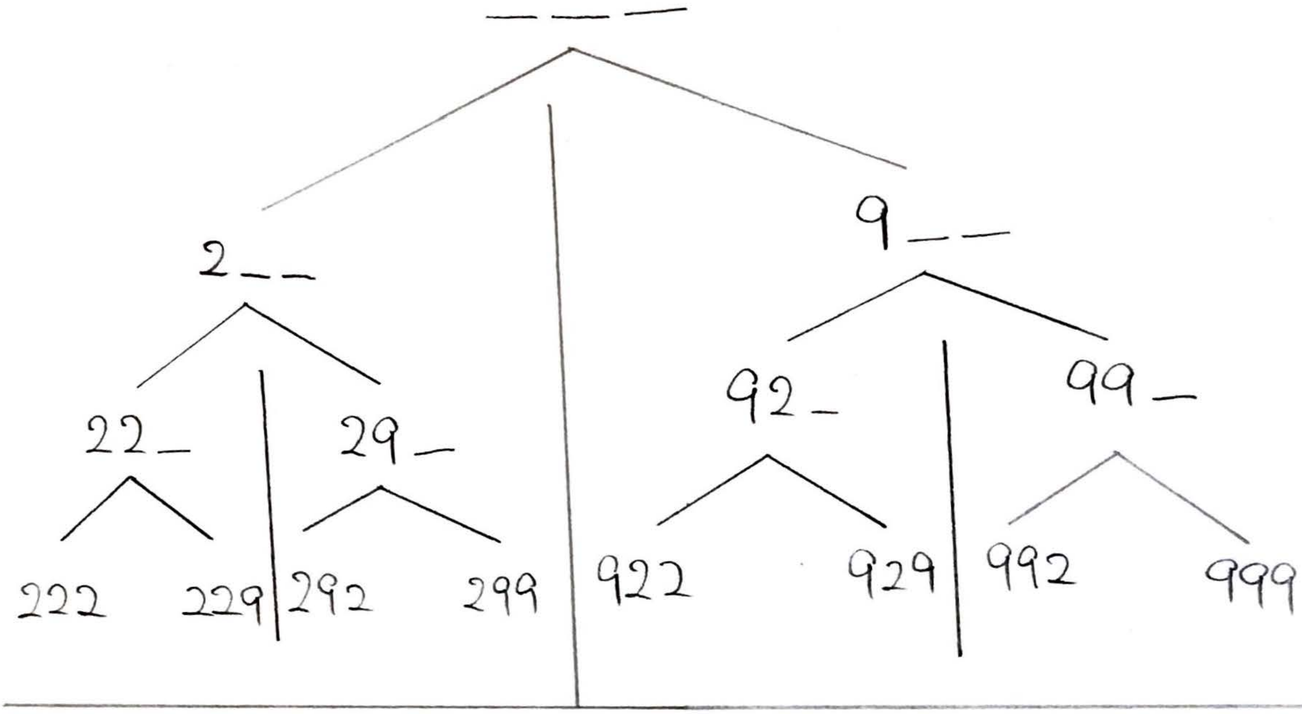
4 (i)

$$\varphi := n \mapsto (ben)(ben+1) = \big((9)n\big)\big((9)n+1\big) : \mathbb{N} \longrightarrow \mathbb{N}$$

