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Ruin probability in a two-dimensional model with correlated Brownian motions

Peter Grandits and Maike Klein

Institute of Statistics and Mathematical Methods in Economics, TU Wien, Wien, Austria

ABSTRACT
We consider two insurance companies with endowment processes given by Brownian motions with drift. The firms can collaborate by transfer payments in order to maximize the probability that none of them goes bankrupt. We show that pushing maximally the company with less endowment is the optimal strategy for the collaboration if the Brownian motions are correlated and the transfer rate can exceed the drift rates. Moreover, we obtain an explicit formula for the minimal ruin probability in case of perfectly positively correlated Brownian motions where we also allow for different diffusion coefficients.

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Introduction

We focus on two insurance companies whose endowment processes are given by correlated Brownian motions with drift. The common aim of the firms is to minimize the probability that at least one of the endowment processes falls below zero and, thus, they collaborate by transfer payments. These payments are assumed to be absolutely continuous with respect to the Lebesgue measure and to be bounded, but they can exceed the drift rates so that a company can be faced with a negative drift rate. Moreover, we assume that transfer payments can be made without friction. The problem can be interpreted as an optimal control problem which consists of minimizing the probability that the two-dimensional endowment process leaves the positive quadrant and in identifying the optimal transfer payments.

If at some point in time, one firm has high endowment and the endowment of the other firm is close to zero then it seems reasonable that the latter is maximally supported and obtains the whole available drift rate. Using this so-called push-bottom strategy turns out to be optimal no matter how big the difference between the endowment processes is: The firm with less endowment receives the maximal drift. To show this result, we use a comparison principle for stochastic differential equations (SDEs) from Ikeda & Watanabe (1977).

If the Brownian motions are perfectly positively correlated, we derive a closed formula for the value function, because only one Brownian motion is involved and we can rewrite the ruin probability in terms for which explicit formulas are available. The arguments also apply to the case where the endowment processes have different diffusion coefficients and the Brownian motions are perfectly positively correlated. The value function turns out to be a classical solution of the corresponding Hamilton-Jacobi-Bellman equation.

CONTACT Maike Klein maike.klein@fam.tuwien.ac.at Institute of Statistics and Mathematical Methods in Economics, TU Wien, Wiedner Hauptstr. 8 / E105-1&-5 FAM, Wien 1040, Austria

This control problem was first studied by McKean & Shepp (2006) for independent Brownian motions and the value function was derived in the case that the transfer payments are at most as high as the drift rates.

In case that each company keeps a given minimal positive drift rate, the value function and the gain of collaboration for independent Brownian motions are obtained in Grandits (2019a) by constructing a classical solution of the associated Hamilton–Jacobi–Bellman equation.

Although the ruin probability is one of the most important evaluation criteria for insurance companies, there are only few articles dealing with two or more companies. For an overview of the one-dimensional case, consult Asmussen and Albrecher (2010). In Chan et al. (2003), a two-dimensional model is analyzed and simple bounds for the ruin probability are obtained by using results from the one-dimensional case. The Laplace transform in the initial endowments of the probability that at least one of the two companies is ruined in finite time is derived in Avram et al. (2008a). Collamore (1996) investigates the probability that a d-dimensional discrete process hits a d-dimensional set A and obtains some large deviation results. For specific choices of the set A, the hitting probability can be seen as a ruin probability. The asymptotic behavior of the ruin probability if the initial endowments both tend to infinity under a light-tails assumption on the claim size distribution is analyzed in Avram et al. (2008b).

Let us emphasize that also for the maximal expected aggregated dividend payments, which is another main evaluation criteria for insurance companies, the literature in the multidimensional setting is scarce. The optimal collaboration for maximizing the total dividend payments of two companies in different models is analyzed for example in Gerber and Shiu (2006), Albrecher et al. (2017), Gu et al. (2018), and Grandits (2019c).

The paper is organized as follows. In Section 1, we introduce our model. We derive the optimal strategy for the transfer payments in order to minimize the ruin probability in Section 2. In Section 3.1, we focus on perfectly positively correlated Brownian motions and compute the value function explicitly. The same arguments are extended to a model with different diffusion coefficients in Section 3.2. Finally, we rewrite the minimal ruin probability for perfectly negatively correlated Brownian motions in terms of the hitting probability of a reflected Brownian motion with drift in Section 4.

1. Model

The endowment processes of the two companies are described by the stochastic processes

$$X_t^x = x + \mu_1 t + W_t + \int_0^t c_s \, \mathrm{d}s,$$
$$Y_t^y = y + \mu_2 t + \widehat{W}_t - \int_0^t c_s \, \mathrm{d}s, \quad t \in [0, \infty),$$

where x, $y > 0$ denote the initial endowments, $\mu_1, \mu_2 > 0$ are constant cash rates, e.g. premium rates, $\widehat{W}_t = \rho W_t + \sqrt{1-\rho^2}\, W_t^{(2)}$, $t \in [0, \infty)$, with $(W_t, W_t^{(2)})_{t\in[0,\infty)}$ a two-dimensional Brownian motion. Here $\rho \in [-1, 1]$ is the correlation coefficient of $(W_t)_{t\in[0,\infty)}$ and $(\widehat{W}_t)_{t\in[0,\infty)}$. The drift rates $(c_s)_{s\in[0,\infty)}$ can be interpreted as transfer payments from one company to the other one. More precisely, if $c_s > 0$, then the first company obtains payments from the second company at time s and if $c_s < 0$ it is vice versa. We say that the companies do not collaborate if $c_s = 0$ for all $s \in [0, \infty)$. We assume that the transfer payments are bounded in such a way that the total drift rate of each company is bounded below by $-\delta$, thus,

$$c_s \in [-\mu_1 - \delta,\ \mu_2 + \delta]$$

for some $\delta > -\min\{\mu_1, \mu_2\}$.

Introducing the control process $u_s := \mu_1 + c_s$, the endowment processes are given by

$$\begin{aligned} X_t^{x,u} &= x + W_t + \int_0^t u_s \, ds, \\ Y_t^{y,u} &= y + \widehat{W}_t + \int_0^t (\bar{\mu} - u_s) \, ds, \end{aligned} \tag{1}$$

where $u_s \in [-\delta, \bar{\mu} + \delta]$ and $\bar{\mu} = \mu_1 + \mu_2$.

We aim at maximizing the probability that both firms survive forever. For this purpose denote by

$$\tau_X(x; u) = \inf\{t \in [0, \infty) \colon X_t^{x,u} \le 0\},$$
$$\tau_Y(y; u) = \inf\{t \in [0, \infty) \colon Y_t^{y,u} \le 0\}$$

the ruin times of the first and second company, respectively, when the control u is used. Let

$$\tau(x, y; u) = \tau_X(x; u) \wedge \tau_Y(y; u).$$

Our target functional is then given by

$$J(x, y; u) = \mathbb{P}[\tau(x, y; u) = \infty]$$

and the value function is

$$V(x, y) = \sup_{u \in \mathcal{U}} J(x, y; u), \tag{2}$$

where $\mathcal{U}$ denotes the set of all admissible controls. More precisely, $\mathcal{U}$ is the set of all progressively measurable processes $(u_s)_{s\in[0,\infty)}$ with respect to the filtration generated by $(W_t, \widehat{W}_t)_{t\in[0,\infty)}$ satisfying $u_s \in [-\delta, \bar{\mu} + \delta]$, $s \in [0, \infty)$.

