Decision Models with Multiple Criteria

Elise Bonzon
elise.bonzon@mi.parisdescartes.fr

LIPADE - Université Paris Descartes
http://www.math-info.univ-paris5.fr/∽bonzon/

1. Introduction

2. Single synthesis criteria

3. Outranking methods

Introduction

Example: holidays

	cost	# days	travel time	hotel	beach dist.	wifi	cultural interest
A	2000€	15	12h	***	45km	Y	++
B	4250€	18	15h	****	0km	N	- -
C	1500€	4	7h	**	250km	N	+
D	3000€	5	10h	***	5km	Y	-

Example: holidays

	cost	# days	travel time	hotel	beach dist.	wifi	cultural interest
A	2000€	15	12h	***	45km	Y	++
B	4250€	18	15h	****	0km	N	- -
C	1500€	4	7h	**	250km	N	+
D	3000€	5	10h	***	5km	Y	-

Problems

- Help the decision-maker to choose his holidays
- Help the decision-maker to structure his preferences: rank the alternatives

Definitions

- Decision-maker: The decision-maker is the person on behalf of whom the decision assistance is done
- Analyst: The person who is in charge of the decision analysis
- Action/Option: "Object" analysed during the decision-making process

Criteria

- A: the set of alternatives, $A = \{a_1, a_2, \ldots, a_m\}$
- F: the set of criteria $F = \{c_1, c_2, \ldots, c_n\}$
- $g_j(a_i)$: the valuation of the alternative i for the criteria j
    - Sufficiency:

$$\forall j, g_j(a) = g_j(b) \Rightarrow \text{ no preferences between } a \text{ and } b$$

Criteria

- A: the set of alternatives, $A = \{a_1, a_2, \dots, a_m\}$
- F: the set of criteria $F = \{c_1, c_2, \dots, c_n\}$
- $g_j(a_i)$: the valuation of the alternative i for the criteria j
    - Sufficiency:

$$\forall j, g_j(a) = g_j(b) \Rightarrow \text{ no preferences between } a \text{ and } b$$

    - Cohesion:

$$\left.\begin{array}{l} \forall j \neq k, g_j(a) = g_j(b) \\ a \text{ preferred to } b \text{ for } g_k \end{array}\right\} \Rightarrow a \text{ preferred to } b$$

Problems

3 types of problems:

- Selection: Choice of a solution (the best one), or of a set of solutions
- Allocation: Allocation of each alternative to a category among n predefined categories
- Ranking: Give a ranking of all the alternatives

Example: social choice theory

- Decision-maker: the voters
- Analyst: the persons who choose the voting procedure
- Actions/Alternatives: candidates
- Problems: selection, ranking

Lexicographical aggregation

- a is preferred to b if
  - a is preferred to b over the most important criteria, OR
  - a and b are indifferent to the most important criteria and a is preferred to b over the second most important criteria, OR
  - a and b are indifferrent to the second most important criteria and a is preferred to b over the third most important criteria
  - ...

Single synthesis criteria

Introduction

- Single synthesis criteria: function g which synthesizes all the criteria:

$$g(a) = f(g_1(a), g_2(a), \ldots, g_n(a))$$

- g allows to compare the alternatives in order to choose one among them, to rank them or to allocate them among categories
- The construction of g is often difficult. Needs to ask a lot of information to the decision-maker
- Criteria are
  - Often contradictory (power and price)
  - Expressed in different unit (power and price)
  - Sometimes difficult to measure in a quantitative way (type of engine). Quantitative criteria rank more than they evaluate

Additive value function model

- Let A be the set of alternatives, g_j $(j = 1, 2, \ldots, n)$ a criteria to maximize

$$\begin{aligned} a \succeq b \quad &\Leftrightarrow \quad u(a) \geq u(b) \\ &\Leftrightarrow \quad \sum_{j=1}^{n} u_j(g_j(a)) \geq \sum_{j=1}^{n} u_j(g_j(b)) \end{aligned}$$

- u is called multi-attribute value function

Example: Hammond, Keeney & Raiffa

Choice of office to rent

- 5 offices have been selected
- 5 attributes are considered:
    - *Transport*: time (minutes)
    - *Customers*: percentage of customers who live near the office
    - *Services*:
        - A (all the services),
        - B (phone and fax machine),
        - C (no service)
    - *Surface area*: square feet ($\simeq 0.1m^2$)
    - *Cost*: $ by month

Example: Hammond, Keeney & Raiffa

	a	*b*	*c*	*d*	*e*
Transport	45	25	20	25	30
Customers	50	80	70	85	75
Services	*A*	*B*	*C*	*A*	*C*
Surface area	800	700	500	950	700
Cost	1950	1700	1500	1900	1750

