Instructions:

Your J-number = 00298436

$a := mnm_{k\in 1..8}\{J_k | (k \in 1..8) \wedge (J_k \neq 0)\} = 2$

$b := mxm(J_k | k \in 1..8) = 9$

$c := \left\lfloor \left(\frac{b+a}{2}\right) = \left\lfloor \left(\frac{2+9}{2}\right) = \left\lfloor \left(1+\frac{9}{2}\right) = \left\lfloor \left(1+4.5\right) = \left\lfloor \left(5.5\right) = 5$

$d :=$ the smallest **odd** number **strictly greater** than c = 7

You may discuss and work with anyone

(i) ⓪

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}(b^k) \qquad n \leftarrow 5$$

$$(2+x)^5 = \sum_{k=0}^{5} \binom{5}{k} (2)^{5-k}(x^k) \qquad a \leftarrow 2, \quad b \leftarrow x$$

$$\begin{aligned}
(2+x)^5 &= \binom{5}{0}(2)^{5-0}(x)^0 + \binom{5}{1}(2)^{5-1}(x)^1 + \binom{5}{2}(2)^{5-2}(x)^2 + \binom{5}{3}(2)^{5-3}(x)^3 + \binom{5}{4}(2)^{5-4}(x)^4 + \binom{5}{5}(2)^{5-5}(x)^5 \\
&= (1)(2)^5(1) + (5)(2)^4(x)^1 + (10)(2)^3(x)^2 + (10)(2)^2(x)^3 + (5)(2)^1(x)^4 + (1)(2)^0(x)^5 \\
&= (1)(32)(1) + (5)(16)(x) + (10)(8)(x)(x) + (10)(4)(x)(x)(x) + (5)(2)(x)(x)(x)(x) + (1)(1)(x)(x)(x)(x)(x) \\
&= 32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5
\end{aligned}$$

(i) ①

$$\binom{5}{0} = \binom{5}{5-0} = \binom{5}{5} = \frac{\cancel{5!}}{\cancel{5!}((5-5)!)} = \frac{1}{1(0!)} = \frac{1}{1} = 1$$

(i) ②

$$\binom{5}{1} = \binom{5}{5-1} = \binom{5}{4} = \frac{5!}{4!((5-4)!)} = \frac{5(\cancel{4!})}{\cancel{4!}(1!)} = \frac{5}{1} = 5$$

(i) ③

$$\binom{5}{2} = \binom{5}{5-2} = \binom{5}{3} = \frac{5!}{3!((5-3)!)} = \frac{5(4)(\cancel{3!})}{\cancel{3!}(2!)} = \frac{5\cdot \cancel{4}^{2}}{\cancel{2}\cdot 1} = \frac{10}{1} = 10$$

6 (1a) Expand the following **EXACTLY AS SHOWN IN CLASS.**

Use the binomial theorem and express your answer as a polynomial with **integer coefficients, ordering the unknowns within each summand numero-alphabetically, writing the summands VERTICALLY in increasing order of degree, and for every degree, ordering the summands lexicographically exactly as shown in class.** Show calculations for all coefficients that occur on the facing page and note only the answer on this page.

3 (i) $(a + x)^c$

My problem: $\left((2) + x\right)^{(5)}$

$$= \left(2 + x\right)^{(5)}$$

$$\begin{aligned} = \; & 32 \\ & + 80x \\ & + 80x^2 \\ & + 40x^3 \\ & + 10x^4 \\ & + x^5 \end{aligned}$$

6

(ii) 0

$$(a+b+c)^n = \sum_{n_1+n_2+n_3=n} \binom{n}{n_1\; n_2\; n_3} a^{n_1}(b^{n_2})(c^{n_3})$$

$n \leftarrow 3$
$a \leftarrow 1$
$b \leftarrow (-x)$
$c \leftarrow y$

$$(1-x+y)^3 = \sum_{n_1+n_2+n_3=3} \binom{3}{n_1\; n_2\; n_3} (1)^{n_1}(-x)^{n_2}(y)^{n_3}$$

$$\begin{aligned}
(1-x+y)^3 = & \binom{3}{003}(1)^0(-x)^0(y)^3 & = (1)(1)(1)(y)^3 & = y(y)(y) & = 1 \\
& +\binom{3}{012}(1)^0(-x)^1(y)^2 & +(3)(1)(-x)(y)^2 & -3(x)(y)(y) & -3x \\
& +\binom{3}{030}(1)^0(-x)^3(y)^0 & +(1)(1)(-x)^3(1) & -x(x)(x) & +3y \\
& +\binom{3}{102}(1)^1(-x)^0(y)^2 & +(3)(1)(1)(y)^2 & +3(y)(y) & +3x^2 \\
& +\binom{3}{111}(1)^1(-x)^1(y)^1 & +(6)(1)(-x)(y) & -6(x)(y) & -6xy \\
& +\binom{3}{120}(1)^1(-x)^2(y)^0 & +(3)(1)(-x)^2(1) & +3(x)(x) & +3y^2 \\
& +\binom{3}{201}(1)^1(-x)^0(y)^1 & +(3)(1)^2(1)(y) & +3(y) & -x^3 \\
& +\binom{3}{210}(1)^2(-x)^1(y)^0 & +(3)(1)^2(-x)(1) & -3(x) & +3x^2y \\
& +\binom{3}{300}(1)^3(-x)^0(y)^0 & +(1)(1)^3(1)(1) & +1 & -3xy^2 \\
& +\binom{3}{021}(1)^0(-x)^2(y)^1 & +(3)(1)(-x)^2(y) & +3(x)(x)(y) & +y^3
\end{aligned}$$

(ii) 1

$$\binom{3}{003} = \binom{3}{030} = \binom{3}{300} = \frac{\cancel{3!}}{\cancel{3!}(0!)(0!)} = \frac{1}{1(1)(1)} = \frac{1}{1} = 1$$

(ii) 2

$$\binom{3}{012} = \binom{3}{120} = \binom{3}{210} = \binom{3}{102} = \binom{3}{201} = \binom{3}{021} = \frac{3!}{0!(2!)(1!)} = \frac{3\cdot\cancel{(2!)}}{(1)\cancel{(2!)}(1)} = \frac{3}{1} = 3$$

(ii) 3

$$\binom{3}{111} = \frac{3!}{(1!)(1!)(1!)} = \frac{3\cdot 2\cdot 1}{1(1)(1)} = 3\cdot 2 = 6$$

(ii) 4

$$\frac{\begin{array}{c} n_1, n_2, n_3 \in \mathbb{N} \\ n_1+n_2+n_3 = 3 \end{array}}{n_1, n_2, n_3 \in 0..3}$$

$$(n_1, n_2, n_3) \in \left\{ \begin{array}{l} (0,0,3), (0,3,0) \\ (3,0,0), (2,1,0), \\ (2,0,1), (1,2,0), \\ (1,0,2), (0,2,1) \\ (0,1,2), (1,1,1) \end{array} \right\}$$

(1a) Expand the following **EXACTLY AS SHOWN IN CLASS.**

Use the multinomial theorem and express your answer as a polynomial with **integer coefficients, ordering the unknowns within each summand numero-alphabetically, writing the summands VERTICALLY in increasing order of degree, and for every degree, ordering the summands lexicographically exactly as shown in class.** Show calculations for all coefficients that occur on the facing page and note only the answer on this page.

3 (ii) $(1 - x + y)^{1+a}$

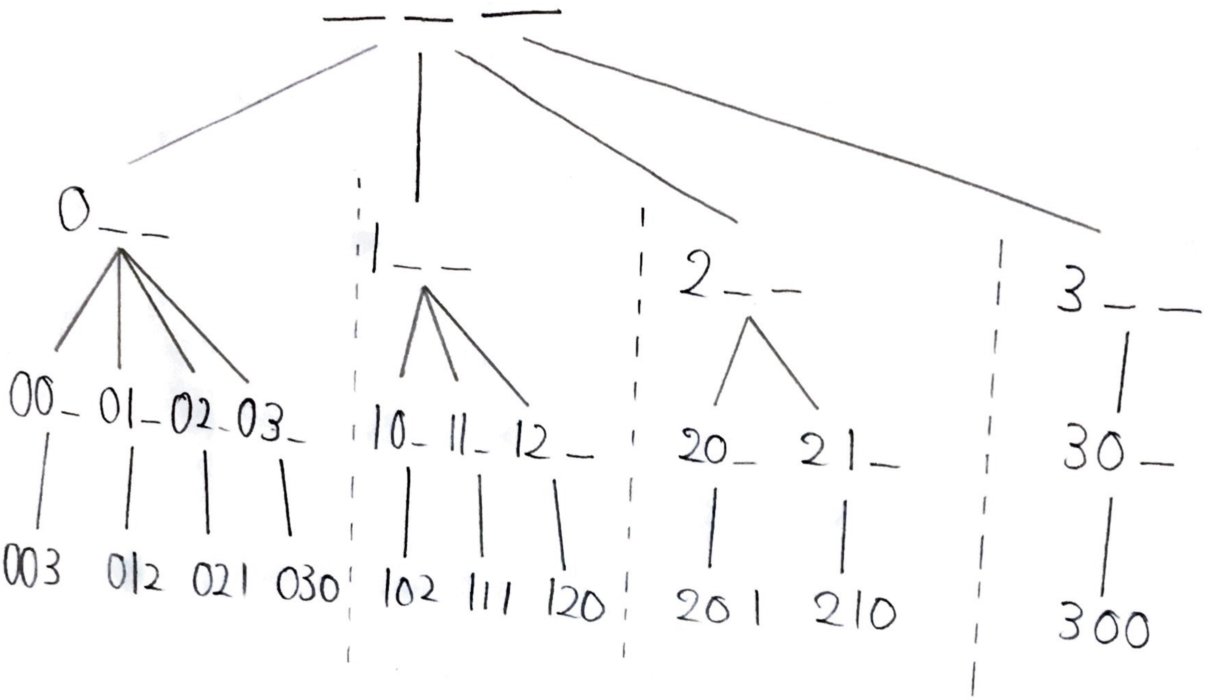
My problem: $$\left((1) + (-1)x + y\right)^{1+(2)}$$

$$= \left(1 + (-1)x + y\right)^{(3)}$$

$$\begin{aligned}
= \quad & 1 \\
& - 3x \\
& + 3y \\
& + 3x^2 \\
& - 6xy \\
& + 3y^2 \\
& - x^3 \\
& + 3x^2y \\
& - 3xy^2 \\
& + y^3
\end{aligned}$$

