(1) ⓪

$$\dfrac{\dfrac{\dfrac{\varphi(n+4) = (n+4)!}{\varphi(0+4) = (0+4)!}\langle n \leftarrow 0 \rangle}{\varphi(4) = 24} \qquad \dfrac{\dfrac{\psi(n+4) = 2^{n+4}}{\psi(0+4) = 2^{0+4}}\langle n \leftarrow 0 \rangle}{\psi(4) = 16}}{\dfrac{\varphi(4) = 24 \geq 16 = \psi(4)}{\varphi(4) \geq \psi(4)}}$$

(1) ① $\underline{\underline{\mathrm{BC}}}$ $\varphi(4) = 24$

$\underline{\underline{\mathrm{RcS}}}$ $\forall n \in \mathbb{N}\ (n \geq 4)\ \varphi(n+1) = (n+1)\varphi(n)$

(1) ②

$\underline{\underline{\mathrm{BC}}}$ $\psi(4) = 16$

$\underline{\underline{\mathrm{RcS}}}$ $\forall n \in \mathbb{N}\ (n \geq 4)\ \psi(n+1) = 2(\psi(n))$

(1) ③ $\forall n \in \mathbb{N}(n \geq 4)\ P(n) = \left(\varphi(n) \geq \psi(n)\right)$

(1) ④ $\underline{\underline{\mathrm{BC}}}$

$$\dfrac{\varphi(4) \geq \psi(4)}{P(4)}$$

(1) ⑤ $\underline{\underline{\mathrm{IS}}}$

$$\dfrac{\dfrac{\forall n \in \mathbb{N}(n \geq 4),\ P(4), P(5) \cdots P(n)}{\begin{aligned} &\varphi(n+1) \quad (\underline{\mathrm{RcS}}) \\ &= (n+1)\varphi(n) \\ &\geqq (n+1)\psi(n) \quad (\underline{\mathrm{IH}}) \\ &= (n+1) \cdot 2^n \\ &\geq (2) \cdot 2^n \\ &= 2^{n+1} \\ &= \psi(n+1) \end{aligned}}}{\dfrac{\forall n \in \mathbb{N}(n \geq 4)\ \varphi(n+1) \geq \psi(n+1)}{P(n+1)}}$$

(1) ⑥

$$\dfrac{\dfrac{P(4), \quad \left(\dfrac{P(4), P(5) \cdots P(n)}{P(n+1)}\right)}{\forall n \in \mathbb{N}\ (n \geq 4)\quad P(n)}}{\forall n \in \mathbb{N}\ \varphi(n+4) \geq \psi(n+4)}$$

2 (1) Find (and recall) a recursive definition for each of the functions below

$$\varphi := n \mapsto n! : \mathbb{N} \longrightarrow \mathbb{N} \quad \text{and}$$

$$\psi := n \mapsto 2^n : \mathbb{N} \longrightarrow \mathbb{N}$$

Prove by induction that: $\forall n \in \mathbb{N} \;\; \varphi(n+4) \geq \psi(n+4)$

BC $\quad \varphi(0) = 1$

RcS $\quad \forall n \in \mathbb{N} \quad \varphi(n+1) = (n+1)(\varphi(n))$

BC $\quad \psi(0) = 1$

RcS $\quad \forall n \in \mathbb{N} \quad \psi(n+1) = 2(\psi(n))$

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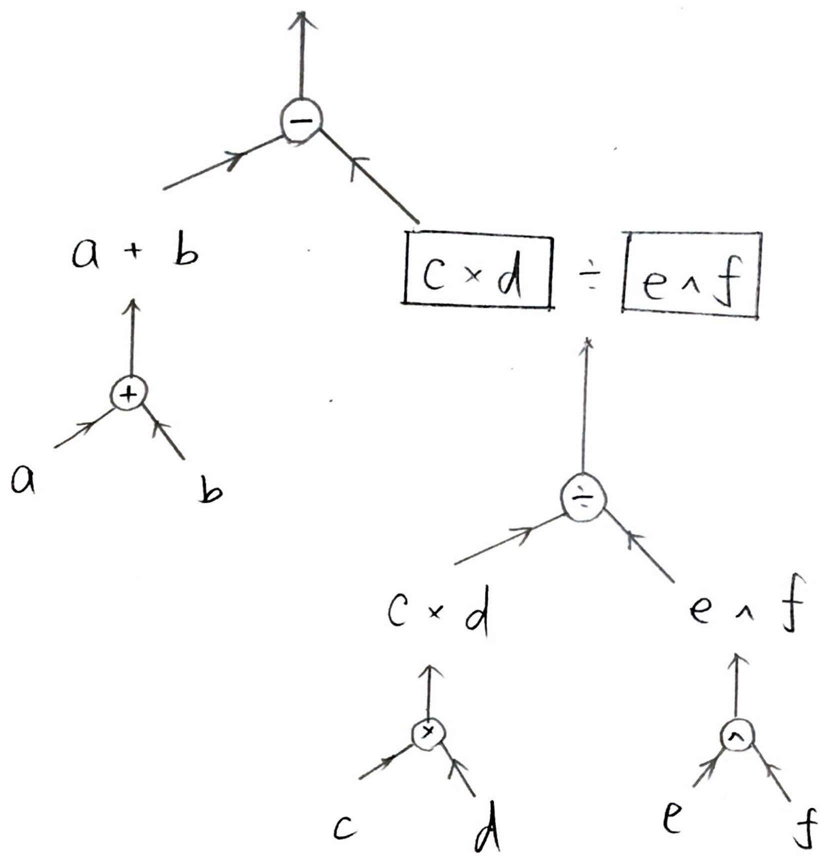
(2) ⓪ <u>TREE</u>

