6 (1a) Expand the following **EXACTLY AS SHOWN IN CLASS.**

Use the binomial theorem and express your answer as a polynomial with **integer coefficients, ordering the unknowns within each summand numero-alphabetically, writing the summands VERTICALLY in increasing order of degree, and for every degree, ordering the summands lexicographically as shown in class.** Show calculations for all coefficients that occur on the facing page.

3 (i) $(b+x)^{1+a}$

My problem: $\left((7)+x\right)^{1+(2)}$

$= \left(7+x\right)^{(3)}$

$= 343$
$+ (147)x$
$+ (21)x^2$
$+ x^3$

3

1a.i.0

$$\frac{(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k}{(7+x)^3 = \sum_{k=0}^{3} \binom{3}{k} 7^{3-k} x^k} \left\langle \begin{array}{l} n \leftarrow 3 \\ a \leftarrow 7 \\ b \leftarrow x \end{array} \right\rangle$$

$$(7+x)^3 = \binom{3}{0}(7)^{3-0}(x)^0 = (1)(7)^3(1) = (1)(343)(1) = 343$$

$$\begin{array}{l|l|l|l}
+\binom{3}{1}(7)^{3-1}(x)^1 & +(3)(7)^2(x)^1 & +(3)(49)(x) & +(147)\,x \\
+\binom{3}{2}(7)^{3-2}(x)^2 & +(3)(7)^1(x)^2 & +(3)(7)(x)(x) & +(21)\,x^2 \\
+\binom{3}{3}(7)^{3-3}(x)^3 & +(1)(7)^0(x)^3 & +(1)(1)(x)(x)(x) & +\,x^3
\end{array}$$

1a.i.1

$$\binom{3}{0} = \binom{3}{3-0} = \binom{3}{3} = 1$$

1a.i.2

$$\binom{3}{1} = \binom{3}{3-1} = \binom{3}{2} = 3$$

1a.i.3

$$\frac{\binom{n}{0} = \binom{n}{n} = 1}{\binom{3}{0} = \binom{3}{3} = 1} \left\langle n \leftarrow 3 \right\rangle$$

1a.i.4

$$\frac{\binom{n}{1} = \binom{n}{n-1} = 3}{\binom{3}{1} = \binom{3}{3-1} = 3} \left\langle n \leftarrow 3 \right\rangle$$

(1a) Expand the following **EXACTLY AS SHOWN IN CLASS.**

Use the multinomial theorem and express your answer as a polynomial with **integer coefficients, ordering the unknowns within each summand numero-alphabetically, writing the summands VERTICALLY in increasing order of degree, and for every degree, ordering the summands lexicographically.** Show calculations for all coefficients that occur on the facing page.

Show calculations for all coefficients that occur on the facing page.

3 (ii) $(b + x + y)^{1+a}$

My problem: $$\left((7) + x + y\right)^{1+(2)}$$

$$= \left(7 + x + y\right)^{(3)}$$

$$\begin{aligned} = \; & 343 \\ & + (147)x \\ & + (21)x^2 \\ & + x^3 \\ & + (3)x^2y \\ & + (42)xy \\ & + (3)xy^2 \\ & + (147)y \\ & + (21)y^2 \\ & + y^3 \end{aligned}$$

B

1a.ii.0

$$(x_1+x_2+x_3)^n = \sum_{p_1+p_2+p_3=n} \binom{n}{p_1\ p_2\ p_3}(x_1^{p_1})(x_2^{p_2})(x_3^{p_3})$$

n

$x_1 \leftarrow 7$

$x_2 \leftarrow x$

$x_3 \leftarrow y$

$$(7+x+y)^3 = \sum_{p_1+p_2+p_3=3} \binom{3}{p_1\ p_2\ p_3}(7)^{p_1}(x)^{p_2}(y)^{p_3}$$

$(7+x+y)^3 = \binom{3}{0\,0\,3}(7)^0(x)^0(y)^3 = (1)(1)(1)(y)^3 = yyy = 343 = 343$

		$+(3)xyy$	$+(147)x$	$+(147)x$
$+\binom{3}{0\,1\,2}(7)^0(x)^1(y)^2$	$+(3)(1)(x)(y)^2$	$+(3)xxy$	$+(21)xx$	$+(21)x^2$
$+\binom{3}{0\,2\,1}(7)^0(x)^2(y)^1$	$+(3)(1)(x)^2(y)$			
$+\binom{3}{0\,3\,0}(7)^0(x)^3(0)^0$	$+(1)(1)(x)^3(1)$	$+xxx$	$+xxx$	$+x^3$
$+\binom{3}{1\,0\,2}(7)^1(x)^0(y)^2$	$+(3)(7)(1)(y)^2$	$+(21)yy$	$+(3)xxy$	$+(3)x^2y$
$+\binom{3}{1\,1\,1}(7)^1(x)^1(y)^1$	$+(6)(7)(x)(y)$	$+(42)xy$	$+(42)xy$	$+(42)xy$
$+\binom{3}{1\,2\,0}(7)^1(x)^2(y)^0$	$+(3)(7)(x)^2(1)$	$+(21)xx$	$+(3)xyy$	$+(3)xy^2$
$+\binom{3}{2\,0\,1}(7)^2(x)^0(y)^1$	$+(3)(7)^2(1)(y)$	$+(147)y$	$+(147)y$	$+(147)y$
$+\binom{3}{2\,1\,0}(7)^2(x)^1(y)^0$	$+(3)(7)^2(x)(1)$	$+(147)x$	$+(21)yy$	$+(21)y^2$
$+\binom{3}{3\,0\,0}(7)^3(x)^0(y)^0$	$+(1)(7)^3(1)(1)$	$+343$	$+yyy$	$+y^3$