For $\bar{\mu} = 1$ and $\rho = 0$, we obtain the same model as in the paper by McKean & Shepp (2006); for $\rho = 0$ and $\delta \in (-\min\{\mu_1, \mu_2\}, 0)$ we are in the setting of Grandits (2019a).

The Hamilton–Jacobi–Bellman equation and the boundary conditions for the optimal control problem (2) are given by

$$\begin{aligned} \left(\frac{1}{2}V_{xx} + \frac{1}{2}V_{yy} + \rho V_{xy} + (\bar{\mu} + \delta)\max\{V_x, V_y\} - \delta \min\{V_x, V_y\}\right)(x, y) = 0 \quad \text{on } (0, \infty) \times (0, \infty), \\ V(x, 0) = V(0, y) = 0, \\ \lim_{x\to\infty} V(x, y) = 1 - \exp\big(-2(\bar{\mu} + \delta)y\big), \\ \lim_{y\to\infty} V(x, y) = 1 - \exp\big(-2(\bar{\mu} + \delta)x\big). \end{aligned}$$

Remark 1.1: We can interpret (2) as a tax policy problem, where the state can influence the endowment of the companies by imposing some kind of taxes, see McKean & Shepp (2006).

Remark 1.2: Throughout this article, we assume that transfer payments can be made frictionless. For future research, it would be interesting to investigate models with friction. There are several ways to add friction. For example, we can modify the underlying processes as follows

$$\begin{aligned} X_t^x &= x + \mu_1 t + W_t + \int_0^t \left(\kappa_1 c_s^+ - c_s^-\right) ds, \\ Y_t^y &= y + \mu_2 t + \widehat{W}_t + \int_0^t \left(\kappa_2 c_s^- - c_s^+\right) ds, \quad t \in [0, \infty), \end{aligned} \tag{3}$$

where $c_s \in [-\mu_1 - \delta, \mu_2 + \delta]$, $s \in [0, \infty)$, for some $\delta > -\min\{\mu_1, \mu_2\}$, c_s^+ and c_s^- denote the positive and negative part of c_s, respectively, and $\kappa_1, \kappa_2 \in [0, 1]$ capture the friction. Hence, the first (second) company only obtains a fraction of κ_1 (κ_2) of every transfer payment which is made by the second (first) company. As far as we can tell this problem cannot be tackled with the arguments we present here. Already the reformulation of (3) to processes similar to those in (1) fails and hence, the transformation we are using in Section 2 for deriving the optimal strategy is not applicable.

2. The optimal strategy for the transfer payments

For deriving the optimal strategy in the control problem (2), we use a comparison theorem for solutions of stochastic differential equations. We focus on the case $|\rho| < 1$ because it is more involved and the arguments simplify for $|\rho| = 1$ and, thus, are omitted.

First, consider the transformation

$$\begin{aligned}
Z_t^{(1)} &= X_t^{x,u} + Y_t^{y,u} = x + y + \bar{\mu}t + (1+\rho)W_t + \sqrt{1-\rho^2}\, W_t^{(2)},\\
Z_t^{(2),\nu} &= Y_t^{y,u} - X_t^{x,u} = y - x + \int_0^t (\bar{\mu} - 2u_s)\,\mathrm{d}s + (\rho - 1)W_t + \sqrt{1-\rho^2}\, W_t^{(2)}\\
&= y - x + \int_0^t \nu_s\,\mathrm{d}s + (\rho-1)W_t + \sqrt{1-\rho^2}\, W_t^{(2)},
\end{aligned}$$

where $\nu_s := \bar{\mu} - 2u_s$ and $\nu_s \in [-(\bar{\mu}+2\delta), \bar{\mu}+2\delta]$, $s \in [0,\infty)$. Observe that $Z^{(1)}$ does not depend on the control $\nu = (\nu_s)_{s\in[0,\infty)}$. Furthermore, it holds that

$$\begin{aligned}
\left\langle (1+\rho)W + \sqrt{1-\rho^2}\, W^{(2)} \right\rangle_t &= 2(1+\rho)t,\\
\left\langle (\rho-1)W + \sqrt{1-\rho^2}\, W^{(2)} \right\rangle_t &= 2(1-\rho)t,
\end{aligned}$$

and

$$\left\langle (1+\rho)W + \sqrt{1-\rho^2}\, W^{(2)}, (\rho-1)W + \sqrt{1-\rho^2}\, W^{(2)} \right\rangle_t = \big((1+\rho)(\rho-1) + 1 - \rho^2\big)t = 0.$$

We rescale the diffusion parts of the processes $Z^{(1)}$ and $Z^{(2),\nu}$ and define

$$B_t^{(1)} = \sqrt{\frac{1+\rho}{2}}\, W_t + \sqrt{\frac{1-\rho}{2}}\, W_t^{(2)}, \quad B_t^{(2)} = -\sqrt{\frac{1-\rho}{2}}\, W_t + \sqrt{\frac{1+\rho}{2}}\, W_t^{(2)}.$$

Hence, we conclude that $(B_t^{(1)})_{t\in[0,\infty)}$, $(B_t^{(2)})_{t\in[0,\infty)}$ are independent Brownian motions and that

$$\begin{aligned}
Z_t^{(1)} &= x + y + \bar{\mu}t + \sqrt{2(1+\rho)}\, B_t^{(1)},\\
Z_t^{(2),\nu} &= y - x + \int_0^t \nu_s\,\mathrm{d}s + \sqrt{2(1-\rho)}\, B_t^{(2)},
\end{aligned}$$

where $(\nu_s)_{s\in[0,\infty)} \in \mathcal{V}$. Here $\mathcal{V}$ denotes the set of all progressively measurable processes $(\tilde{\nu}_s)_{s\in[0,\infty)}$ with respect to the filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$, which is generated by $(\sqrt{2(1+\rho)}B_t^{(1)})_{t\in[0,\infty)}$ and $(\sqrt{2(1-\rho)}B_t^{(2)})_{t\in[0,\infty)}$, and $(\tilde{\nu}_s)_{s\in[0,\infty)}$ satisfies $\tilde{\nu}_s \in [-(\bar{\mu}+2\delta), \bar{\mu}+2\delta]$, $s \in [0,\infty)$. Note that $(\mathcal{F}_t)_{t\in[0,\infty)}$ and the filtration generated by $(W_t, \widehat{W}_t)_{t\in[0,\infty)}$ coincide.

So far all arguments hold for $\rho \in [-1,1]$. Now let $|\rho| < 1$. Then $(\mathcal{F}_t)_{t\in[0,\infty)}$ also coincides with the filtration generated by the Brownian motion $(B_t^{(1)}, B_t^{(2)})_{t\in[0,\infty)}$. In particular, there exists a measurable function

$$\nu\colon [0,\infty) \times C\big([0,\infty);\mathbb{R}\big) \times C\big([0,\infty);\mathbb{R}\big) \to [-(\bar{\mu}+2\delta), \bar{\mu}+2\delta]$$

such that for every $s \in [0,\infty)$

$$\nu_s = \nu\left(s, (B_{r\wedge s}^{(1)})_{r\in[0,\infty)}, (B_{r\wedge s}^{(2)})_{r\in[0,\infty)}\right).$$

We rewrite $\tau(x,y;u)$ in terms of $Z^{(1)}$ and $Z^{(2),\bar{\mu}-2u}$. More precisely,

$$\tau(x,y;u) = \inf\left\{t \in [0,\infty) \colon \left|Z_t^{(2),\bar{\mu}-2u}\right| \geq Z_t^{(1)}\right\} =: \tau^{\bar{\mu}-2u}.$$

Hence,

$$V(x,y) = \sup_{\nu \in \mathcal{V}} \mathbb{P}[\tau^\nu = \infty] \tag{4}$$

and for every optimal $\nu^* \in \mathcal{V}$ in (4) we obtain an optimal $u^* \in \mathcal{U}$ for (2) by setting $u^* = (\bar{\mu} - \nu^*)/2$.

