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# HITTING LINES WITH TWO-DIMENSIONAL BROWNIAN MOTION*

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**Abstract.** We use a simple model for the spontaneous activity of two neurons to arrive at a correlated two-dimensional Brownian motion, $(X_1(t), X_2(t))$. We then compute the joint distribution of $\tau_i = \inf\{t: X_i(t) = a_i\}$, $i = 1, 2$, for the driftless case, and give an expression for Brownian motion with drift.

**AMS (MOS) 1980 subject classifications.** Primary 60J65; secondary 60J70

**Key words.** Brownian motion, hitting times and places, inverse Gaussian distribution, diffusion equation, neural activity

**1. Introduction.** This paper consists of the computation of several hitting time and hitting place distributions for two-dimensional Brownian motion. The motivation for this study is two-fold: first, to get a diffusion model for the firing behavior of a simple network of neurons, and second, to get an interesting two-dimensional version of the inverse Gaussian distribution.

Fienberg (1974) has reviewed various models for the firing of single neurons. A classical model of Gerstein and Mandelbrot (1964) says that if the electrical state (or potential) of the neural membrane is specified by a single number, which moves towards or away from the firing potential as the neuron receives excitatory or inhibitory input, respectively, then the time to firing can be approximated by the first hitting time of a certain level for a Brownian motion with drift. The authors showed that this model could be used to provide a satisfactory fit to some data that they observed; more importantly, they showed by Monte Carlo methods that neural activity in the presence of stimuli could also be well duplicated by a modification of the above random walk model.

Next, the review by Folks and Chhikara (1978) shows that the inverse Gaussian distribution has many nice statistical properties which, to a large extent mirror those of the Gaussian distribution. It is natural, then, to ask whether there is a multivariate inverse Gaussian whose statistical properties are similar to those of the multivariate Gaussian.

Several proposals for a bivariate inverse Gaussian have already appeared in the literature. Barndorff-Nielsen and Blaesild (1983) define reproductive exponential families and propose a bivariate inverse Gaussian model; they claim that their generalization has nice statistical properties (i.e., affords tractable estimation and analysis of variance), but their proposal does not have inverse Gaussian marginals. Wasan (1969), (1972) proposes several bivariate inverse Gaussians but does not develop their properties.

**2. A simple neural network.** Consider the three neurons of Fig. 1. Neuron $A$ sends predominantly excitatory signals, $s$, to $B$ and $C$. $B$ and $C$ share a common noise, $n$, and they also have independent sources of noise, $n_1$ and $n_2$, as shown. If the electrical states of $B$ and $C$ are encoded by single numbers, $X_1(t)$ and $X_2(t)$, respectively, then $X_i(t)$ has three components; the common noise $n$, the particular noise $n_i$, and the

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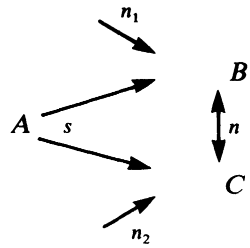


FIG. 1.

signal $s$. Let the noise variances be $\sigma^2$ for $n$ and $\sigma_i^2$ for $n_i$; then it is easy to see that

$$\operatorname{corr}(X_1(t), X_2(t)) = \left[\left(1+\frac{\sigma_1^2}{\sigma^2}\right)\left(1+\frac{\sigma_2^2}{\sigma^2}\right)\right]^{-1/2}$$

which is a function of the noise ratios. Also, we may allow the drifts of $X_1(t)$ and $X_2(t)$ (due to the signal, $s$) to be different since $B$ and $C$ may accept the same input but integrate it differently. When either Brownian reaches the firing threshold, the appropriate neuron fires, returns to its resting state, and the process starts afresh. What are of interest, then, are the firing times or the first hitting times for the Brownian motions. Alternatively, this model can be used to study a single neuron: if we postulate that the neuron has two interacting trigger zones (Gerstein et al. (1964)), then the components of the Brownian motion describe the electrical state of each zone. Mathematically, the two problems are the same.

