Instructions:

Your J-number = 00266502

Replace every 0 in your J-number by 1 to get

Your non-zero J number := 11266512

Define $\forall k \in 1..8$,

$a_k :=$ the k[th] entry in your non-zero J-number counting from the left. Therefore,

$a_1 :=$ 1

$a_2 :=$ 1

$a_3 :=$ 2

$a_4 :=$ 6

$a_5 :=$ 6

$a_6 :=$ 5

$a_7 :=$ 1

$a_8 :=$ 2

$a := mnm_{k\in 1..8}\{J_k | (k \in 1..8) \wedge (J_k \neq 0)\} =$ 2

$b := mxm(J_k | k \in 1..8) =$ 6

$c := \left\lfloor\left(\frac{b+a}{2}\right)\right. = \left\lfloor\left(\frac{6+2}{2}\right)\right. = \left\lfloor\left(\frac{8}{2}\right)\right. = \left\lfloor\left(\frac{\cancel{2}(4)}{\cancel{2}}\right)\right. = \lfloor(4) = 4$

$d :=$ the smallest **odd** number **strictly greater** than c = 5

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6 (1a) Expand the following **EXACTLY AS SHOWN IN CLASS.**

Use the binomial theorem and express your answer as a polynomial with **integer coefficients, writing the terms VERTICALLY, in increasing order of degree, ordering the unknowns within each term numero-alphabetically, and the terms lexicographically.** Show calculations for all coefficients that occur on the facing page.

3 (i) $(b+x)^{1+a}$

$= (6+x)^{1+2}$

$= (6+x)^{3}$

$= 216$

$+ (108)x$

$+ (18)x^2$

$+ x^3$

2

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(i) $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

$n \leftarrow 3$

$a \leftarrow 6$

$b \leftarrow x$

$((6)+(x))^3 = (6+x)^3 = \sum_{k=0}^{3} \binom{3}{k} (6)^{3-k} (x)^k$

$$
\begin{aligned}
(6+x)^3 &= \binom{3}{0}(6)^{3-0}(x)^0 + \binom{3}{1}(6)^{3-1}(x)^1 + \binom{3}{2}(6)^{3-2}(x)^2 + \binom{3}{3}(6)^{3-3}(x)^3 \\
&= (1)(6)^3(1) + (3)(6)^2(x) + (3)(6)^1(x)^2 + (1)(6)^0(x)^3 \\
&= (1)\,216\,(1) + (3)(36)(x) + (3)(6)(x)(x) + (1)(1)(x)(x)(x) \\
&= 216 + 108(x) + 18(x)(x) + (x)(x)(x) \\
&= 216 + (108)x + (18)x^2 + x^3
\end{aligned}
$$

(i) $\binom{3}{2}$

$= \binom{3}{3-2}$

$= \binom{3}{1}$

$= C(3,1)$

$= 3$

(1a) Expand the following **EXACTLY AS SHOWN IN CLASS.**

Use the multinomial theorem and express your answer as a polynomial with **integer coefficients, writing the terms VERTICALLY, in increasing order of degree, ordering the unknowns within each term numero-alphabetically, and the terms lexicographically.** Show calculations for all coefficients that occur on the facing page.

3 (ii) $(b + x + y)^{1+a}$

$= (6 + x + y)^{1+2}$

$= (6 + x + y)^{3}$

$= 216$

$+ (108)x$

$+ (18)x^2$

$+ x^3$

$+ (3)x^2y$

$+ (36)xy$

$+ (3)xy^2$

$+ (108)y$

$+ (18)y^2$

$+ y^3$

-1

2

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(iii) $(x_1 + x_2 + x_3)^n = \sum\limits_{p_1+p_2+p_3=n} \binom{n}{p_1\ p_2\ p_3} x_1^{p_1} x_2^{p_2} x_3^{p_3}$

$n \leftarrow 3$
$x_1 \leftarrow 6$
$x_2 \leftarrow x$
$x_3 \leftarrow y$

$((6)+(x)+(y))^3 = (6+x+y)^3 = \sum\limits_{p_1+p_2+p_3=3} \binom{3}{p_1\ p_2\ p_3} (6)^{p_1}(x)^{p_2}(y)^{p_3}$