To characterize an optimal control ν^* for (4) denote by $\mathcal{F}^{(1)}$ the filtration generated by $(B_t^{(1)})_{t\in[0,\infty)}$, i.e.

$$\mathcal{F}^{(1)} = \sigma\left(B_s^{(1)} \colon s \in [0,\infty)\right),$$

and observe that

$$\begin{aligned}\mathbb{P}[\tau^\nu = \infty] &= \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\left\{\left|Z_t^{(2),\nu}\right| < Z_t^{(1)}\ \forall\, t\in[0,\infty)\right\}}\middle|\mathcal{F}^{(1)}\right]\right]\\ &= \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\left\{\left|y-x+\int_0^t \nu\left(s,\left(B_{r\wedge s}^{(1)}\right)_{r\in[0,\infty)},\left(B_{r\wedge s}^{(2)}\right)_{r\in[0,\infty)}\right)\mathrm{d}s+\sqrt{2(1-\rho)}\,B_t^{(2)}\right| < Z_t^{(1)}\ \forall\, t\in[0,\infty)\right\}}\middle|\mathcal{F}^{(1)}\right]\right].\end{aligned}$$

Note that $(B_t^{(1)})_{t\in[0,\infty)}$ and $(Z_t^{(1)})_{t\in[0,\infty)}$ are measurable with respect to $\mathcal{F}^{(1)}$ and that $(B_t^{(2)})_{t\in[0,\infty)}$ is independent of $\mathcal{F}^{(1)}$. Now choose a realization of $B^{(1)}$. In particular, $B_t^{(1)} := f(t)$ and $Z_t^{(1)} := g(t)$ are fixed. Furthermore, let

$$\hat{\nu}\big(s, (B_{r\wedge s}^{(2)})_{r\in[0,\infty)}\big) = \nu\big(s, (f(r\wedge s))_{r\in[0,\infty)}, (B_{r\wedge s}^{(2)})_{r\in[0,\infty)}\big).$$

If the optimal strategy $\hat{\nu}^*$ for maximizing

$$\mathbb{E}\left[\mathbb{1}_{\left\{\left|y-x+\int_0^t \hat{\nu}\left(s,(B_{r\wedge s}^{(2)})_{r\in[0,\infty)}\right)\mathrm{d}s+\sqrt{2(1-\rho)}\,B_t^{(2)}\right|<g(t)\ \forall t\in[0,\infty)\right\}}\right] \tag{5}$$

is independent of f and g, it is also optimal for maximizing $\mathbb{P}[\tau^\nu = \infty]$ over all $\nu \in \mathcal{V}$. To identify $\hat{\nu}^*$ one can extend the arguments from Theorem 2.1 in Ikeda & Watanabe (1977) and its corollary to the case, where the control does not take values in $[-1,1]$ but in $[-(\bar{\mu}+2\delta), \bar{\mu}+2\delta]$ and the diffusion coefficient equals $\sqrt{2(1-\rho)}$ instead of 1. Observe that the required filtration is given by $(\mathcal{F}_t)_{t\in[0,\infty)}$ and that $(B_t^{(2)})_{t\in[0,\infty)}$ is an $(\mathcal{F}_t)_{t\in[0,\infty)}$-Brownian motion. Then for all $\hat{\nu}$ we have

$$\begin{aligned}\mathbb{P}\left[\left|y-x+\int_0^t \hat{\nu}\big(s,(B_{r\wedge s}^{(2)})_{r\in[0,\infty)}\big)\,\mathrm{d}s + \sqrt{2(1-\rho)}\,B_t^{(2)}\right| < g(t)\ \forall t\in[0,\infty)\right]&\\ \leq \mathbb{P}\left[Z_t^* < g(t)\ \forall\, t\in[0,\infty)\right]&,\end{aligned}$$

where $(Z_t^*)_{t\in[0,\infty)}$ satisfies

$$\mathrm{d}Z_t^* = -(\bar{\mu}+2\delta)\,\mathrm{sign}(Z_t^*)\,\mathrm{d}t + \sqrt{2(1-\rho)}\,\mathrm{d}B_t^{(2)}, \quad Z_0^* = y - x \tag{6}$$

with

$$\mathrm{sign}(z) = \begin{cases}1, & z>0,\\ 0, & z=0,\\ -1, & z<0.\end{cases}$$

The existence of a strong solution of (6) which is strongly unique follows from Theorem 1 in Veretennikov (1983). In particular, there exists a measurable function F such that

$$Z_t^* = F\big((B^{(2)}_{s\wedge t})_{s\in[0,\infty)}\big), \quad t \in [0,\infty),$$

and, thus, the drift of Z_t^* is a measurable function of the Brownian motion $B^{(2)}$ up to time $t \in [0,\infty)$. Hence, the optimal control $\hat{v}^*$ for maximizing (5) is given by

$$\hat{v}^*\big(s, (B^{(2)}_{r\wedge s})_{r\in[0,\infty)}\big) = -(\bar{\mu} + 2\delta)\,\mathrm{sign}\Big(F\left((B^{(2)}_{r\wedge s})_{r\in[0,\infty)}\right)\Big), \quad s \in [0,\infty),$$

and, in particular, it is independent of f and g. Therefore,

$$v_s^* = -(\bar{\mu} + 2\delta)\,\mathrm{sign}\big(Z_s^{(2),v^*}\big), \quad s \in [0,\infty), \tag{7}$$

is optimal for (4). By setting $u_s^* = \frac{\bar{\mu} - v_s^*}{2}$, we obtain an optimal control for (2).

For $|\rho| = 1$ one also uses the arguments from Ikeda & Watanabe (1977) and obtains the same optimal strategy.

Remark 2.1: Observe that one can change the definition of the optimal strategy u^* on the set $\{X_t^{x,u^*} = Y_t^{y,u^*}\} = \{Z_t^* = 0\}$, $t \in [0,\infty)$, and then obtains an indistinguishable process, because with probability one the set $\{t \in [0,\infty) : Z_t^* = 0\}$, where $(Z_t^*)_{t\in[0,\infty)}$ is a solution of (6), has Lebesgue measure zero, for details see Appendix C in Beneš (1976).

We now summarize the result in the following theorem.

Theorem 2.2: *Let $\rho \in [-1,1]$. The optimal drift rate $(u_s^*)_{s\in[0,\infty)}$ in (2) for minimizing the ruin probability is given by*

$$u_s^* = u_s^*\left(X_s^{x,u^*}, Y_s^{y,u^*}\right) = (\bar{\mu} + \delta)\mathbb{1}_{\left\{X_s^{x,u^*} \le Y_s^{y,u^*}\right\}} - \delta\mathbb{1}_{\left\{X_s^{x,u^*} > Y_s^{y,u^*}\right\}}, \quad s \in [0,\infty).$$

Moreover, the optimal transfer rate is $c_s^* = u_s^* - \mu_1$.