**3. Preliminaries.** We start with a correlated driftless Brownian motion $X(t)$ with $EX(t)=0$ and $\operatorname{var} X(t) = t\Sigma$. Here

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \quad \text{and} \quad \Sigma^{1/2} = \begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

where $\rho = \sin(2\beta)$, $|\beta| \leqq \pi/4$. Thus $X(t) = \Sigma^{1/2} Z(t)$ where $Z(t)$ is a standard Brownian motion: $\operatorname{var} Z(t) = tI$. Also define the two stopping times $\tau_i = \inf\{t\colon X_i(t) = a_i\} = \inf\{t\colon Z(t) \in L_i\}$. Here $a_i > 0$ without loss of generality, and $L_i$ is the line $\{v \in \mathbb{R}^2\colon v' \cdot \Sigma^{1/2} e_i = a_i\}$ and $\{e_1, e_2\}$ is the standard basis for $\mathbb{R}^2$. By the scale invariance of Brownian motion, we can take $a_1 = 1$. Finally, by elementary methods, we arrive at the following problem (see Fig. 2): start a Brownian motion at $x = (x_1, x_2) = (r_0 \cos\theta_0, r_0 \sin\theta_0)$, and study the stopping times and places associated with $\tau_1$ and $\tau_2$. Here $\tau_i$ is the first hitting time of $L_i$, and $\alpha = \pi/2 + \sin^{-1}\rho$. Also, let $W = \{(r, \theta)\colon r > 0,$

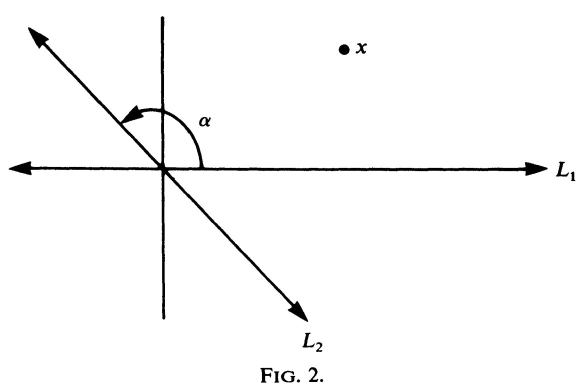


FIG. 2.

$0<\theta<\alpha\}$, and $\tau'=\tau_1\wedge\tau_2$ be the first hitting time of $\partial W$. Our aim is to get the joint density of $(\tau_1, \tau_2)$, and on the way we compute other quantities that are also of interest. In particular, we study the following quantities:

a) $P^x(\tau'>t, Z(t)\in B)$, $B\subset W$,
b) $P^x(\tau'>t)$,
c) $P^x(Z(\tau')\in A)$, $A\subset\partial W$,
d) $P^x(\tau'\in dt, Z(t)\in da)$, $a\in\partial W$,
e) $P^x(\tau_1\in ds, \tau_2\in dt)$,
f) the above quantities in the presence of drift.

Here $P^x$ is the measure associated with standard Brownian motion starting at $x$; $E^x$ will denote the corresponding expectation. Note that the marginal distributions of $\tau_i$ are easy:

$$P^x(\tau_2\in dt)=\frac{x_2}{t\sqrt{t}}\phi\left(\frac{x_2}{\sqrt{t}}\right)dt \quad\text{and}\quad P^x(\tau_1\in dt)=\frac{u}{t\sqrt{t}}\phi\left(\frac{u}{\sqrt{t}}\right)dt$$

where $u=x_1\sin\alpha-x_2\cos\alpha$ and $\phi(x)=(2\pi)^{-1/2}e^{-x^2/2}$.