3

(ii) 5 <u>TREE</u> (n_1, n_2, n_3)



(i) ⓪

BC $\rho(0) = 1, \quad \rho(1) = 2$

RcS $\forall n \in \mathbb{N}$

$$\rho(n+2) - 11(\cancel{\rho(n+1)}) + 18(\rho(n)) = 0$$
$$+ 11(\cancel{\rho(n+1)}) \qquad + 11(\rho(n+1))$$

$$\rho(n+2) + 18(\cancel{\rho(n)}) = 11(\rho(n+1))$$
$$- \cancel{18}(\rho(n)) \qquad - 18(\rho(n))$$

$$\rho(n+2) = 11(\rho(n+1)) - 18(\rho(n))$$

(i) ①

$$p(x^2) - q(x) - r = 0 \quad \langle p \leftarrow 1,\ q \leftarrow 11,\ r \leftarrow (-18) \rangle$$

$$(1)x^2 - (11)(x) - (-18) = 0$$

$$x^2 - 11x + 18 = 0$$

$$(x-9)(x-2) = 0$$

$$x - \cancel{9} = 0 \qquad x - \cancel{2} = 0$$
$$+\cancel{9} \ +9 \qquad +\cancel{2} \ +2$$

$$x = 9 \qquad x = 2$$

$$(x=9) \vee (x=2)$$

$$(r_1 = 9) \neq (r_2 = 2)$$

$$r_1 \neq r_2$$

$$\forall n \in \mathbb{N} \quad \rho(n) = f(9)^n + g(2)^n$$

(i) ②

$$\rho(n) = f(9)^n + g(2)^n \quad \langle n \leftarrow 0 \rangle$$

$$\rho(0) = f(9)^0 + g(2)^0$$

$$1 = f + \cancel{g}$$
$$- g \qquad \cancel{-g}$$

$$1 - g = f$$

6 (1b) Solve the following recurrence relations with the indicated initial conditions, using the formula for second-order recurrences given in class, using substitution. Prove by substitution that your answer is correct.

3 (i)
$$\begin{cases} \rho(0) := 1 \\ \rho(1) := 2 \\ \forall n \in \mathbb{N} \quad \Big(\rho(n+2)\Big) - (a+b)\Big(\rho(n+1)\Big) + (ab)\Big(\rho(n)\Big) = 0 \end{cases}$$

First rewrite the recurrence with your values of a and b.

My problem:

$$\Big(\rho(n+2)\Big) - \Big((2)+(9)\Big)\Big(\rho(n+1)\Big) + \Big((2)(9)\Big)\Big(\rho(n)\Big) = 0$$

My problem:

$$\Big(1\Big)\Big(\rho(n+2)\Big) + \Big(-11\Big)\Big(\rho(n+1)\Big) + (18)\Big(\rho(n)\Big) = 0$$

My solution: $\forall n \in \mathbb{N}$

$\rho(n)$

$= (2)^n$

36

(1) ③ $P(n) = f(9)^n + g(2)^n$ $\langle n \leftarrow 1 \rangle$

$P(1) = f(9)^1 + g(2)^1$

$2 = 9f + 2g, \quad f = 1 - g$

$2 = 9(1-g) + 2g$

$2 = 9 - 9g + 2g$

$2 = \cancel{9} - 7g$

$-9 \qquad -\cancel{9}$

$\cancel{-7} = \cancel{-7}g$

$\cancel{\div(-7)} \qquad \cancel{\div(-7)}$

$1 = g, \quad f = 1 - g$

$f = 0, \quad g = 1$

$P(n) = 0(9)^n + 1(2)^n$

$P(n) = (2)^n$

(i) ④ $P(0) := \left(P(0) = (2)^0\right)$

$P(1) := \left(P(1) = (2)^1\right)$

$P(n+2) := \left(P(n+2) = (2)^{n+2}\right), \quad P(n) := \left(P(n) = (2)^n\right)$

(i) ⑤ BS

(0)	(1)
LS	LS
$= P(0)$ (BC)	$= P(1)$ (BC)
$= 1$	$= 2$
$= (2)^0$	$= (2)^1$
$=$ RS	$=$ RS
$P(0)$	$P(1)$

(i) 6

IS

$$\frac{P(0),\ P(1),\ \cdots\ P(n+1)}{\begin{aligned} &LS \\ &= P(n+2) \quad (\underline{RS}) \\ &= 11\left(P(n+1)\right) - 18\left(P(n)\right) \quad (\underline{IH}) \\ &= 11(2)^{n+1} - 18(2)^n \\ &= 11(2)(2)^n - 18(2)^n \\ &= 22(2)^n - 18(2)^n \\ &= 4(2)^n \\ &= (2)^2(2)^n \\ &= (2)^{n+2} \\ &= RS \end{aligned}}$$

$$P(n+2)$$

(i) 7

$$\frac{\dfrac{P(0),\ P(1),\ \dfrac{P(0),\ P(1),\ \cdots\ P(n+1)}{P(n+2)}}{\forall n \in \mathbb{N}\quad P(n)}}{\forall n \in \mathbb{N}\ \left(P(n) = (2)^n\right)}$$

(ii) ⓪

BC $\rho_{(0)} = 1$, $\rho_{(1)} = 2$

RC $\forall n \in \mathbb{N}$

$$4(\rho_{(n+2)}) - 36(\rho_{(n+1)}) + 81(\rho_{(n)}) = 0$$

$$+36(\rho_{(n+1)}) \qquad +36(\rho_{(n+1)})$$

$$4(\rho_{(n+2)}) + 81(\rho_{(n)}) = 36(\rho_{(n+1)})$$

$$-81(\rho_{(n)}) \qquad -81(\rho_{(n)})$$

$$\cancel{4}(\rho_{(n+2)}) = 36(\rho_{(n+1)}) - 81(\rho_{(n)})$$

$$\div\cancel{4} \qquad \div 4$$

$$\rho_{(n+2)} = 9(\rho_{(n+1)}) - \frac{81}{4}(\rho_{(n)})$$

(ii) ①

$$p(x^2) - q(x) - r = 0 \qquad \langle p \leftarrow 4,\ q \leftarrow 36,\ r \leftarrow (-81) \rangle$$

$$(4)x^2 - (36)x - (-81) = 0$$

$$4x^2 - 36x + 81 = 0$$

$$(2x - 9)^2 = 0$$

$$(\ )^{\frac{1}{2}} \qquad (\ )^{\frac{1}{2}}$$

$$2x - \cancel{9} = 0$$

$$+\cancel{9} \qquad +9$$

$$\cancel{2}x = 9$$

$$\div\cancel{2} \qquad \div 2$$

$$x = \frac{9}{2}$$

$$\frac{9}{2} = r_1 = r_2 = \frac{9}{2}$$

$$r_1 = r_2 = r$$

$$\forall n \in \mathbb{N}\ \rho_{(n)} = f\left(\frac{9}{2}\right)^n + g\,n\left(\frac{9}{2}\right)^n$$

(ii) ②

$$\rho_{(n)} = f\left(\frac{9}{2}\right)^n + g\left(\frac{9}{2}\right)^n$$

$$\rho_{(0)} = f\left(\frac{9}{2}\right)^0 + g(0)\left(\frac{9}{2}\right)^0 \qquad \langle n \leftarrow 0 \rangle$$

$$1 = f + 0$$

$$1 = f$$

(ii) ③

$$\rho_{(n)} = f\left(\frac{9}{2}\right)^n + gn\left(\frac{9}{2}\right)^n$$

$$\rho_{(1)} = f\left(\frac{9}{2}\right)^1 + g(1)\left(\frac{9}{2}\right)^1 \qquad \langle n \leftarrow 1 \rangle$$

$$2 = \frac{9}{2}f + \frac{9}{2}g$$

$$\times\frac{2}{9} \qquad \times\frac{2}{9}$$

$$\frac{4}{9} = \cancel{\frac{9}{2}}\times\cancel{\frac{2}{9}}f + \cancel{\frac{9}{2}}\times\cancel{\frac{2}{9}}g$$

$$\frac{4}{9} = f + g,\quad f = 1$$

$$\frac{4}{9} = \cancel{1} + g$$

$$-1 \qquad -\cancel{1}$$

$$-\frac{5}{9} = g$$

(1b) Continued. Solve the following recurrence relations with the indicated initial conditions, using the formula for second-order recurrences given in class, using substitution. Prove by substitution that your answer is correct.

3 (ii)
$$\begin{cases} \rho(0) := 1 \\ \rho(1) := 2 \\ \forall n \in \mathbb{N} \quad (a^2)\Big(\rho(n+2)\Big) - (2ab)\Big(\rho(n+1)\Big) + (b^2)\Big(\rho(n)\Big) = 0 \end{cases}$$

First rewrite with your values of a and b.