Theorem 2.2 implies that the company with less endowment is as much supported as possible, i.e. this company receives a drift of $\bar{\mu} + \delta$ until the endowment processes of the two companies are equal: The push-bottom strategy is optimal.

Remark 2.3: If one aims at maximizing the expected number of surviving firms instead of the joint survival probability, then Grandits (2019b) shows that for independent Brownian motions it is not always optimal to use the push-bottom strategy. This fact was first observed by McKean & Shepp (2006) who solved the associated Hamilton-Jacobi-Bellman equation numerically and identified regions where it seems optimal to push the top company.

Remark 2.4: In Fernholz et al. (2013a) and Fernholz et al. (2013b), the authors analyze two diffusion processes where the drift and diffusion coefficients are rank-dependent and the Brownian motions are independent. The leader obtains a negative drift coefficient and the laggard a positive one. For the isotropic case, i.e. for the same diffusion coefficients, this corresponds to the optimal controlled processes X^{x,u^*} and Y^{y,u^*} for $\rho = 0$ in our setting.

Remark 2.5: For independent Brownian motions, i.e. $\rho = 0$, the value function can be computed explicitly. McKean & Shepp (2006) show that for $\delta = 0$ and $\bar{\mu} = 1$ the value function is given by

$$V(x,y) = 1 - e^{-2\min\{x,y\}} - 2\min\{x,y\}\,e^{-x-y}.$$

For $\delta \in (-\min\{\mu_1, \mu_2\}, 0)$ Grandits (2019a) obtains in Theorem 4.1 the following value function

$$V(x, y) = 1 - e^{-2(\bar{\mu}+\delta)\min\{x,y\}} - \frac{\bar{\mu}+\delta}{\delta} e^{-\bar{\mu}(x+y)} \left(1 - e^{-2\delta \min\{x,y\}}\right).$$

Remark 2.6: Even though an optimal strategy for (2) is identified in Theorem 2.2, so far only in the special cases $\rho = 0$ and $\rho = 1$ a closed form for the value function can be derived, see Remark 2.5 and Section 3.

Hence, it seems difficult to guess an optimal strategy and prove its optimality by a verification result. For $\rho = 0$, McKean and Shepp (2006) use a verification argument to confirm the guessed value function and the optimal strategy. Also in the case $\rho = 1$, a verification approach can be applied, see Remark 3.3.

3. Perfectly positive correlation: $\rho = 1$

In this section, we consider perfectly positively correlated Brownian motions, i.e. $\rho = 1$, which implies $\widehat{W}_t = W_t$, and derive an explicit formula for the value function (2). In Section 3.1, we deal with the simplest case where $\delta = 0$ and $\bar{\mu} = 1$. For $\rho = 1$, the same arguments can be used for endowment processes X and Y having different diffusion coefficients. Hence, we extend our model and state the value function for different diffusion coefficients $\sigma_1 > 0$ and $\sigma_2 > 0$ in Section 3.2. In both sections, we also compute the gain of collaboration.

3.1. Deriving the value function for $\delta = 0$ and $\bar{\mu} = 1$

We now derive the value function for $\delta = 0$ and $\bar{\mu} = 1$. The same arguments extend to the case $\delta > -\min\{\mu_1, \mu_2\}$ with $\bar{\mu} > 0$ but lead to more complicated terms. Therefore, we first focus on this simple case.

Let $\delta = 0$, $\bar{\mu} = 1$. As in Section 2, consider the processes

$$Z_t^{(1)} = X_t^{x,u} + Y_t^{y,u} = x + y + 2W_t + t,$$
$$Z_t^{(2),v} = Y_t^{y,u} - X_t^{x,u} = y - x + \int_0^t (1 - 2u_s)\, ds = y - x + \int_0^t v_s\, ds,$$

where $v_s := 1 - 2u_s$ and $v_s \in [-1, 1]$, $s \in [0, \infty)$. Recall that $Z^{(1)}$ does not depend on the control $v = (v_s)_{s\in[0,\infty)}$,

$$\tau(x, y; u) = \inf\left\{t \in [0, \infty) : Z_t^{(1)} \le \left|Z_t^{(2),1-2u}\right|\right\} =: \tau^{1-2u}$$

and that

$$V(x, y) = \sup_{v\in\mathcal{V}} \mathbb{P}[\tau^v = \infty].$$

Here $\mathcal{V}$ is the set of all progressively measurable processes $(v_s)_{s\in[0,\infty)}$ with respect to the filtration generated by $(W_t)_{t\in[0,\infty)}$ satisfying $v_s \in [-1, 1]$ for all $s \in [0, \infty)$.

Since $Z^{(1)}$ is a Brownian motion with drift and it is not controllable by v and $Z^{(2),v}$ does not depend on the Brownian motion $(W_t)_{t\in[0,\infty)}$, the best strategy is to control $Z^{(2),v}$ in such a way that it is going to be zero with the highest possible rate and then let $Z^{(2),v}$ stay zero. In particular,

$$v_s^* = -\operatorname{sign}\left(Z_s^{(2),v^*}\right), \quad s \in [0, \infty),$$

is optimal. Note that the corresponding optimal strategy u^* in (2) is given by $u_s^* = \mathbb{1}_{\{X_s^{x,u^*} \le Y_s^{y,u^*}\}}$, which we have already seen in Theorem 2.2; also recall Remark 2.1.

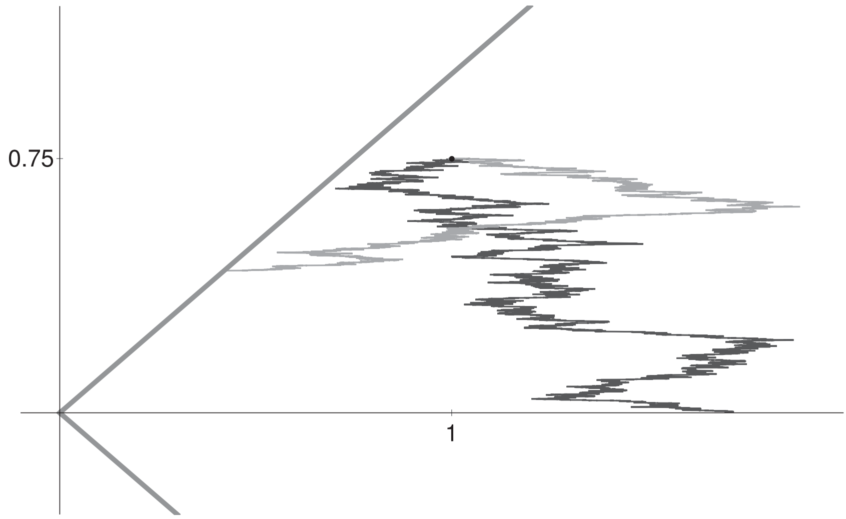


Figure 1. Two trajectories of $(Z^{(1)}, Z^{(2),\nu^*})$. In case of the light gray trajectory ruin occurs before $Z^{(2),\nu^*}$ becomes zero, i.e. before time $y-x$. In the other case, ruin has not occurred until time $y-x$.