**4. Brownian motion in the wedge.** The main result of this section is contained in (8). Most of the subsequent results follow from it. If we have a positive bounded continuous function $f$ defined on $W$, and which vanishes on $\partial W$, then

$$u(t,x)=E^xf(Z(\tau'\wedge t))=\int_W f(y)P^x(\tau'>t, Z(t)\in dy) \tag{1}$$

satisfies the heat equation with boundary and initial conditions

$$u_t=\tfrac{1}{2}\Delta u \quad\text{in } W,\qquad u(0,x)=f(x),\quad x\in W,\qquad u(t,z)=0,\quad z\in\partial W. \tag{2}$$

We can solve (2) when $\alpha=\pi/m$, $m=1,2,\cdots$ by the method of images. That is, if we let $T_0=I$, $F_j$ be the matrix representing the reflection across the line $y=x\tan(\pi j/m)$, and $T_j=F_j\circ T_{j-1}$, and let $\tilde f(y)=(-1)^kf(T_k^{-1}y)$ for $y\in T_k(W)$, we have the initial value problem

$$\tilde u_t=\tfrac{1}{2}\Delta\tilde u \quad\text{in } \mathbb{R}^2,\qquad \tilde u(0,x)=\tilde f(x),\quad x\in\mathbb{R}^2. \tag{3}$$

The solution is

$$\tilde u(t,x)=E^x\tilde f(Z(t))=\int_W f(y)\sum_{k=0}^{2m-1}(-1)^k\phi_2\left(\frac{x-T_ky}{\sqrt{t}}\right)\frac{dy}{t}$$

where $\phi_2(x)=(2\pi)^{-1}\exp(-x'x/2)$. It is easy to see that

$$P^x(\tau'>t, Z(t)\in dy)=\frac{1}{t}\sum_{k=0}^{2m-1}(-1)^k\phi_2\left(\frac{x-T_ky}{\sqrt{t}}\right)dy. \tag{4}$$

While (4) is appropriate only for the special angles $\theta_0=\pi/m$, the following argument gives us the result in general. To facilitate this, we use polar coordinates: let $x=(r_0\cos\theta_0, r_0\sin\theta_0)$, $y=(r\cos\theta, r\sin\theta)$ to get

$$P^x(\tau'>t, Z(t)\in dy)=\frac{1}{2\pi t}e^{-(r^2+r_0^2)/2t}\sum_{k=0}^{2m-1}(-1)^k e^{(rr_0/t)\cos(\theta-\theta_k)} \tag{5}$$

where $\theta_k$ is the argument of $T_ky$. Note that (Magnus et al. (1966))

$$e^{\gamma z}=2\sum_{n=0}^{\infty}T_n(\gamma)I_n(z) \tag{6}$$

where $T_n$ is the $n$th Chebyshev polynomial and $I_n$ is the modified Bessel function of order $n$. Recalling that $T_n(\cos\theta)=\cos(n\theta)$, and using the fact that

$$\sum_{k=0}^{2m-1}(-1)^k\cos n(\theta-\theta_k)=2m\sin(n\theta)\sin(n\theta_0) \tag{7}$$

if $m$ divides $n$ and zero otherwise, we get

$$P^x(\tau'>t, Z(t)\in dy)=\frac{2r}{t\alpha}e^{-(r^2+r_0^2)/2t}\sum_{n=0}^{\infty}\sin\frac{n\pi\theta}{\alpha}\sin\frac{n\pi\theta_0}{\alpha}I_{n\pi/\alpha}\left(\frac{rr_0}{t}\right)dr\,d\theta \tag{8}$$

whenever $\alpha=\pi/m$. But formula (8) is valid for all $\alpha$, and it is easy to see that it solves problem (2).

Expression (8) has also been essentially derived by Sommerfeld (1894) by an extension of the method of images. See also Carslaw and Jaeger (1959) and Buckholtz and Wasan (1979).