$$v(S) = a + b - c \times d \div e \wedge f$$

$$= \boxed{a+b} - \boxed{\boxed{c \times d} \div \boxed{e \wedge f}}$$



(2) ①

$a := v(A)$

$b := v(B)$

$c := v(C)$

$d := v(D)$

$e := v(E)$

$f := v(F)$

(2) ②

$A \cap B = \{\}$

(2) ③

$\left(C \times D \div \mathrm{Fnc}(F, e)\right) \leq (A \cap B)$

(2) ④

$$\frac{A \cap B = \{\}}{A \cup B \approx A \amalg B}$$

2 (2) Starting with sets A, B, C, D, E and F with $a := \nu(A)$, $b := \nu(B), a := \nu(C), d := \nu(D), e := \nu(E)$ and $f := \nu(F)$, use as many of the operations in the list **below** as many times as you wish:

$$\{(\ )^c, \cup, \cap, \setminus, \prod\ , \coprod\ , Fnc(A, B), (\ ), \tilde{-}\}$$

(i) **to construct a set S whose cardinality appears below,** and

$$\nu(S) = a + b - c \times d \div e \wedge f$$

(ii) **to construct word-problems whose solutions require counting the number of elements of the sets you constructed in (i).**

Note that the numbers are such that $\nu(S) \in \mathbb{N}$

Draw a formation-tree for the expression on the facing side.

Draw a formation-tree for the expression that you have constructed.

Prove that the cardinality of the set that you have constructed equals the given expression.

Write the word problem step by step respecting the formation-tree of the expression that you have constructed.

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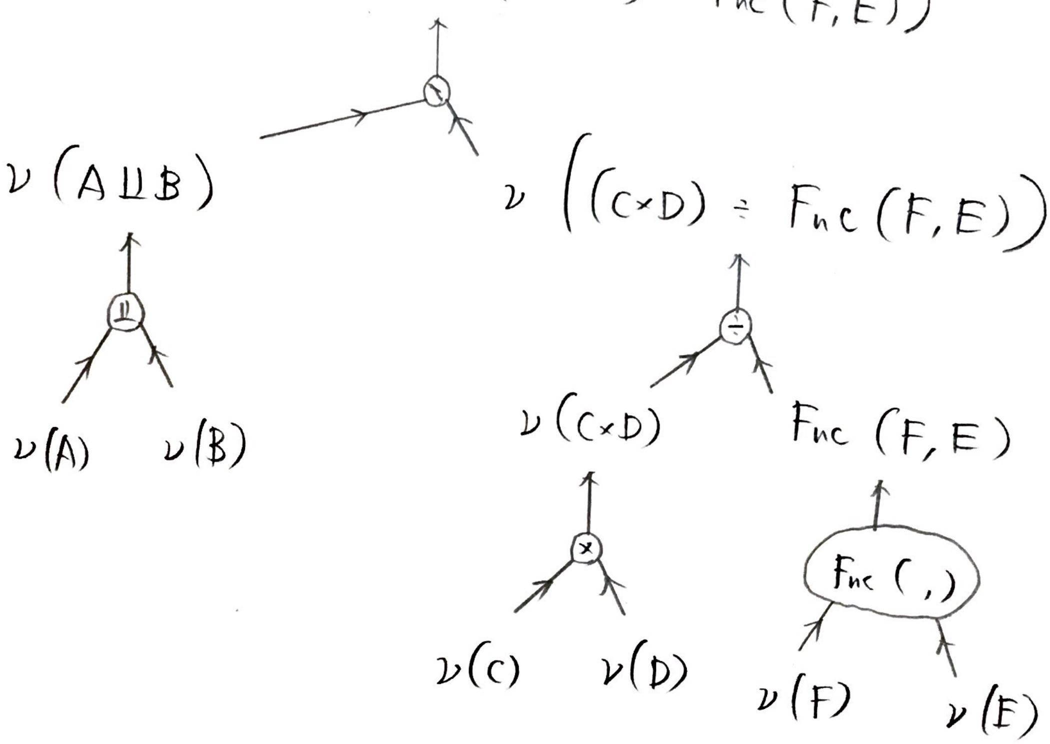
(2) ⑤ $\nu(S)$