$$
\begin{aligned}
(6+x+y)^3 = {} & \binom{3}{0\ 0\ 3}(6)^0(x)^0(y)^3 + \binom{3}{0\ 1\ 2}(6)^0(x)^1(y)^2 + \binom{3}{0\ 2\ 1}(6)^0(x)^2(y)^1 + \binom{3}{0\ 3\ 0}(6)^0(x)^3(y)^0 \\
& + \binom{3}{1\ 0\ 2}(6)^1(x)^0(y)^2 + \binom{3}{1\ 1\ 1}(6)^1(x)^1(y)^1 + \binom{3}{1\ 2\ 0}(6)^1(x)^2(y)^0 \\
& + \binom{3}{2\ 0\ 1}(6)^2(x)^0(y)^1 + \binom{3}{2\ 1\ 0}(6)^2(x)^1(y)^0 + \binom{3}{3\ 0\ 0}(6)^3(x)^0(y)^0 \\
= {} & (1)(1)(1)(y)^3 + (3)(1)(x)(y)^2 + (3)(1)(x)^2(y) + (1)(1)(x)^3(1) + (3)(6)(1)(y)^2 \\
& + (6)(6)(x)(y) + (3)(6)(x)^2(1) + (3)(6)^2(1)(y) + (3)(6)^2(x)(1) + (1)(6)^3(1)(1) \\
= {} & yyy + (3)xyy + (3)xxy + xxx + (18)yy + (36)xy + (18)xx + (108)y + (108)x + 216 \\
= {} & 216 + (108)x + (18)xx + xxx + (3)xxy + (36)xy + (3)xyy + (108)y + (18)yy + yyy \\
= {} & 216 + (108)x + (18)x^2 + x^3 + (3)x^2y + (36)xy + (3)xy^2 + (108)y + (18)y^2 + y^3
\end{aligned}
$$

(ii) $\binom{3}{0\ 0\ 3} = \binom{3}{0\ 3\ 0} = \binom{3}{3\ 0\ 0} = \frac{\cancel{3!}}{\cancel{3!}(0!)(0!)} = \frac{1}{(1)(1)(1)} = 1$

(ii) $\binom{3}{0\ 1\ 2} = \binom{3}{0\ 2\ 1} = \binom{3}{1\ 0\ 2} = \binom{3}{1\ 2\ 0} = \binom{3}{2\ 0\ 1} = \binom{3}{2\ 1\ 0} = \frac{3!}{(2!)(1!)(0!)} = \frac{3\cancel{(2)}}{\cancel{(2)}(1)(1)} = \frac{3}{1} = 3$

(ii) $\binom{3}{1\ 1\ 1} = \frac{3!}{(1!)(1!)(1!)} = \frac{3(2)(1)}{(1)(1)(1)} = 6$

6 (1b) Solve the following recurrence relations with the indicated initial conditions, using the algorithm for second-order recurrences given in class, and **prove by induction that your answers satisfy the recurrence:**

3 (i)
$$\begin{cases} & \rho(0) := a \\ & \rho(1) := b \\ \forall n \in \mathbb{N} & (\rho(n+2)) - (a+b)(\rho(n+1)) + (ab)(\rho(n)) = 0 \end{cases}$$

First rewrite the recurrence with your values of a and b.

My problem:

$$\begin{cases} & \rho(0) := 2 \\ & \rho(1) := 6 \\ \forall n \in \mathbb{N} & (\rho(n+2)) - (2+6)(\rho(n+1)) + (2)(6)(\rho(n)) = 0 \end{cases}$$

$$\begin{cases} & \rho(0) := 2 \\ & \rho(1) := 6 \\ \forall n \in \mathbb{N} & (\rho(n+2)) - 8(\rho(n+1)) + 12(\rho(n)) = 0 \end{cases}$$