We next compute $P^x(\tau'>t)$. If we integrate out $\theta$ and $r$ in (8) and use the following identities:

$$\begin{gathered}2I'_\nu(x)=I_{\nu-1}(x)+I_{\nu+1}(x),\\ \int_0^\infty e^{-\beta t^2}I_\nu(\alpha t)\,dt=\frac{1}{2}\sqrt{\frac{\pi}{\beta}}\,e^{-\alpha^2/8\beta}I_{\nu/2}\left(\frac{\alpha^2}{8\beta}\right)\end{gathered} \tag{9}$$

(Magnus et al. (1966)), we get[1]

$$P^x(\tau'>t)=\frac{2r_0}{\sqrt{2\pi t}}e^{-3r_0^2/4t}\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{1}{n}\sin\frac{n\pi\theta_0}{\alpha}\left\{I_{(\nu-1)/2}\left(\frac{r_0^2}{4t}\right)+I_{(\nu+1)/2}\left(\frac{r_0^2}{4t}\right)\right\} \tag{10}$$

where $\nu=n\pi/\alpha$.

When the wedge angle is special—$\alpha=\pi/m$—we have the alternate expression

$$P^x(\tau'>t)=2\sum_{k=0}^{2m-1}(-1)^{k+1}F\left(\frac{k\pi}{m}\right) \tag{11}$$

where

$$F(u)=\int_0^u\frac{r_0}{\sqrt{t}}\cos(\theta-\theta_0)R\left(-\frac{r_0}{\sqrt{t}}\cos(\theta-\theta_0)\right)d\theta. \tag{12}$$

Here, $R$ is Mills' ratio: $R(x)=\Phi(-x)/\phi(x)$, where $\Phi$ and $\phi$ are the normal distribution and density functions, respectively. We omit the details of this computation.

The quantity $P^x(\tau'>t)$ was also studied by Spitzer (1958), who computed its transform. Checking the asymptotics of modified Bessel functions, it is easy to see that $E^x(\tau')^\beta=\int_0^\infty\tau^{\beta-1}P^x(\tau'>t)\,dt>\infty$ if and only if $\alpha\beta<\pi/2$, independent of $x$.

The distribution of $Z(\tau')$ is also of interest. Now $u(x;A)=P^x(Z(\tau')\in A)$ satisfies Laplace's equation $\Delta u=0$ with boundary condition $u(x,A)=I\{x\in A\}$. The Green's function for the wedge is easily computed, and we have

$$P^x(Z(\tau')\in da)=\frac{1}{\alpha}\frac{(a/r_0)^{\pi/\alpha-1}\sin(\pi\theta_0/\alpha)}{\sin^2(\pi\theta_0/\alpha)+(1\pm\cos(\pi\theta_0/\alpha))^2}\frac{da}{r_0} \tag{13}$$

where we use the plus sign for $\tau_2<\tau_1$ and the minus sign otherwise. Using elementary estimates, it is easy to see that $E^xZ_i(\tau')^\beta$ exists iff $\alpha\beta<\pi$, again independent of the initial position, $x$.

[1] Expression (10) corrects a mistake in Wasan and Buckholtz (1979).

**5. Joint distribution of $(\tau_1, \tau_2)$.** Using an argument similar to that of Daniels (1982)[2], it can be shown that if $P^x(\tau > t, Z(t') \in dy) = f(t, x, y)\, dy$, then

$$P^x(\tau' \in dt, Z(\tau') \in da) = \frac{1}{2}\left[\frac{\partial}{\partial n} f(t, x, y)\Big|_{y=a}\right] da\, dt \tag{14}$$

where $\partial/\partial n$ denotes the derivative in the inward normal direction. See Fig. 3. When $\theta = \phi(\alpha)$, $\partial/\partial n = (1/r)\,\partial/\partial\theta(-(1/r)\,\partial/\partial\theta)$, so from (8) we get

$$P^x(\tau' \in dt, Z(\tau') \in da) = \frac{\pi}{\alpha^2 t a} e^{-(a^2+r_0^2)/2t} \sum_{n=0}^{\infty} \delta_n n \sin \frac{n\pi\theta_0}{\alpha} I_{n\pi/\alpha}\left(\frac{a r_0}{t}\right) \tag{15}$$