$$
\begin{aligned}
&= a + b - c \times d \div e \wedge f \\
&= \nu(A) + \nu(B) - \nu(C) \times \nu(D) \div \nu(E) \wedge \nu(F) \\
&= \nu(A \amalg B) - \nu(C) \times \nu(D) \div \mathrm{Fnc}(F, E) \\
&= \left(\nu(A \amalg B)\right) - \left(\nu(C \times D) \div \mathrm{Fnc}(F, E)\right) \\
&= \nu\left((A \amalg B) \setminus \left((C \times D) \div \mathrm{Fnc}(F, E)\right)\right)
\end{aligned}
$$

---

(2) ⑥ <u>TREE</u>

$\nu\left((A \amalg B) \setminus \left((C \times D) \div \mathrm{Fnc}(F, E)\right)\right)$

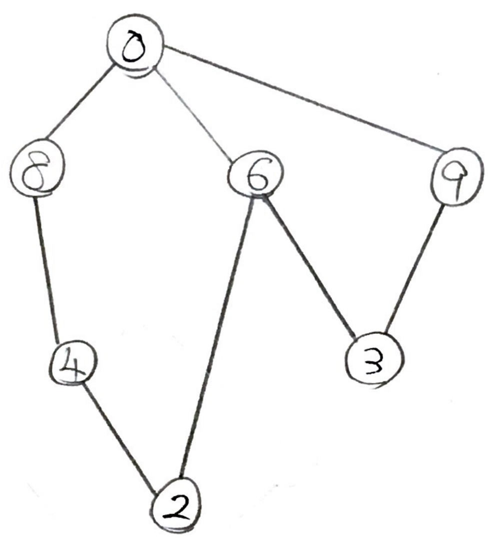


(3) ⓪

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall a \in \mathbb{N}}{((a,a)\in \mathbb{N}) \wedge a|a}}{aRa}}{\forall a \in \mathbb{N}\ (aRa)}}{Rflx(R)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\forall a,b \in \mathbb{N},\ aRb,\ bRa}{(a|b) \wedge (b|a)}}{a=b}}{\forall a,b \in \mathbb{N} \left(\dfrac{(aRb)\wedge(bRa)}{a=b}\right)}}{Asym(R)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall d,e,f,\ dRe,\ eRf}{(d|e) \wedge (e|d)}}{d|f}}{dRf}}{\forall d,e,f \left(\dfrac{(dRe)\wedge(eRf)}{dRf}\right)}}{Trns(R)}}{PO(R,\mathbb{N})}$$

---

(3) ① J = 00298436



$$HsDgrmD\left(\left(\{J_k \mid k \in 1..8\},\ |\right)\right)$$

2 (3) Define a relation on $\mathbb{N}$ as follows:

$$\forall\, x, y \in \mathbb{N} \quad (xRy) :\Leftrightarrow (x|y)$$

Prove that: $PO(R, \mathbb{N})$

Draw the Hasse Diagram for the $\left\{ J_k \middle| k \in 1..8 \right\}$ under divisibility ordering|,

that is: $$HsDgrmD\left(\left(\left\{ J_k \middle| k \in 1..8 \right\}, |\right)\right)$$

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(4) 6

$\forall x, y \in \mathbb{Z} \quad (x R y) :\Longleftrightarrow (x^2 = y^2)$

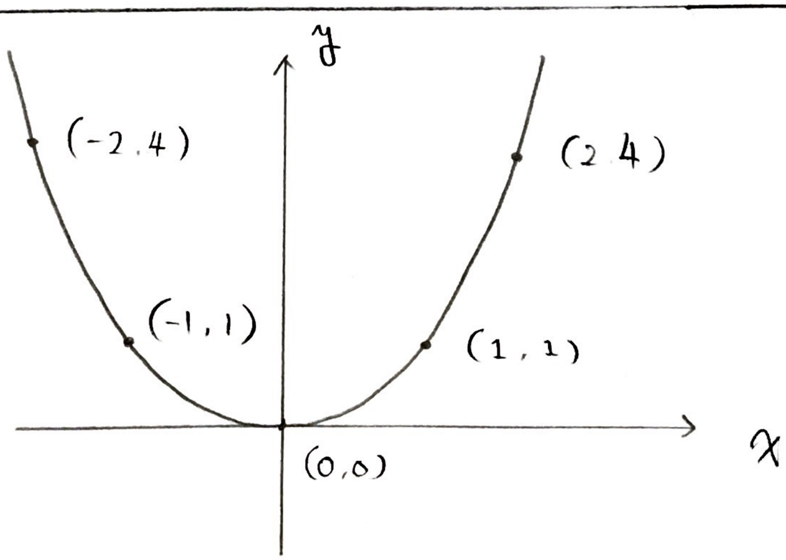
$x^2 = y^2$	x	y	x^2	y^2
0	0	0	0^2	0^2
1	1	1	1^2	1^2
	(-1)	(-1)	$(-1)^2$	$(-1)^2$
4	2	2	2^2	2^2
	(-2)	(-2)	$(-2)^2$	$(-2)^2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$\frac{\mathbb{Z}}{R} = \bigcup_{k \in 1..n \in \mathbb{N}} \left\{ \{-k, k\} \right\}$$