1a.ii.1

$$\binom{3}{0\,0\,3} = \binom{3}{0\,3\,0} = \binom{3}{3\,0\,0} = \frac{\cancel{3!}}{\cancel{3!}(0!)(0!)} = \frac{1}{(1)(1)(1)} = 1$$

1a.ii.2

$$\binom{3}{0\,1\,2} = \binom{3}{0\,2\,1} = \binom{3}{1\,0\,2} = \binom{3}{1\,2\,0} = \binom{3}{2\,0\,1} = \binom{3}{2\,1\,0} = \frac{3!}{(2!)(1!)(0!)} = \frac{3!\cancel{2!}}{\cancel{(2!)}(1!)(0!)} = \frac{3}{1} = 3$$

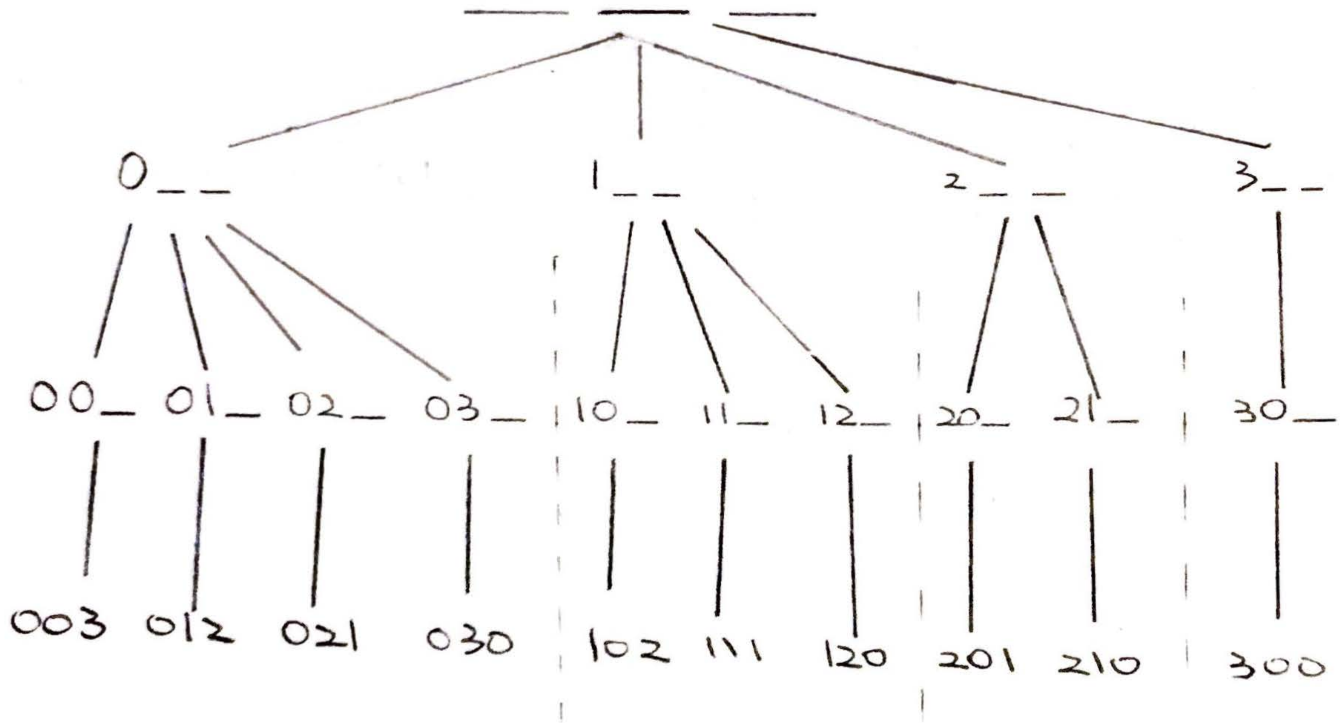
1a.ii.3

$$\binom{3}{1\,1\,1} = \frac{3!}{(1!)(1!)(1!)} = \frac{(3)(2)(1)}{(1)(1)(1)} = \frac{6}{1} = 6$$

(1a.ii.4)

$P_1 + P_2 + P_3 = 3$

Tree



6 (1b) Solve the following recurrence relations with the indicated initial conditions, using the algorithm for second-order recurrences given in class, and **prove by induction that your answers satisfy the recurrence:**

3 (i)
$$\begin{cases} \rho(0) := 0 \\ \rho(1) := 10 \\ \forall n \in \mathbb{N} \quad \Big(\rho(n+2)\Big) - (a+b)\Big(\rho(n+1)\Big) + (ab)\Big(\rho(n)\Big) = 0 \end{cases}$$

First rewrite the recurrence with your values of a and b.