Example: Hammond, Keeney & Raiffa

	a	*b*	*c*	*d*	*e*
↓*Transport*	45	25	20	25	30
↑*Customers*	50	80	70	85	75
↑*Services*	*A*	*B*	*C*	*A*	*C*
↑*Surface area*	800	700	500	950	700
↓*Cost*	1950	1700	1500	1900	1750

We want to

- Minimize criteria *Transport* and *Cost*
- Maximize criteria *Customers*, *Services* and *Surface area*

Example: Hammond, Keeney & Raiffa

	a	b	c	d	e
↓*Transport*	45	25	20	25	30
↑*Customers*	50	80	70	85	75
↑*Services*	A	B	C	A	C
↑*Surface area*	800	700	500	950	700
↓*Cost*	1950	1700	1500	1900	1750

- Alternative b dominates alternative e

Example: Hammond, Keeney & Raiffa

	a	b	c	d	e
$\downarrow$*Transport*	45	25	20	25	30
$\uparrow$*Customers*	50	80	70	85	75
$\uparrow$*Services*	A	B	C	A	C
$\uparrow$*Surface area*	800	700	500	950	700
$\downarrow$*Cost*	1950	1700	1500	1900	1750

- Alternative b dominates alternative e
- d dominates a

Example: Hammond, Keeney & Raiffa

	b	c	d
↓*Transport*	25	20	25
↑*Customers*	80	70	85
↑*Services*	B	C	A
↑*Surface area*	700	500	950
↓*Cost*	1700	1500	1900

- Alternative b dominates alternative e
- d dominates a
- Divide and conquer: eliminate alternatives
    - Elimination of a and e

Example: Hammond, Keeney & Raiffa

	b	c	d
$\downarrow$*Transport*	25	20	25
$\uparrow$*Customers*	80	70	85
$\uparrow$*Services*	B	C	A
$\uparrow$*Surface area*	700	500	950
$\downarrow$*Cost*	1700	1500	1900

- No more domination
- We will try to find compromises

Example: Hammond, Keeney & Raiffa

	b	c	d
$\downarrow$*Transport*	25	20	25
$\uparrow$*Customers*	80	70	85
$\uparrow$*Services*	B	C	A
$\uparrow$*Surface area*	700	500	950
$\downarrow$*Cost*	1700	1500	1900

- No more domination
- We will try to find compromises
- All the alternatives except c have the same value for the criteria *Transport*

Example: Hammond, Keeney & Raiffa

	b	c	d
$\downarrow$*Transport*	25	20	25
$\uparrow$*Customers*	80	70	85
$\uparrow$*Services*	B	C	A
$\uparrow$*Surface area*	700	500	950
$\downarrow$*Cost*	1700	1500	1900

- No more domination
- We will try to find compromises
- All the alternatives except c have the same value for the criteria *Transport*
- Modify c in such a way the value of this criteria is the same than the other alternatives

Example: Hammond, Keeney & Raiffa

	b	c	d
$\downarrow$*Transport*	25	20	25
$\uparrow$*Customers*	80	70	85
$\uparrow$*Services*	B	C	A
$\uparrow$*Surface area*	700	500	950
$\downarrow$*Cost*	1700	1500	1900

- No more domination
- We will try to find compromises
- All the alternatives except c have the same value for the criteria *Transport*
- Modify c in such a way the value of this criteria is the same than the other alternatives
  - What increase on the criteria *Customer* would allow to compensate for exactly a loss of 5mn in *Transport* time for the alternative c?

Example: Hammond, Keeney & Raiffa

	b	c	d
$\downarrow$ *Transport*	25	20	25
$\uparrow$ *Customers*	80	70	85
$\uparrow$ *Services*	B	C	A
$\uparrow$ *Surface area*	700	500	950
$\downarrow$ *Cost*	1700	1500	1900

- No more domination
- We will try to find compromises
- All the alternatives except c have the same value for the criteria *Transport*
- Modify c in such a way the value of this criteria is the same than the other alternatives
    - What increase on the criteria *Customer* would allow to compensate for exactly a loss of 5mn in *Transport* time for the alternative c?
    - Difficult but central question!

Example: Hammond, Keeney & Raiffa

	c	c'
$\downarrow$*Transport*	20	**25**
$\uparrow$*Customers*	70	$\mathbf{70+}\,\delta$
$\uparrow$*Services*	C	C
$\uparrow$*Surface area*	500	500
$\downarrow$*Cost*	1500	1500

- We want to find δ which would make c and c' equivalent

Example: Hammond, Keeney & Raiffa

	c	c'
$\downarrow$*Transport*	20	**25**
$\uparrow$*Customers*	70	$\mathbf{70} + \delta$
$\uparrow$*Services*	*C*	*C*
$\uparrow$*Surface area*	500	500
$\downarrow$*Cost*	1500	1500

- We want to find δ which would make c and c' equivalent
- For $\delta = 8$, the decision-maker said that he is indifferent between c and c'