$$= \{0\} \cup \{-1, 1\} \cup \{-2, 4\} \cup \dots \cup \{-n, n^2\}$$

$Eqv(R, \mathbb{Z})$

(4) 1



2 (4) Define a relation on $\mathbb{Z}$ as follows:

$$\forall\, x, y \in \mathbb{Z} \quad (xRy) :\Leftrightarrow (x^2 = y^2)$$

Prove that: $Eqv(R, \mathbb{Z})$ and find the equivalence classes.

(5) ⓐ

$$\forall (a,b),(c,d) \in \mathbb{N}\times\mathbb{N} \left(\left((a,b) R (c,d) \right) :\Leftrightarrow \left(a+b = c+d \right) \right)$$

$x+y$	x	y
0	0	0
1	0	1
	1	0
2	0	2
	1	1
⋮	2	0
	⋮	⋮

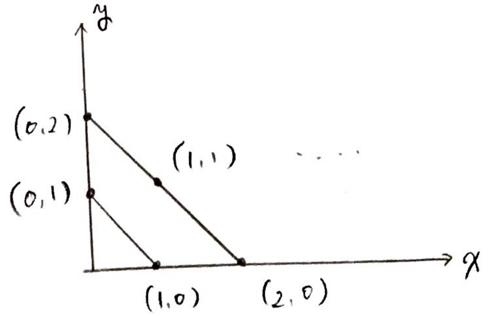
$$\frac{\mathbb{N}\times\mathbb{N}}{R} = \{0,0\} \cup \{(0,1),(1,0)\} \cup \{(0,2),(1,1),(2,0)\} \cup \ldots$$

$$\cup \{(0,n),(1,n-1),\ldots,(n,0)\}$$

$$= \bigcup_{k\in 0..n} \{(k,n-k)\}$$

$$Eqv(R, \mathbb{N}\times\mathbb{N})$$

(5) ①



2 (5) Define a relation on $\mathbb{N} \times \mathbb{N}$ as follows:

$$\forall\,(a,b),(c,d) \in \mathbb{N} \times \mathbb{N} \quad \Big((a,b)R(c,d)\Big) :\Leftrightarrow \Big(a+b=c+d\Big)$$

Prove that: $Eqv(R, \mathbb{N} \times \mathbb{N})$ and find the equivalence classes.

(6) ⓪

$$\forall (a,b),(c,d) \in \mathbb{N}\times\mathbb{N} \left(\left((a,b) R (c,d)\right) :\Leftrightarrow \left(a^2+b^2 = c^2+d^2\right) \right)$$

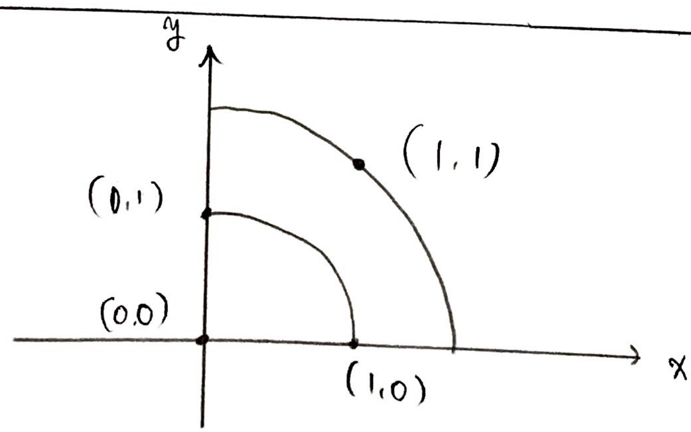
x^2+y^2	x	y	x^2	y^2
0	0	0	0^2	0^2
1	0	1	0^2	1^2
	1	0	1^2	0^2
2	1	1	1^2	1^2
4	0	2	0^2	2^2
	2	0	2^2	0^2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$\frac{\mathbb{N}\times\mathbb{N}}{R} = \{0,0\} \cup \{(0,1),(1,0)\} \cup \dots \cup \{(n,n)\}$$

$$= \bigcup_{k=0..n} \{(k,n)\}$$

$$Eqv(R, \mathbb{N}\times\mathbb{N})$$

(6) ①



2 (6) Define a relation on $\mathbb{N} \times \mathbb{N}$ as follows:

$$\forall (a,b),(c,d) \in \mathbb{N} \times \mathbb{N} \quad \Big((a,b)R(c,d)\Big) :\Leftrightarrow \Big(a^2 + b^2 = c^2 + d^2\Big)$$

Prove that: $Eqv(R, \mathbb{N} \times \mathbb{N})$ and find the equivalence classes.