Now assume that $y > x$. Then the optimal controlled process $Z^{(2),\nu^*}$ is given by

$$Z_t^{(2),\nu^*} = \max\{y-x-t, 0\} = \begin{cases} y-x-t, & t \le y-x, \\ 0, & t \ge y-x. \end{cases}$$

For computing the value function, we focus on $\mathbb{P}[\tau^{\nu^*} < \infty]$. On the set $\{\tau^{\nu^*} < \infty\}$ the stopping time τ^{ν^*} either occurs before the process $Z^{(2),\nu^*}$ becomes zero or afterwards. For two possible trajectories, see Figure 1. Hence,

$$\begin{aligned}\mathbb{P}[\tau^{\nu^*} < \infty] &= \mathbb{P}[\tau^{\nu^*} \le y-x] + \mathbb{P}[y-x < \tau^{\nu^*} < \infty] \\ &= \mathbb{P}\left[Z_t^{(1)} \le Z_t^{(2),\nu^*} \text{for some } t \in [0, y-x]\right] + \int_{(0,\infty)} \mathbb{P}\left[y-x < \tau^{\nu^*} < \infty,\, Z_{y-x}^{(1)} \in \mathrm{d}w\right].\end{aligned}$$

We compute the two summands separately.

$$\begin{aligned}\mathbb{P}\left[Z_t^{(1)} \le Z_t^{(2),\nu^*} \text{for some } t \in [0, y-x]\right] &= \mathbb{P}\left[\inf_{0 \le t \le y-x} \{W_t + t\} \le -x\right] \\ &= 1 - \Phi\left(\frac{y}{\sqrt{y-x}}\right) + \exp(-2x)\, \Phi\left(\frac{y-2x}{\sqrt{y-x}}\right), \qquad (8)\end{aligned}$$

where Φ denotes the cumulative distribution function of a standard normal distribution. The last equality follows from Formula 1.2.4 in Part II, Chapter 2 of Borodin & Salminen (2002).

For the second summand, we obtain

$$\begin{aligned}&\int_{(0,\infty)} \mathbb{P}\left[y-x < \tau^{\nu^*} < \infty,\, Z_{y-x}^{(1)} \in \mathrm{d}w\right] \\ &= \int_{(0,\infty)} \mathbb{P}\left[\tau^{\nu^*} < \infty \,\middle|\, \tau^{\nu^*} > y-x,\, Z_{y-x}^{(1)} = w\right] \mathbb{P}\left[\tau^{\nu^*} > y-x,\, Z_{y-x}^{(1)} \in \mathrm{d}w\right]\end{aligned}$$

$$= \int_{(0,\infty)} \mathbb{P}\left[Z_t^{(1)} \leq 0 \text{ for some } t > y-x \,\middle|\, \tau^{\nu^*} > y-x, Z_{y-x}^{(1)} = w\right] \mathbb{P}\left[\tau^{\nu^*} > y-x,\, Z_{y-x}^{(1)} \in \mathrm{d}w\right]$$

$$= \int_{(0,\infty)} \mathbb{P}\left[\inf_{t\geq 0}\left\{w + t + 2\left(W_{y-x+t} - W_{y-x}\right)\right\} \leq 0\right] \mathbb{P}\left[\tau^{\nu^*} > y-x,\, Z_{y-x}^{(1)} \in \mathrm{d}w\right]$$

$$= \int_{(0,\infty)} \exp\left(-\frac{w}{2}\right) \mathbb{P}\left[\tau^{\nu^*} > y-x,\, Z_{y-x}^{(1)} \in \mathrm{d}w\right], \tag{9}$$

where the last equality follows from Formula 1.2.4 (1) in Part II, Chapter 2 of Borodin & Salminen (2002). For the remaining probability, we have

$$\begin{aligned}
&\mathbb{P}[\tau^{\nu^*} > y-x,\, Z_{y-x}^{(1)} \in \mathrm{d}w] \\
&\quad = \mathbb{P}\left[Z_t^{(1)} > Z_t^{(2),\nu^*} \text{ for all } t \in [0, y-x],\, Z_{y-x}^{(1)} \in \mathrm{d}w\right] \\
&\quad = \mathbb{P}\left[\inf_{0\leq t\leq y-x}\{W_t + t\} > -x,\, y + W_{y-x} \in \frac{\mathrm{d}w}{2}\right] \\
&\quad = \mathbb{P}\left[y + W_{y-x} \in \frac{\mathrm{d}w}{2}\right] - \mathbb{P}\left[\inf_{0\leq t\leq y-x}\{W_t + t\} \leq -x,\, W_{y-x} + y - x \in \frac{\mathrm{d}w}{2} - x\right] \\
&\quad = \frac{1}{2\sqrt{2\pi(y-x)}}\left[\exp\left(-\frac{(w-2y)^2}{8(y-x)}\right) - \exp\left(\frac{w-x-y}{2} - \frac{(w+2x)^2}{8(y-x)}\right)\right]\mathrm{d}w,
\end{aligned} \tag{10}$$

where we use Formula 1.2.8 in Part II, Chapter 2 of Borodin & Salminen (2002) in the last equality. Hence, combining (9) and (10) yields

$$\int_{(0,\infty)} \mathbb{P}\left[y - x < \tau^{\nu^*} < \infty,\, Z_{y-x}^{(1)} \in \mathrm{d}w\right] = \exp\left(-\frac{x+y}{2}\right)\left[2\Phi\left(\frac{x}{\sqrt{y-x}}\right) - 1\right]. \tag{11}$$

Therefore, we conclude from (8) and (11) that for $y > x$ it holds that

$$V(x,y) = \Phi\left(\frac{y}{\sqrt{y-x}}\right) - \exp(-2x)\,\Phi\left(\frac{y-2x}{\sqrt{y-x}}\right) - \exp\left(-\frac{x+y}{2}\right)\left[2\Phi\left(\frac{x}{\sqrt{y-x}}\right) - 1\right].$$

For $y = x$ we have that $Z_t^{(2),\nu^*} = 0$ for all $t \in [0,\infty)$. Therefore,

$$V(x,x) = \mathbb{P}\left[Z_t^{(1)} > 0 \text{ for all } t \in [0,\infty)\right] = 1 - \exp(-x).$$

For $y < x$, we use the symmetry of the problem and conclude that $V(x,y) = V(y,x)$.

To summarize, we have shown the following result.

Theorem 3.1: *For $\rho = 1$, $\delta = 0$ and $\bar{\mu} = 1$ the value function* (2) *is given by*

$$V(x,y) = \begin{cases} 1 - \exp(-x) = 1 - \exp(-y), & y = x, \\ \Phi\left(\frac{\max\{x,y\}}{\sqrt{|y-x|}}\right) - \exp\left(-2\min\{x,y\}\right)\Phi\left(\frac{|y-x| - \min\{x,y\}}{\sqrt{|y-x|}}\right) & y \neq x. \\ \quad - \exp\left(-\frac{x+y}{2}\right)\left(2\Phi\left(\frac{\min\{x,y\}}{\sqrt{|y-x|}}\right) - 1\right), & \end{cases}$$

Figure 2 depicts the value function of Theorem 3.1.