Example: Hammond, Keeney & Raiffa

	c	c'
$\downarrow$*Transport*	20	**25**
$\uparrow$*Customers*	70	$\mathbf{70+}\,\delta$
$\uparrow$*Services*	C	C
$\uparrow$*Surface area*	500	500
$\downarrow$*Cost*	1500	1500

- We want to find δ which would make c and c' equivalent
- For $\delta = 8$, the decision-maker said that he is indifferent between c and c'
- Replace c by c'

Example: Hammond, Keeney & Raiffa

	b	c'	d
$\downarrow$*Transport*	25	25	25
$\uparrow$*Customers*	80	78	85
$\uparrow$*Services*	B	C	A
$\uparrow$*Surface area*	700	500	950
$\downarrow$*Cost*	1700	1500	1900

- All the alternatives have the same evaluation for the criteria *Transport*
- Divide and conquer: Eliminate criteria
    - Elimination of the attribute *Transport*

Example: Hammond, Keeney & Raiffa

	b	c'	d
$\uparrow$*Customers*	80	78	85
$\uparrow$*Services*	B	C	A
$\uparrow$*Surface area*	700	500	950
$\downarrow$*Cost*	1700	1500	1900

- All the alternatives have the same evaluation for the criteria *Transport*
- Divide and conquer: Eliminate criteria
  - Elimination of the attribute *Transport*

Example: Hammond, Keeney & Raiffa

	b	c'	d
$\uparrow$*Customers*	80	78	85
$\uparrow$*Services*	*B*	*C*	*A*
$\uparrow$*Surface area*	700	500	950
$\downarrow$*Cost*	1700	1500	1900

- Check again the dominations

Example: Hammond, Keeney & Raiffa

	b	c'	d
$\uparrow$*Customers*	80	78	85
$\uparrow$*Services*	B	C	A
$\uparrow$*Surface area*	700	500	950
$\downarrow$*Cost*	1700	1500	1900

- Check again the dominations
- No domination

Example: Hammond, Keeney & Raiffa

	b	c'	d
↑*Customers*	80	78	85
↑*Services*	*B*	*C*	*A*
↑*Surface area*	700	500	950
↓*Cost*	1700	1500	1900

- Check again the dominations
- No domination
- Find a compromise

Example: Hammond, Keeney & Raiffa

	b	c'	d
$\uparrow$*Customers*	80	78	85
$\uparrow$*Services*	B	C	A
$\uparrow$*Surface area*	700	500	950
$\downarrow$*Cost*	1700	1500	1900

- Check again the dominations
- No domination
- Find a compromise

$\Rightarrow$ Try to "suppress" *Services* using the criteria *Cost* in reference

Example: Hammond, Keeney & Raiffa

	b	c'	d
$\uparrow$*Customers*	80	78	85
$\uparrow$*Services*	B	C	A
$\uparrow$*Surface area*	700	500	950
$\downarrow$*Cost*	1700	1500	1900

- What maximum increase of rent would you accept to pay to go from service C to service B for the alternative c'?

Example: Hammond, Keeney & Raiffa

	b	c'	d
↑*Customers*	80	78	85
↑*Services*	B	C	A
↑*Surface area*	700	500	950
↓*Cost*	1700	1500	1900

- What maximum increase of rent would you accept to pay to go from service C to service B for the alternative c'?
  - Answer: 250$

Example: Hammond, Keeney & Raiffa

	b	c'	$c"$	d
$\uparrow$*Customers*	80	78	78	85
$\uparrow$*Services*	B	C	B	A
$\uparrow$*Surface area*	700	500	500	950
$\downarrow$*Cost*	1700	1500	$1500 + \mathbf{250}$	1900

- What maximum increase of rent would you accept to pay to go from service C to service B for the alternative c'?
  - Answer: 250$

Example: Hammond, Keeney & Raiffa

	b	c'	c"	d
↑*Customers*	80	78	78	85
↑*Services*	B	C	B	A
↑*Surface area*	700	500	500	950
↓*Cost*	1700	1500	$1500 + \mathbf{250}$	1900

- What maximum increase of rent would you accept to pay to go from service C to service B for the alternative c'?
  - Answer: 250$
- How much should the rent decrease if the service for alternative d would go from A to B?