$$\forall n \in \mathbb{N} \quad \rho(n) = \frac{1}{2}(6)^n + \frac{3}{2}(2)^n$$

3

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(i) BC (0) $\rho(0) := 2$

(1) $\rho(1) := 6$

RcS $\forall n \in \mathbb{N}$

$$\rho(n+2) - 8(\rho(n+1)) + 12(\rho(n)) = 0$$

$$+8(\rho(n+1)) \qquad +8(\rho(n+1))$$

$$\rho(n+2) + 12(\rho(n)) = 8(\rho(n+1))$$

$$-12(\rho(n)) \qquad -12(\rho(n))$$

$$\rho(n+2) = 8(\rho(n+1)) - 12(\rho(n))$$

$$(1)(\rho(n+2)) = 8(\rho(n+1)) + (-12)(\rho(n))$$

$p \leftarrow 1 \qquad q \leftarrow 8 \qquad r \leftarrow (-12)$

(0) $p(x^2) - q(x) - r = 0$

$p \leftarrow 1$, $q \leftarrow 8$, $r \leftarrow (-12)$

$$(1)x^2 - 8x - (-12) = 0$$

$$x^2 - 8x + 12 = 0$$

$$(x-6)(x-2) = 0$$

$$(x-6=0) \vee (x-2=0)$$

$$+6 \quad +6 \qquad +2 \quad +2$$

$$(x=6) \vee (x=2)$$

(1) $r_1 = 6 \neq 2 = r_2$

$\forall n \in \mathbb{N} \quad \rho(n) = f(6)^n + g(2)^n$

(3) $\rho(0) = 2$

$$f(6)^0 + g(2)^0 = 2$$

$$f(1) + g(1) = 2$$

$$f + g = 2$$

$$\times(-2) \qquad \times(-2)$$

$$(-2)f + (-2)(g) = -4$$

$$\frac{1}{2} + g = 2$$

$$-\frac{1}{2} \qquad -\frac{1}{2}$$

$$g = \frac{3}{2}$$

$\rho(1) = 6$

$$f(6)^1 + g(2)^1 = 6$$

$$f(6) + g(2) = 6$$

$$(-2)f + (-2)(g) = -4$$

$$4(f) = 2$$

$$\div 4 \qquad \div 4$$

$$\frac{\cancel{4}(f)}{\cancel{4}} = \frac{2}{4}$$

$$f = \frac{1}{2}$$

RcS

LS

$= f(n+2)$

$= \frac{1}{2}(6)^{n+2} + \frac{3}{2}(2)^{n+2}$

$= \frac{1}{2}(6)^n(6)^2 + \frac{3}{2}(2)^n(2)^2$

$= \frac{1}{2}(6)(6)(6)^n + \frac{3}{\cancel{2}}(\cancel{2})(2)(2)^n$

$= \frac{1}{\cancel{2}}(\cancel{2})(3)(6)(6)^n + 6(2)^n$

$= 18(6)^n + 6(2)^n$

RS

$= 8(f(n+1)) - 12(f(n))$

$= 8\left(\frac{1}{2}(6)^{n+1} + \frac{3}{2}(2)^{n+1}\right) - 12\left(\frac{1}{2}(6)^n + \frac{3}{2}(2)^n\right)$

$= 8\left(\frac{1}{2}\right)(6)^n(6)^1 + 8\left(\frac{3}{2}\right)(2)^n(2)^1 - 12\left(\frac{1}{2}\right)6^n - 12\left(\frac{3}{2}\right)2^n$

$= \cancel{2}(4)\left(\frac{1}{\cancel{2}}\right)(6)(6)^n + 8\left(\frac{3}{2}\right)(2)(2)^n - \cancel{2}(6)\left(\frac{1}{\cancel{2}}\right)6^n - \cancel{2}(6)\left(\frac{3}{\cancel{2}}\right)2^n$

$= 24(6)^n + 24(2)^n - 6(6)^n - 18(2)^n$

$= 24(6)^n + 24(2)^n + (-6)(6)^n + (-18)(2)^n$

$= 18(6)^n + 6(2)^n$

$LS = 18(6)^n + 6(2)^n = RS$

$LS = RS$

3

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(i) (1) $\forall n \in \mathbb{N} \quad \rho(n) = f(6)^n + g(2)^n$

$f = \frac{1}{2} \qquad g = \frac{3}{2}$

$\forall n \in \mathbb{N} \quad \rho(n) = \frac{1}{2}(6)^n + \frac{3}{2}(2)^n$

(i) (4) BC (0) LS

$= \rho(0)$

$= \frac{1}{2}(6)^0 + \frac{3}{2}(2)^0$

$= \frac{1}{2}(1) + \frac{3}{2}(1)$

$= \frac{1+3}{2}$

$= \frac{4}{2}$

$= 2$

$=$ RS

BC (1) LS

$= \rho(1)$

$= \frac{1}{2}(6)^1 + \frac{3}{2}(2)^1$

$= \frac{1}{2}(6) + \frac{3}{\cancel{2}}(\cancel{2})$

$= \frac{1}{\cancel{2}}(3)(\cancel{2}) + 3$

$= 3 + 3$

$= 6$

$=$ RS

(1b) Continued. Solve the following recurrence relations with the indicated initial conditions, using the algorithm for second-order recurrences given in class, and **prove by induction that your answers satisfy the recurrence:**

3 (ii)
$$\begin{cases} & \rho(0) := a \\ & \rho(1) := b \\ \forall n \in \mathbb{N} & (a^2)(\rho(n+2)) - (2ab)(\rho(n+1)) + (b^2)(\rho(n)) = 0 \end{cases}$$

First rewrite with your values of a and b.

My problem:

$$\begin{cases} & \rho(0) := 2 \\ & \rho(1) := 6 \\ \forall n \in \mathbb{N} & (2)^2(\rho(n+2)) - 2(2)(6)(\rho(n+1)) + (6)^2(\rho(n)) = 0 \end{cases}$$

$$\begin{cases} & \rho(0) := 2 \\ & \rho(1) := 6 \\ \forall n \in \mathbb{N} & 4(\rho(n+2)) - 24(\rho(n+1)) + 36(\rho(n)) = 0 \end{cases}$$

$\forall n \in \mathbb{N} \quad \rho(n) = 2(3)^n$

3

You are completely free to discuss and work with anyone.