My problem:

$$\Big((2)^2\Big)\Big(\rho(n+2)\Big) - \Big(2(2)(9)\Big)\Big(\rho(n+1)\Big) + \Big((9)^2\Big)\Big(\rho(n)\Big) = 0$$

My problem:

$$\Big(4\Big)\Big(\rho(n+2)\Big) + \Big(-36\Big)\Big(\rho(n+1)\Big) + (81)\Big(\rho(n)\Big) = 0$$

My solution: $\forall n \in \mathbb{N}$

$$\rho(n)$$

$$= \left(\frac{9}{2}\right)^n - \frac{5n}{9}\left(\frac{9}{2}\right)^n$$

3

You may discuss and work with anyone

(ii) ④

$$\rho(n) = f\left(\frac{9}{2}\right)^n + gn\left(\frac{9}{2}\right)^n,\quad f = 1,\quad g = \left(-\frac{5}{9}\right)$$

$$\rho_{(n)} = (1)\left(\frac{9}{2}\right)^n + \left(-\frac{5}{9}\right)n\left(\frac{9}{2}\right)^n$$

$$\rho_{(n)} = \left(\frac{9}{2}\right)^n - \frac{5n}{9}\left(\frac{9}{2}\right)^n$$

(ii) ⑤

$$P(0) := \left(\rho_{(0)} = \left(\frac{9}{2}\right)^0 - \frac{5(0)}{9}\left(\frac{9}{2}\right)^0\right)$$

$$P(1) := \left(\rho_{(1)} = \left(\frac{9}{2}\right)^1 - \frac{5(1)}{9}\left(\frac{9}{2}\right)^1\right)$$

$$P(n+2) := \left(\rho_{(n+2)} = \left(\frac{9}{2}\right)^{n+2} - \frac{5(n+2)}{9}\left(\frac{9}{2}\right)^{n+2}\right)$$

$$P(n) := \left(\rho_{(n)} = \left(\frac{9}{2}\right)^n - \frac{5n}{9}\left(\frac{9}{2}\right)^n\right)$$

(ii) ⑥ BS

(0) LS

$= \rho(0)$ (BC)

$= 1$

$= \left(\frac{9}{2}\right)^0 - \frac{5(0)}{9}\left(\frac{9}{2}\right)^0$

$=$ RS

$P(0)$

(1) LS

$= \rho(1)$ (BC)

$= 2$

$= \frac{9}{2} - \frac{5}{2}$

$= \frac{9}{2} - \frac{5}{9}\left(\frac{9}{2}\right)$

$= \left(\frac{9}{2}\right)^1 - \frac{5(1)}{9}\left(\frac{9}{2}\right)^1$

$=$ RS

$P(1)$

(ii) ⑦ IS

$P(0), P(1), \ldots P(n+1)$

LS

$= \rho(n+2)$ (RcS)

$= 9(\rho_{(n+1)}) - \frac{81}{4}(\rho_{(n)})$ (IH)

$= 9\left(\left(\frac{9}{2}\right)^{n+1} - \frac{5(n+1)}{9}\left(\frac{9}{2}\right)^{n+1}\right) - \frac{81}{4}\left(\left(\frac{9}{2}\right)^n - \frac{5n}{9}\left(\frac{9}{2}\right)\right.$

$= 9\left(\frac{9}{2}\right)\left(\frac{9}{2}\right)^n - 5(n+1)\left(\frac{9}{2}\right)\left(\frac{9}{2}\right)^n - \frac{81}{4}\left(\frac{9}{2}\right)^n - \frac{45n}{4}\left(\right.$

$= \left(\frac{81}{4}\right)\left(\frac{9}{2}\right)^n - \left(\frac{10+5n}{9}\right)\left(\frac{81}{4}\right)\left(\frac{9}{2}\right)^n$

$= \left(\frac{9}{2}\right)^2\left(\frac{9}{2}\right)^n - \frac{5(n+2)}{9}\left(\frac{9}{2}\right)^2\left(\frac{9}{2}\right)^n$

$= \left(\frac{9}{2}\right)^{n+2} - \frac{5(n+2)}{9}\left(\frac{9}{2}\right)^{n+2}$

$=$ RS

$P(n+2)$

(ii) ⑧

$$\frac{P(0),\ P(1),\ \dfrac{P(0), P(1), \ldots, P(n+1)}{P(n+2)}}{\forall n \in \mathbb{N}\quad P(n)}$$

$$\forall n \in \mathbb{N}\left(\rho_{(n)} = \left(\frac{9}{2}\right)^n - \frac{5n}{9}\left(\frac{9}{2}\right)^n\right)$$

5 (1c) Starting with two sets A nd B with $a := \nu(A)$ and $b := \nu(B)$, that is use as many of the operations on sets that you have learnt from the set:

$$\left\{(\ )^c, \cup, \cap, \setminus, \prod\ , \coprod\ , Fnc(A,B), (\ ), \underset{\sim}{-}\right\}$$

(i) **to construct a set S whose cardinalities are the following,** and

(ii) **to construct word-problems whose solutions require counting the number of elements of the sets you constructed in (i).**

<u>You will get 0, if you calculate numerical values of any of the expressions or subexpressions in (1C)</u>.

Draw a formation-tree for the expression on the facing side.

Draw a formation-tree for the expression-set that you have constructed on the facing side.

Prove on the facing side that the cardinality of the set that you have constructed equals the given expression.

On the right side write the word problem step by step respecting the formation-tree of the expression that you have constructed exactly as shown in class.

(i) 0

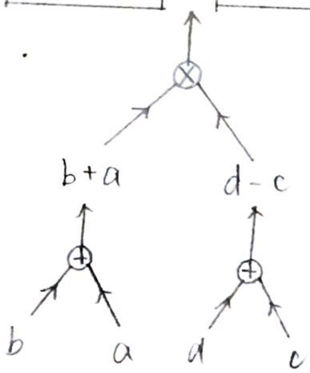
$$(b+a)(b-a)$$
$$= \left((b+a)(d-c)\right)\left\langle \begin{matrix} d \leftarrow b \\ c \leftarrow a \end{matrix} \right\rangle$$

(i) 1

$$\nu(A) = a$$
$$\nu(B) = b$$
$$\nu(C) = c$$
$$\nu(D) = d$$

(i) 2 TREE

$$(b+a)(d-c)$$
$$= \boxed{b+a} \times \boxed{d-c}$$



(i) 3

$$A \cap B = \{\}$$

(i) 4

$$C \subseteq D$$

(i) 5

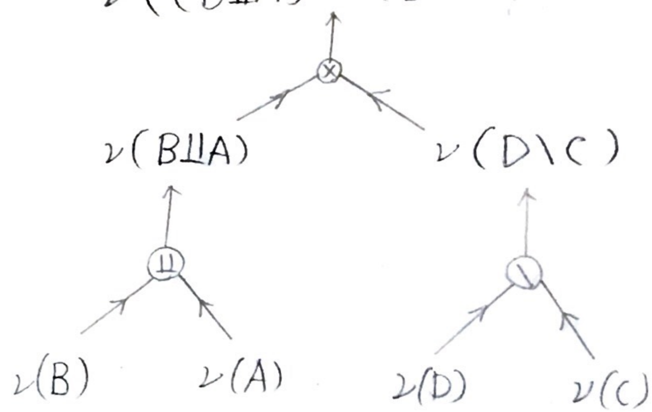
$$\frac{A \cap B = \{\}}{A \cup B \approx A \amalg B}$$

(i) 6

$$(b+a) \times (d-c)$$
$$= (\nu(B) + \nu(A)) \times (\nu(D) - \nu(C))$$
$$= (\nu(B \amalg A)) \times (\nu(D) - \nu(C))$$
$$= (\nu(B \amalg A))(\nu(D \setminus C))$$
$$= \nu((B \amalg A) \times (D \setminus C))$$

(i) 7 TREE

$$\nu((B \amalg A) \times (D \setminus C))$$



(1c) Continued

1 (i) $(b+a)(b-a)$

My expression:

A := Set of JR trains which leave Tokyo for Osaka at January 1st, 2019 10am.

B := Set of Express busses which leave Tokyo for Osaka at January 1st, 2019 10am.

C := Set of ANA air planes which leave Osaka for Fukuoka at January 2nd, 2019 11am.

D := Set of ANA air planes which were supposed to leave Osaka for Fukuoka at January 2nd, 2019 11am, but canceled.

Stn := X wants to go to Fukuoka from Tokyo in two days at January 1st 10am and wants to use JR train between Tokyo and Osaka, and a plane between Osaka to Fukuoka.

Q. How many distinct ways does X have to go to Fukuoka from Tokyo?

1

(ii) 0

b^{a+b}

$= b^\wedge (a+b)$

$= ((c \wedge (a+b)) \langle c \leftarrow b \rangle$

(ii) 1

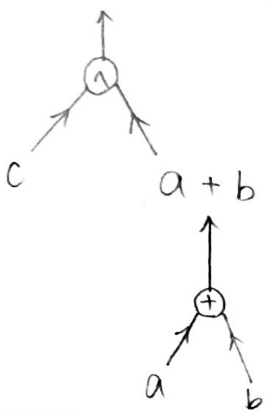
$\nu(A) = a$

$\nu(B) = b$

$\nu(C) = c$

(ii) 2 <u>TREE</u>

$c \wedge \boxed{a+b}$



(ii) 3

$\nu(A) = a$

$\nu(B) = b$

$\nu(C) = c$

(ii) 4 $A \cap B = \{\}$

(ii) 5 $\dfrac{A \cap B = \{\}}{A \cup B \approx A \amalg B}$

(ii) 6

$c \wedge (a+b)$

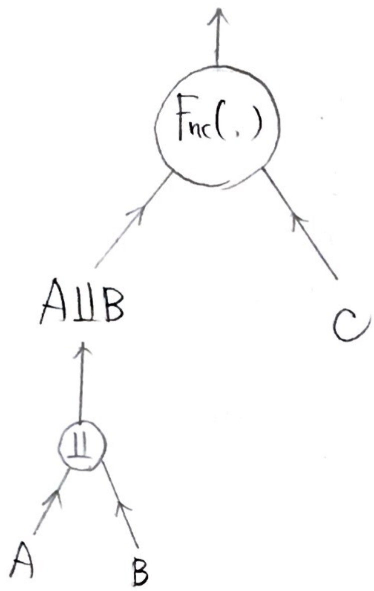
$= \nu(C) \wedge (\nu(A) + \nu(B))$

$= \nu(C) \wedge (\nu(A \amalg B))$

$= \nu(\mathrm{Fnc}((A \amalg B), C))$

(ii) 7 <u>TREE</u>

$\mathrm{Fnc}((A \amalg B), C)$



(1c) Continued

1 (ii) b^{a+b}

My expression:

A := Set of men who are attending Dr. Bagchi's special party.

B := Set of women who are attending Dr. Bagchi's special party.

C := Set of all kinds of meals which are provided to people in the party.

Stu := In the party, every body must choose only one meal.

Q, How many distinct ways are there that each person can choose only one meal in the party?