My problem:

$$\Big(\rho(n+2)\Big) - \Big((2)+(7)\Big)\Big(\rho(n+1)\Big) + \Big((2)(7)\Big)\Big(\rho(n)\Big) = 0$$

My solution: $\forall n \in \mathbb{N}$

$$\rho(n)$$

$$= (2)(7)^n + (-2)(2)^n$$

3

(1b.i.0) **BC** (0) $P(0)=0$

(1) $P(1)=10$

RcS $\forall n \in \mathbb{N}$

$$P(n+2)-9(P(n+1))+14(P(n))=0$$
$$+9(P(n+1)) \qquad +9(P(n+1))$$

$$P(n+2)+14(P(n))=9(P(n+1))$$
$$-14(P(n)) \qquad -14(P(n))$$

$$(1)P(n+2)=9(P(n+1))+(-14(P(n))$$

$$p \leftarrow 1 \qquad q \leftarrow 9 \qquad r \leftarrow (-14)$$

$$P(x^2)-9(x)-r=0 \quad \langle p \leftarrow 1,\ q \leftarrow 9,\ r \leftarrow (-14) \rangle$$

$$(1)\,x^2-(9)x-(-14)=0$$

$$x^2-9x+14=0$$

$$(x-7)\,(x-2)=0$$

$$x-7=0 \qquad x-2=0$$
$$+7 \;\; +7 \qquad +2 \;\; +2$$

$$x=7 \qquad x=2$$

$$(x=7) \vee (x=2)$$

$$(r_1=7) \neq (r_2=2)$$

$$r_1 \neq r_2$$

$$\forall n \in \mathbb{N} \quad P(n)=f(7)^n+g(2)^n$$

(1b.i.1)

$$P(n)=f(7)^n+g(2)^n \quad \langle n \leftarrow 0 \rangle$$

$$P(0)=f(7)^0+g(2)^0 \qquad P(0)=0$$

$$0=f+g$$
$$-g \qquad -g$$

$$-g=f$$

$$-g=f$$

(1b.i.2)

$$P(n)=f(7)^n+g(2)^n \quad \langle n \leftarrow 1 \rangle$$

$$P(1)=f(7)^1+g(2)^1 \qquad P(1)=10$$

$$10=f(7)+g(2) \qquad -g=f$$

$$10=-g(7)+g(2)$$

$$10=(-7+2)g$$

$$10=(-5)g$$
$$\div(-5) \qquad \div(-5)$$

$$-2\;\frac{\cancel{10}}{\cancel{(-5)}} \qquad \frac{\cancel{(-5)}g}{\cancel{(-5)}}$$

$$-2=g$$

$$-2=g$$

$$2=-g \qquad -g=f$$

$$2=f$$

$$P(n)=(2)(7)^n+(-2)(2)^n$$

1b.i.3

$$\forall n \in \mathbb{N}\ P(n) = f(7)^n + g(2)^n$$

$$f = 2 \qquad g = -2$$

$$\forall n \in \mathbb{N}\ P(n) = (2)(7)^n + (-2)(2)^n$$

1b.i.4

$\underline{\underline{BC}}$ (0) LS

$= P(0)$

$= (2)(7)^0 + (-2)(2)^0$

$= (2)(1) + (-2)(1)$

$= 2 + (-2)$

$= 0$

$= RS$

$\underline{\underline{BC}}$ (1) LS

$= P(1)$

$= (2)(7)^1 + (-2)(2)^1$

$= 14 + (-4)$

$= 10$

$= RS$

.i.5 **RcS**

LS

$= P(n+2)$

$= (2)(7)^{n+2} + (-2)(2)^{n+2}$

$= (2)(7)^n (7)^2 + (-2)(2)^n (2)^2$

$= (2)(7)(7)(7)^n + (-2)(2)(2)(2)^n$

$= 98(7)^n + (-8)((2)^n)$

RS

$= 9((P(n+1)) - 14(P(n))$

$= 9((2)(7)^{n+1} + (-2)(2)^{n+1}) - 14((2)(7)^n + (-2)(2)^n)$

$= 9(2)(7)^n(7)^1 + 9(-2)(2)^n(2)^1 - 14(2)(7)^n - 14(-2)(2)^n$

$= 126(7)^n - 36(2)^n - 28(7)^n + 28(2)^n$

$= 98((7)^n) + (-8)((2)^n)$

$$LS = 98(7)^n + (-8)((2)^n) = RS$$

$$LS = RS$$

(1b) Continued. Solve the following recurrence relations with the indicated initial conditions, using the algorithm for second-order recurrences given in class, and **prove by induction that your answers satisfy the recurrence:**

3 (ii)
$$\begin{cases} \rho(0) := 0 \\ \rho(1) := 10 \\ \forall n \in \mathbb{N} \quad (a^2)\Big(\rho(n+2)\Big) - (2ab)\Big(\rho(n+1)\Big) + (b^2)\Big(\rho(n)\Big) = 0 \end{cases}$$

First rewrite with your values of a and b.