(ii) BC (0) $f(0) = 2$

(1) $f(1) := 6$

RCS $\forall n \in \mathbb{N}$

$$4(f(n+2)) - 24(f(n+1)) + 36(f(n)) = 0$$
$$+24(f(n+1)) \qquad +24(f(n+1))$$

$$4(f(n+2)) + 36(f(n)) = 24(f(n+1))$$
$$-36(f(n)) \qquad -36(f(n))$$

$$4(f(n+2)) = 24(f(n+1)) - 36(f(n))$$
$$\div 4 \qquad \div 4$$

$$\frac{\cancel{4}}{\cancel{4}}(f(n+2)) = \frac{{}^{6}\cancel{24}}{\cancel{4}}(f(n+1)) - \frac{{}^{9}\cancel{36}}{\cancel{4}}(f(n))$$

$$f(n+2) = 6(f(n+1)) - 9(f(n))$$

$$(1)\, f(n+2) = 6(f(n+1)) + (-9)(f(n))$$

$$p \leftarrow 1 \qquad q \leftarrow 6 \qquad r \leftarrow (-9)$$

(0) $p(x^2) - q(x) - r = 0$

$$(1)\, x^2 - (6)x - (-9) = 0 \qquad \begin{pmatrix} p \leftarrow 1 \\ q \leftarrow 6 \\ r \leftarrow (-9) \end{pmatrix}$$

$$x^2 - 6(x) + 9 = 0$$

$$(x)^2 - 2(3)(x) + (3)^2 = 0$$

$$(x-3)^2 = 0$$

$$x - 3 = 0$$
$$+3 \qquad +3$$

$$x = 3$$

(2) $r_1 = r_2 = 3$

$\forall n \in \mathbb{N} \quad f(n) = f(3)^n + g(n)(3)^n$

(3) $f(0) = 2$

$$f(3)^0 + g(0)(3)^0 = 2$$

$$f(1) + 0 = 2$$

$$f = 2$$

$f(1) = 6$

$$f(3)^1 + g(1)(3)^1 = 6$$

$$2(3) + g(3) = 6$$

$$6 + 3(g) = 6$$
$$-6 \qquad -6$$

$$3(g) = 0$$
$$\div 3 \qquad \div 3$$

$$\frac{\cancel{3}}{\cancel{3}}(g) = 0$$

$$g = 0$$

6 (2a) Solve the following recurrence relation for $\varphi(n)$, noting that $\alpha(n)$ is just an arithmetic progression, and **prove that your answer satisfies the recursion**:

$$\begin{cases} & \alpha(0) := a \\ \forall n \in \mathbb{N} & \alpha(n+1) := \alpha(n) + b \\ \forall n \in \mathbb{N} & \varphi(n) := (\alpha(n))c^n \end{cases}$$

Show calculations on the facing side.

Rewrite the recursion using your values of $a, b,$ and c.

My problem:

$$\begin{cases} & \alpha(0) := 2 \\ \forall n \in \mathbb{N} & \alpha(n+1) := \alpha(n) + 6 \\ \forall n \in \mathbb{N} & \varphi(n) := (\alpha(n))4^n \end{cases}$$

(i) Compute, showing full computations:

$\alpha(0)$
$= 2$

$\varphi(0)$
$= 2$

$\alpha(1)$
$= 2 + 6$

$\varphi(1)$
$= (2 + 6)4$

$\alpha(2)$
$= 2 + 6(2)$

$\varphi(2)$
$= (2 + 6(2))4^2$

$\alpha(3)$
$= 2 + 6(3)$

$\varphi(3)$
$= (2 + 6(3))4^3$

6

Conjecture:

$\alpha(n)$
$= 2 + 6(n)$

$\varphi(n)$
$= (2 + 6(n))4^n$

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(i) $\alpha(0)$

$= 2$

(i) $\alpha(1)$

$= \alpha(0+1)$

$= \alpha(0) + 6$

$= 2 + 6$

(i) $\alpha(2)$

$= \alpha(1+1)$

$= \alpha(1) + 6$

$= 2 + 6 + 6$

$= 2 + 6(2)$

(i) $\alpha(3)$

$= \alpha(2+1)$

$= \alpha(2) + 6$

$= 2 + 6(2) + 6$

$= 2 + 6(3)$

(i) $\varphi(0)$

$= (\alpha(0))4^0$

$= 2(1)$

$= 2$

(i) $\varphi(1)$

$= (\alpha(1))4^1$

$= (2+6)(4)$

(i) $\varphi(2)$

$= (\alpha(2))4^2$

$= (2 + 6(2))4^2$

(i) $\varphi(3)$

$= (\alpha(3))4^3$

$= (2 + 6(3))4^3$

(2a)