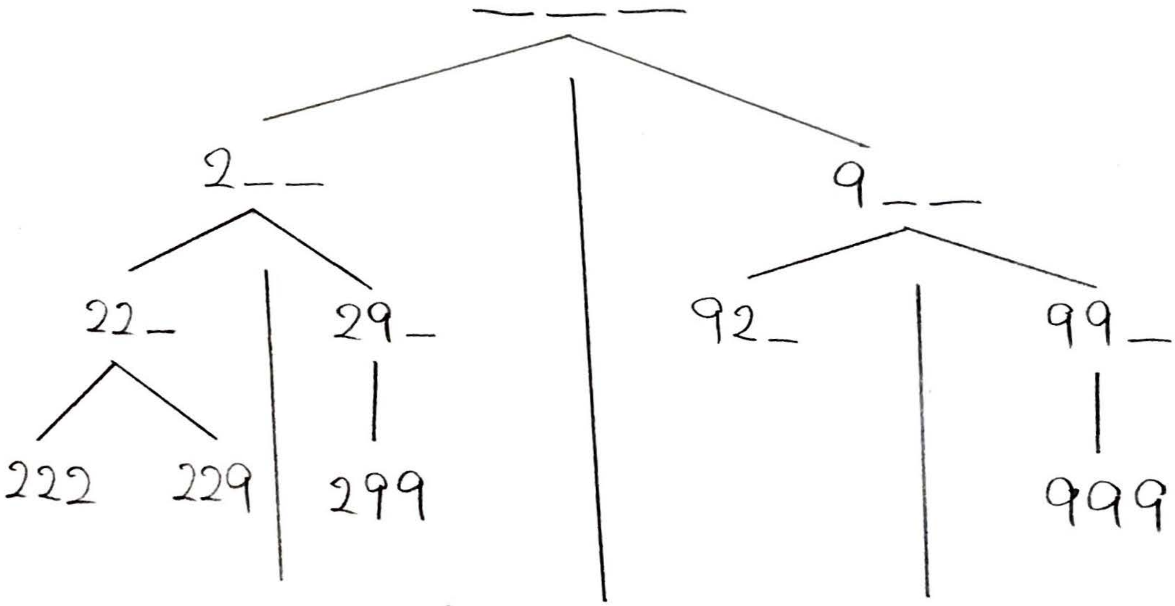
BC $\varphi(0) = 0$

RcS $\varphi(n+1) = 81n^2 + 171n + 90$

(i) TREE



(ii) TREE



6 (2a) **Use a tree (which you should draw on the facing page)** to **generate, list,** and **count** the number of elements in the set of all **3-letter words** in the alphabet $\{s,b\}$ such that

My $\{s,b\} = \{\ 2, 9\ \}$

3 (i) order matters and repetition is allowed, and

$Wrd((3,\{2,9\},OM,RA))$

$= \{222, 229, 292, 299, 922, 929, 992, 999\}$

$v(Wrd((3,\{2,9\},OM,RA)))$

$= 8$

3 (ii) order does **not** matter and repetition is allowed

~~OM~~ RA

$Wrd((3,\{2,9\},\cancel{OM,RA}))$

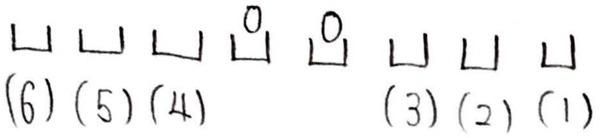
$= \{222, 229, 299, 999\}$

~~OM~~, RA

$v(Wrd((3,\{2,9\},\cancel{OM,RA})))$

$= 4$

8



$$P((8-2),(8-2))$$

$$= P(6,6)$$

$$= \left(P(n,k)\right)\left\langle \begin{matrix} n \leftarrow 6 \\ k \leftarrow 6 \end{matrix} \right\rangle$$

$$= \left(\frac{n!}{(n-k)!}\right)\langle \text{''} \rangle$$

$$= \frac{6!}{(6-6)!}$$

$$= \frac{6!}{0!}$$

$$= \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1}$$

$$= 720$$

6 (2b) Count the number of **distinct** 8-digit numbers that may be made by **permuting**:

$$MSJ \ := \ \left\{ J_k : n_k \,\middle|\, k \in 1..\left(\nu(SJ)\right) \right\}$$

$$= \ \left\{ 0:2, 2:1, 3:1, 4:1, 6:1, 8:1, 9:1 \right\}$$

given that:

- 0 **must NOT occur** in the **three** leftmost places

and

- 0 **must occur** in the fourth place and in the fifth place counted from the left.