where $\delta_n$ is 1 if $\theta = 0$ and $(-1)^{n+1}$ if $\theta = \alpha$. It is clear by symmetry that $P^x(\tau' \in dt, X(\tau') \in da) = P^{\tilde{x}}(\tau' \in dt, Z(\tau') \in da')$ where $\tilde{x}$ is the reflection of $x$ across the line $y = x \tan(\alpha/2)$. And for the special angle $\alpha = \pi/m$, a simpler formula is available:

$$P^x(\tau' \in dt, Z(\tau') \in da) = \frac{1}{t^2} \sum_{0}^{2m-1} (-1)^k \phi_2\left(\frac{x - T_k y}{\sqrt{t}}\right)\Big|_{y_2=0} (x' T_k e_2) \tag{16}$$

with $P^x(\tau' \in dt, Z(\tau') \in da')$ computed by symmetry.

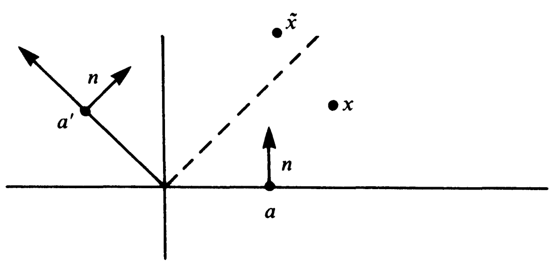


FIG. 3.

Finally, we can compute $P^x(\tau_1 \in ds, \tau_2 \in dt)$, the joint density of $(\tau_1, \tau_2)$. By the strong Markov property we have for $s < t$,

$$\begin{aligned} P^x(\tau_1 \in ds, \tau_2 \in dt) &= \int_{\partial W} P^x(\tau_1 \in ds, Z(\tau') \in da, \tau_2 \in dt) \\ &= \int_{\partial W} P^x(\tau_1 = \tau' \in ds, Z(\tau') \in da) P^{a \sin \alpha}(\tau_2 \in dt - s). \end{aligned} \tag{17}$$

But the first term of the integrand is given by (15) and the second term is just

$$P^{a \sin \alpha}(\tau_2 \in dt - s) = \frac{a \sin \alpha}{(t-s)^{3/2}} \phi\left(\frac{a \sin \alpha}{\sqrt{t-s}}\right) dt, \tag{18}$$

since $\tau_2$ is just a one-dimensional inverse Gaussian. After some computation and (9) we get

$$\begin{aligned} &P^x(\tau_1 \in ds, \tau_2 \in dt) \\ &= \frac{\pi \sin \alpha}{2\alpha^2 \sqrt{s}(t-s)\sqrt{t - s \cos^2 \alpha}} \exp\left(-\frac{r_0^2}{2s} \frac{2(t-s) + (t - s\cos 2\alpha)}{(t-s) + (t - s \cos 2\alpha)}\right) \\ &\quad \times \sum_{n=0}^{\infty} (-1)^{n+1} n \sin \frac{n\pi\theta_0}{\alpha} I_{n\pi/2\alpha}\left(\frac{r_0^2(t-s)/s}{2(t-s) + 2(t - s\cos 2\alpha)}\right). \end{aligned} \tag{19}$$

[2] I thank Professor D. Siegmund for this reference.