1

(iii) ⓪

$$\begin{pmatrix} b+a \\ b-a \end{pmatrix}$$

$$= C(b+a,\ b-a)$$

$$= \left(C(b+a,\ d-c)\right)\left\langle \begin{matrix} d \leftarrow b \\ c \leftarrow a \end{matrix} \right\rangle$$

(iii) ①

$\nu(A) = a$

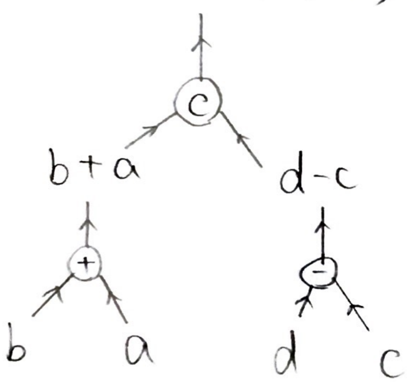
$\nu(B) = b$

$\nu(C) = c$

$\nu(D) = d$

(iii) ② <u>TREE</u>

$C(b+a,\ d-c)$



(iii) ③

$$\frac{d-c \in 0..(b+a)}{d-c \leq b+a}$$

(iii) ④ $A \cap B = \{\}$

(iii) ③

$$\frac{A \cap B = \{\}}{A \cup B \approx A \sqcup B}$$

(iii) ⑥

$C \leq D$

(iii) ⑦

$c(b+a,\ d-c)$

$= c(\nu(B) + \nu(A),\ \nu(D) - \nu(C))$

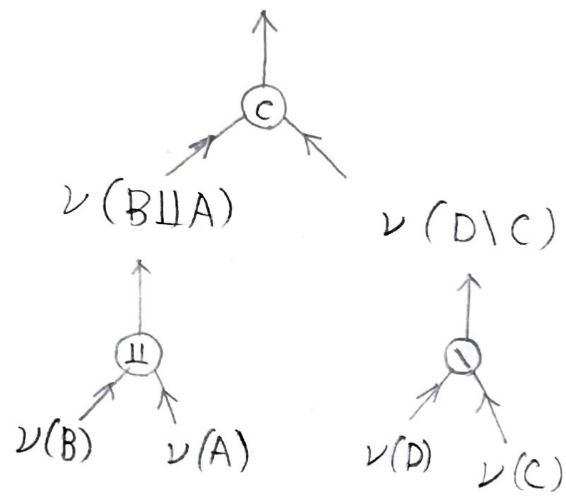
$= c(\nu(B \sqcup A),\ \nu(D) - \nu(C))$

$= c(\nu(B \sqcup A),\ \nu(D \setminus C))$

(iii) ⑧

<u>TREE</u>

$c(\nu(B \sqcup A),\ \nu(D \setminus C))$



(1c) Continued

1 (iii) $\binom{b+a}{b-a}$

My expression:

A := Set of caps in the store.

B := Set of hats in the store

C := Set of people who are in the store.

D := Set of salesman who are working in the store.

Stu := In the store, only customers must choose one cap or one hat. In the store, there are only customers and salesman.

Q. How many ways are there the only customers choose one cap or one hat?

1

(iv) ⓪

$$\begin{pmatrix} \binom{2b}{b} \\ \binom{2a}{a} \end{pmatrix}$$

$$= c\left(\binom{2b}{b}, \binom{2a}{a}\right)$$

$$= c\left(c(2b, b), c(2a, a)\right)$$

$$= c\left(c(c, b), c(d, a)\right) \left\langle \begin{matrix} c \leftarrow 2b \\ d \leftarrow 2a \end{matrix} \right\rangle$$

(iv) ①

$\nu(A) = a$

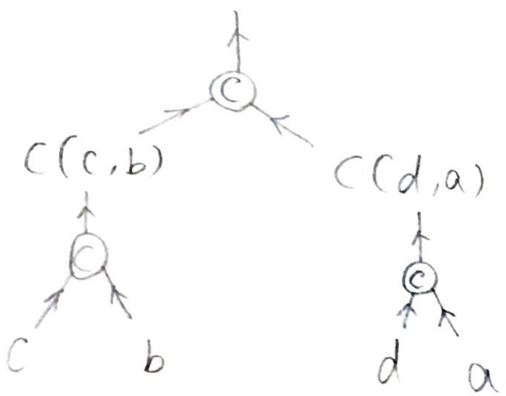
$\nu(B) = b$

$\nu(C) = c$

$\nu(D) = d$

(iv) ② TREE

$c(c(c,b), c(d,a))$



(iv) ③

$$\frac{b \in 0..c}{b \leq c}$$

(iv) ④

$$\frac{a \in 0..d}{a \leq d}$$

(iv) ⑤

$$\frac{c(d,a) \in 0..c(c,b)}{c(d,a) \leq c(c,b)}$$

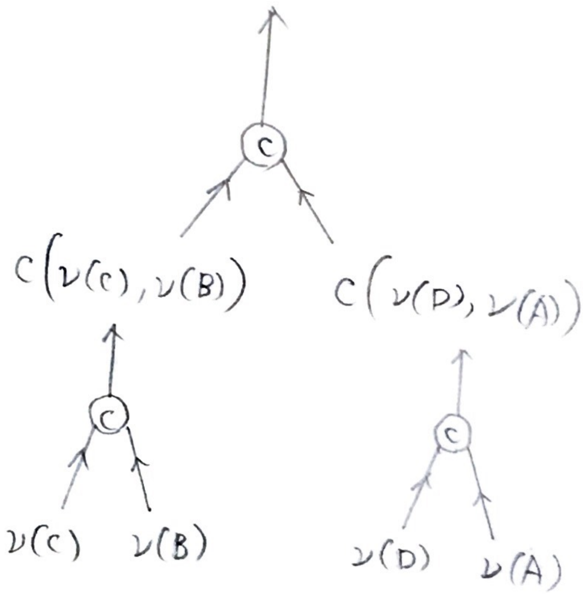
(iv) ⑥

$$c(c(c,b), c(d,a))$$

$$= c\left(c(\nu(C), \nu(B)), c(\nu(D), \nu(A))\right)$$

(iv) ⑦

$c(c(\nu(C), \nu(B)), c(\nu(D), \nu(A)))$



(1c) Continued

2 (iv) $\binom{\binom{2b}{b}}{\binom{2a}{a}}$

My expression:

A := Set of garages which a woman has.

B := Set of dishes which a man has.

C := Set of fruits on the table.

D := Set of cars in the store.

Stu := First of all, the man chooses as many fruits on the table as the dishes he has. At the same time and a different place, the woman chooses as many cars in the store as the garages that she has.

Then you will choose as many books, the numbers of whom is as many as the way of his fruits, as the way that she chooses cars.

Q. How many ways are there that you choose books?

2

(2a) ⓪ $\alpha(0) = 9$

(2a) ①

$$\varphi(0) = \alpha(0) + 7 = 9 + 7 = 16$$

(2a) ②

$$\alpha(1) = (\varphi(0))(5) = 16(5) = 80$$

(2a) ③

$$\varphi(1) = (\alpha(0))(5) + 7 = (9)(5) + 7 = 45 + 7 = 52$$

(2a) ④

$$\alpha(2) = (\varphi(1))(5) = 52(5) = 260$$

(2a) ⑤

$$\varphi(2) = (\alpha(1))(5) + 7 = 80(5) + 7 = 400 + 7 = 407$$

(2a) ⑥

$$\alpha(3) = (\varphi(2))(5) = 407(5) = 2035$$

(2a) ⑦

$$\varphi(3) = (\alpha(2))(5) + 7 = 260(5) + 7 = 1300 + 7 = 1307$$

(2a) ⑧

$$\alpha(n+2) = (\varphi(n+1))(5) = ((\alpha(n))(5) + 7)(5) = (5)(5)(\alpha(n)) + 7(5) = 25(\alpha(n)) + 35$$

$$\alpha(n+2) = 25(\alpha(n)) + 35$$

$$+\frac{35}{24} \qquad +\frac{35}{24}$$

$$\alpha(n+2) + \frac{35}{24} = 25(\alpha(n)) + 35 + \frac{35}{24}$$

$$\alpha(n+2) + \frac{35}{24} = 25(\alpha(n)) + \frac{35}{24}\cdot 25$$

$$\alpha(n+2) + \frac{35}{24} = 25\left(\alpha(n) + \frac{35}{24}\right)$$

(2a) ⑨

$$\alpha(n+2) + \frac{35}{24} = 25\left(\alpha(n) + \frac{35}{24}\right), \ \alpha(0) = 9, \ \alpha(1) = 80$$

$$\alpha(n) = \left(\left(9 + \frac{35}{24}\right)(25)^{\frac{n}{2}} - \frac{35}{24}\right)\left(\frac{1+(-1)^n}{2}\right) + \left(\left(80 + \frac{35}{24}\right)(25)^{\frac{n-1}{2}} - \frac{35}{24}\right)\left(\frac{1+(-1)^{n+1}}{2}\right)$$

$$\alpha(n) = \left(\left(\frac{251}{24}\right)(5)^n - \frac{35}{24}\right)\left(\frac{1+(-1)^n}{2}\right) + \left(\left(\frac{1955}{24}\right)(5)^{n-1} - \frac{35}{24}\right)\left(\frac{1+(-1)^{n+1}}{2}\right)$$

6 (2a) Consider the following recurrence relation for $\varphi(n)$ and $\alpha(n)$ and **answer the questions**:

$$\begin{cases} & \alpha(0) := b \\ \forall n \in \mathbb{N} & \alpha(n+1) := (\varphi(n))c \end{cases} \qquad \begin{cases} & \varphi(0) := \alpha(0) + d \\ \forall n \in \mathbb{N} & \varphi(n+1) := (\alpha(n))c + d \end{cases}$$

Show calculations on the facing side.

Rewrite the recursion using your values of $b, c,$ and d.

