6 (1a) Compute, $\forall n \in \mathbb{N}$, $\forall k \in 0..n$ the following, and express the result as a polynomial in n. **Show computations or substitutions into appropriate formulas exactly as shown in class on the facing page and write the answers on this page.**

1 (i) $P(P(n,n),P(n,n))$

$= ((n!)!)$

1 (ii) $P(C(n,n),C(n,n))$

$= 1$

1 (iii) $C(P(n,n),P(n,n))$

$= 1$

1 (iv) $C(C(n,n),C(n,n))$

$= 1$

1 (v) $P(P(n,n),C(n,n))$

$= n!$

1 (vi) $P(C(n,n),P(n,n))$

$= P(1, n!)$

5

(i) $P(n,n) = n!$ $\quad \left\langle n \leftarrow P(n,n) \right\rangle$

$P(P(n,n),P(n,n)) = ((P(n,n))!)$

$P(P(n,n),P(n,n)) = ((n!)!)$

(ii) $P(n,n) = n!$ $\quad \left\langle n \leftarrow C(n,n) \right\rangle$

$P(C(n,n),C(n,n)) = ((C(n,n))!)$

$P(C(n,n),C(n,n)) = 1! = 1$

(iii) $C(n,n) = 1$ $\quad \left\langle n \leftarrow P(n,n) \right\rangle$

$C(P(n,n),P(n,n)) = 1$

(iv) $C(n,n) = 1$ $\quad \left\langle n \leftarrow C(n,n) \right\rangle$

$C(C(n,n),C(n,n)) = 1$

(v) $P(n,1) = n$ $\quad \left\langle \begin{array}{l} n \leftarrow P(n,n) \\ 1 \leftarrow C(n,n) \end{array} \right\rangle$

$P(P(n,n),C(n,n)) = P(n,n) = n!$

(vi) $P(C(n,n),P(n,n))$

$= P(1, n!)$

6 (1b) Either **provide complete reasoning,** or **make a systematic, exhaustive list** to count the following.

3 (i) the total number of **non-symmetric relations**

$$R \in Rln(\{a,b,c\},\{a,b,c\})$$

that is, compute:

$$\nu(\{R \in Rln(\{a,b,c\},\{a,b,c\}) | (\cancel{Sym}(R))\}) = \underline{\quad 448 \quad}$$

2

3 (ii) the total number of **constant functions** $\alpha: o..n \longrightarrow o..n$

that is, compute: $\nu(\{\alpha \in Fnc(0..n, 0..n) | Cnst(\alpha)\})$

$$\nu(\{\alpha \in Fnc(0..n, 0..n) | Cnst(\alpha)\}) = \underline{\quad n+1 \quad}$$

(i) $S := \{a, b, c\}$

$T := \{a, b, c\}$

$m := \nu(S) = 3$

$n := \nu(T) = 3$

$\nu(Rln(S,T)) =$

$= 2^{(m)(n)}$

$= 2^{(3)(3)}$

$= 2^9$

No.

$\nu(SymRln(S))$

$= 2^{\left(\frac{m^2+m}{2}\right)}$

$= 2^{\frac{3^2+3}{2}}$

$= 2^{\frac{12}{2}}$

$= 2^6$

3(S)

$\nu(R \in Rln(\{a,b,c\}, \{a,b,c\}) \mid (\cancel{Sym}(R))\})$

$= \nu(R \in Rln(S, T)) \mid \cancel{Sym}(R))\})$

$= \nu(Rln(S,T)) - \nu(SymRl(S))$

$= 2^9 - 2^6$

$= 512 - 64$

$= 448$

5 (1c) Decide if:

$$\forall n \in \mathbb{N}$$

$$\nu\big(Bij(0..n, 0..n)\big) = \nu\Big(Bij\big(1..(n+1), 1..(n+1)\big)\Big)$$

$S_1 := 0..n$

$S_2 := 1..(n+1)$

$s_1 := \nu(S_1) = n+1$

$s_2 := \nu(S_2) = n+1$

LS

$= \nu(Bij(0..n, 0..n))$

$= \nu(Bij(S_1, S_1)$

$= s_1!$

$= (n+1)!$

RS

$= \nu(BijFnc(1..(n+1), 1..(n+1))$

$= \nu(BijFnc(S_2, S_2)$

$= s_2!$

$= (n+1)!$

5

$LS = (n+1)! = RS$

$\nu(Bij(0..n, 0..n)) = \nu(Bij(1..(n+1), 1..(n+1)))$

6 (2a) **Use a tree (which you should draw on the facing page)** to **generate, list,** and **count** the number of elements in the set W of all **4-letter words** in the alphabet $\{a, b\}$ such that

3 (i) order matters and repetition is allowed, and

$$Wrd((4, \{a,b\}, OM, RA))$$

$$= \left\{ \begin{array}{l} aaaa, aaab, aaba, aabb, abaa, abab, abba, abbb, \\ baaa, baab, baba, babb, bbaa, bbab, bbba, bbbb \end{array} \right\}$$

$$v\left(Wrd((4, \{a,b\}, OM, RA))\right)$$

$$= 16$$

3 (ii) order does **not** matter and repetition is allowed

$$Wrd((4, \{a,b\}, \cancel{OM}, RA))$$

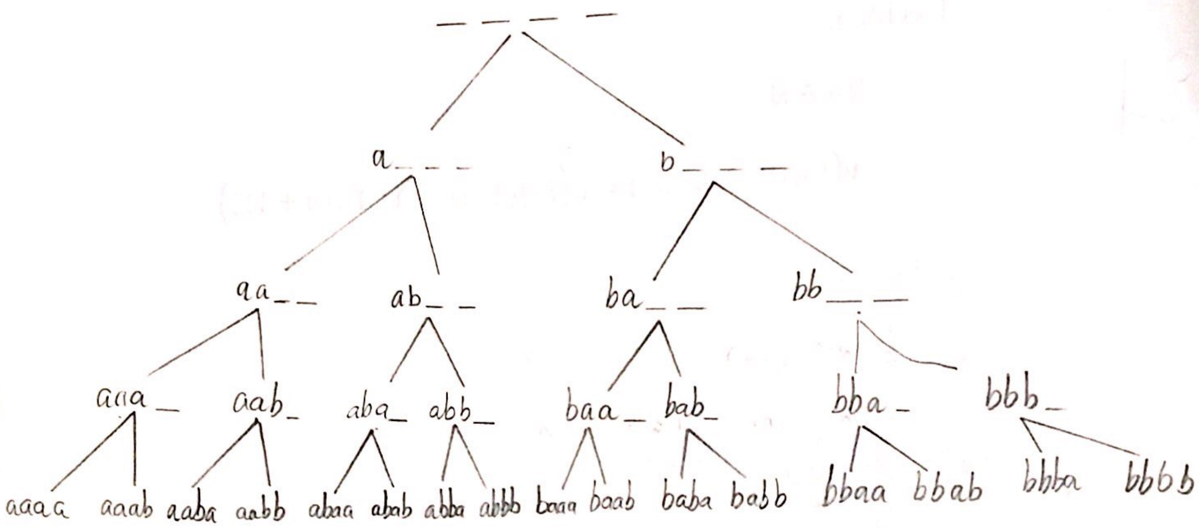
$$= \left\{ \begin{array}{l} aaaa, aaab, aabb, abbb, \\ bbbb \end{array} \right\}$$