My problem:

$$\Big((2)^2\Big)\Big(\rho(n+2)\Big) - \Big(2(2)(7)\Big)\Big(\rho(n+1)\Big) + \Big((7)^2\Big)\Big(\rho(n)\Big) = 0$$

My solution: $\forall n \in \mathbb{N}$

$$\rho(n)$$

$$= \left(\frac{20}{7}\right)(n)\left(\frac{7}{2}\right)^n$$

3

1b.ii.0

BC

⓪ $P(0)=0$

① $P(1)=1$

RS $\forall n \in \mathbb{N}$

$$4(P(n+2))-28(P(n+1))+49(P(n))=0$$
$$+28(P(n+1)) \qquad +28(P(n+1))$$
$$4(P(n+2))+49(P(n))=28(P(n+1))$$
$$-49(P(n)) \qquad -49(P(n))$$
$$4(P(n+2))=28(P(n+1))-49(P(n))$$
$$\div 4 \qquad \div 4 \qquad \div 4$$
$$(1)(P(n+2))=7(P(n+1))-\left(\tfrac{49}{4}\right)(P(n))$$

1b.ii.1

$$px^2-qx-r=0 \qquad \left\langle \begin{array}{l} p \leftarrow 1 \\ q \leftarrow 7 \\ r \leftarrow -\tfrac{49}{4} \end{array} \right\rangle$$
$$x^2-7x-\left(-\tfrac{49}{4}\right)=0$$
$$x^2-7x+\tfrac{49}{4}=0$$
$$x^2-7x+\tfrac{49}{4}=0$$
$$\times 4 \quad \times 4 \qquad \times 4$$
$$4(x^2)-4(7x)+\tfrac{49}{4}(4)=0$$
$$(4)(x)^2-(4)(7)x+(7)^2=0$$
$$(2x-7)^2=0$$
$$2x-7=0$$
$$+7 \quad +7$$
$$2x=7$$
$$\div 2 \quad \div 2$$
$$\frac{\cancel{2}x}{\cancel{2}} \quad \frac{7}{2}$$
$$x=\tfrac{7}{2}$$
$$r_1=\tfrac{7}{2}=r_2$$
$$r_1=r_2=r$$
$$\forall n \in \mathbb{N} \quad P(n)=f\left(\tfrac{7}{2}\right)^n+gn\left(\tfrac{7}{2}\right)^n$$

1b.ii.2

$$P(n)=f\left(\tfrac{7}{2}\right)^n+gn\left(\tfrac{7}{2}\right)^n \qquad \langle n \leftarrow 0 \rangle$$
$$P(0)=f\left(\tfrac{7}{2}\right)^0+g(0)\left(\tfrac{7}{2}\right)^0 \quad P(0)=0$$
$$0=f(1)+0$$
$$0=f$$

1b.ii.3

$$P(n)=f\left(\tfrac{7}{2}\right)^n+g(n)\left(\tfrac{7}{2}\right)^n \qquad \langle n \leftarrow 1 \rangle$$
$$P(1)=f\left(\tfrac{7}{2}\right)^1+g(1)\left(\tfrac{7}{2}\right)^1 \quad P(1)=10$$
$$10=f\left(\tfrac{7}{2}\right)+g\left(\tfrac{7}{2}\right) \quad f=0$$
$$10=0+g\left(\tfrac{7}{2}\right)$$
$$10=g\left(\tfrac{7}{2}\right)$$
$$\div\tfrac{7}{2} \quad \div\tfrac{7}{2}$$
$$\left(\tfrac{2}{7}\right)10=g\left(\tfrac{7}{2}\right)\left(\tfrac{2}{7}\right)$$
$$\tfrac{20}{7}=g$$

1b.ii.4

$$P(n)=0\left(\tfrac{7}{2}\right)+\left(\tfrac{20}{7}\right)(n)\left(\tfrac{7}{2}\right)^n$$
$$=0+\left(\tfrac{20}{7}\right)(n)\left(\tfrac{7}{2}\right)^n$$
$$=\left(\tfrac{20}{7}\right)(n)\left(\tfrac{7}{2}\right)^n$$

Due at 1300 (day) or 1730 (evening) on 20180326

(1c) Continued

1 (i) $(b+a)(b-a)$

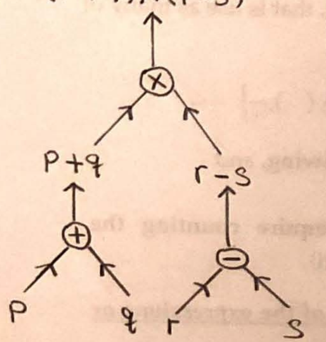
My expression:

1c.i.0 $(b+a)(b-a)$

$= (b+a)\times(b-a)$

1c.i.1 $\dfrac{(b+a)\times(b-a)}{= ((p+q)\times(r-s))}$ $\left\langle \begin{array}{l} p \leftarrow b \\ q \leftarrow a \\ r \leftarrow b \\ s \leftarrow a \end{array} \right\rangle$