720 #

6

5 (2c) Recall your recursive definitions for the following functions from (1c).

$$\varphi := n \longmapsto (ben)(ben+1) = ((9)n)((9)n+1) : \mathbb{N} \longrightarrow \mathbb{N}$$

BC $\varphi(0) = 0$

RcS $\varphi(n+1) = 81n^2 + 171n + 90$

5 (i) Prove by induction that:

$$\forall n \in \mathbb{N} \qquad Evn\Big(\varphi(n)\Big)$$

0

(3a)

$s = 2,\ b = 9$

$2b$	$b+s$	$b-s$
$= 18$	$= 11$	$= 7$

(i)

$$\mathrm{BnmlCfn}\left(x^{b+s}y^{b-s},\ (x+y)^{2b}\right)$$

$$= \mathrm{BnmlCfn}\left(x^{11}y^{7},\ (x+y)^{18}\right)$$

$$= \left(\mathrm{BnmlCfn}\left(a^{n-k}b^{k},\ (a+b)^{n}\right)\right)\left\langle \begin{matrix} a \leftarrow x \\ b \leftarrow y \\ n \leftarrow 18 \\ k \leftarrow 7 \end{matrix} \right\rangle$$

$$= \binom{n}{k} \langle \ " \ \rangle$$

$$= \binom{18}{7}$$

$$= C(n, k) \left\langle \begin{matrix} n \leftarrow 18 \\ k \leftarrow 7 \end{matrix} \right\rangle$$

$$= \left(\frac{n!}{k!((n-k)!)}\right) \langle \ " \ \rangle$$

✓

$$= \frac{18!}{7!((18-7)!)}$$

$$= \frac{\cancel{18}(17)(16)\overset{3}{\cancel{(15)}}\cancel{(14)}(13)\overset{3}{\cancel{(12)}}\cancel{(11!)}}{\cancel{7}\cancel{(6)}\cancel{(5)}\cancel{(4)}\cancel{(3)}\cancel{(2)}(1)\cancel{(11!)}}$$

$$= \frac{17(16)(3)(13)(3)}{1}$$

$$= 31824$$

6 (3a) Answer the following:

D $s := mnm_{k\in 1..8}\{U_k \in 0..9 | U_k \neq 0\} = 2$

D $b := mxm_{k\in 1..8}\{U_k \in 0..9 | k \in 1..8\} = 9$

$2b = 2(9) = 18$

1 (i) Compute, using the **binomial theorem**, the coefficient of x^{b+s} in the expansion of: $(x+y)^{2b}$

$\binom{2b}{b-s}$

$$BnmlCfn\left(\underset{y^{b-s}}{x^{b+s}}, (x+y)^{2b}\right) = 31824$$

1 (ii) Compute, using the **binomial theorem**, the coefficient of y^{b+s} in the expansion of: $(x+y)^{2b}$

$\binom{2b}{b+s}$

$$BnmlCfn\left(\underset{y^{b+s}}{\cancel{y^{b-s}}}, (x+y)^{2b}\right) = 31824$$

$\binom{2b}{b-s\quad b+s}$

1 (iii) Compute, using the **multinomial theorem**, the coefficient of x^{b+s} in the expansion of: $(x+y)^{2b}$

$\binom{2b}{b+s\quad b-s}$

$$MltnmlCfn\left(x^{b+s}y^{b-s}, (x+y)^{2b}\right) = 31824$$

1 (iv) Compute, using the **multinomial theorem**, the coefficient of x^{b+s} in the expansion of: $(x+y)^{2b}$

$$MltnmlCfn\left(x^{b-s}y^{b+s}, (x+y)^{2b}\right) = 31824$$

1 (v) The answers in (i), (ii), (iii), and (iv) equal

[✓ Y] N [✓ Pf] W

1 (vi) Does the answer to (v) depend on the values of s and b?