(i) **BC** $\varphi(0) := 2$

RcS $\forall n \in \mathbb{N} \quad \varphi(n+1) := (\alpha(n+1))4^{n+1} = (\alpha(n)+6)(4)^n 4 = (\alpha(n))4^n(4) + 6(4)^n 4 = (\varphi(n))4 + 6(4)^{n+1}$

$\forall n \in \mathbb{N} \quad \varphi(n) = (2 + 6(n))\,4^n$

$P(0) := (\varphi(0) = (2 + 6(0))\,4^0)$

$P(n) := (\varphi(n) = (2 + 6(n))\,4^n)$

$P(n+1) := (\varphi(n+1) = (2 + 6(n+1))\,4^{n+1})$

BC $\varphi(0)$

$= 2$

$= 2 + 0$

$= 2 + 6(0)$

$= (2 + 6(0))(1)$

$= (2 + 6(0))\,4^0$

$$\frac{\qquad\qquad}{P(0)}$$

IS $$\frac{P(0) \quad P(1) \quad \dots \quad P(n)}{\varphi(n+1)}$$

$= (\varphi(n))4 + 6(4)^{n+1}$ (RcS)

$= (2 + 6(n))\,4^n(4) + 6(4)^{n+1}$ (P(n))

$= (2 + 6(n))4^{n+1} + 6(4)^{n+1}$

$= (2 + 6(n) + 6)(4)^{n+1}$

$= (2 + 6(n+1))\,4^{n+1}$

$$\frac{\qquad\qquad}{P(n+1)}$$

Snths

$$\frac{P(0) \qquad \dfrac{P(0) \quad P(1) \quad \dots P(n)}{P(n+1)}}{\forall n \in \mathbb{N} \quad P(n)}$$

6 (2b) Find a conjecture for the sum: $Sm(\varphi,n) := S(\varphi,n) := \sum_{k=0}^{n} \varphi(k)$ where $\varphi(n)$ is defined, as in (2a) by the recursion appearing below, by writing out the terms, multiplying $S(\varphi,n)$ by a suitable factor, shifting by a term to the right and subtracting term by term:

$$\begin{cases} & \alpha(0) := a \\ \forall n \in \mathbb{N} & \alpha(n+1) := \alpha(n) + b \\ \forall n \in \mathbb{N} & \varphi(n) := (\alpha(n))c^n \end{cases}$$

Show computations on the facing side.

Rewrite the recursion using your values of $a, b,$ and c.

My problem:

$$\begin{cases} & \alpha(0) := 2 \\ \forall n \in \mathbb{N} & \alpha(n+1) := \alpha(n) + 6 \\ \forall n \in \mathbb{N} & \varphi(n) := (\alpha(n))4^n \end{cases}$$

Conjecture: $\forall n \in \mathbb{N}$ $S(\varphi,n)$

$= 2 + 2(4^{n+1})n$

6

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$$\varphi(0) = 2 = 2(1)$$
$$\varphi(1) = (2 + 6(1))\,4^1 = 2(4) + 6(1)4$$
$$\varphi(2) = (2 + 6(2))\,4^2 = 2(4^2) + 6(2)4^2$$
$$\dots$$
$$\varphi(n) = (2 + 6(n))\,4^n = 2(4^n) + 6(n)4^n$$

$$S(\varphi,n) = 2(1 + 4^1 + 4^2 + \dots + 4^n) + 6(1)4 + 6(2)4^2 + \dots + 6(n)4^n$$

$$S(\varphi,n) = 2\left((1)\frac{1-4^{n+1}}{1-4}\right) + 6(1)4 + 6(2)4^2 + \dots + 6(n)4^n$$

$$S(\varphi,n) = -2\left(\frac{1-4^{n+1}}{3}\right) + 6(4) + 6(2)4^2 + \dots + 6(n)4^n$$

$$\times\left(-\frac{1}{4}\right) \qquad \times\left(-\frac{1}{4}\right)$$

$$-\frac{1}{4}(S(\varphi,n)) = \frac{\cancel{2}}{{}_{2}\cancel{4}}\left(\frac{1-4^{n+1}}{3}\right) - 6 - 6(2)4^1 - \dots - 6(n)4^{n-1}$$

$$-\frac{1}{4}(S(\varphi,n)) = \frac{1}{2}\left(\frac{1-4^{n+1}}{3}\right) - 6 - 6(2)4^1 - 6(3)4^2 - \dots - 6(n)4^{n-1}$$

$$-\frac{1}{4}(S(\varphi,n)) = \frac{1}{2}\left(\frac{1-4^{n+1}}{3}\right) - 6 - 6\cancel{(4)} - 6(4) - 6\cancel{(2)}4^2 - 6(4^2) - \dots - 6\cancel{(n-1)}4^{n-1} - 6(4^{n-1})$$