My problem:

$$\begin{cases} & \alpha(0) := (9) \\ \forall n \in \mathbb{N} & \alpha(n+1) := (\varphi(n))(5) \end{cases} \qquad \begin{cases} & \varphi(0) := \alpha(0) + d = (16) \\ \forall n \in \mathbb{N} & \varphi(n+1) := (\alpha(n))(5) + (7) \end{cases}$$

Show full computations on the facing side for all values and your conjecture:

$\alpha(0)$	$\varphi(0)$
$= 9$	$= 16$
$\alpha(1)$	$\varphi(1)$
$= 80$	$= 52$
$\alpha(2)$	$\varphi(2)$
$= 260$	$= 407$
$\alpha(3)$	$\varphi(3)$
$= 2035$	$= 1307$

6

Conjecture : $\forall n \in \mathbb{N}$

$\alpha(n)$

$$= \left(\left(\frac{251}{24}\right)(5)^n - \frac{35}{24}\right)\left(\frac{1+(-1)^n}{2}\right) + \left(\left(\frac{1955}{24}\right)(5)^{n-1} - \frac{35}{24}\right)\left(\frac{1+(-1)^{n+1}}{2}\right)$$

$\varphi(n)$

$$= \left(\left(\frac{391}{24}\right)(5)^n - \frac{7}{24}\right)\left(\frac{1+(-1)^n}{2}\right) + \left(\left(\frac{1255}{24}\right)(5)^{n-1} - \frac{7}{24}\right)\left(\frac{1+(-1)^{n+1}}{2}\right)$$

You may discuss and work with anyone

(2a) 10 $\varphi(n+2)$

$= (\alpha(n+1))(5) + 7$

$= ((\varphi(n))(5))(5) + 7$

$= 5(5)(\varphi(n)) + 7$

$= 25(\varphi(n)) + 7$

$$\varphi(n+2) = 25(\varphi(n)) + 7$$

$$+\frac{7}{24} \qquad +\frac{7}{24}$$

$$\varphi(n+2) + \frac{7}{24} = 25(\varphi(n)) + \frac{25\times 7}{24}$$

$$\varphi(n+2) + \frac{7}{24} = 25\left(\varphi(n) + \frac{7}{24}\right)$$

(2a) 11 $\varphi(n+2) + \frac{7}{24} = 25\left(\varphi(n) + \frac{7}{24}\right)$, $\varphi(0) = 16$, $\varphi(1) = 52$

$$\varphi(n) = \left(\left(16 + \frac{7}{24}\right)(25)^{\frac{n}{2}} - \frac{7}{24}\right)\left(\frac{1+(-1)^n}{2}\right) + \left(\left(52 + \frac{7}{24}\right)(25)^{\frac{n-1}{2}} - \frac{7}{24}\right)\left(\frac{1+(-1)^{n+1}}{2}\right)$$

$$\varphi(n) = \left(\left(\frac{391}{24}\right)(5)^n - \frac{7}{24}\right)\left(\frac{1+(-1)^n}{2}\right) + \left(\left(\frac{1255}{24}\right)(5)^{n-1} - \frac{7}{24}\right)\left(\frac{1+(-1)^{n+1}}{2}\right)$$

(1) ⓪

$$k := \frac{n-1}{2}$$

$$K(0) = \alpha(0) + \alpha(1)$$
$$K(1) = \alpha(2) + \alpha(3)$$
$$K(2) = \alpha(4) + \alpha(5)$$
$$\vdots$$
$$K(k) = \alpha(n-1) + \alpha(n)$$

(1) ③

$$l := \frac{n}{2} - 1$$

$$L(0) = \alpha(1) + \alpha(2)$$
$$L(1) = \alpha(3) + \alpha(4)$$
$$L(2) = \alpha(5) + \alpha(6)$$
$$\vdots$$
$$L(l) = \alpha(n-1) + \alpha(n)$$

(1) ④

$$L(l) = \alpha(n-1) + \alpha(n)$$
$$= \left(\left(\frac{251}{24}\right)(5)^{n} - \frac{35}{24}\right) + \left(\left(\frac{1955}{24}\right)(5)^{n-2} - \frac{35}{24}\right)$$
$$= \left(\frac{6275}{600}\right)(5)^{n} + \left(\frac{1955}{600}\right)(5)^{n} - \frac{35}{12}$$
$$= \left(\frac{8230}{600}\right)(5)^{n} - \frac{35}{12}$$
$$= \left(\frac{823}{60}\right)(5)^{2(l+1)} - \frac{35}{12}$$
$$= \left(\frac{823}{60}\right)(25)^{l+1} - \frac{35}{12}$$
$$= \left(\frac{4115}{12}\right)(25)^{l} - \frac{35}{12}$$

(1) ①

$$K(k) = \alpha(n-1) + \alpha(n)$$
$$= \left(\left(\frac{251}{24}\right)(5)^{n-1} - \frac{35}{24}\right) + \left(\left(\frac{1955}{24}\right)(5)^{n-1} - \frac{35}{24}\right)$$
$$= \frac{2206}{24}(5)^{n-1} - \frac{35}{12}$$
$$= \frac{1103}{12}(5)^{2k} - \frac{35}{12}$$
$$= \frac{1103}{12}(25)^{k} - \frac{35}{12}$$

(1) ⑤

$$\sum_{i=0}^{l} L(i) = \frac{\left(\frac{4115}{12}\right)(25^{l+1} - 1)}{25-1} - \frac{35}{12}(l+1)$$
$$= \frac{1}{24}\left(\frac{4115}{12}\right)(25^{l+1} - 1) - \frac{35}{12}(l+1)$$
$$= \left(\frac{4115}{288}\right)(25^{l+1} - 1) - \frac{35}{12}(l+1)$$
$$= \left(\frac{4115}{288}\right)\left((25)^{\frac{n}{2}} - 1\right) - \frac{35}{12}\left(\frac{n}{2}\right)$$
$$= \left(\frac{4115}{288}\right)(5^{n} - 1) - \frac{35}{24}n$$

(1) ②

$$\sum_{i=0}^{k} K(i) = \frac{\left(\frac{1103}{12}\right)(25^{k+1} - 1)}{25-1} - \frac{35}{12}(k+1)$$
$$= \frac{1}{24}\left(\frac{1103}{12}\right)(25^{k+1} - 1) - \frac{35}{12}(k+1)$$
$$= \left(\frac{1103}{288}\right)(25^{k+1} - 1) - \frac{35}{12}(k+1)$$
$$= \left(\frac{1103}{288}\right)\left((25)^{\frac{n-1}{2}+1} - 1\right) - \frac{35}{12}\left(\frac{n-1}{2} + 1\right)$$
$$= \left(\frac{1103}{288}\right)(5^{n+1} - 1) - \frac{35}{24}(n+1)$$

(1) ⑥

$$S_m(\alpha, n) = \left(\sum_{i=0}^{l} L(i) + \alpha(0)\right)\left(\frac{1+(-1)^n}{2}\right) + \left(\sum_{i=0}^{k} K(k)\right)\left(\frac{1+(-1)^{n+1}}{2}\right)$$

(1) ⑦ $\alpha(0) = 9$

$$S_m(\alpha, n) = \left(\left(\frac{4115}{288}\right)(5^{n} - 1) - \frac{35}{24}n + 9\right)\left(\frac{1+(-1)^n}{2}\right) + \left(\left(\frac{1103}{288}\right)(5^{n+1} - 1) - \frac{35}{24}(n+1)\right)\left(\frac{1+(-1)^{n+1}}{2}\right)$$

6 (2b) Find conjectures for the sums:

$$Sm(\alpha, n) := S(\alpha, n) := \sum_{k=0}^{n} \alpha(k) \quad \text{and}$$

$$Sm(\varphi, n) := S(\varphi, n) := \sum_{k=0}^{n} \varphi(k)$$

where $\alpha(n)$ and $\varphi(n)$ are defined, as in (2a) by the recursion appearing below,

$$\begin{cases} & \alpha(0) := b \\ \forall n \in \mathbb{N} & \alpha(n+1) := (\varphi(n))c \end{cases} \qquad \begin{cases} & \varphi(0) := \alpha(0) + d \\ \forall n \in \mathbb{N} & \varphi(n+1) := (\alpha(n))c + d \end{cases}$$

Show computations on the facing side to indicate how you arrive at your conjecture.

Rewrite the recursion using your values of a, b, and c.

My problem:

$$\begin{cases} & \alpha(0) := (9) \\ \forall n \in \mathbb{N} & \alpha(n+1) := (\varphi(n))(5) \end{cases} \qquad \begin{cases} & \varphi(0) := \overset{\alpha(0)+d}{\cancel{\alpha(0)}} = (16) \\ \forall n \in \mathbb{N} & \varphi(n+1) := (\alpha(n))(5) + (7) \end{cases}$$

<u>My Conjectures</u> :

6

(1) $\forall n \in \mathbb{N}$ $Sm(\alpha, n)$

$$= \left(\left(\frac{4115}{288}\right)(5^n - 1) - \frac{35}{24}n + 9\right)\left(\frac{1+(-1)^n}{2}\right) + \left(\left(\frac{1103}{288}\right)((5)^{n+1} - 1) - \frac{35}{24}(n+1)\right)\left(\frac{1+(-1)^{n+1}}{2}\right)$$

(2) $\forall n \in \mathbb{N}$ $Sm(\varphi, n)$

$$= \left(\left(\frac{5515}{288}\right)(5^n - 1) - \frac{7}{24}n + 16\right)\left(\frac{1+(-1)^n}{2}\right) + \left(\frac{823}{288}(5^{n+1} - 1) - \frac{7}{24}(n+1)\right)\left(\frac{1+(-1)^{n+1}}{2}\right)$$

(2) ⓪

$$k := \frac{n-1}{2}$$

$$K(0) = \varphi(0) + \varphi(1)$$
$$K(1) = \varphi(2) + \varphi(3)$$
$$K(2) = \varphi(4) + \varphi(5)$$
$$\vdots$$
$$K(k) = \varphi(n-1) + \varphi(n)$$

(2) ③ $l := \frac{n}{2} - 1$

$$L(0) = \varphi(1) + \varphi(2)$$
$$L(1) = \varphi(3) + \varphi(4)$$
$$L(2) = \varphi(5) + \varphi(6)$$
$$\vdots$$
$$L(l) = \varphi(n-1) + \varphi(n)$$

(2) ①

$$K(k) = \varphi(n-1) + \varphi(n)$$
$$= \left(\left(\frac{391}{24}\right)(5)^{n-1} - \frac{7}{24}\right) + \left(\left(\frac{1255}{24}\right)(5)^{n-1} - \frac{7}{24}\right)$$
$$= \left(\frac{1646}{24}\right)(5)^{n-1} - \frac{7}{24} \times 2$$
$$= \left(\frac{1646}{60}\right)(5)^{n} - \frac{7}{12}$$
$$= \left(\frac{1646}{60}\right)(5)^{2k+1} - \frac{7}{12}$$
$$= \left(\frac{823}{72}\right)(25)^{k} - \frac{7}{12}$$