Remark 3.2: One can show that the function V stated in Theorem 3.1 satisfies

$$V \in C^2\left((0,\infty)^2\right) \cap C\left([0,\infty)^2\right),$$

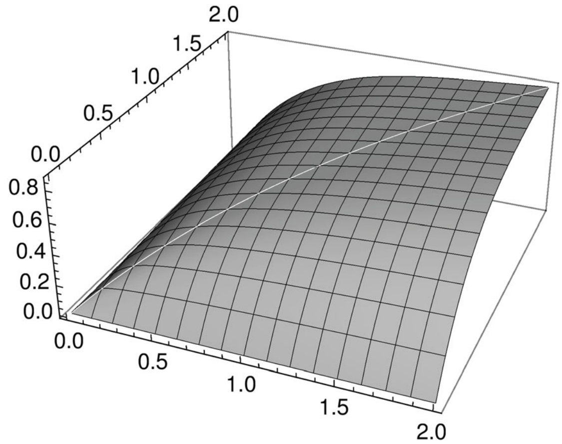


Figure 2. The value function V for $\rho = 1, \delta = 0$ and $\bar{\mu} = 1$.

V solves

$$\begin{aligned} \frac{1}{2}V_{xx}(x,y) + \frac{1}{2}V_{yy}(x,y) + V_{xy}(x,y) + \max\{V_x(x,y), V_y(x,y)\} &= 0, \\ V(x,0) = V(0,y) &= 0, \\ \lim_{x\to\infty} V(x,y) &= 1-\exp(-2y), \\ \lim_{y\to\infty} V(x,y) &= 1-\exp(-2x), \end{aligned}$$

and

$$V_x(x,y) - V_y(x,y) = \begin{cases} 2\exp(-2x)\Phi\left(\frac{y-2x}{\sqrt{y-x}}\right) > 0, & x < y, \\ -2\exp(-2y)\Phi\left(\frac{x-2y}{\sqrt{x-y}}\right) < 0, & x > y, \\ 0, & x = y. \end{cases}$$

Therefore,

$$\begin{aligned} \left\{(x,y) \in (0,\infty)^2 \colon V_x(x,y) > V_y(x,y)\right\} &= \left\{(x,y) \in (0,\infty)^2 \colon x < y\right\}, \\ \left\{(x,y) \in (0,\infty)^2 \colon V_x(x,y) < V_y(x,y)\right\} &= \left\{(x,y) \in (0,\infty)^2 \colon x > y\right\}, \\ \left\{(x,y) \in (0,\infty)^2 \colon V_x(x,y) = V_y(x,y)\right\} &= \left\{(x,y) \in (0,\infty)^2 \colon x = y\right\}. \end{aligned}$$

Remark 3.3: Another possibility for proving Theorem 3.1 is to use a classical verification theorem. Here one proceeds as follows:

- Guess a candidate for the optimal strategy.

  (The intuitive optimal solution (v_s^*) in (7) is indeed optimal, see Theorem 2.2.)
- Compute the value function $w(x,y)$ of the chosen strategy.

  (For (v_s^*) use the same arguments as in the proof of Theorem 3.1.)
- Show that $w \in C^2((0,\infty)^2) \cap C([0,\infty)^2)$.
- Apply a classical verification theorem.

We now compare the gain of collaboration. If the two firms do not collaborate, i.e. $u_s = \mu_1$ for all $s \in [0,\infty)$, then the survival probability of both firms is given by

$$\mathbb{P}[X_t^{x,\mu_1} > 0,\ Y_t^{y,\mu_1} > 0 \text{ for all } t \in [0,\infty)]. \tag{12}$$

For the cases $\mu_1 = \mu_2$, $\mu_1 > \mu_2$ with $x \geq y$ and $\mu_1 < \mu_2$ with $x \leq y$, the endowment of one company is for all $t \in [0,\infty)$ lower than the other company's endowment. Thus, (12) is just the survival probability of the firm with lower endowment and we have

$$\mathbb{P}[X_t^{x,\mu_1} > 0,\ Y_t^{y,\mu_1} > 0 \text{ for all } t \in [0,\infty)] = 1 - \exp\left(-2\min\left\{\mu_1 x, \mu_2 y\right\}\right).$$

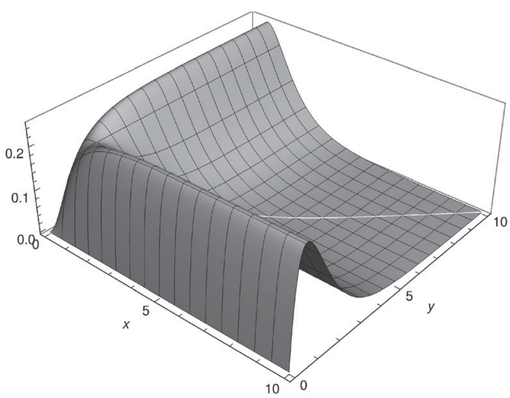
For $\mu_1 > \mu_2$ with $x < y$ and $\mu_1 < \mu_2$ with $x > y$ the company with lower initial endowment has less endowment until time $\frac{y-x}{\mu_1-\mu_2}$ and afterwards its endowment process is always larger than the process of the company with higher initial endowment. Therefore, it becomes more involved to compute (12). We apply similar arguments as in the derivation of the value function in Theorem 3.1, in particular, we use Formulas 1.2.4, 1.2.4 (1) and 1.2.8 in Part II, Chapter 2 of Borodin & Salminen (2002).

Assume that $\mu_1 > \mu_2$ and $x < y$. Then it holds that

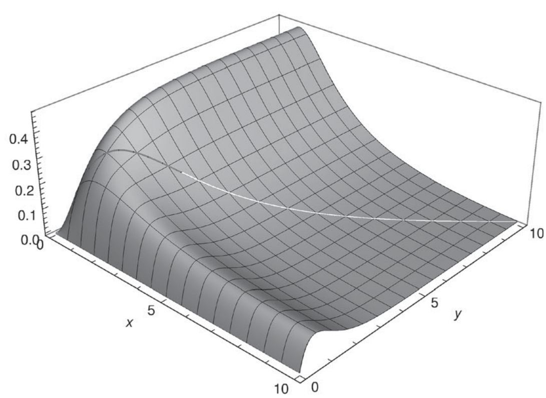
$$\begin{aligned}
&\mathbb{P}[X_t^{x,\mu_1} > 0,\ Y_t^{y,\mu_1} > 0 \text{ for all } t \in [0,\infty)] \\
&= 1 - \mathbb{P}\left[\inf_{0\leq t\leq \frac{y-x}{\mu_1-\mu_2}} X_t^{x,\mu_1} \leq 0\right] - \mathbb{P}\left[\inf_{0\leq t\leq \frac{y-x}{\mu_1-\mu_2}} X_t^{x,\mu_1} > 0,\ \inf_{t\geq \frac{y-x}{\mu_1-\mu_2}} Y_t^{y,\mu_1} \leq 0\right] \\
&= \Phi\left(\frac{\mu_1 y - \mu_2 x}{\sqrt{(y-x)(\mu_1-\mu_2)}}\right) + \exp\left((4\mu_2 - 2\mu_1)x - 2\mu_2 y\right)\Phi\left(\frac{(3\mu_2 - 2\mu_1)x + (\mu_1 - 2\mu_2)y}{\sqrt{(y-x)(\mu_1-\mu_2)}}\right) \\
&\quad - \exp(-2\mu_1 x)\,\Phi\left(\frac{\mu_1 y + (\mu_2 - 2\mu_1)x}{\sqrt{(y-x)(\mu_1-\mu_2)}}\right) - \exp\left(-2\mu_2 y\right)\Phi\left(\frac{\mu_2 x + (\mu_1 - 2\mu_2)y}{\sqrt{(y-x)(\mu_1-\mu_2)}}\right).
\end{aligned}$$

For $\mu_1 < \mu_2$ with $x > y$ change the role of x and y and the role of μ_1 and μ_2.