$$S(\varphi,n) = -2\left(\frac{1-4^{n+1}}{3}\right) + \cancel{6(4)} + 6\cancel{(2)}4^2 + \dots + 6\cancel{(n-1)}4^{n-1} + 6(n)4^n$$

$$\left(1-\frac{1}{4}\right)(S(\varphi,n)) = \left(\frac{1}{2}-2\right)\left(\frac{1-4^{n+1}}{3}\right) - 6 - 6(4) - 6(4^2) - \dots - 6(4^{n-1}) + 6(n)4^n$$

$$\frac{3}{4}(S(\varphi,n)) = -\frac{\cancel{3}}{2}\left(\frac{1-4^{n+1}}{\cancel{3}}\right) - 6(1 + 4 + 4^2 + \dots + 4^{n-1}) + 6(n)4^n$$

$$\frac{3}{4}(S(\varphi,n)) = \frac{4^{n+1}-1}{2} - 6\left((1)\frac{1-4^n}{1-4}\right) + 6(n)4^n$$

$$\frac{3}{4}(S(\varphi,n)) = \frac{4^{n+1}-1}{2} - {}^{2}\cancel{6}\left(\frac{1-4^n}{(-\cancel{3})}\right) + 6(n)4^n$$

$$\frac{3}{4}(S(\varphi,n)) = \frac{4^{n+1}-1}{2} + 2 - 2(4^n) + 6(n)4^n$$

$$\frac{3}{4}(S(\varphi,n)) = \frac{4^{n+1}-1}{2} + \frac{4}{2} - \frac{4(4^n)}{2} + \frac{12(n)4^n}{2}$$

$$\frac{3}{4}(S(\varphi,n)) = \frac{4^{n+1} - 1 + 4 - 4(4^n) + 12(n)4^n}{2}$$

$$\frac{3}{4}(S(\varphi,n)) = \frac{4^{n+1} + 3 - 4^{n+1} + 12(n)4^n}{2}$$

$$\frac{3}{4}(S(\varphi,n)) = \frac{3(1 + 4(n)4^n)}{2}$$

$$\div\frac{3}{4} \qquad \div\frac{3}{4}$$

$$\frac{\cancel{3}}{\cancel{4}}\left(\frac{\cancel{4}}{\cancel{3}}\right)S(\varphi,n) = \frac{\cancel{3}(1+(4^{n+1})n)}{\cancel{2}}\left(\frac{{}^{2}\cancel{4}}{\cancel{3}}\right)$$

$$S(\varphi,n) = 2 + 2(4^{n+1})n$$

6 (3a) Define the function $\alpha: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ recursively as follows:

$$\forall n \in \mathbb{N}, \alpha(n, a+b) := \begin{cases} \alpha(n-a-b, a+b) + ab & if\, n \geq a+b \\ 0 & if\, n < a+b \end{cases}$$

Rewrite the definition above with your values of a and b:

My problem:

$$\forall n \in \mathbb{N}, \alpha(n, 8) := \begin{cases} \alpha(n-8, 8) + 12 & \text{if } n \geq 8 \\ 0 & \text{if } n < 8 \end{cases}$$

Compute **showing complete calculations** on the facing page:

1 (i) $\alpha(0, a+b)$

$= 0$

1 (ii) $\alpha(1, a+b)$

$= 0$

1 (iii) $\alpha(10, a+b)$

$= 12$

1 (iv) $\alpha(100, a+b)$

$= 12\ (12)$

1 (v) $\alpha(1000, a+b)$

$= 125\ (12)$

1(vi) Make a conjecture regarding what the function α calculates and prove your conjecture.

Conjecture :

$$\forall n \in \mathbb{N} \quad \alpha(n, 8) := \left(\left\lfloor \frac{n}{8} \right\rfloor\right) 12$$

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(3a) $n - a - b$	(3a) $a + b$	(3a) ab
$= n - 2 - 6$	$= 2 + 6$	$= 2(6)$
$= n - 8$	$= 8$	$= 12$

(i) $\alpha(0, a+b)$
$= \alpha(0, 8)$
$= 0$

(ii) $\alpha(1, a+b)$
$= \alpha(1, 8)$
$= 0$

(iii) $\alpha(10, a+b)$
$= \alpha(10, 8)$
$= \alpha(10-8, 8) + 12$
$= \alpha(2, 8) + 12$
$= 0 + 12$
$= 12$

(iv) $\alpha(100, a+b)$
$= \alpha(100, 8)$
$= \left(\left\lfloor\left(\frac{100}{8}\right)\right\rfloor\right) 12$
$= 12(12)$