Finally, using the fact that (see Fig. 3)

$$P^x(\tau_1 \in ds, \tau_2 \in dt) = P^{\tilde{x}}(\tau_1 \in dt, \tau_2 \in ds), \qquad s < t$$

the joint density of $(\tau_1, \tau_2)$ is determined for $s \neq t$. In (19) we can let $t \to s$ and use the fact that as $z \to 0$, $I_\nu(z) \sim (z/2)^\nu / \Gamma(\nu+1)$ to get

$$P^x(\tau_1 \in ds, \tau_2 \in dt) \to \begin{cases} 0 & \text{if } 0 < \alpha < \dfrac{\pi}{2}, \\ \infty & \text{if } \dfrac{\pi}{2} < \alpha < \pi, \\ \dfrac{r_0^2 \sin 2\theta_0}{4\pi s^3} \exp\left(-\dfrac{r_0^2}{2s}\right) & \text{if } \alpha = \dfrac{\pi}{2}. \end{cases} \tag{20}$$

Thus the joint density of $(\tau_1, \tau_2)$ is discontinuous on the line $s = t$ only when the original Brownian motion $X(t)$ is positively correlated. Of course, we could have started with (16) to get a simpler expression for the joint density of $(\tau_1, \tau_2)$ for the special angles $\alpha = \pi/m$; we omit the straightforward calculation.

**6. Brownian motion with drift.** Of course, the case of Brownian motion with drift is of more interest. The analysis, however, is considerably more complicated and so it is not always possible to evaluate the integrals that arise. In this section, we extend the results of the previous sections to Brownian motion with drift.

If, in § 3, the correlated process $X(t)$ has drift $\theta = (\theta_1, \theta_2)'$ where $\theta_i > 0$, it is easy to see that for the corresponding process $Z(t)$, in the wedge $W$, we have drift $\mu = (\mu_1, \mu_2)' = (\rho\theta_2 - \theta_1, -\theta_2(1-\rho^2)^{1/2})$. Let

$$f(t; a, b) = \sqrt{\frac{b}{2\pi t^3}} \exp\left(-\frac{b}{2a^2} \frac{(t-a)^2}{t}\right) \tag{21}$$

be the inverse Gaussian density in its usual form [6, p. 263]. Then it is easy to see that $\tau_1$ has density

$$f\left(t; \frac{|x_1 \sin\alpha - x_2 \cos\alpha|}{\theta_1\sqrt{1-\rho^2}}, (x_1 \sin\alpha - x_2\cos\alpha)^2\right)$$

and $\tau_2$ has density

$$f\left(t; \frac{x_2}{\theta_2\sqrt{1-\rho^2}}, x_2^2\right).$$

Let $P_\mu^x$ be the measure associated with uncorrelated Brownian motion starting at $x$ and with drift $\mu$:

$$P_\mu^x(Z(t_1) \in A_1, \cdots, Z(t_n) \in A_n) = P_0^x(Z(t_1) + \mu t_1 \in A_1, \cdots, Z(t_n) + \mu t_n \in A_n) \tag{22}$$

for all $n$, for all $t_1 < \cdots < t_n$, and for all Borel sets $A_i$. Our basic tool is the exponential (likelihood ratio) martingale

$$\frac{dP_\mu^x}{dP_0^x} = \exp\left(\mu'(Z(t) - x) - t|\mu|^2/2\right) \tag{23}$$

on $\mathscr{F}_t$, the sigma field generated by $\{Z(s): s \leqq t\}$. Thus, we have that

$$v(t, x, y) = P_\mu^x(\tau' > t, Z(t) \in dy) = e^{\mu'(y-x) - |\mu|^2 t/2} P_0^x(\tau' > t, Z(t) \in dy) \tag{24}$$

is a solution to the diffusion equation with convection or drift:

$$v_t = \tfrac{1}{2}\Delta v + \mu' \nabla v, \quad v(0, x, y) = \delta_{x-y}, \quad v(t, a, y) = 0, \quad a \in \partial W \tag{25}$$

where $\delta$ is the Dirac delta. Of course, in (24), $P_0^x(\tau' > t, Z(t) \in dy)$ is given by (8). The expressions for $P_\mu^x(\tau' > t)$ and $P_\mu^x(Z(\tau') \in A)$ do not seem to be convenient as they were for the driftless case ((10) and (13) in § 4), but the joint density of $\tau'$ and $Z(\tau')$ is available. In fact, Daniels' argument gives (see Fig. 3)