$$v\left(Wrd((4, \{a,b\}, \cancel{OM}, \overset{RA}{\cancel{RA}}))\right)$$

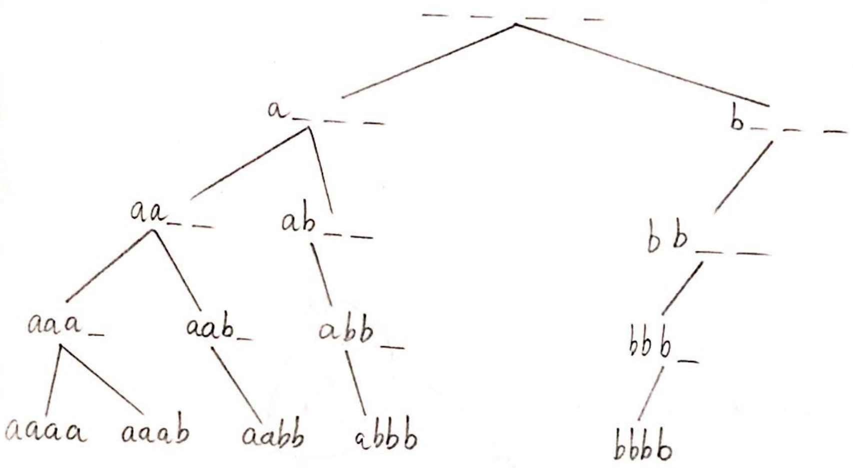
$$= 4$$

6

(i) Tree



(ii) Tree



6 (2b) Count the number of **distinct** quadruples (a, b, c, d) such that $a, b, c, d \in 1..9$ and

$$10 \nmid (a + b + c + d)$$

D $(a, b, c, d) = (p, q, r, s) :\Leftrightarrow (a = p) \wedge (b = q) \wedge (c = r) \wedge (d = s)$

D $(a, b, c, d), (p, q, r, s)$ are said to be **distinct** $:\Leftrightarrow (a, b, c, d) \neq (p, q, r, s)$

R Note that a number may NOT be used more than once.

R You may, if you wish, make a complete and exhaustive list of the possibilities.

$$\underbrace{\nu(\{(a, b, c, d) \in (1..9)^4 \mid 10 \nmid (a + b + c + d)\})}_{\bar{K}_{10}} =$$

$K = \{(a,b,c,d) \in (1..9)^4\}$

2/23/2018. $K_{10} = \{(a,b,c,d) \in (1..9)^4 \mid a+b+c+d = 10\}$

$K_{20} = \{ \qquad = 20\}$

$K_{30} = \{(a,b,c,d) \qquad = 30\}$

$\nu(\bar{K}_{10}) = \nu(K) - (\nu(K_{10}) + \nu(K_{20}) + \nu(K_{30}))$

$\nu(K) = 9.8.7.6$ (b/c a number may not be used more than once)

$\nu(K_{10}) = P(4,4)$

$\nu(K_{20}) = C(4,2)\,P(4,4)$

$\nu(K_{30}) = P(4)$

$\nu(\bar{K}_{10}) = 9.8.7.6 - (P(4,4) + C(4,2)(P(4,4)) + P(4))$

5

1,4 + 2,3 = 10

10

1,9 2,8 3,7 4,6

P(4,4) P(4,4)

P(4,4)

C(4,2) . (P(4,4))

15

9,6 8,7

5 (2c) Find recursive definitions for the functions:

$$\alpha := n \mapsto P(n,n) : \mathbb{N} \longrightarrow \mathbb{N}$$

$$\varphi := n \mapsto C(n,n) : \mathbb{N} \longrightarrow \mathbb{N}$$

3 (i) You must indicate the base case(s) and recursive step(s) explicitly.

2 (ii) Prove directly (without using induction) that:

$$\forall n \in \mathbb{N} \left((\alpha(n)) \geq (\varphi(n)) \right)$$

BC: $\alpha(0) = P(0,0) = 0! = 1$

RcS: $\alpha(n+1) = P(n+1, n+1) = (n+1)! = (n+1)\alpha(n)$

the -

BC: $\varphi(0) = C(0,0) = 1$

RcS: $\varphi(n+1) = C(n+1, n+1) = 1$

the '

Pf $\alpha(n) \div \varphi(n)$

$$= \frac{P(n,n)}{C(n,n)}$$

$$= \frac{n! \quad n!((n-n)!)}{(n-n)! \cdot n!}$$

$=$

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6 (3a) Answer the following:

1 (i) Compute, using the **binomial theorem**, the coefficient of $x^n y^n$ in the expansion of: $(x+y)^{2n}$

$$cfn(x^n y^n)$$
$$= \binom{2n}{n}$$

1 (ii) Compute, using the **multinomial theorem**, the coefficient of $x^n y^n$ in the expansion of: $(x+y)^{2n}$

$$cfn(x^n y^n)$$
$$= \binom{2n}{n\ n}$$

1 (iii) Prove that your answers in (i) and (ii) are equal

3

1 (iv) Compute the number of permutations of the word **abacus** such that consecutive letters of the alphabet **remain consecutive** in the permutation. You must either list or provide complete reasoning on the facing page and write the answer below. **Do not compute the numerical value of your result.**

2 (v) Compute the number of permutations of the word **abacus** such that consecutive letters of the alphabet **do not remain consecutive** in the permutation. You must either list or provide complete reasoning on the facing page and write the answer below. **Do not compute the numerical value of your result.**

(iii)

Pf

$$\binom{2n}{n}$$

$$= \frac{((2n)!)}{n!\,((2n-n)!)}$$

$$= \frac{((2n)!)}{n!\; n!}$$

$$= \binom{2n}{n\;n}$$

$$\binom{2n}{n} = \binom{2n}{n\;n}$$

fg

(iv)

$$a := \left\{\begin{array}{l} a:2 \\ b:1 \\ c:1 \\ u:1 \\ s:1 \end{array}\right\}$$

(v)

$$k := \left\{\begin{array}{l} a:2 \\ b:1 \\ c:1 \\ u:1 \\ s:1 \end{array}\right\}$$

$\nu(k) =$

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6 (3b) Use the binomial theorem to find prove that:

$\forall p \in \mathbb{N}, \forall n \in \mathbb{N}, \forall k \in 0..n$

$$\sum_{k=1}^{n} \binom{n}{k} p^k = (p+1)^n - 1$$

PF

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$a \leftarrow 1$

$b \leftarrow p$

$$(1+p)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} p^k$$

$$(1+p)^n = \binom{n}{0} 1^{n-0} p^0 + \sum_{k=1}^{n} \binom{n}{k} p^k$$

$$(1+p)^n = \frac{n!}{0!((n-0)!)} 1 + \sum_{k=1}^{n} \binom{n}{k} p^k$$

$$(1+p)^n = 1 + \sum_{k=1}^{n} \binom{n}{k} p^k$$

$-1 \quad -1$

-1

$$(p+1)^n - 1 = \sum_{k=1}^{n} \binom{n}{k} p^k$$

$$(\forall p \in N)(\forall n \in N)(\forall k \in 0..n)\left(\sum_{k=1}^{n} \binom{n}{k} p^k = (p+1)^n - 1\right)$$

∎

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5 (3c) Recall the recursive definitions for the following functions from (3b).

$$\alpha := n \mapsto P(n,n) \colon \mathbb{N} \longrightarrow \mathbb{N}$$

$$\varphi := n \mapsto C(n,n) \colon \mathbb{N} \longrightarrow \mathbb{N}$$

Use your recursive definition(s) from (3b) to **prove by induction** that:

$$\forall n \in \mathbb{N} \left((\alpha(n)) \geq (\varphi(n)) \right)$$

BC $\alpha(0)$

$= P(0,0)$

$= 0!$

$= 1$

≥ 1

$= C(0,0)$

$= \varphi(0)$

$\alpha(0) \geq \varphi(0)$

$P(0)$

IH: Assume $P(0)$ $P(1)$ ... $(P(n))$

IS: $\alpha(n+1) \div \varphi(n+1)$

$= P(n+1, n+1) \div C(n+1, n+1)$

$= (n+1)! \div 1$

≥ 1

$\alpha(n+1) \geq \varphi(n+1)$

$P(n+1)$

$P(0) \left(\dfrac{P(0) \quad P(1) \quad \ldots \quad (P(n))}{P(n+1)} \right)$

$\forall n \in N \; (P(n))$

$\forall n \in N \quad P(n,n) \geq C(n,n)$

$\forall n \in N \left((\alpha(n)) \geq (\varphi(n)) \right)$

0

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