(2) ②

$$\sum_{i=0}^{k} K(i) = \frac{\left(\frac{823}{12}\right)\left(25^{k+1} - 1\right)}{25 - 1} - \frac{7}{12}(k+1)$$
$$= \left(\frac{823}{12 \times 24}\right)\left(25^{k+1} - 1\right) - \frac{7}{12}(k+1)$$
$$= \left(\frac{823}{288}\right)\left(25^{\frac{n-1}{2}+1} - 1\right) - \frac{7}{12}\left(\frac{n-1}{2} + 1\right)$$
$$= \left(\frac{823}{288}\right)\left(5^{n+1} - 1\right) - \frac{7}{24}(n+1)$$

(2) ④

$$L(l) = \varphi(n-1) + \varphi(n)$$
$$= \left(\left(\frac{391}{24}\right)(5)^{n} - \frac{7}{24}\right) + \left(\left(\frac{1255}{24}\right)(5)^{n-2} - \frac{7}{24}\right)$$
$$= \left(\frac{1103}{60}\right)(5)^{n} - \frac{7}{24} \times 2$$
$$= \left(\frac{1103}{60}\right)(5)^{2(l+1)} - \frac{7}{12}$$
$$= \left(\frac{1103}{60}\right)(25)^{l+1} - \frac{7}{12}$$
$$= \left(\frac{5515}{12}\right)(25)^{l} - \frac{7}{12}$$

(2) ⑤

$$\sum_{i=0}^{l} L(i) = \frac{\left(\frac{5515}{12}\right)\left(25^{l+1} - 1\right)}{25 - 1} - \frac{7}{12}(l+1)$$
$$= \left(\frac{5515}{24 \times 12}\right)\left(25^{l+1} - 1\right) - \frac{7}{12}(l+1)$$
$$= \left(\frac{5515}{288}\right)\left(25^{\frac{n}{2}} - 1\right) - \frac{7}{12}\left(\frac{n}{2}\right)$$
$$= \left(\frac{5515}{288}\right)\left(5^{n} - 1\right) - \frac{7}{24}n$$

(2) ⑥

$$Sm(\varphi, n) = \left(\sum_{i=0}^{l} L(i) + \varphi(0)\right)\left(\frac{1+(-1)^n}{2}\right) + \left(\sum_{i=0}^{k} K(i)\right)\left(\frac{1+(-1)^{n+1}}{2}\right)$$

(2) ⑦ $\varphi(0) = 16$

$$Sm(\varphi, n) = \left(\left(\frac{5515}{288}\right)\left(5^n - 1\right) - \frac{7}{24}n + 16\right)\left(\frac{1+(-1)^n}{2}\right) + \left(\left(\frac{823}{288}\right)\left(5^{n+1} - 1\right) - \frac{7}{24}(n+1)\right)\left(\frac{1+(-1)^{n+1}}{2}\right)$$

(2c)

⓪ $P(0) := \left(\alpha(0) = \left(\left(\frac{251}{24}\right)(5)^0 - \frac{35}{24}\right)\left(\frac{1+(-1)^0}{2}\right) + \left(\left(\frac{1955}{24}\right)(5)^{0-1} - \frac{35}{24}\right)\left(\frac{1+(-1)^{0+1}}{2}\right)\right)$

$P(n+1) := \left(\alpha(n+1) = \left(\left(\frac{251}{24}\right)(5)^{n+1} - \frac{35}{24}\right)\left(\frac{1+(-1)^{n+1}}{2}\right) + \left(\left(\frac{1955}{24}\right)(5)^{n+1-1} - \frac{35}{24}\right)\left(\frac{1+(-1)^{n+1+1}}{2}\right)\right)$

(2c)

① BS

LS

$= \alpha(0)$ (BC)

$= 9$

$= \frac{216}{24}$

$= \left(\frac{251}{24} - \frac{35}{24}\right)(1) + \left(\left(\frac{1955}{24}\right)\left(\frac{1}{5}\right) - \frac{35}{24}\right)(0)$

$= \left(\left(\frac{251}{24}\right)(5)^0 - \frac{35}{24}\right)\left(\frac{1+1}{2}\right) + \left(\left(\frac{1955}{24}\right)(5)^{-1} - \frac{35}{24}\right)\left(\frac{1-1}{2}\right)$

$= \left(\left(\frac{251}{24}\right)(5)^0 - \frac{35}{24}\right)\left(\frac{1+(-1)^0}{2}\right) + \left(\left(\frac{1955}{24}\right)(5)^{0-1} - \frac{35}{24}\right)\left(\frac{1+(-1)^{0+1}}{2}\right)$

$= RS$

$P(0)$

(2c) ② IS $\quad P(0), P(1), P(2), \ldots P(n)$

LS

$= \alpha(n+1)$ (RcD)

$= (\alpha(n))(5)$ (IH)

$= ((\alpha(n-1))(5) + 7)(5)$

$= 25(\alpha(n-1)) + 35$

$= 25\left(\left(\left(\frac{251}{24}\right)(5)^{n-1} - \frac{35}{24}\right)\left(\frac{1+(-1)^{n-1}}{2}\right) + \left(\left(\frac{1955}{24}\right)(5)^{n-1-1} - \frac{35}{24}\right)\left(\frac{1+(-1)^{n-1+1}}{2}\right)\right) + 35$

$= \left(\left(\frac{251}{24}\right)(5)^{n+1} - \frac{35}{24} \times 25\right)\left(\frac{1+(-1)^{n-1+2}}{2}\right) + \left(\left(\frac{1955}{24}\right)(5)^{n} - \frac{35}{24} \times 25\right)\left(\frac{1+(-1)^{n-1+1+2}}{2}\right) + 35$

$= \left(\left(\frac{251}{24}\right)(5)^{n+1} - \frac{35 \times 25}{24} + 35\right)\left(\frac{1+(-1)^{n+1}}{2}\right) + \left(\left(\frac{1955}{24}\right)(5)^{n+1-1} - \frac{35}{24} \times 25 + 35\right)\left(\frac{1+(-1)^{n+1+1}}{2}\right)$

$= \left(\left(\frac{251}{24}\right)(5)^{n+1} - \frac{35}{24}\right)\left(\frac{1+(-1)^{n+1}}{2}\right) + \left(\left(\frac{1955}{24}\right)(5)^{n+1-1} - \frac{35}{24}\right)\left(\frac{1+(-1)^{n+1+1}}{2}\right)$

$= RS$

$P(n+1)$

5 (2c) Prove by induction your conjectures from (2a) and (2b) are valid.

(2c) ③
$$\frac{P(0), \quad \dfrac{P(1), P(2), \ldots P(n)}{P(n+1)}}{\forall n \in \mathbb{N} \quad P(n)}$$

(2c) ④

$$P(0) := \left(\varphi(0) = \left(\left(\frac{391}{24}\right)(5)^0 - \frac{7}{24} \right) \left(\frac{1+(-1)^0}{2} \right) + \left(\left(\frac{1255}{24}\right)(5)^{0-1} - \frac{7}{24} \right) \left(\frac{1+(-1)^{0+1}}{2} \right) \right)$$

$$P(n+1) := \left(\varphi(n+1) = \left(\left(\frac{391}{24}\right)(5)^{n+1} - \frac{7}{24} \right) \left(\frac{1+(-1)^{n+1}}{2} \right) + \left(\left(\frac{1255}{24}\right)(5)^{n+1-1} - \frac{7}{24} \right) \left(\frac{1+(-1)^{n+1+1}}{2} \right) \right)$$

(2c) ⑤ BS

LS

$= \varphi(0)$ (BC)

$= 16$

$= \dfrac{384}{24}$

$= \left(\dfrac{391}{24}\right)(1) - \dfrac{7}{24}$

$= \left(\left(\dfrac{391}{24}\right)(5)^0 - \dfrac{7}{24} \right) \left(\dfrac{1+(-1)^0}{2} \right) + \left(\left(\dfrac{1255}{24}\right)(5)^{0-1} - \dfrac{7}{24} \right) \left(\dfrac{1+(-1)^{0+1}}{2} \right) \right)$

$=$ RS

$$\overline{\quad P(0) \quad}$$

(2c) ⑥ IS

$$\frac{P(0), P(1), P(2), \dots P(n)}{}$$

LS

$= \varphi(n+1)$ (RcD)

$= (\alpha(n))(5) + 7$ (IH)

$= ((\varphi(n-1))(5))(5) + 7$

$= 25(\varphi(n-1)) + 7$

$= 25\left[\left(\left(\frac{391}{24}\right)(5)^{n-1} - \frac{7}{24}\right)\left(\frac{1+(-1)^{n-1}}{2}\right) + \left(\left(\frac{1255}{24}\right)(5)^{n-1-1} - \frac{7}{24}\right)\left(\frac{1+(-1)^{n-1+1}}{2}\right)\right] + 7$

$= \left(\left(\frac{391}{24}\right)(5)^{n+1} - \frac{7}{24} \times 25 - 7\right)\left(\frac{1+(-1)^{n+1}}{2}\right)$

$+ \left(\left(\frac{1255}{24}\right)(5)^{n+1-1} - \frac{7}{24} \times 25 - 7\right)\left(\frac{1+(-1)^{n+1+1}}{2}\right)$

$= \left(\left(\frac{391}{24}\right)(5)^{n+1} - \frac{7}{24}\right)\left(\frac{1+(-1)^{n+1}}{2}\right)$

$+ \left(\left(\frac{1255}{24}\right)(5)^{n+1-1} - \frac{7}{24}\right)\left(\frac{1+(-1)^{n+1+1}}{2}\right)$

$= RS$

$$\overline{}$$

$P(n+1)$

(2c) ⑦ $P(0)$, $\dfrac{P(0), P(1), P(2), \dots P(n)}{P(n+1)}$

$$\overline{}$$

$\forall n \in \mathbb{N}\ P(n)$

(2c) 8 $P(0) :=$

$$\left(Sm(\alpha,0) = \left(\left(\frac{4115}{288}\right)(5^0-1) - \frac{35}{24}(0)+9\right)\left(\frac{1+(-1)^0}{2}\right) + \left(\left(\frac{1103}{288}\right)(5^{0+1}-1) - \frac{35}{24}(0+1)\right)\left(\frac{1+(-1)^{0+1}}{2}\right)\right)$$