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(T) ⓪

BC $\rho(0) = 0,\ \rho(0) = 1$

RcS $\forall n \in \mathbb{N}$

$$\rho(n+2) - \cancel{\rho(n+1)} - \rho(n) = 0$$
$$\cancel{+\rho(n+1)} \qquad + \rho(n+1)$$

$$\rho(n+2) - \cancel{\rho(n)} = \rho(n+1)$$
$$\cancel{+\rho(n)} \qquad + \rho(n)$$

$$\rho(n+2) = \rho(n+1) + \rho(n)$$

(T) ①

$$p(x^2) - q(x) - r = 0$$

$$(1)(x^2) - (1)(x) - (1) = 0 \quad \left\langle \begin{matrix} p \leftarrow 1 \\ q \leftarrow 1 \\ r \leftarrow 1 \end{matrix} \right\rangle$$

$$x^2 - x - 1 = 0$$

$$\left(x = \frac{1-\sqrt{5}}{2}\right) \vee \left(x = \frac{1+\sqrt{5}}{2}\right)$$

$$\left(r_1 = \frac{1-\sqrt{5}}{2}\right) \neq \left(r_2 = \frac{1+\sqrt{5}}{2}\right)$$

$$r_1 \neq r_2$$

$$\forall n \in \mathbb{N}\ \rho(n) = f\left(\frac{1-\sqrt{5}}{2}\right)^n + g\left(\frac{1+\sqrt{5}}{2}\right)^n$$

2 (7) Find a formula for the Fibonacci Numbers by solving the recurrence below. The Fibonacci numbers are defined recursively as follows:

$$\begin{cases} & \rho(0) := 0 \\ & \rho(1) := 1 \\ \forall n \in \mathbb{N} & \rho(n+2) - \rho(n+1) - \rho(n) = 0 \end{cases}$$

(7) ② $\rho(n) = f\left(\frac{1-\sqrt{5}}{2}\right)^n + g\left(\frac{1+\sqrt{5}}{2}\right)^n$ $\langle n \leftarrow 0 \rangle$

$\rho(0) = f\left(\frac{1-\sqrt{5}}{2}\right)^0 + g\left(\frac{1+\sqrt{5}}{2}\right)^0$

$0 = f + \cancel{g}$

$-g \qquad \cancel{-g}$

$-g = f$

(7) ③ $\rho(n) = f\left(\frac{1-\sqrt{5}}{2}\right)^n + g\left(\frac{1+\sqrt{5}}{2}\right)^n$ $\langle n \leftarrow 1 \rangle$

$\rho(1) = f\left(\frac{1-\sqrt{5}}{2}\right)^1 + g\left(\frac{1+\sqrt{5}}{2}\right)^1$

$1 = f\left(\frac{1-\sqrt{5}}{2}\right) + g\left(\frac{1+\sqrt{5}}{2}\right)$, $-g = f$

$1 = -g\left(\frac{1-\sqrt{5}}{2}\right) + g\left(\frac{1+\sqrt{5}}{2}\right)$

$1 = g\left(\frac{\cancel{1}+\sqrt{5} \cancel{-1} + \sqrt{5}}{2}\right)$

$1 = g(\sqrt{5})$

$1 = \cancel{\sqrt{5}}\, g$

$\div\sqrt{5} \qquad \cancel{\div\sqrt{5}}$

$\frac{1}{\sqrt{5}} = g$

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(7) ④ $\frac{1}{\sqrt{5}} = g, \quad -g = f$

$$f = -\frac{1}{\sqrt{5}}, \quad g = \frac{1}{\sqrt{5}}$$

---

(7) ⑤ $P(n) = f\left(\frac{1-\sqrt{5}}{2}\right)^n + g\left(\frac{1+\sqrt{5}}{2}\right)^n, \quad f = -\frac{1}{\sqrt{5}}, \quad g = \frac{1}{\sqrt{5}}$

$$P(n) = \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$

---

(7) ⑥

$$P(0) = \left(P(0) = \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^0 - \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^0\right)$$

$$P(1) = \left(P(1) = \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^1\right)$$

$$P(n+2) = \left(P(n+2) = \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right)$$

---

(7) ⑦ BS

(0) LS

$= P(0)$

$= 0$

$= \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^0 - \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^0$