Y [✓ N] [✓ Pf] W

-1

5

(ii)$^{(3a)}$

$$\mathrm{BnmlCfu}\left(x^{b+5}y^{b-5}, (x+b)^{2b}\right)$$

$$= \mathrm{BnmlCfu}\left(x^{11}y^{7}, (x+y)^{18}\right)$$

$$= 31824$$

(3a) (iii)

$$\mathrm{MltnmlCfu}\left(x^{b+5}y^{b-5}, (x+y)^{2b}\right)$$

$$= \mathrm{MltnmlCfu}\left(x^{11}y^{7}, (x+y)^{18}\right)$$

$$= \left(\mathrm{MltnmlCfu}\left(a^{n-k}b^{k}, (a+b)^{n}\right) \left\langle \begin{matrix} a \leftarrow x \\ b \leftarrow y \\ n \leftarrow 18 \\ k \leftarrow 7 \end{matrix} \right\rangle\right.$$

$$= \left\langle \begin{matrix} 18 \\ (18-7), 7 \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} 18 \\ 11, 7 \end{matrix} \right\rangle$$

$$= \frac{18!}{11!(7!)}$$

$$= \frac{\cancel{18}(17)(16)(\overset{3}{\cancel{15}})(\cancel{14})(13)(\overset{3}{\cancel{12}})(\cancel{11!})}{\cancel{11!}(\cancel{7})(\cancel{6})(\cancel{5})(\cancel{4})(\cancel{3})(\cancel{2})(1)}$$

$$= \frac{17(16)(3)(13)(3)}{1}$$

$$= 31824$$

(3a) (iv)

$$\mathrm{MltnmlCfu}\left(x^{b-5}y^{b+5}, (x+y)^{2b}\right)$$

$$= \mathrm{MltnmlCfu}\left(x^{7}y^{11}, (x+y)^{18}\right)$$

$$= \left(\mathrm{MltnmlCfu}\left(a^{n-k}b^{n}, (a+b)^{n}\right) \left\langle \begin{matrix} a \leftarrow x \\ b \leftarrow y \\ n \leftarrow 18 \\ k \leftarrow 11 \end{matrix} \right\rangle\right.$$

$$= \left\langle \begin{matrix} 18 \\ (18-11), 11 \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} 18 \\ 7, 11 \end{matrix} \right\rangle$$

$$= \frac{18!}{7!(11!)}$$

$$= \frac{18!}{11!(7!)}$$

$$= 31824$$

(3a) (v)

$$\mathrm{BnmlCfu}\left(x^{b+5}y^{b-5}, (x+y)^{2b} = 31824,\right.$$

$$\mathrm{BnmlCfu}\left(x^{b+5}y^{b-5}, (x+y)^{2b} = 31824,\right.$$

$$\mathrm{MltnmlCfu}\left(x^{b+5}y^{b-5}, (x+y)^{2b}\right) = 31824,$$

$$\mathrm{MltnmlCfu}\left(x^{b-5}y^{b+5}, (x+y)^{2b}\right) = 31824$$

$$\mathrm{BnmlCfu}\left(x^{b+5}y^{b-5}, (x+y)^{2b}\right)$$

$$= \mathrm{BnmlCfu}\left(x^{b+5}y^{b-5}, (x+y)^{2b}\right)$$

$$= \mathrm{MltnmlCfu}\left(x^{b+5}y^{b-5}, (x+y)^{2b}\right)$$

$$= \mathrm{MltnmlCfu}\left(x^{b-5}y^{b+5}, (x+y)^{2b}\right)$$

(3a) (vi)

6 (3b) Show your work on the facing side to compute the value of:

$$\sum_{n_1+n_2+n_3+n_4=5} \binom{5}{n_1 \quad n_2 \quad n_3 \quad n_4}$$

=

0

5 (3c) Recall the from (1c) and (2c)

$$\varphi := \; n \longmapsto \; (ben)(ben + 1) = \big((9)n\big)\big((9)n + 1\big) : \mathbb{N} \longrightarrow \mathbb{N}$$

Prove directly that:

$$\forall n \in \mathbb{N} \qquad Evn\left(\varphi(n)\right)$$