$$\begin{aligned} &P_\mu^x(\tau' \in dt, Z(\tau') \in da) = e^{-x'\mu - t|\mu|^2/2 + \mu' a} P_0^x(\tau' \in dt, Z(\tau') \in da), \\ &P_\mu^x(\tau' \in dt, Z(\tau') \in da') = e^{-x'\mu - t|\mu|^2/2 + a'(\mu_1 \cos\alpha + \mu_2 \sin\alpha)} P_0^x(\tau' \in dt, Z(\tau') \in da'), \end{aligned} \tag{26}$$

where we use (15) for $P_0^x(\tau' \in dt, Z(\tau') \in da)$.

Finally, we have for $s < t$

$$P_\mu^x(\tau_1 \in ds, \tau_2 \in dt) = \int_{\partial W} P_\mu^x(\tau_1 \in ds, Z(\tau_1) \in da) P^{a \sin\alpha}(\tau_2 \in dt - s). \tag{27}$$

Note that $P_\mu^{a \sin\alpha}(\tau_2 \in dt - s)$ is just the inverse Gaussian density and $P_\mu^x(\tau_1 \in ds, Z(\tau_1) \in da)$ is given by (21). The integral involved, however, does not seem to be tractable.

**7. Concluding remarks.** Clearly, the $\mu = 0$ case is much easier than the $\mu \neq 0$ case; however, the physical motivation demands $\mu \neq 0$. For higher dimensional problems, similar methods can be used to get the joint distribution of $\tau = (\tau_1, \cdots, \tau_p)$, where $\tau_i = \inf\{t > 0 \colon X_i(t) = a_i\}$, $a_i > 0$, and $X(t) \sim N(0, t\Sigma)$ is a driftless correlated Brownian motion. The geometry in $\mathbb{R}^p$ is quite complicated though; for certain patterned $\Sigma$ (e.g., $\Sigma_{ij} \equiv \rho$), the transformation to independence can be done in closed form, and the joint distribution of $\tau$ is available. The same problem for general $\Sigma$ and for $\mu \neq 0$ requires further investigation, and will be the subject of a subsequent paper.

REFERENCES

[1] O. Barndorff-Nielsen and P. Blaesild (1983), *Reproductive exponential families*, Ann. Stat., 11, pp. 770–782.

[2] P. Buckholtz and M. Wasan (1979), *First passage probabilities of a two dimensional Brownian motion in an anisotropic medium*, Sankhyā A, 41, pp. 198–206.

[3] H. Carslaw and J. Jaeger (1959), *Conduction of Heat in Solids*, Oxford Univ. Press, Cambridge.

[4] H. Daniels (1982), *Sequential tests constructed from images*, Ann. Stat., 10, pp. 394–400.

[5] S. Fienberg (1974), *Stochastic models for single neuron firing trains: a survey*, Biometrics, 30, pp. 399–427.

[6] L. Folks and R. Chhikara (1978), *The inverse Gaussian distribution and its statistical application—A review*, JRSS B, 40, pp. 263–289.

[7] G. Gerstein and B. Mandelbrot (1964), *Random walk models for the spike activity of a single neuron*, Biophys. J., 4, pp. 41–68.

[8] W. Magnus, F. Oberhettinger and R. Soni (1966), *Formulas and Theorems for the Special Functions of Mathematical Physics*, Springer-Verlag, Berlin.

[9] A. Sommerfeld (1894), in Mathematische Annalen, 45, pp. 263ff.

[10] F. Spitzer (1958), *Some theorems concerning two dimensional Brownian motion*, Trans. Amer. Math. Soc., 87, pp. 187–197.

[11] M. T. Wasan (1969), *First Passage Time Distribution of Brownian Motion with Positive Drift*, Queen's Univ. Papers on Pure and Applied Mathematics, 19.

[12] ——— (1973), *Differential representation of a bivariate inverse Gaussian process*, J. Mult. Analysis, 3, pp. 243–247.