The gain of collaboration is then given by $V(x,y) - \mathbb{P}[X_t^{x,\mu_1} > 0,\ Y_t^{y,\mu_1} > 0 \text{ for all } t \in [0,\infty)]$. See Figure 3 for the gain of collaboration for different drift rates μ_1 and μ_2.



(a)



(b)

Figure 3. The gain of collaboration for $\rho = 1, \delta = 0, \bar{\mu} = 1$ and different drift rates μ_1 and μ_2. (a) $\mu_1 = \mu_2 = \frac{1}{2}$. and (b) $\mu_1 = \frac{1}{4}$, $\mu_2 = \frac{3}{4}$.

3.2. Different diffusion coefficients

We now extend the model (1) and allow for different diffusion coefficients for X and Y. More precisely, for $\sigma_1, \sigma_2 > 0$ let

$$\begin{aligned}\widetilde{X}_t^{x,u} &= x + \sigma_1 W_t + \int_0^t u_s \,\mathrm{d}s,\\ \widetilde{Y}_t^{y,u} &= y + \sigma_2 W_t + \int_0^t (\bar{\mu} - u_s)\,\mathrm{d}s, \quad t \in [0,\infty),\end{aligned} \tag{13}$$

where $u_s \in [-\delta\sigma_1,\ \bar{\mu} + \sigma_2\delta]$, $\delta > -\frac{\bar{\mu}}{\sigma_1+\sigma_2}$, $\bar{\mu} > 0$. Thus, the relative drift rates $\frac{u_s}{\sigma_1}$ and $\frac{\bar{\mu}-u_s}{\sigma_2}$ are bounded below by $-\delta$.

Define $\widetilde{\tau}(x,y;\sigma_1,\sigma_2;u) = \tau_{\widetilde{X}}(x;u) \wedge \tau_{\widetilde{Y}}(y;u)$ and denote by

$$V(x,y;\sigma_1,\sigma_2) = \sup_{u\in\widetilde{\mathcal{U}}} \mathbb{P}[\,\widetilde{\tau}(x,y;\sigma_1,\sigma_2;u) = \infty]$$

the value function in the extended model (13). Here $\widetilde{\mathcal{U}}$ denotes the set of progressively measurable processes $(\tilde{u}_s)_{s\in[0,\infty)}$ with respect to the filtration generated by $(W_t)_{t\in[0,\infty)}$ such that $\tilde{u}_s \in [-\delta\sigma_1, \bar{\mu} + \delta\sigma_2]$ for all $s \in [0,\infty)$. Consider the transformation

$$\begin{aligned}\widetilde{Z}_t^{(1)} &= \widetilde{X}_t^{x,u} + \widetilde{Y}_t^{y,u} = x + y + (\sigma_1+\sigma_2)\,W_t + \bar{\mu}t,\\ \widetilde{Z}_t^{(2),v} &= \sigma_1\widetilde{Y}_t^{y,u} - \sigma_2\widetilde{X}_t^{x,u} = \sigma_1 y - \sigma_2 x + \int_0^t \big(\sigma_1\bar{\mu} - (\sigma_1+\sigma_2)u_s\big)\,\mathrm{d}s = \sigma_1 y - \sigma_2 x + \int_0^t v_s\,\mathrm{d}s,\end{aligned}$$

where $v_s := \sigma_1\bar{\mu} - (\sigma_1+\sigma_2)u_s$ and $v_s \in [-\sigma_2\eta, \sigma_1\eta]$, $s \in [0,\infty)$ with $\eta = \bar{\mu} + \delta(\sigma_1+\sigma_2) > 0$. Observe that also in the extended model $\widetilde{Z}^{(1)}$ does not depend on the control $v = (v_s)_{s\in[0,\infty)}$ and $\widetilde{Z}^{(2),v}$ does not depend on the Brownian motion $(W_t)_{t\in[0,\infty)}$.

Using the same arguments as in Section 3.1 (but with more lengthy computations), we obtain the following result.

Theorem 3.4: *Let $y > \frac{\sigma_2}{\sigma_1}x$. Then the value function for the extended model* (13) *is given by*

$$\begin{aligned}V(x,y;\sigma_1,\sigma_2) &= \Phi\left(\frac{\delta\sigma_2\,x + (\bar{\mu}+\delta\sigma_2)\,y}{N}\right) - \exp\left(-\frac{2\bar{\mu}\,(x+y)}{(\sigma_1+\sigma_2)^2}\right)\Phi\left(\frac{A\sigma_2\,x + B\,y}{(\sigma_1+\sigma_2)\,N}\right)\\ &\quad - \exp\left(-\frac{2(\bar{\mu}+\delta\sigma_2)\,x}{\sigma_1^2}\right)\Phi\left(\frac{(\bar{\mu}+\delta\sigma_2)\,y - (A+\delta\sigma_2)\,\frac{\sigma_2}{\sigma_1}\,x}{N}\right)\\ &\quad + \exp\left(-\frac{2\bar{\mu}\,y}{(\sigma_1+\sigma_2)^2} - \frac{2\sigma_2\,x}{\sigma_1^2}\left(\frac{\bar{\mu}\,\sigma_2}{(\sigma_1+\sigma_2)^2} + \delta\right)\right)\Phi\left(\frac{B\,y - C\,\frac{\sigma_2}{\sigma_1}\,x}{(\sigma_1+\sigma_2)\,N}\right),\end{aligned}$$

where

$$\begin{aligned}N &= \sqrt{\sigma_2(\sigma_1 y - \sigma_2 x)\big(\bar{\mu} + \delta(\sigma_1+\sigma_2)\big)},\\ A &= 2\bar{\mu} + \delta(\sigma_1+\sigma_2),\\ B &= (\bar{\mu}+\delta\sigma_2)(\sigma_1+\sigma_2) - 2\bar{\mu}\sigma_1,\\ C &= \delta\sigma_1^2 + 3\delta\sigma_1\sigma_2 + 2(\bar{\mu}+\delta\sigma_2)\sigma_2.\end{aligned}$$

For $y = \frac{\sigma_2}{\sigma_1}x$ it holds that

$$V\left(x, \frac{\sigma_2}{\sigma_1}x;\sigma_1,\sigma_2\right) = 1 - \exp\left(-\frac{2\,\bar{\mu}\,x}{\sigma_1(\sigma_1+\sigma_2)}\right) = 1 - \exp\left(-\frac{2\,\bar{\mu}\,y}{\sigma_2(\sigma_1+\sigma_2)}\right).$$

For $y < \frac{\sigma_2}{\sigma_1} x$ *we have*

$$V(x, y; \sigma_1, \sigma_2) = V(y, x; \sigma_2, \sigma_1).$$

The optimal strategy for $V(x, y; \sigma_1, \sigma_2)$ *is given by*

$$u_s^* = u_s^*\left(\widetilde{X}_s^{x,u^*}, \widetilde{Y}_s^{y,u^*}\right) = \begin{cases} \bar{\mu} + \delta\sigma_2, & \widetilde{Y}_s^{y,u^*} \geq \frac{\sigma_2}{\sigma_1} \widetilde{X}_s^{x,u^*}, \\ -\delta\sigma_1, & \widetilde{Y}_s^{y,u^*} < \frac{\sigma_2}{\sigma_1} \widetilde{X}_s^{x,u^*}. \end{cases}$$