Example: Hammond, Keeney & Raiffa

	b	c'	c"	d
↑*Customers*	80	78	78	85
↑*Services*	B	C	B	A
↑*Surface area*	700	500	500	950
↓*Cost*	1700	1500	1500 + **250**	1900

- What maximum increase of rent would you accept to pay to go from service C to service B for the alternative c'?
  - Answer: 250$
- How much should the rent decrease if the service for alternative d would go from A to B?
  - Answer: 100$

Example: Hammond, Keeney & Raiffa

	b	c'	c''	d	d'
↑*Customers*	80	78	78	85	85
↑*Services*	B	C	B	A	B
↑*Surface area*	700	500	500	950	950
↓*Cost*	1700	1500	1500 **+ 250**	1900	1900 - **100**

- What maximum increase of rent would you accept to pay to go from service C to service B for the alternative c'?
  - Answer: 250$
- How much should the rent decrease if the service for alternative d would go from A to B?
  - Answer: 100$

Example: Hammond, Keeney & Raiffa

	b	c'	c"	d	d'
$\uparrow$*Customers*	80	78	78	85	85
$\uparrow$*Services*	B	C	B	A	B
$\uparrow$*Surface area*	700	500	500	950	950
$\downarrow$*Cost*	1700	1500	1500 + **250**	1900	1900 - **100**

- What maximum increase of rent would you accept to pay to go from service C to service B for the alternative c'?
- How much should the rent decrease if the service for alternative d would go from A to B?
- We replace c' by c", and d by d'

Example: Hammond, Keeney & Raiffa

	b	c"	d'
$\uparrow$*Customers*	80	78	85
$\uparrow$*Services*	B	B	B
$\uparrow$*Surface area*	700	500	950
$\downarrow$*Cost*	1700	1750	1800

- What maximum increase of rent would you accept to pay to go from service C to service B for the alternative c'?
- How much should the rent decrease if the service for alternative d would go from A to B?
- We replace c' by c", and d by d'

Example: Hammond, Keeney & Raiffa

	b	c"	d'
↑*Customers*	80	78	85
↑*Services*	B	B	B
↑*Surface area*	700	500	950
↓*Cost*	1700	1750	1800

- What maximum increase of rent would you accept to pay to go from service C to service B for the alternative c'?
- How much should the rent decrease if the service for alternative d would go from A to B?
- We replace c' by c", and d by d'
- We can suppress the criteria *Services*

Example: Hammond, Keeney & Raiffa

	b	c"	d'
↑*Customers*	80	78	85
↑*Surface area*	700	500	950
↓*Cost*	1700	1750	1800

- What maximum increase of rent would you accept to pay to go from service C to service B for the alternative c'?
- How much should the rent decrease if the service for alternative d would go from A to B?
- We replace c' by c", and d by d'
- We can suppress the criteria *Services*

Example: Hammond, Keeney & Raiffa

	b	c"	d'
$\uparrow$*Customers*	80	78	85
$\uparrow$*Surface area*	700	500	950
$\downarrow$*Cost*	1700	1750	1800

- Check again dominations

Example: Hammond, Keeney & Raiffa

	b	c"	d'
↑*Customers*	80	78	85
↑*Surface area*	700	500	950
↓*Cost*	1700	1750	1800

- Check again dominations
- c" can be eliminated

Example: Hammond, Keeney & Raiffa

	b	d'
↑*Customers*	80	85
↑*Surface area*	700	950
↓*Cost*	1700	1800

- Check again dominations
- c" can be eliminated

Example: Hammond, Keeney & Raiffa

	b	d'
$\uparrow$*Customers*	80	85
$\uparrow$*Surface area*	700	950
$\downarrow$*Cost*	1700	1800

- No more domination
- How much would you accept to pay to increase the surface area of 250sf for the alternative b?

Example: Hammond, Keeney & Raiffa

	b	d'
$\uparrow$*Customers*	80	85
$\uparrow$*Surface area*	700	950
$\downarrow$*Cost*	1700	1800

- No more domination
- How much would you accept to pay to increase the surface area of 250\$ for the alternative b?
  - Answer: 250\$

Example: Hammond, Keeney & Raiffa

	b	b'	d'
$\uparrow$*Customers*	80	80	85
$\uparrow$*Surface area*	700	700 + **250**	950
$\downarrow$*Cost*	1700	1700 + **250**	1800

- No more domination
- How much would you accept to pay to increase the surface area of 250$ for the alternative b?
    - Answer: 250$

Example: Hammond, Keeney & Raiffa

	b'	d'
$\uparrow$*Customers*	80	85
$\uparrow$*Surface area*	950	950
$\downarrow$*Cost*	1950	1800

- No more domination
- How much would you accept to pay to increase the surface area of 250$ for the alternative b?
  - Answer: 250$
- Replace b by b'

Example: Hammond, Keeney & Raiffa

	b'	d'
↑*Customers*	80	85
↑*Surface area*	950	950
↓*Cost*	1950	1800

- No more domination
- How much would you accept to pay to increase the surface area of 250$ for the alternative b?
    - Answer: 250$
- Replace b by b'
- Suppress the alternative *Surface area*