(v) $\alpha(1000, a+b)$
$= \alpha(1000, 8)$
$= \left(\left\lfloor\left(\frac{1000}{8}\right)\right\rfloor\right) 12$
$= (125)(12)$

BC $\alpha(0,8) := 0$

RCS $\forall n \in \mathbb{N}$ $\alpha(n+1,8) := \begin{cases} \alpha((n+1)-8,8)+12 & \text{if } n+1 \geq 8 \\ 0 & \text{if } n+1 < 8 \end{cases}$

$= \begin{cases} \alpha(n+1-8,8)+12 & \text{if } n+1-1 \geq 8-1 \\ 0 & \text{if } n+1 < 8 \end{cases}$

$= \begin{cases} \alpha(n-7,8)+12 & \text{if } n \geq 7 \\ 0 & \text{if } n+1 < 8 \end{cases}$

$\forall n \in \mathbb{N}$ $\alpha(n,8) = \left(\left\lfloor \frac{n}{8} \right\rfloor\right)12$

$P(0) := \left(\alpha(0,8) = \left(\left\lfloor \frac{0}{8} \right\rfloor\right)12\right)$

$P(n) := \left(\alpha(n,8) = \left(\left\lfloor \frac{n}{8} \right\rfloor\right)12\right)$

$P(n+1) := \left(\alpha(n+1,8) = \left(\left\lfloor \frac{n+1}{8} \right\rfloor\right)12\right)$

BC $\alpha(0,8)$

$= 0$

$= 0(12)$

$= \left(\left\lfloor \frac{0}{8} \right\rfloor\right)12$

$$\frac{\qquad\qquad}{P(0)}$$

IS $\dfrac{P(0) \quad P(1) \quad \ldots \quad P(n)}{\quad}$

$\forall n \in \mathbb{N}$ $n \geq 7$

$\alpha(n+1,8)$

$= \alpha(n-7,8) + 12$

$= \left(\left\lfloor \frac{n-7}{8} \right\rfloor\right)12 + 12$ (IH n ($n-7 < n$))

$= \left(\left(\left\lfloor \frac{n-7}{8} \right\rfloor\right) + 1\right)12$

$= \left(\left\lfloor \frac{n-7}{8} + 1 \right\rfloor\right)12$

$\forall n \in \mathbb{N}$ $n+1 < 8$

$\alpha(n+1,8)$

$= 0$

$= 0(12)$

$= \left(\lfloor 0 \rfloor\right)12$

$= \left(\left\lfloor \frac{n+1}{8} \right\rfloor\right)12$ $(n+1 < 8)$

$$\frac{\qquad\qquad}{P(n+1)}$$

(vi)

$$= \left(\left\lfloor \left(\frac{n+7}{8} + \frac{8}{8} \right) \right\rfloor \right) 12$$

$$= \left(\left\lfloor \left(\frac{n+7+8}{8} \right) \right\rfloor \right) 12$$

$$= \left(\left\lfloor \left(\frac{n+1}{8} \right) \right\rfloor \right) 12$$

$$\frac{\qquad}{P(n+1)}$$

$$P(n+1)$$

Snths

$$\frac{P(0) \qquad \dfrac{P(0) \quad P(1) \quad \ldots \quad P(n)}{P(n+1)}}{\forall n \in \mathbb{N} \quad P(n)}$$

6 (3b) Prove, following the method shown in class, that: $(\boldsymbol{a}+\boldsymbol{b}+\mathbf{1})^{\frac{1}{a+b}}$ **is irrational,** that is: $\boldsymbol{Rtl}\left((\boldsymbol{a}+\boldsymbol{b}+\mathbf{1})^{\frac{1}{a+b}}\right)$.

First rewrite the problem using your values of a and b.

My Problem:

$Rtl\left((2+6+1)^{\frac{1}{2+6}}\right)$

$Rtl\left((9)^{\frac{1}{8}}\right)$

$Rtl\left(\sqrt[4]{3}\right)$

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3b) $(9)^{\frac{1}{8}}$

$= (3^2)^{\frac{1}{8}}$

$= (3)^{\frac{\cancel{2}}{\cancel{8}4}}$

$= (3)^{\frac{1}{4}}$

$= \sqrt[4]{3}$

(3b) Pf

$\neg(Rtl(\sqrt[4]{3}))$

$Rtl(\sqrt[4]{3})$

$\sqrt[4]{3} \in \mathbb{Q}$ $\quad \sqrt[4]{3} \neq 0$

$\exists p_u \in \mathbb{N} \setminus \{0\}$ $\quad \exists q_u \in \mathbb{N} \setminus \{0\}$ $\quad \sqrt[4]{3} = \frac{p_u}{q_u}$