$P(n+1) :=$

$$\left(Sm(\alpha,n+1) = \left(\left(\frac{4115}{288}\right)(5^{n+1}-1) - \frac{35}{24}(n+1)+9\right)\left(\frac{1+(-1)^{n+1}}{2}\right) + \left(\left(\frac{1103}{288}\right)(5^{n+1+1}-1) - \frac{35}{24}(n+1+1)\right)\left(\frac{1+(-1)^{n+1+1}}{2}\right)\right)$$

(2c) 9 BS

LS

$= Sm(\alpha,0)$ (BC)

$= \sum_{k=0}^{0} \alpha(k)$

$= \alpha(0)$

$= 9$

$= 9 + \left(\frac{4115}{288}\right)(0) - \frac{35}{24}(0)$

$= \left(\left(\frac{4115}{288}\right)(5^0-1) - \frac{35}{24}(0)+9\right)\left(\frac{1+(-1)^0}{2}\right) + \left(\left(\frac{1103}{288}\right)(5^{0+1}-1) - \frac{35}{24}(0+1)\right)\left(\frac{1+(-1)^0}{2}\right)$

$= RS$

$P(0)$

(2c) (10) IS

$$\frac{P(0),\ P(1),\ P(2),\ \ldots\ P(n)}{\begin{aligned}
&LS \\
&= Sm(\alpha, n+1) \quad (RcD) \\
&= \sum_{k=1}^{n+1} \alpha(k) \quad (\underline{IH}) \\
&= Sm(\alpha, n) + \alpha(n+1) \\
&= \left(\left(\frac{4115}{288}\right)(5^n - 1) - \frac{35}{24}(n) + 9\right)\left(\frac{1+(-1)^n}{2}\right) + \left(\left(\frac{1103}{288}\right)(5^{n+1} - 1) - \frac{35}{24}(n+1)\right)\left(\frac{1+(-1)^{n+1}}{2}\right) \\
&\quad + \left(\left(\frac{201}{24}\right)(5)^n - \frac{35}{24}\right)\left(\frac{1+(-1)^n}{2}\right) + \left(\left(\frac{1955}{24}\right)(5^{n-1}) - \frac{35}{24}\right)\left(\frac{1+(-1)^{n+1}}{2}\right) \\
&= \left(\left(\frac{4115}{288} + \frac{3012}{288}\right)(5)^n - \frac{4115}{288} - \frac{35}{24}(n+1) + 9\right)\left(\frac{1+(-1)^{n+1}}{2}\right) \\
&\quad + \left(\left(\frac{1103}{288} + \frac{23460}{288}\right)(5)^n - \frac{1103}{288} - \frac{35}{24}(n+1)\right)\left(\frac{1+(-1)^{n+2}}{2}\right) \\
&= \left(\left(\frac{4115}{288}\right)(5^{n+1} - 1) - \frac{35}{24}(n+1) + 9\right)\left(\frac{1+(-1)^{n+1}}{2}\right) \\
&\quad + \left(\left(\frac{1103}{288}\right)(5^{n+1} - 1) - \frac{35}{24}(n+1)\right)\left(\frac{1+(-1)^{n+1+1}}{2}\right) \\
&= RS
\end{aligned}}$$

$$P(n+1)$$

(2c) (11)

$$\frac{P(0), \quad \dfrac{P(0), P(1) \ldots P(n)}{P(n+1)}}{\forall n \in \mathbb{N} \quad P(n)}$$

(2c) (12)

$$P(0) := \left(Sm(\varphi, 0) = \left(\left(\frac{5515}{288}\right)(5^0 - 1) - \frac{7}{24}(0) + 16\right)\left(\frac{1+(-1)^0}{2}\right) + \left(\left(\frac{223}{288}\right)(5^{0+1} - 1) - \frac{7}{24}(0+1)\right)\left(\frac{1+(-1)^{0+1}}{2}\right)\right)$$

$$P(n+1) := \left(Sm(\varphi, n+1) = \left(\left(\frac{5515}{288}\right)(5^{n+1} - 1) - \frac{7}{24}(n+1) + 16\right)\left(\frac{1+(-1)^{n+1}}{2}\right) + \left(\frac{223}{288}\right)(5^{n+1+1} - 1) - \frac{7}{24}(n+1+1)\right)\left(\frac{1+(-1)^{n+1+1}}{2}\right)\right)$$

(2c) 13 $\underline{BS}$

$$
\begin{aligned}
&LS\\
&= Sm(\varphi, 0) \quad (\underline{BC})\\
&= \varphi(0)\\
&= 16\\
&= 0 + 16 + 0\\
&= \left(\left(\left(\frac{5515}{288}\right)(5^0 - 1) - \frac{7}{24}(0) + 16\right)\left(\frac{1+(-1)^0}{2}\right)\right.\\
&\qquad \left. + \left(\left(\frac{823}{288}\right)(5^{0+1} - 1) - \frac{7}{24}(0+1)\right)\left(\frac{1+(-1)^{0+1}}{2}\right)\right)\\
&= RS
\end{aligned}
$$

$$\frac{}{P(0)}$$

(2c) 14 $\underline{IS}$

$$\frac{P(0), P(1), P(2), \cdots P(n)}{}$$

$$
\begin{aligned}
&LS\\
&= Sm(\varphi, n+1) \quad (\underline{RcD})\\
&= Sm(\varphi, n) + \varphi(n+1) \quad (\underline{IH})\\
&= \left(\left(\frac{5515}{288}\right)(5^n - 1) - \frac{7}{24}(n) + 16\right)\left(\frac{1+(-1)^n}{2}\right) + \left(\left(\frac{823}{288}\right)(5^{n+1} - 1) - \frac{7}{24}(n+1)\right)\left(\frac{1+(-1)^{n+1}}{2}\right)\\
&\qquad + \left(\left(\frac{391}{24}\right)(5)^{n+1} - \frac{7}{24}\right)\left(\frac{1+(-1)^{n+1}}{2}\right) + \left(\left(\frac{1255}{24}\right)(5)^n - \frac{7}{24}\right)\left(\frac{1+(-1)^{n+1+1}}{2}\right)\\
&= \left(\left(\frac{5515}{288} + \frac{4692}{288}\right)(5^n) + \left(\frac{391}{24}\right)(5-1) - \frac{7}{24}(n+1) + 16\right)\left(\frac{1+(-1)^{n+1}}{2}\right)\\
&\qquad + \left(\left(\frac{823}{288} + \frac{15060}{288}\right)(5^n) + \left(\frac{823}{288}\right)(5-1) - \frac{7}{24}(n+1+1)\right)\left(\frac{1+(-1)^{n+1+1}}{2}\right)\\
&= \left(\left(\frac{5515}{288}\right)(5^{n+1} - 1) - \frac{7}{24}(n+1) + 16\right)\left(\frac{1+(-1)^{n+1}}{2}\right)\\
&\qquad + \left(\left(\frac{823}{288}\right)(5^{n+1+1} - 1) - \frac{7}{24}(n+1+1)\right)\left(\frac{1+(-1)^{n+1+1}}{2}\right)\\
&= RS
\end{aligned}
$$

$$\frac{}{P(n+1)}$$

(2c) 15

$$\frac{P(0), \quad \dfrac{P(0), P(1), P(2), \cdots P(n)}{P(n+1)}}{\forall n \in \mathbb{N} \ P(n)}$$

(2i) ⓪

BC $\varphi(0) := 2$

RcS $\varphi(n+1) := (\varphi(n))(5) + 7$

(2i) ①

$$\varphi(n+1) = (\varphi(n))(5) + 7$$
$$+\frac{7}{4} \qquad\qquad +\frac{7}{4}$$
$$\varphi(n+1) + \frac{7}{4} = (\varphi(n))(5) + \frac{35}{4}$$
$$\varphi(n+1) + \frac{7}{4} = 5\left(\varphi(n) + \frac{7}{4}\right)$$

(2i) ②

$$\varphi(n+1) + \frac{7}{4} = 5\left(\varphi(n) + \frac{7}{4}\right)$$
$$\varphi(n) = \left(\varphi(0) + \frac{7}{4}\right)\cdot 5^n - \frac{7}{4}, \quad \varphi(0) = 2$$
$$\varphi(n) = \left(2 + \frac{7}{4}\right)\cdot 5^n - \frac{7}{4}$$
$$\varphi(n) = \left(\frac{15}{4}\right)(5)^n - \frac{7}{4}$$

(4i) ⓪

$P(0) := \left(\varphi(0) = \frac{15}{4}(5)^0 - \frac{7}{4}\right)$

$P(n+1) := \left(\varphi(n+1) = \frac{15}{4}(5)^{n+1} - \frac{7}{4}\right)$

(4i) ① BC

LS

$$= \varphi(0) \quad (\text{BC})$$
$$= 2$$
$$= \frac{8}{4}$$
$$= \frac{15}{4} - \frac{7}{4}$$
$$= \frac{15}{4}(1) - \frac{7}{4}$$
$$= \frac{15}{4}(5)^0 - \frac{7}{4}$$
$$= \text{RS}$$

$P(0)$

(4i) ② IS $\quad \dfrac{P(0), P(1), P(2), \dots P(n)}{}$

LS

$$= \varphi(n+1) \quad (\text{RcD})$$
$$= (\varphi(n))(5) + 7 \quad (\text{IH})$$
$$= \left(\left(\frac{15}{4}\right)(5)^n - \frac{7}{4}\right)(5) + 7$$
$$= \left(\frac{15}{4}\right)(5)^{n+1} - \frac{7}{4}(5) + 7$$
$$= \left(\frac{15}{4}\right)(5)^{n+1} - \frac{35}{4} + \frac{28}{4}$$
$$= \left(\frac{15}{4}\right)(5)^{n+1} - \frac{7}{4}$$

$P(n+1)$

(4i) ③

$$\frac{P(0), \quad \dfrac{P(0), P(1), P(2) \dots P(n)}{P(n+1)}}{P(n)}$$

6 (3a) Consider the following recurrence relation for $\varphi(n)$ and $\alpha(n)$ and **answer the questions**:

$$\begin{cases} & \varphi(0) := a \\ \forall n \in \mathbb{N} & \varphi(n+1) := (\varphi(n))c + d \end{cases}$$

Show calculations on the facing side.