1c.i.2 $(p+q)\times(r-s)$



1c.i.3 $S \subseteq R$

$P, Q, R, S \in \mathcal{U}$

$p := \nu(P)$

$q := \nu(Q)$

$r := \nu(R)$

$s := \nu(S)$

1c.i.4 $p+q$

$= \nu(P) + \nu(Q)$

$= \nu(P \amalg Q)$

1c.i.5 $\dfrac{S \subseteq R}{r-s}$

$= \nu(R) - \nu(S)$

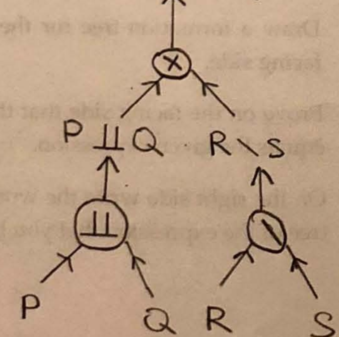
$= \nu(R \setminus S)$

1c.i.6 $(p+q)\times(r-s)$

$= (\nu(P \amalg Q))\times(\nu(R\setminus S))$

$= \nu((R \amalg Q)\times(R\setminus S))$

1c.i.7 $(P \amalg Q)\times(R\setminus S)$



1.c.ii.0

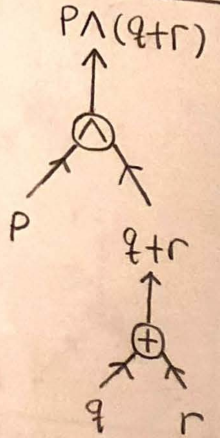
b^{a+b}

$= b\wedge(a+b)$

1.c.ii.1

$$\frac{b\wedge(a+b)}{=p\wedge(q+r)}\left\langle\begin{array}{l}p\leftarrow b\\ q\leftarrow a\\ r\leftarrow b\end{array}\right\rangle$$

1.c.ii.2



1.c.ii.3

$p := v(P)$

$q := v(Q)$

$r := v(R)$

1.c.ii.4

$q+r$

$= v(Q)+v(R)$

$= v(Q \amalg R)$

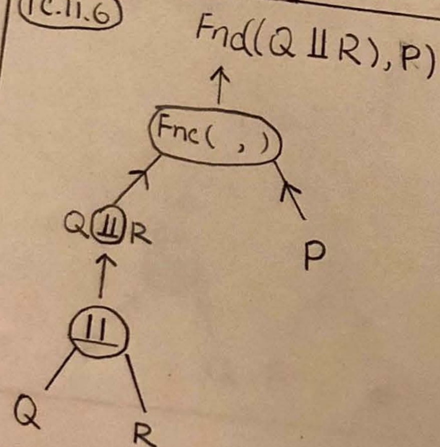
1c.ii.5

$p^{(q+r)}$

$= (v(P))^{v(Q\amalg R)}$

$= v(Fnc((Q\amalg R),P))$

1c.ii.6



Due at 1300 (day) or 1730 (evening) on 20180326

(1c) Continued

1 (ii) b^{a+b}

My expression:

6 (2a) Solve the following recurrence relation for $\varphi(n)$, noting that $\alpha(n)$ is just an arithmetic progression, and **prove by induction that your conjectures are valid:**

$$\begin{cases} & \alpha(0) := b \\ \forall n \in \mathbb{N} & \alpha(n+1) := (\alpha(n))c \end{cases} \qquad \begin{cases} & \varphi(0) := \cancel{0}\alpha(0) \\ \forall n \in \mathbb{N} & \varphi(n+1) := \cancel{(\alpha(n)) + d} \; (\varphi(n))c+d \end{cases}$$

Show calculations on the facing side.

Rewrite the recursion using your values of b, c, and d.

My problem:

$$\begin{cases} & \alpha(0) := 7 \\ \forall n \in \mathbb{N} & \alpha(n+1) := (\alpha(n))4 \end{cases} \qquad \begin{cases} & \varphi(0) := 7 \\ \forall n \in \mathbb{N} & \varphi(n+1) := (\varphi(n))4+5 \end{cases}$$

(i) Compute, showing full computations:

$\alpha(0)$

$= 7$

$\alpha(1)$

$= 7(4)$

$\alpha(2)$

$= 7(4^2)$

$\alpha(3)$

$= 7(4^3)$

$\varphi(0)$

$= 7$

$\varphi(1)$

$= 7(4)+5$

$\varphi(2)$

$= 7(4)^2+5(4+1)$

$\varphi(3)$

$= 7(4)^3+5(4^2+4+1)$

<u>Conjecture</u>:

$\alpha(n)$

$= 7(4^n)$

$\varphi(n)$

$= 7(4)^n+\frac{5}{3}(4^n-1)$

6

You may discuss and work with anyone

1c.i.0 $(b+a)(b-a)$

Note: $\alpha(0)=7$

2a.i.0
$$\begin{aligned}\alpha(1) &= \alpha(0+1)\\ &= (\alpha(0))4\\ &= 7(4)\end{aligned}$$

2a.i.1
$$\begin{aligned}\alpha(2) &= \alpha(1+1)\\ &= (\alpha(1))4\\ &= 7(4)(4)\\ &= 7(4^2)\end{aligned}$$

2a.i.2
$$\begin{aligned}\alpha(3) &= \alpha(2+1)\\ &= (\alpha(2))4\\ &= 7(4^2)4\\ &= 7(4^3)\end{aligned}$$

2a.i.3
$$\begin{aligned}\varphi(0) &= \alpha(0)\\ &= 7\end{aligned}$$

2a.i.4
$$\begin{aligned}\varphi(1) &= \varphi(0+1)\\ &= (\varphi(0))4+5\\ &= 7(4)+5\end{aligned}$$

2a.i.6
$$\begin{aligned}\varphi(2) &= \varphi(1+1)\\ &= (\varphi(1))4+5\\ &= (7(4)+5)4+5\\ &= 7(4)^2+(5)4+5\\ &= 7(4)^2+5(4+1)\end{aligned}$$