$\sqrt[4]{3} = \frac{p}{q}$ $\quad \gcd(p,q) = 1$

$\times q \quad \times q$

$\sqrt[4]{3}\, q = \frac{p}{\cancel{q}}(\cancel{q})$

$\sqrt[4]{3}\, q = p$

$(\,)^4 \quad (\,)^4$

$(\sqrt[4]{3}\, q)^4 = p^4$

$3q^4 = p^4$ $\quad Prm(3)$

$3 \mid p^4$

$3 \mid p$

$\exists! r \quad \frac{p}{3} = r$

$\times 3 \quad \times 3$

$\frac{p}{\cancel{3}}(\cancel{3}) = 3r$

$p = 3r$

$(\,)^4 \quad (\,)^4$

$p^4 = (3r)^4$ $\quad p^4 = 3q^4$

$3q^4 = 81r^4$

$\div 3 \quad \div 3$

$\frac{\cancel{3}q^4}{\cancel{3}} = \frac{\cancel{81}^{27} r^4}{\cancel{3}}$

$q^4 = 27r^4$

$q^4 = 3(9)r^4$ $\quad Prm(3)$

$3 \mid q^4$

$3 \mid q$ $\quad 3 \mid p$

$3 \mid \gcd(p,q)$ $\quad \gcd(p,q) = 1$

$3 \mid 1$

$\perp$

$\neg(Rtl(\sqrt[4]{3}))$

~~Rtl~~ $(\sqrt[4]{3})$

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5 (3c) Consider the numbers defined below, in which the dot indicates a decimal point, and the string in parentheses is the repeating part of a repeating decimal.

(A) $x := .(ab)$ $= .(26)$

(B) $y := .aababbabbba\ldots$ $= .2262662 6662\ldots$,

where for every $\forall n \in \mathbb{N}$ there are n b's between the (n + 1)st and the (n+2)nd a.

2 (i) Find a recursive definition for x and prove that: $\boldsymbol{Rtl(x)}$, that is, **x is rational.**

$\forall n \in N$ $\quad x := \sum_{k=0}^{n} (26)\, 10^{(-2-2k)}$.

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(i) $x = .(26)$

$\times 100 \quad \times 100$

$100x = 26.(26)$

$100x = 26 + .(26)$

$100x = 26 + x$

$-x \quad -x$

$99x = 26$

$\div 99 \quad \div 99$

$\frac{\cancel{99}}{\cancel{99}} x = \frac{26}{99}$

$x = \frac{26}{99} \quad 26 \in \mathbb{Z} \quad 99 \in \mathbb{N} \setminus \{0\}$

$x \in \mathbb{Q}$

$RHL(x)$

(i) $x_0 := .26 = 26(10^{-2}) = 26(10^{-2-0}) = 26(10^{-2-2(0)})$

$x_1 := .2626 = 26(10^{-2}) + 26(10^{-4}) = 26(10^{-2-2(0)}) + 26(10^{-2-2(1)})$

$x_2 := .262626 = 26(10^{-2}) + 26(10^{-4}) + 26(10^{-6}) = 26(10^{-2-2(0)}) + 26(10^{-2-2(1)}) + 26(10^{-2-2(2)})$

...

$x_n := 26(10^{-2-2(0)}) + 26(10^{-2-2(1)}) + 26(10^{-2-2(2)}) + \dots + 26(10^{-2-2(n)})$

$\forall n \in \mathbb{N} \quad x := \sum_{k=0}^{n} 26(10^{-2-2(k)})$

(1b) (ii)

(2) $\forall n \in \mathbb{N}\ \rho(n) = f(3)^n + g(n)(3)^n$

$f = 2 \quad g = 0$

$\forall n \in \mathbb{N}\ \rho(n) = 2(3)^n + (0)(n)(3)^n$

$= 2(3)^n + 0$

$= 2(3)^n$

(1b)(ii)

(4) BC (0) LS

$= \rho(0)$

$= 2(3)^0$

$= 2(1)$

$= 2$

$=$ RS

BC (1) LS

$= \rho(1)$

$= 2(3)^1$

$= 2(3)$

$= 6$

$=$ RS

RcS

LS

$= \rho(n+2)$

$= 2(3)^{n+2}$

$= 2(3)^n(3)^2$

$= 2(9)(3)^n$

$= 18(3)^n$

RS

$= 6(\rho(n+1)) - 9(\rho(n))$

$= 6(2(3)^{n+1}) - 9(2(3)^n)$

$= 6(2)(3)^n(3)^1 - 9(2)(3)^n$

$= 12(3)(3)^n - 18(3)^n$

$= 36(3)^n - 18(3)^n$

$= 36(3)^n + (-18)(3)^n$

$= 18(3)^n$

LS $= 18(3)^n =$ RS

LS $=$ RS