Example: Hammond, Keeney & Raiffa

	b'	d'
$\uparrow$*Customers*	80	85
$\downarrow$*Cost*	1950	1800

- No more domination
- How much would you accept to pay to increase the surface area of 250$ for the alternative b?
  - Answer: 250$
- Replace b by b'
- Suppress the alternative *Surface area*

Example: Hammond, Keeney & Raiffa

	b'	d'
$\uparrow$*Customers*	80	85
$\downarrow$*Cost*	1950	1800

- Check again dominations

Example: Hammond, Keeney & Raiffa

	b'	d'
↑*Customers*	80	85
↓*Cost*	1950	1800

- Check again dominations
- d' dominates b'

Example: Hammond, Keeney & Raiffa

	b'	d'
$\uparrow$*Customers*	80	85
$\downarrow$*Cost*	1950	1800

- Check again dominations
- d' dominates b'

$\Rightarrow$ We advise d in final choice

Example: Hammond, Keeney & Raiffa

- We have to prove than $d \succ a$, $d \succ b$, $d \succ c$ and $d \succ e$
- Domination: $b \succ c''$, $c \sim c'$, $c' \sim c''$, $d' \sim d$, $b' \sim b$, $d' \succ b'$
- Compromises and dominations:
  - $b \succ e$, $d \succ a$
  - $c \sim c'$, $c' \sim c''$, $b \succ c'' \Rightarrow b \succ c$
  - $d' \sim d$, $b' \sim b$, $d' \succ b' \Rightarrow d \succ b$

Conjoint Measurement

- Very simple process
- No question about the “intensity of the preferences”

Conjoint Measurement

- Very simple process
- No question about the "intensity of the preferences"

But, a few problems:

- The set of alternatives has to be small
  - Otherwise, too much questions to ask
- If a new alternative appears, the process has to be start all over again

Weighted sum

- Let A be the set of alternatives and g_j $(j = 1, 2, \ldots, n)$ be the criteria
- Let w_j be the weight of g_j, for all j.

$$
\begin{aligned}
a \succeq b \quad &\Leftrightarrow \quad u(a) \geq u(b) \\
&\Leftrightarrow \quad \sum_{j=1}^{n} u_j(g_j(a)) \geq \sum_{j=1}^{n} u_j(g_j(b)) \\
&\Leftrightarrow \quad \sum_{j=1}^{n} (w_j \times g_j(a)) \geq \sum_{j=1}^{n} (w_j \times g_j(b))
\end{aligned}
$$

Weighted sum

	a	b	w_j
Profit (€)	60 000	48 000	0.6
Time-savings (mn)	60	70	0.4
Weighted sum			

Weighted sum

	a	b	w_j
Profit (€)	60 000	48 000	0.6
Time-savings (mn)	60	70	0.4
Weighted sum	36 024	28 828	

Weighted sum

	a	b	w_j
Profit (€)	60 000	48 000	0.6
Time-savings (mn)	60	70	0.4
Weighted sum	36 024	28 828	

- Normalization

Weighted sum

	a	b	w_j
Profit (€)	60	48	0.6
Time-savings (mn)	60	70	0.4
Weighted sum	60	56.8	

- Normalization of the profit: divided by 1000

Weighted sum

	a	b	w_j
Profit (€)	30	24	0.6
Time-savings (mn)	60	70	0.4
Weighted sum	42	**42.4**	

- Normalization of the profit: divided by 2000

Weighted sum

	a	b	w_j
Profit (€)	30	24	0.6
Time-savings (mn)	60	70	0.4
Weighted sum	42	**42.4**	

- Normalization of the profit: divided by 2000
- The ranking depends on how the normalization is done
- The alternatives have to be normalize independently
- The result has to be independent of the normalization

Weighted sum

- Implicitly, the scales are considered to be linear
- Previous example: The scale of the criteria “Time-savings” is not linear
- The importance of one minute of time is not the same to go from 0 to 2000€ of profit, or to go from 200 000 to 202 000€

Additive multi-attribute value model

- "Natural" extension of the weighted sum which takes into account the non-linearity of preferences
- Let A be the set of alternatives, g_j $(j = 1, 2, \ldots, n)$ a criteria and w_j the weight of g_j, for all j

$$a \succeq b \Leftrightarrow u(a) \geq u(b) \text{ with } u(a) = f(g_1(a), \ldots, g_n(a))$$

- Specific case: additive form

$$u(a) = \sum_{j=1}^{n} w_j \times u_j(g_j(a))$$

where $u_j(g_j^{min}) = 0$, $u_j(g_j^{max}) = 100$ and $\sum_{j=1}^{n} w_j = 1$

How to construct the single-attribute value functions?