2a.i.7
$$\begin{aligned}\varphi(3) &= \varphi(2+1)\\ &= (\varphi(2))4+5\\ &= (7(4)^2+5(4+1))4+5\\ &= 7(4)^3+5(4^2+4+1)\end{aligned}$$

2a.i.7
$$\begin{aligned}\varphi(n) &= 7(4)^n+5(1+4^1+4^2+4^3+\ldots+4^{n-1})\\ &= 7(4)^n+5\left(\frac{1-4^{n-1+1}}{1-4}\right)\\ &= 7(4)^n+5\left(\frac{1-4^n}{-3}\right)\\ &= 7(4)^n+\frac{5}{3}(4^n-1)\end{aligned}$$

2a.i.8 **BC** $\varphi(0)=7$

RcS $\forall n \in \mathbb{N}\ \varphi(n+1)=(\varphi(n))4+5$

$\forall n \in \mathbb{N}\ \varphi(n)=7(4)^n+\frac{5}{3}(4^n-1)$

$P(0) := (\varphi(0)=7(4)^0+\frac{5}{3}(4^0-1))$

$P(n) := (\varphi(n)=7(4)^n+\frac{5}{3}(4^n-1))$

$P(n+1) := (\varphi(n+1)=7(4)^{n+1}+\frac{5}{3}(4^{n+1}-1))$

BC
$$\begin{aligned}\text{LS} &= \varphi(0)\\ &= 7\\ &= 7(1)\\ &= 7(4^0)\\ &= 7(4)^0+0\\ &= 7(4)^0+\tfrac{5}{3}(0)\\ &= 7(4)^0+\tfrac{5}{3}(1-1)\\ &= 7(4)^0+\tfrac{5}{3}(4^0-1)\\ &= \text{RS}\end{aligned}$$

$$\overline{P(0)}$$

2a.i.9 **IS**
$$\frac{P(0)\ P(1)\ldots P(n)}{}$$
$$\begin{aligned}\text{LS} &= \varphi(n+1)\\ &= (\varphi(n))4+5 && (\text{RcS})\\ &= (7(4)^n+\tfrac{5}{3}(4^n-1))4+5 && (\varphi(n))\\ &= 7(4)^n(4)+\tfrac{5}{3}(4^n-1)4+5\\ &= 7(4)^{n+1}+\tfrac{5}{3}(4^{n+1}-4)+5\\ &= 7(4)^{n+1}+\tfrac{5}{3}4^{n+1}-4(\tfrac{5}{3})+5\\ &= 7(4)^{n+1}+\tfrac{5}{3}4^{n+1}-\tfrac{20}{3}+5(1)\\ &= 7(4)^{n+1}+\tfrac{5}{3}4^{n+1}-\tfrac{20}{3}+5(\tfrac{3}{3})\\ &= 7(4)^{n+1}+\tfrac{5}{3}4^{n+1}-\tfrac{20}{3}+\tfrac{15}{3}\\ &= 7(4)^{n+1}+\tfrac{5}{3}4^{n+1}-\tfrac{5}{3}\\ &= 7(4)^{n+1}+\tfrac{5}{3}(4^{n+1}-1)\\ &= \text{RS}\end{aligned}$$

$$\overline{P(n+1)}$$

2a.i.10 **Snths**
$$\frac{\dfrac{P(0)\ P(1)\ldots P(n)}{P(n+1)}}{\forall n \in \mathbb{N}\ P(n)}$$

6 (2b) Find a conjecture for the sum: $Sm(\varphi,n) := S(\varphi,n) := \sum_{k=0}^{n} \varphi(k)$ where $\varphi(n)$ is defined, as in (2a) by the recursion appearing below, by writing out the terms, multiplying $S(\varphi,n)$ by a suitable factor, shifting by a term to the right and subtracting term by term:

$$\begin{cases} & \alpha(0) := b \\ \forall n \in \mathbb{N} & \alpha(n+1) := (\alpha(n))c \end{cases} \qquad \begin{cases} & \varphi(0) := \alpha(0) \\ \forall n \in \mathbb{N} & \varphi(n+1) := \cancel{(\varphi(n)) + d} \end{cases}$$

$(\varphi(n))c+d$

Show computations on the facing side.

Rewrite the recursion using your values of $a, b,$ and c.

My problem:

$$\begin{cases} & \alpha(0) := (7) \\ \forall n \in \mathbb{N} & \alpha(n+1) := (\alpha(n))(4) \end{cases} \qquad \begin{cases} & \varphi(0) := \alpha(0) = (7) \\ \forall n \in \mathbb{N} & \varphi(n+1) := (\varphi(n))\cancel{+(\;)} \end{cases}$$

$(4)+(5)$

<u>My Conjecture</u>:

$$\forall n \in \mathbb{N} \qquad S(\varphi,n)$$

$$= \frac{104}{9}4^n - \frac{5}{3}n - \frac{41}{9}$$

6

(b.0) $\varphi(0)=7$

$\varphi(1)=7(4)+5$

$\varphi(2)=7(4)^2+5(1+4)$

$\varphi(3)=7(4)^3+5(1+4+4^2)$

$\varphi(n)=7(4)^n+5(1+4+4^2+\dots+4^{n-1})$

(b.1) $S(\varphi(n))=7(1+4+4^2+4^3+\dots+4^n)+5+5(1+4)+5(1+4+4^2)+\dots+5(1+4+4^2+\dots+4^{n-1})$