Rewrite the recursion using your values of $b, c,$ and d.

$$\begin{cases} & \varphi(0) := a \\ \forall n \in \mathbb{N} & \varphi(n+1) := (\varphi(n))(5) + (7) \end{cases}$$

Show computations on the facing side for making a conjecture:

2 (i) Conjecture:

$$\forall n \in \mathbb{N} \qquad \varphi(n) = \frac{15}{4}(5)^n - \frac{7}{4}$$

4 (i) Prove your conjecture by induction:

0

(3b) ⓞ

$$\neg\left(\cancel{RtI}\left(\left(13\right)^{\frac{1}{12}}\right)\right)$$

$$\neg\left(\neg\left(Rtl\left(\left(13\right)^{\frac{1}{12}}\right)\right)\right), \quad \neg(\neg p) = p$$

$$Rtl\left(\left(13\right)^{\frac{1}{12}}\right)$$

$$\exists n, d \in \mathbb{N}\left(\left(13\right)^{\frac{1}{12}} = \frac{n}{d}\right), \quad \gcd(n, d) = 1$$

$$\left(13\right)^{\frac{1}{12}} = \frac{n}{d}$$

$$\times d \qquad \times d$$

$$\left(\left(13\right)^{\frac{1}{12}}\right) d = \left(\frac{n}{\cancel{d}}\right)\cancel{d}$$

$$\left(\left(13\right)^{\frac{1}{12}}\right) d = n$$

$$(\ )^{12} \qquad (\ )^{12}$$

$$\left(\left(\left(13\right)^{\frac{1}{12}}\right) d\right)^{12} = (n)^{12}$$

$$13 d^{12} = n^{12}, \quad prm(13)$$

$$13 \mid n^{12} \quad \left(\frac{prm(p) \quad p \mid n^{12}}{p \mid n}\right)$$

$$13 \mid n$$

$$\exists q \in \mathbb{N} \quad n = 13(q) + 0$$

$$n = 13(q)$$

$$(\ )^{12} \qquad (\ )^{12}$$

$$n^{12} = (13)^{12} q^{12}$$

$$n^{12} = (13)^{12} q^{12}, \quad 13 d^{12} = n^{12}$$

$$13 d^{12} = (13)^{12} q^{12}$$

$$\div 13 \qquad \div 13$$

$$d^{12} = (13)^{11} q^{12}$$

6 (3b) Prove, following the method shown in class, that: $(a+b+2)^{\frac{1}{a+b+1}}$ is **irrational**, that is: $\cancel{Rtl}\left((a+b+2)^{\frac{1}{a+b+1}}\right)$.

First rewrite the problem using your values of a and b.

My Problem:

$$\cancel{Rtl}\left(((2)+(9)+2)^{\frac{1}{(2)+(9)+1}}\right)$$

that is: $\cancel{Rtl}\left((13)^{\frac{1}{(12)}}\right)$

(3b) ①

$$d^{12} = (13)^{11} q^{12}$$

$$13 \mid d^{2} \quad \text{prem}(13)$$

$$13 \mid d$$

$$13 \mid n, \ 13 \mid d$$

$$13 \mid (\gcd(n,d)), \ \gcd(n,d) = 1$$

$$13 \mid 1$$

$$\perp$$

$$\neg\left(Rtl\left((13)^{\frac{1}{12}}\right)\right)$$

$$\cancel{Rtl}\left((13)^{\frac{1}{12}}\right)$$

6

) $x := .(ab) = .(92)$

$x_0 = .92$

$x_1 = .9292 = .92 + .0092 = .92 + \frac{92}{10000} = x_0 + \frac{92}{10^4}$

$x_2 = .929292 = .9292 + .000092 = .9292 + \frac{92}{1000000} = x_1 + \frac{92}{10^6}$

$x_3 = .92929292 = .929292 + 0.0000000092 = .929292 + \frac{92}{100000000} = x_2 + \frac{92}{10^8}$

$\vdots$

$\forall n \in \mathbb{N} \quad x_{n+1} = x_n + \frac{92}{10^{2(n+2)}}$

BC $\quad x_0 = .92$

RcS $\quad \forall n \in \mathbb{N} \quad x_{n+1} = x_n + \frac{92}{10^{2(n+2)}}$

(ii)

$x = .(92)$

$\times 100 \qquad \times 100$

$100x = 92.(92)$

$100x = 92 + .(92)$

$100x = 92 + \cancel{x}$

$-x \qquad \cancel{-x}$

$99x = 92$

$\div 99 \qquad \div 99$

$\frac{\cancel{99}}{\cancel{99}}x = \frac{92}{99}$

$x = \frac{92}{99}.$

$x \in \mathbb{Q}$

$Rtl(x)$

2 (3c1) Consider the numbers defined below, in which the dot indicates a decimal point, and the string in parentheses is the repeating part of a repeating decimal.

$$x := .(ba) \qquad = \quad .(92)$$

Show **computations on the facing side** and record answers on this side.

1 (i) Find a recursive definition for x and check that it is correct.

BC $\quad x_0 = .92$

RcS $\quad \forall n \in \mathbb{N} \quad x_{n+1} = x_n + \dfrac{92}{10^{2(n+2)}}$

1 ~~(i)~~ (ii) Prove that: $Rtl(x)$, that is, x is rational.

So
no
reality
wheel is
wasted.

2

You may discuss and work with anyone

(iii) $y := .bbabaabaaab\ldots = .99292292229\ldots$

$y_0 = .99$

$y_1 = .9929 = .99 + .0029 = y_0 + \frac{29}{10000} = y_0 + \frac{29}{10^4}$

$y_2 = .9929229 = .9929 + .0000229 = y_1 + \frac{229}{10000000} = y_1 + \frac{229}{10^7}$

$y_3 = .9929229229 = .9929229 + .00000002229 = y_2 + \frac{2229}{100000000000} = y_2 + \frac{2229}{10^{11}}$

$$\forall n \in \mathbb{N} \quad y_{n+1} = y_n + \frac{1}{10^{\left(4+\frac{n^2+5n}{2}\right)}}\left(29 + \frac{200(10^n-1)}{9}\right)$$

BC $\quad y_0 = .99$

RS $\quad \forall n \in \mathbb{N} \quad y_{n+1} = y_n + \frac{1}{10^{\left(4+\frac{n^2+5n}{2}\right)}}\left(29 + \frac{200(10^n-1)}{9}\right)$

(iv) ⓪ $\forall n, p \in \mathbb{N} \left(f(x) = \frac{1}{n!} p^n x^n (y-x)^n \right)$

(iv) ①

$$\frac{Z = x(y-x)}{Z \leq \frac{y^2}{4}}$$

$$0 < f(x) = \frac{1}{n!} p^n x^n (y-x)^n$$
$$= \frac{p^n}{n!} \left(x(y-x)\right)^n$$
$$< \frac{p^n}{n!} \left(\frac{y^2}{4}\right)^n$$

(iv) ②

$$\frac{0 \leq x \leq y}{0 \leq \sin x < 1}$$

$$0 < (\sin x) f(x) < \frac{1}{n!} \left(\frac{py^2}{4}\right)^n$$

$$0 < \int_0^y (\sin x) f(x)\, dx < \frac{1}{n!}\left(\frac{py^2}{4}\right)^n y, \quad \lim_{n\to\infty} \frac{a^n}{n!} = 0$$

$$0 < \int_0^y \sin x\, f(x)\, dx < 1$$

3 (3c2) Consider the numbers defined below, in which the dot indicates a decimal point, and the string in parentheses is the repeating part of a repeating decimal.

$$y := .bbabaabaaab\ldots \quad = .9929922922229\ldots$$

where for every $\forall n \in \mathbb{N}$ there are n a's between the $(n+1)^{st}$ and the$(n+2)^{nd}$ b.

Show **computations on the facing side** and record answers on this side.

1 (iii) Find a recursive definition for y and check that it is correct.

BC $y_0 = .99$

RcS $y_{n+1} = y_n + \dfrac{1}{10^{\left(4+\frac{n^2+5n}{2}\right)}}\left(29 + \dfrac{200(10^n-1)}{9}\right)$

2 (iv) Prove that: ~~Rtl~~(y), that is, **y is irrational.**

(iv)③ $\int_0^y \sin x\, f(x)\,dx$

$= -\cos x\, f(x)\Big|_0^y + \int_0^y \cos x\, f'(x)\,dx$

$= f(y) - f(0) + \int_0^y \cos x\, f'(x)\,dx$

$= f(y) - f(0) + \sin x\, f'(x)\Big|_0^y - \int_0^y \sin x\, f''(x)\,dx$

$= \sum_{k=0}^{n} (-1)^k \left(f^{(2k)}(y) + f^{(2k)}(0)\right)$

$= 2\sum_{k=0}^{n} (-1)^k f^{(2k)}(0)$

Very complicated proof.

Please talk to me.

(iv) ④

$$f(x) = \frac{1}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}p^n\pi^{n-k}x^{n+k}$$

$$f(0) = f'(0) = f''(0) \cdots = f^{(n-1)}(0) = 0$$

$$f^{(n+k)}(x) = \frac{1}{n!}(-1)^k\binom{n}{k}p^n y^{n-k}(n+k)! + g(k)$$

$$f^{(n+k)}(0) = (-1)\binom{n}{k}\frac{(n+k)!}{n!}p^n y^{n-k}, \qquad \neg(\cancel{Rtl}(y))$$

$$Rtl(y)$$

$$\exists p,q \in \mathbb{N}\left(y = \frac{q}{p}\right), \ gcd(p,q) = 1$$

$$p^n y^{n-k} = p^n \cdot \frac{q^{n-k}}{p^{n-k}} = p^k q^{n-k}$$

$$\int_0^y \sin x\, f(x)\,dx = \mathbb{Z}, \quad 0 < \int_0^y \sin x\, f(x)\,dx < 1$$

$$\perp$$

$$\neg(Rtl(y))$$

$$\cancel{Rtl}(y)$$