- In order to specify an additive model, we have to define the functions u_i and the weights w_i for all $i \in F$
- Several methods exist to construct the u_i and the w_i
- These methods have to be applied several times to construct each function
- Example: Let the problem of "choosing a car" over three criteria {comfort, price, acceleration}

Construction of the functions u_i

- Method 1: when the number of values on the scale E_i is finite
  1. Rank the elements of E_i
  2. Rank the intervals between two consecutive elements in the previous ranking
  3. Assign values which respect information obtain in both previous steps
- Example: comfort criteria g_1
  1. "very comfortable" $\succ$ "comfortable" $\succ$ "quite comfortable" $\succ$ "rather uncomfortable" $\succ$ "uncomfortable"

     $\Rightarrow \ (e_1^1 \succ e_1^2 \succ e_1^3 \succ e_1^4 \succ e_1^5)$
  2. $(e_1^2 \ominus_1 e_1^3) \succ (e_1^4 \ominus_1 e_1^3) \succ (e_1^1 \ominus_1 e_1^2) \sim (e_1^4 \ominus_1 e_1^5)$
  3. $u_1(e_1^1) = 100,\ u_1(e_1^2) = 85,\ u_1(e_1^3) = 45,\ u_1(e_1^4) = 15,\ u_1(e_1^5) = 0$

Construction of the functions u_i

- Method 2: price criteria g_2 (from 10 to 20 k€)
  1. Discretization of the scale:
     $\Rightarrow$ $e_2^1 = 20k€$, $e_2^2 = 18k€$, $e_2^3 = 16k€$, $e_2^4 = 14k€$, $e_2^5 = 12k€$, $e_2^6 = 10k€$
  2. $(e_2^2 \ominus_2 e_2^1) \prec (e_2^3 \ominus_2 e_2^2) \prec (e_2^4 \ominus_2 e_2^3) \prec (e_2^5 \ominus_2 e_2^4) \sim (e_2^6 \ominus_2 e_2^5)$
  3. $u_2(e_2^1) = 0$, $u_2(e_2^2) = 10$, $u_2(e_2^3) = 25$, $u_2(e_2^4) = 45$, $u_2(e_2^5) = 70$, $u_2(e_2^6) = 100$
  4. We assume that the function is piecewise linear

Construction of the functions u_i

- Method 2: Acceleration criteria g_3 (from 28 to 31s for 1km standing start)
  1. Discretization of the scale:
     $\Rightarrow$ $e_3^1 = 28s$, $e_3^2 = 28.5s$, $e_3^3 = 29s$, $e_3^4 = 29.5s$, $e_3^5 = 30s$, $e_3^6 = 30.5s$, $e_3^7 = 31s$
  2. $(e_3^2 \ominus_3 e_3^1) \succ (e_3^3 \ominus_3 e_3^2) \succ (e_3^4 \ominus_3 e_3^3) \succ (e_3^5 \ominus_3 e_3^4) \succ (e_3^6 \ominus_3 e_3^5) \sim (e_3^7 \ominus_3 e_3^6)$
  3. $u_3(e_3^1) = 100$, $u_3(e_3^2) = 50$, $u_3(e_3^3) = 30$, $u_3(e_3^4) = 12.5$, $u_3(e_3^5) = 8$, $u_3(e_3^6) = 4$, $u_3(e_3^7) = 0$
  4. We assume that the function is piecewise linear

Construction of the weights w_i

1. We construct one alternative *per* criteria, such that the alternative i has the best evaluation for criteria i, and the worst evaluation for all other criteria:
    - Let b_i be an alternative such that $\forall j \neq i$, $g_j(b_i) = g_j^{min}$, and $g_i(b_i) = g_j^{max}$
    - Rank all the alternatives b_j following the preferences of the decision-maker
    - If for example this ranking is $b_n \succ \ldots \succ b_1$, we deduce that $w_n \geq \ldots \geq w_1$

Construction of the weights w_i

2. Choose an alternative b such that:
   - $g_i(b) = g_i^{min} \; \forall i \neq n$
   - Specify $g_n(b)$ such that $b_1 \sim b$. We have:

$$\begin{aligned} & u(b_1) = u(b) \\ \Rightarrow \quad & \textstyle\sum_{i=1}^{n} u_i(b_1) = \sum_{i=1}^{n} u_i(b) \\ \Rightarrow \quad & 100 \times w_1 = u_n(g_n(b)) \times w_n \\ \Rightarrow \quad & \frac{w_n}{w_1} = \frac{100}{u_n(g_n(b))} \end{aligned}$$

3. Proceed on the same way for $g_2, \ldots, g_{n-1}$
4. We obtain thus $\frac{w_n}{w_i}$, $\forall i \in \{1, \ldots, n-1\}$

Construction of the weights w_i: example

1. (uncomfortable, $20k€$, $\mathbf{28s}$) $\succ$ (uncomfortable, $\mathbf{10k€}$, $31s$) $\succ$ (**very comfortable**, $20k€$, $31s$). We have:

Construction of the weights w_i: example

1. $(\text{uncomfortable}, 20k€, \mathbf{28s}) \succ (\text{uncomfortable}, \mathbf{10k€}, 31s) \succ (\textbf{very comfortable}, 20k€, 31s)$. We have:
    - $b_3 \succ b_2 \succ b_1$, and thus $u(b_3) > u(b_2) > u(b_1)$,

Construction of the weights w_i: example

1. (uncomfortable, $20k€$, **28s**) $\succ$ (uncomfortable, **10k€**, $31s$) $\succ$ (**very comfortable**, $20k€$, $31s$). We have:
   - $b_3 \succ b_2 \succ b_1$, and thus $u(b_3) > u(b_2) > u(b_1)$,
   - So, $w_3 > w_2 > w_1$

Construction of the weights w_i: example

1. $(\text{uncomfortable}, 20k€, \mathbf{28s}) \succ (\text{uncomfortable}, \mathbf{10k€}, 31s) \succ (\textbf{very comfortable}, 20k€, 31s)$. We have:
   - $b_3 \succ b_2 \succ b_1$, and thus $u(b_3) > u(b_2) > u(b_1)$,
   - So, $w_3 > w_2 > w_1$
2. $(\text{very comfortable}, 20k€, 31s) \sim (\text{uncomfortable}, 20k€, 29.5s)$
   So, $100 \times w_1 = u_3(29.5s) \times w_3$, thus

$$\frac{w_3}{w_1} = \frac{100}{u3(29.5s)} = \frac{100}{12.5} = 8$$

Construction of the weights w_i: example

1. $(\text{uncomfortable}, 20k€, \mathbf{28s}) \succ (\text{uncomfortable}, \mathbf{10k€}, 31s) \succ (\textbf{very comfortable}, 20k€, 31s)$. We have:
   - $b_3 \succ b_2 \succ b_1$, and thus $u(b_3) > u(b_2) > u(b_1)$,
   - So, $w_3 > w_2 > w_1$
2. $(\text{very comfortable}, 20k€, 31s) \sim (\text{uncomfortable}, 20k€, 29.5s)$
   So, $100 \times w_1 = u_3(29.5s) \times w_3$, thus
   $$\frac{w_3}{w_1} = \frac{100}{u3(29.5s)} = \frac{100}{12.5} = 8$$
3. $(\text{uncomfortable}, 10k€, 31s) \sim (\text{uncomfortable}, 20k€, 28.5)$
   So, $100 \times w_2 = u_3(28.5s) \times w_3$, thus
   $$\frac{w_3}{w_2} = \frac{100}{u3(28.5s)} = \frac{100}{50} = 2$$

Construction of the weights w_i: example

1. (uncomfortable, $20k€$, **28s**) $\succ$ (uncomfortable, **10k€**, $31s$) $\succ$ (**very comfortable**, $20k€$, $31s$). We have:
   - $b_3 \succ b_2 \succ b_1$, and thus $u(b_3) > u(b_2) > u(b_1)$,
   - So, $w_3 > w_2 > w_1$
2. (very comfortable, $20k€$, $31s$) $\sim$ (uncomfortable, $20k€$, $29.5s$)
   So, $100 \times w_1 = u_3(29.5s) \times w_3$, thus
   $$\frac{w_3}{w_1} = \frac{100}{u3(29.5s)} = \frac{100}{12.5} = 8$$
3. (uncomfortable, $10k€$, $31s$) $\sim$ (uncomfortable, $20k€$, 28.5)
   So, $100 \times w_2 = u_3(28.5s) \times w_3$, thus
   $$\frac{w_3}{w_2} = \frac{100}{u3(28.5s)} = \frac{100}{50} = 2$$
4. Let's say that $w_3 = 8$, $w_1 = 1$ and $w_2 = 4$. As we want $\sum_{i=1}^{3} w_i = 1$, we have

Construction of the weights w_i: example

1. (uncomfortable, $20k€$, **28s**) $\succ$ (uncomfortable, **10k€**, $31s$) $\succ$ (**very comfortable**, $20k€$, $31s$). We have:
   - $b_3 \succ b_2 \succ b_1$, and thus $u(b_3) > u(b_2) > u(b_1)$,
   - So, $w_3 > w_2 > w_1$
2. (very comfortable, $20k€$, $31s$) $\sim$ (uncomfortable, $20k€$, $29.5s$) So, $100 \times w_1 = u_3(29.5s) \times w_3$, thus
$$\frac{w_3}{w_1} = \frac{100}{u3(29.5s)} = \frac{100}{12.5} = 8$$
3. (uncomfortable, $10k€$, $31s$) $\sim$ (uncomfortable, $20k€$, 28.5) So, $100 \times w_2 = u_3(28.5s) \times w_3$, thus
$$\frac{w_3}{w_2} = \frac{100}{u3(28.5s)} = \frac{100}{50} = 2$$
4. Let's say that $w_3 = 8$, $w_1 = 1$ and $w_2 = 4$. As we want $\sum_{i=1}^{3} w_i = 1$, we have
   - $w_1 = \frac{1}{13}$, $w_2 = \frac{4}{13}$, $w_3 = \frac{8}{13}$