$=7\left(\frac{4^{n+1}-1}{1-4}\right)+5+5\left(\frac{4^2-1}{4-1}\right)+5\left(\frac{4^3-1}{4-1}\right)+\dots 5\left(\frac{4^n-1}{4-1}\right)$

$=\frac{7}{3}(4^{n+1}-1)+5+\frac{5}{3}(4^2-1)+\frac{5}{3}(4^3-1)+\dots+\frac{5}{3}(4^n-1)$

$=\frac{7}{3}(4^{n+1}-1)+5+\frac{5}{3}(4)^2-\frac{5}{3}+\frac{5}{3}(4)^3-\frac{5}{3}+\dots+\frac{5}{3}(4)^n-\frac{5}{3}$

$=\frac{7}{3}(4^{n+1}-1)+5+\frac{5}{3}(4^2+4^3+\dots+4^n)-\frac{5}{3}-\frac{5}{3}-\dots-\frac{5}{3}$

(2b.2) $S=\frac{7}{3}(4^{n+1}-1)+5+\frac{5}{3}4^2(1+4+4^2+\dots+4^{n-2})-\frac{5}{3}(n-1)$

$=\frac{7}{3}(4)4^n-\frac{7}{3}+5+\frac{80}{3}\left(\frac{4^{n-2+1}-1}{4-1}\right)-\frac{5}{3}(n)+\frac{5}{3}$

$=\frac{7}{3}(4)(4^n)-\frac{7}{3}+\frac{15}{3}+\frac{80}{9}(4^{n-1}-1)-\frac{5}{3}n+\frac{5}{3}$

$=\frac{28}{3}4^n+\frac{13}{3}+\cancel{\frac{80}{9}}^{20}\left(\frac{1}{4}\right)4^n-\frac{80}{9}-\frac{5}{3}n$

$=4^n\left(\frac{28}{3}+\frac{20}{9}\right)-\frac{41}{9}-\frac{5}{3}n$

$=\frac{104}{9}4^n-\frac{5}{3}n-\frac{41}{9}$

5 (2c) Prove by induction your conjecture from (2b) is valid.

$$LS = \frac{104}{9}(4)^{n+1} - \frac{5}{3}(n+1) - \frac{41}{9} = RS$$

$$LS = RS$$

8

(2C.0) BC $\forall n \in \mathbb{N}\ S(\varphi(n))$

$\frac{104}{9}4^n - \frac{5}{3}n - \frac{41}{9}$

(2C.1) Rcs

LS

$= S(\varphi, n+1)$

$= S(\varphi, n) + \varphi(n+1)$ (Rcs)

$= \frac{104}{9}4^n - \frac{5}{3}n - \frac{41}{9} + 7(4)^{n+1} - \frac{5}{3}(1 - 4^{n+1})$ (P(n))

$= \frac{104}{9}4^n - \frac{5}{3}n - \frac{41}{9} + 7(4)(4^n) - \frac{5}{3} + \frac{5}{3}(4)(4)^n$

$= (\frac{104}{9} + 28 + \frac{20}{3})4^n - \frac{5}{3}n - \frac{5}{3} - \frac{41}{9}$

$= (\frac{104}{9} + \frac{252}{9} + \frac{60}{9})4^n - \frac{5}{3}(n+1) - \frac{41}{9}$

$= \frac{416}{9}4^n - \frac{5}{3}(n+1) - \frac{41}{9}$

$= \frac{104}{9}(4)(4^n) - \frac{5}{3}(n+1) - \frac{41}{9}$

$= \frac{104}{9}(4)^{n+1} - \frac{5}{3}(n+1) - \frac{41}{9}$

$= RS$

$LS = \frac{104}{9}(4)^{n+1} - \frac{5}{3}(n+1) - \frac{41}{9} = RS$

$LS = RS$

6 (3a) Define the function $\alpha: \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ 'recursively' as follows:

$$\forall n \in \mathbb{N}, \alpha(n, b, c, d) := \begin{cases} \alpha(cn - b, b, c, d) + d & if\, cn \geq b \\ 0 & if\, cn < b \end{cases}$$

Rewrite the definition above with your values of a and b:

My problem:

$$\forall n \in \mathbb{N}, \alpha(n, 7, 4, 5) := \begin{cases} \alpha((4)n - 7, 7, 4, 5) + & if\,(4)\,n \geq \\ 0 & if\,(4)\,n < 7 \end{cases}$$

Compute **showing complete calculations** on the facing page:

1 (i) $\alpha(0, b, c, d)$

$= 0$

1 (ii) $\alpha(1, b, c, d)$

$= 0$

1 (iii) $\alpha(10, b, c, d)$

= Does not make sense

1 (iv) $\alpha(100, b, c, d)$

= Does not make sense

1 (v) $\alpha(1000, b, c, d)$

= Does not make sense

1(vi) Make a conjecture regarding what the function α calculates and prove your conjecture.