$=$ RS

$P(0)$

(1) LS

$= P(1)$

$= 1$

$= \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^1$

$=$ RS

$P(1)$

(7) 8

IS

$$\frac{P(0),\ P(1),\ \cdots\ P(n+1)}{P(n+2)}$$

LS

$= P(n+2)$ (RS)

$= P(n+1) + P(n)$ (IH)

$= \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} + \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{n}$

$= \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n}\left(\frac{1+\sqrt{5}}{2} + 1\right) - \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{n}\left(\frac{1-\sqrt{5}}{2} + 1\right)$

$= \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{2+2\sqrt{5}}{4} + \frac{4}{4}\right) - \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{n}\left(\frac{2-2\sqrt{5}}{4} + \frac{4}{4}\right)$

$= \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n}\left(\frac{1+2\sqrt{5}+5}{4}\right) - \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{n}\left(\frac{1-2\sqrt{5}+5}{4}\right)$

$= \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n}\left(\frac{1+\sqrt{5}}{2}\right)^{2} - \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{n}\left(\frac{1-\sqrt{5}}{2}\right)^{2}$

$= \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{n+2}$

$= RS$

$$\overline{P(n+2)}$$

(17) 9

$$\frac{P(0),\ P(1),\ \dfrac{P(0),\ P(1)\ \cdots\ P(n+1)}{P(n+2)}}{\forall n \in \mathbb{N}\quad P(n)}$$

(8) ⓪

BC $S(2,0) = \sum_{k=0}^{0} k^2$

$= 0$

RcS $S(2,n+1) = \sum_{k=0}^{n+1} k^2$

$= (n+1)^2 + \sum_{k=0}^{n} k^2$

$= (n+1)^2 + S(2,n)$

---

(8) ①

$P(0) := \left(S(2,0) = \frac{0(0+1)(2(0)+1)}{6} \right)$

$P(n+1) := \left(S(2,n+1) = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6} \right)$

---

(8) ②

BS LS

$= S(2,0)$ (BC)

$= 0$

$= \frac{0(0+1)(2(0)+1)}{6}$

$=$ RS

$P(0)$

2 (8) Prove by induction that:

$$S(2,n) \quad := \quad \sum_{k=0}^{n} k^2 \quad = \quad \frac{n(n+1)(2n+1)}{6}$$

(8) 3

IS

$$\frac{P(0),\ P(1) \ldots\ P(n)}{LS}$$

$$= S(2, n+1) \quad (\underline{RcS})$$

$$= (n+1)^2 + S(2,n) \quad (\underline{IH})$$

$$= (n+1)^2 + \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{6(n+1)(n+1) + n(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(6n+6+2n^2+n)}{6}$$

$$= \frac{(n+1)(n+2)(2n+3)}{6}$$

$$= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$$

$$= \frac{RS}{P(n+1)}$$

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(8) (4)

$$\frac{\frac{P(0), \quad \left(\frac{P(0),\ P(1),\ \cdots\ P(n)}{P(n+1)}\right)}{\forall n \in \mathbb{N} \quad P(n)}}{\forall n \in \mathbb{N} \quad S(2,n) = \frac{n(n+1)(2n+1)}{6}}$$

(9) ⓐ

BC $S(3,0) = \sum_{k=0}^{0} k^3$

$= 0$

RcS $S(3, n+1) = \sum_{k=0}^{n+1} k^3$

$= (n+1)^3 + \sum_{k=0}^{n} k^3$

$= (n+1)^3 + S(3, n)$

---

(9) ①

$P(0) = \left(S(3,0) = \left(S(1,0)\right)^2 \right)$

$P(n+1) = \left(S(3,n+1) = \left(S(1,n+1)\right)^2 \right)$

---

(9) ② BS

LS

$= S(3,0)$ (BC)

$= 0$

$= \left(\frac{1}{2}(0)(0+1)\right)^2$

$= \left(S(1,0)\right)^2$

$= RS$

$P(0)$

2 (9) Use the method of differences shown in class to find a recursive definition of:

$$S(3,n) := \sum_{k=0}^{n} k^3$$

Prove by induction that:

$$S(3,n) := \sum_{k=0}^{n} k^3 = \left(S(1,n)\right)^2$$

(9) 3 IS

$P(0), P(1), \ldots P(n)$

LS

$= S(3, n+1)$ (R.S)

$= (n+1)^3 + S(3, n)$ (IH)

$= (n+1)^3 + (S(1,n))^2$

$= (n+1)^3 + (\frac{1}{2} n(n+1))^2$

$= (n+1)(n^2 + 2n + 1) + \frac{1}{4}(n+1)(n^3 + n^2)$

$= \frac{1}{4}(n+1)(4n^2 + 8n + 4 + n^3 + n^2)$

$= \frac{1}{4}(n+1)(n+1)(n^2 + 4n + 4)$

$= \frac{1}{4}(n+1)^2(n+2)^2$

$= (\frac{1}{2}(n+1)(n+2))^2$

$= (S(1, n+1))^2$

$=$ RS

$P(n+1)$

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(9) ④

$$\dfrac{\dfrac{P(0), \quad \dfrac{P(0), P(1), \cdots P(n)}{P(n+1)}}{\forall n \in \mathbb{N}\ P(n)}}{\forall n \in \mathbb{N}\ S(3,n) = \left(S(1,n)\right)^2}$$