Outranking methods

Outranking methods

- Outranking relation $O(x, y)$: "x is at least as good than y"

$$O(x, y) \Leftrightarrow C(x, y) \land \neg D(x, y)$$

- We say that an action x **outranks** an action y if:
  - x is as least as good than y for a majority of criteria: **concordance condition** $C(x, y)$
  - without beeing to much worse with respect to the other criteria: **no discordance condition** $\neg D(x, y)$,

    $\rightarrow$ that is, there is no criteria veto for $O(x, y)$.

Outranking methods

- Example of a concordance index $C(x, y)$:

$$C(x, y) \Leftrightarrow \frac{\sum_{j \in >_{xy}} w_j}{\sum_i w_i} \geq \gamma$$

with

- $>_{xy}$ the set of criteria for which $x > y$
- w_j the weight of criteria j
- γ the the **concordance threshold** (majority threshold)

Outranking methods

- Example of a discordance index $D(x, y)$:

$$D(x, y) \Leftrightarrow \exists j : g_j(y) - g_j(x) > v_j$$

  with

  - $g_j(x)$ the evaluation of x for the criteria j
  - v_j is the **veto threshold** for the criteria j

Example

Two alternatives (a and b), five criteria and $\gamma = 0.60$ (concordance threshold)

	Cr1	Cr2	Cr3	Cr4	Cr5
a	10	50	1000	72	60
b	7	40	950	69	74
w_j	0.1	0.3	0.1	0.2	0.3
veto		15			20

- $C(a,b)$: $\sum_{j \in >_{xy}} w_j = 0.1 + 0.3 + 0.1 + 0.2 = 0.7 \geq 0.6$
- $\neg D(a,b)$: $g_5(b) - g_5(a) = 14 < 20$

$\Rightarrow$ $O(a,b)$

Example

Two alternatives (a and b), five criteria and $\gamma = 0.60$ (concordance threshold)

	Cr1	Cr2	Cr3	Cr4	Cr5
a	10	50	1000	72	60
b	7	40	950	69	84
w_j	0.1	0.3	0.1	0.2	0.3
veto		15			20

- $C(a,b)$: $\sum_{j \in >_{xy}} w_j = 0.1 + 0.3 + 0.1 + 0.2 = 0.7 \geq 0.6$
- $D(a,b)$: $g_5(b) - g_5(a) = 24 > 20$

$\Rightarrow \neg O(a,b)$

Outranking methods and semi order

- It is possible to add an indifference threshold q_j for each criteria j
- The associated preference order is then a semi order

Outranking methods and semi order

- It is possible to add an indifference threshold q_j for each criteria j
- The associated preference order is then a semi order
- Concordance index $C(x, y)$:

$$C(x, y) \Leftrightarrow \frac{\sum_{j \in >_{xy}} w_j}{\sum_i w_i} \geq \gamma$$

- $j \in >_{xy}$ iff

$$g_j(x) > g_j(y) + q_j$$

Outranking methods

- Outranking methods allow to:
    - Choose an alternative
    - Rank the alternatives
    - Allocate alternatives to different categories

Choice problematic

- The outranking relation can be not transitive: it is possible to have cycles
- The outranking relation is not complete

Choice problematic

- The outranking relation can be not transitive: it is possible to have cycles
- The outranking relation is not complete

$\Rightarrow$ Reduction of the cycles and/or computation of the core

Choice problematic

- The outranking relation can be not transitive: it is possible to have cycles
- The outranking relation is not complete

$\Rightarrow$ Reduction of the cycles and/or computation of the core

  - Reduction of the cycles: every alternatives in a cycle are considered indifferent
  - Core

Choice problematic

Core of a graph:

- Every not outranked action is in the core of the graph
- No action in the core can outrank another action of the core
- Every action which is not in the core of the graph has to be outranked by an action of the core, otherwise it has to be in the core
- A graph can have several cores. If the graph does not contain any cycle, there is a unique core

Ranking problematic

- Rank the alternatives
    - Alternatives which are not outranked are the best alternatives
    - We eliminate those. The alternatives which are now not outranked are the second best ones
    - We eliminate those and proceed again...

Allocation problematic

- Allocate each alternative to a pre-defined category (partition of A)
- We do not want to compare the alternatives, but to evaluate their own individual value
- The allocation is ordered
- Three steps:
    1. Define some reference alternatives to characterize the categories (limit profiles),
    2. Compare each alternative $a \in A$ with the reference alternatives
    3. Apply a procedure of allocation