Conjecture : Does not make sense

3

You may discuss and work with anyone

3a.0 $cn-b = 4n-7$	3a.1 $b = 7$	3a.2 $c = 4$	$= 5$

3a.i.0 $0(4) = 0 < 7$

3a.i.1 $d(0,b,c,d)$
$= d(0,7,4,5)$
$= 0$

3a.ii.0 $1(4) = 4 < 7$

3a.ii.1 $d(1,b,c,d)$
$= d(1,7,4,5)$
$= 0$

3a.iii.0 $10(4) = 40 \not< 7$

3a.iv.0 $100(4) = 400 \not< 7$

3a.v.0 $1000(4) = 4000 \not< 7$

6 (3b) Prove, following the method shown in class, that: $(a+b+1)^{\frac{1}{a+b}}$ **is irrational,** that is: $\cancel{Rtl}\left((a+b+1)^{\frac{1}{a+b}}\right)$.

First rewrite the problem using your values of a and b.

My Problem:

$$\cancel{Rtl}\left(((2)+(7)+1)^{\frac{1}{(2)+(7)}}\right)$$

that is: $\cancel{Rtl}\left((10)^{\frac{1}{(9)}}\right)$

$\cancel{Rtl}\,(\sqrt[9]{10})$

6

3b.0 $(10)^{\frac{1}{9}}$

$= \sqrt[9]{10}$

3b.1 <u>Pf</u>

$\neg(\cancel{Rtl}(\sqrt[9]{10}))$

$Rtl(\sqrt[9]{10})$

$\sqrt[9]{10} \in \mathbb{Q} \quad \sqrt[9]{10} \neq 0$

$\exists p \in \mathbb{N}\setminus\{0\} \quad \exists q \in \mathbb{N}\setminus\{0\} \quad \sqrt[9]{10} = \frac{p}{q} \quad gcd(p,q) = 1$

$\frac{p}{q} = \sqrt[9]{10}$

$\times q \qquad \times q$

$(\cancel{q})\frac{p}{\cancel{q}} = \sqrt[9]{10}\,(q)$

$p = \sqrt[9]{10}\,q$

$(\ )^9 \qquad (\ )^9$

$p^9 = (\sqrt[9]{10}\,q)^9$

$p^9 = 10q^9$

$10 \mid p^9$

$(2)(5) \mid p^9 \quad Prm(2) \quad Prm(5)$

$2 \mid p \quad 5 \mid p$

$\exists r \in \mathbb{R} \quad p = 2r$

$10 \mid p$

$\exists r \in \mathbb{R} \quad \frac{p}{r} = 10$

$\times r \qquad \times r$

$p = 10r$

$(\ )^9 \qquad (\ )^9$

$p^9 = 10^9 r^9 \qquad p^9 = 10q^9$

$10q^9 = 10^9 r^9$

$\div 10 \qquad \div 10$

$q^9 = 10^8 r^9$

$q^9 = 10(10^7)r^9$

$10 \mid q^9$

$(2)(5) \mid q^9 \quad Prm(2) \quad Prm(5)$

$2 \mid q \quad 5 \mid q$

$10 \mid q \qquad 10 \mid p$

$10 \mid (gcd(p,q)) \quad gcd(p,q) = 1$

$10 \mid 1$

$\perp$

$\neg(\neg(\cancel{Rtl}(\sqrt[9]{10})))$

$\cancel{Rtl}(\sqrt[9]{10})$

5 (3c) Consider the numbers defined below, in which the dot indicates a decimal point, and the string in parentheses is the repeating part of a repeating decimal.

(A) $x := .(ab)$ $= .(27)$

(B) $y := .aababbabbba\ldots$ $= .227277277772\ldots$,

where for every $\forall n \in \mathbb{N}$ there are n b's between the $(n + 1)^{st}$ and the $(n+2)^{nd}$ a.

2 (i) Find a recursive definition for x and prove that: $Rtl(x)$, that is, **x is rational.**

$$x_n := x_{n-1} + \frac{27}{10^{2(n+1)}}$$

2

3c.i.0

$x = .(27)$

$\times 100 \quad \times 100$

$100x = 27.(27)$

$100x = 27 + .(27)$

$100x = 27 + x$

$-x \quad -x$

$99x = 27$

$\div 99 \quad \div 99$

$\frac{\cancel{99}}{\cancel{99}}x = \frac{\cancel{27}^{3}}{\cancel{99}_{11}}$

$x = \frac{3}{11} \quad 3 \in \mathbb{Z} \quad 11 \in \mathbb{N} \setminus \{0\}$

$x \in \mathbb{Q}$

Rtl (x)

3c.i.1

$x = .ab = .27$

$x_0 = .27$

$x_1 = .2727 = .27 + .0027 = .27 + \frac{27}{10000} = .27 + \frac{27}{10^4} = x_0 + \frac{27}{10^4}$

$x_2 = .272727 = .2727 + .000027 = .2727 + \frac{27}{1000000} = x_1 + \frac{27}{10^6}$

$x_3 = .27272727 = .272727 + \frac{27}{100000000} = .272727 + \frac{27}{10^8} = x_2 + \frac{27}{10^8}$

$\vdots$

$x_n = x_{n-1} + \frac{27}{10^{2(n+1)}}$

BC $\quad x_0 = .27$

RCS $\quad \forall n \in \mathbb{N}$

$x_n := x_{n-1} + \frac{27}{10^{2(n+1)}}$