(10) ⓪

BC $\beta(0) = (0)!$

$= 0$

RcS $\beta(n+1) = (n+1)!$

$= (n+1)(n)!$

$= (n+1)\,\beta(n)$

BC $\gamma(0) = 0^0$

$= 1$

RcS $\gamma(n+1) = (n+1)^{n+1}$

$= \sum_{k \in 0..n+1} \binom{n+1}{k} n^k$

$= \sum_{k \in 0..n-1} \binom{n+1}{k} n^k$

$+ \binom{n+1}{n} n^n$

$+ \binom{n+1}{n+1} n^{n+1}$

$= \sum_{k \in 0..n-1} \binom{n+1}{k} n^k + (n+1)\,\gamma(n) + n\,\gamma(n)$

$= \sum_{k \in 0..n+1} \binom{n+1}{k} n^k + (2n+1)\,\gamma(n)$

(10) ①

$$(a+b)^n = \sum_{k \in a..n} \binom{n}{k} a^{n-k} b^k$$

$\langle a \leftarrow 1,\ b \leftarrow n,\ n \leftarrow n+1 \rangle$

$$(n+1)^{n+1} = \sum_{k \in a..n+1} \binom{n+1}{k} n^k$$

(10) ②

$$P(0) = (\gamma(0) \geq \beta(0))$$

$$P(n+1) = (\gamma(n+1) \geq \beta(n+1))$$

2 (10) Find (and recall) a recursive definition for each of the functions below

$$\beta := n \mapsto n! : \mathbb{N} \longrightarrow \mathbb{N} \quad \text{and}$$

$$\gamma := n \mapsto n^n : \mathbb{N} \longrightarrow \mathbb{N}$$

Prove by induction that: $\forall n \in \mathbb{N} \;\; \gamma(n) \geq \beta(n)$

(10) 3

BS

LS

$= \gamma(0)$ (BC)

$= 1$

≥ 0

$= \beta(0)$

$= RS$

$$\frac{LS = \gamma(0) \geq \beta(0) = RS}{P(0)}$$

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(10) ④

IS $\quad P(0), P(1), P(2), \ldots P(n)$

LS

$= \gamma(n+1) \quad$ (RcS)

$= \sum_{k=0}^{n+1} \binom{n+1}{k} n^k + (2n+1)\gamma(n) \quad$ (IH)

$\geq \sum_{k=0}^{n+1} \binom{n+1}{k} n^k + (2n+1)\beta(n)$

$= \sum_{k=0}^{n+1} \binom{n+1}{k} n^k + (2n+1)(n!)$

$\geq (2n+1)(n!)$

$\geq (n+1)(n!)$

$= (n+1)!$

$= \beta(n+1)$

$=$ RS

$LS = \gamma(n+1) \geq \beta(n+1) = RS$

$P(n+1)$

(10) 5

$$\frac{\dfrac{P(0), \quad \left(\dfrac{P(0), P(1), \cdots P(n)}{P(n+1)}\right)}{\forall n \in \mathbb{N} \quad P(n)}}{\forall n \in \mathbb{N} \left(\gamma(n) \geq \beta(n)\right)}$$

(11)

P	q	PC_1q	PC_2q	PC_3q	PC_4q	PC_5q	PC_6q	PC_7q	PC_8q
0	0	0	0	0	0	1	0	1	1
0	1	0	0	0	1	0	0	1	0
1	0	0	0	1	0	0	1	0	1
1	1	0	1	0	0	0	1	0	0

PC_9q	$PC_{10}q$	$PC_{11}q$	$PC_{12}q$	$PC_{13}q$	$PC_{14}q$	$PC_{15}q$	$PC_{16}q$
0	0	1	1	1	1	0	1
1	1	0	1	1	0	1	1
0	1	0	1	0	1	1	1
1	0	1	0	1	1	1	1

2 (11) Find truth-tables for each of 16 binary connectives $\left(c_k \middle| k \in 1..(16)\right)$ for Boolean Logic and express each formula pc_kq in words.

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2 (12) Define that set of truth-values $TV := \left\{-1, 0, 1\right\}$

(i) Define truth-tables for negation, disjunction, conjunction, and implication and justify your definitions on the basis of your interpretation of the truth-values.

(ii) How many types arise as analogues of the types: tautology, contradiction, and contingency, and how can they be interpreted?

(iii) Is implication definable in terms on negation and disjunction in your system?

(13) ⓪

$$\neg(p \to r)$$

$$= \neg((\neg p) \vee r) \qquad ((x \to y) = (\neg x \vee y)$$

$$= (\neg(\neg p) \wedge (\neg r)) \qquad (\neg(x \vee y) = (\neg x \vee \neg y)$$

$$= p \wedge (\neg r)$$

(13) ①

$$\neg(p \leftrightarrow q) \qquad ((x \leftrightarrow y) = (x \to y) \wedge (y \to x))$$

$$= \neg((p \to q) \wedge (q \to p)) \qquad (\neg(x \wedge y) = (\neg x \vee \neg y))$$

$$= (\neg(p \to q) \vee \neg(q \to p))$$

$$= ((p \wedge (\neg q)) \vee (q \wedge (\neg p))$$

(13) ②

$$\neg(r \leftrightarrow s) \qquad (\neg(x \leftrightarrow y) = ((x \wedge (\neg y)) \vee (y \wedge (\neg x))$$

$$= ((r \wedge (\neg s)) \vee (s \wedge (\neg r))$$

⑬ ③

$$((p \leftrightarrow q) \wedge (p \to r) \wedge (r \leftrightarrow s) \to (q \to s)$$

$$= \neg((p \leftrightarrow q) \wedge (p \to r) \wedge (r \leftrightarrow s)) \vee ((\neg q) \vee s)$$

$$= (\neg(p \leftrightarrow q) \vee (\neg(p \to r)) \vee (\neg(r \leftrightarrow s))) \vee ((\neg q) \vee s)$$

$$= (\neg q) \vee ((\neg p) \vee (p \wedge (\neg q)) \vee (p \wedge (\neg r)) \vee s \vee r \vee (s \wedge (\neg r))$$

$$= (\neg q) \vee (\neg p) \vee s \vee r \vee (p \wedge (\neg r))$$

$$= (\neg q) \vee s \vee r \vee ((\neg p) \vee p) \qquad ((\neg x) \vee x = T)$$

$$= (\neg q) \vee s \vee r \vee T \qquad (x \vee T = T)$$

$$= T$$

2 (13) Define a relation Q on the set of propositions Prp in Propositional Calculus as follows:

$$\forall p, q \in Prp \quad pQq :\Leftrightarrow (p \leftrightarrow q)$$

(i) Prove that Q is and equivalence relation on Prp, that is, $Eqv(Q)$.

Define $\mathbb{F} := \frac{Prp}{Q}$, the set of equivalence classes $EqvCls(Prp, Q)$

(ii) Prove that: $\forall p, q, r, s \in Prp \quad \left(\frac{pQq \quad p \longrightarrow r \quad rQs}{q \longrightarrow s}\right)$

(iii) Define, in view of (ii) a relation $\mathcal{E}$ on $\mathbb{F}$ as follows:

$$\forall F, G \in \mathbb{F} \qquad F\mathcal{E}G :\Leftrightarrow (F \Longleftarrow\!\!\Rightarrow G)$$

noting that $\Rightarrow$ means that every proposition in the equivalence class F implies (in the ordinary sense $\longrightarrow$ of PC) every proposition in the equivalence class G.

(iv) Prove that $PO(\mathcal{E}, \mathbb{F})$, that is, $\mathcal{E}$ is a partial order on $\mathbb{F}$.

2 (14) Consider the functions:

$$\alpha := n \mapsto (n^n)! : \mathbb{N} \longrightarrow \mathbb{N}$$

$$\beta := n \mapsto (n!)^n : \mathbb{N} \longrightarrow \mathbb{N}$$

$$\gamma := n \mapsto (n!)! : \mathbb{N} \longrightarrow \mathbb{N}$$

$$\delta := n \mapsto (n)^{n!} : \mathbb{N} \longrightarrow \mathbb{N}$$

Formulate a conjecture about the eventual **behaviour** of these functions with respect to value-wise ordering and **prove** your conjecture by induction.

$n = 0 \; (\alpha(0) = \beta(0) = \gamma(0) > \delta(0))$

$n = 1 \; (\alpha(1) = \beta(1) = \gamma(1) = \delta(1))$

$n = 2 \; (\alpha(2) > \beta(2) = \delta(2) > \gamma(2))$

$n = 3 \; (\alpha(3) > \delta(3) > \gamma(3) > \beta(3))$

$\forall n \in \mathbb{N} \; (n \geq 4) \; (\alpha(n) > \gamma(n) > \delta(n) > \beta(n))$

2 (15) Use the method of differences shown in class to find a recursive definition

for: $S(p,n) := \sum_{k=0}^{n} k^p$ where $p, n \in \mathbb{N}$

Using your recursive definition, find a formula for $S(p,n)$ and **prove** by induction that your formula is valid.