Remark 3.5: Observe that the Hamilton–Jacobi–Bellman equation in the extended model (13) is given by

$$\left(\frac{\sigma_1^2}{2} V_{xx} + \frac{\sigma_2^2}{2} V_{yy} + \sigma_1\sigma_2 V_{xy} + \max_{a \in [-\delta\sigma_1, \bar{\mu}+\delta\sigma_2]} \left\{ a V_x + (\bar{\mu} - a) V_y \right\}\right)(x, y; \sigma_1, \sigma_2) = 0 \tag{14}$$

for $x, y > 0$ with boundary conditions

$$V(x, 0; \sigma_1, \sigma_2) = V(0, y; \sigma_1, \sigma_2) = 0, \tag{15}$$

$$\lim_{x\to\infty} V(x, y; \sigma_1, \sigma_2) = 1 - \exp\left(-\frac{2(\bar{\mu} + \delta\sigma_1)\, y}{\sigma_2^2}\right), \tag{16}$$

$$\lim_{y\to\infty} V(x, y; \sigma_1, \sigma_2) = 1 - \exp\left(-\frac{2(\bar{\mu} + \delta\sigma_2)\, x}{\sigma_1^2}\right). \tag{17}$$

One can prove that $V(\cdot, \cdot; \sigma_1, \sigma_2) \in C^2((0,\infty)^2) \cap C([0,\infty)^2)$ and that $V(\cdot, \cdot; \sigma_1, \sigma_2)$ solves (14) with boundary conditions (15), (16) and (17). Furthermore,

$$\left\{(x, y) \in (0,\infty)^2 \colon V_x(x, y; \sigma_1, \sigma_2) > V_y(x, y; \sigma_1, \sigma_2)\right\} = \left\{(x, y) \in (0,\infty)^2 \colon y > \frac{\sigma_2}{\sigma_1} x\right\}, \tag{18}$$

$$\left\{(x, y) \in (0,\infty)^2 \colon V_x(x, y; \sigma_1, \sigma_2) < V_y(x, y; \sigma_1, \sigma_2)\right\} = \left\{(x, y) \in (0,\infty)^2 \colon y < \frac{\sigma_2}{\sigma_1} x\right\},$$

$$\left\{(x, y) \in (0,\infty)^2 \colon V_x(x, y; \sigma_1, \sigma_2) = V_y(x, y; \sigma_1, \sigma_2)\right\} = \left\{(x, y) \in (0,\infty)^2 \colon y = \frac{\sigma_2}{\sigma_1} x\right\}. \tag{19}$$

To see that (18) holds true, note that for $y > \frac{\sigma_2}{\sigma_1} x$ we have

$$\begin{aligned} &V_x(x, y; \sigma_1, \sigma_2) - V_y(x, y; \sigma_1, \sigma_2) \\ &\quad = \frac{2(\bar{\mu} + \delta\sigma_2)}{\sigma_1^2} \exp\left(-\frac{2(\bar{\mu} + \delta\sigma_2)\, x}{\sigma_1^2}\right) \Phi\left(\frac{(\bar{\mu} + \delta\sigma_2)\, y - (A + \delta\sigma_2)\frac{\sigma_2}{\sigma_1} x}{N}\right) \\ &\qquad + 2\left(\frac{\bar{\mu}(\sigma_1 - \sigma_2)}{\sigma_1^2(\sigma_1 + \sigma_2)} - \frac{\delta\sigma_2}{\sigma_1^2}\right) \exp\left(-\frac{2\bar{\mu} y}{(\sigma_1 + \sigma_2)^2} - \frac{2\sigma_2 x}{\sigma_1^2}\left(\frac{\bar{\mu}\sigma_2}{(\sigma_1 + \sigma_2)^2} + \delta\right)\right) \Phi\left(\frac{By - C\frac{\sigma_2}{\sigma_1} x}{(\sigma_1 + \sigma_2)N}\right). \end{aligned}$$

If $\delta \in \left(-\frac{\bar{\mu}}{\sigma_1+\sigma_2}, \frac{\bar{\mu}(\sigma_1-\sigma_2)}{\sigma_2(\sigma_1+\sigma_2)}\right]$, then one can directly conclude that $(V_x - V_y)(x, y; \sigma_1, \sigma_2) > 0$ for all $y > \frac{\sigma_2}{\sigma_1} x$. If $\delta > \frac{\bar{\mu}(\sigma_1-\sigma_2)}{\sigma_2(\sigma_1+\sigma_2)}$, then observe that

$$\lim_{y \downarrow \frac{\sigma_2}{\sigma_1} x} (V_x - V_y)(x, y; \sigma_1, \sigma_2) = 0$$

and

$$
\begin{aligned}
&(V_x - V_y)_y(x, y; \sigma_1, \sigma_2) \\
&= \sqrt{\frac{2}{\pi}}\left(2\sigma_1 y + (\sigma_1 - \sigma_2)\,x\right)\frac{\bar{\mu}\sqrt{\sigma_2(\bar{\mu} + \delta(\sigma_1 + \sigma_2))}}{\sigma_1(\sigma_1 + \sigma_2)^2(\sigma_1 y - \sigma_2 x)^{\frac{3}{2}}} \exp\left(-\frac{\left(\delta\sigma_2\, x + (\bar{\mu} + \delta\sigma_2)\, y\right)^2}{2N^2}\right) \\
&\quad + \frac{4\bar{\mu}\,(\delta\sigma_2(\sigma_1 + \sigma_2) + \bar{\mu}(\sigma_2 - \sigma_1))}{\sigma_1^2(\sigma_1 + \sigma_2)^3} \exp\left(-\frac{2\bar{\mu}\, y}{(\sigma_1 + \sigma_2)^2} - \frac{2\sigma_2\, x}{\sigma_1^2}\left(\frac{\bar{\mu}\,\sigma_2}{(\sigma_1 + \sigma_2)^2} + \delta\right)\right) \\
&\quad \times \Phi\left(\frac{B\,y - C\,\frac{\sigma_2}{\sigma_1}\,x}{(\sigma_1 + \sigma_2)\,N}\right) > 0
\end{aligned}
$$

for $y > \frac{\sigma_2}{\sigma_1}\,x$. Hence, also in this case we have $(V_x - V_y)(x, y; \sigma_1, \sigma_2) > 0$.

Similarly, one derives (19).

Figure 4 depicts the value function $V(x, y; \sigma_1, \sigma_2)$ for different diffusion rates and different δ.

If we consider two insurance companies whose endowment processes are Brownian motions with drift $\mu_i > 0$, diffusion coefficients $\sigma_i > 0$, $i = 1, 2$, and initial endowments $x, y > 0$, respectively, then the survival probability can be derived similarly to the case $\sigma_1 = \sigma_2 = 1$ and is given by

$$
\mathbb{P}[x + \mu_1\, t + \sigma_1 W_t > 0,\ y + \mu_2\, t + \sigma_2 W_t > 0 \text{ for all } t \in [0, \infty)]
$$

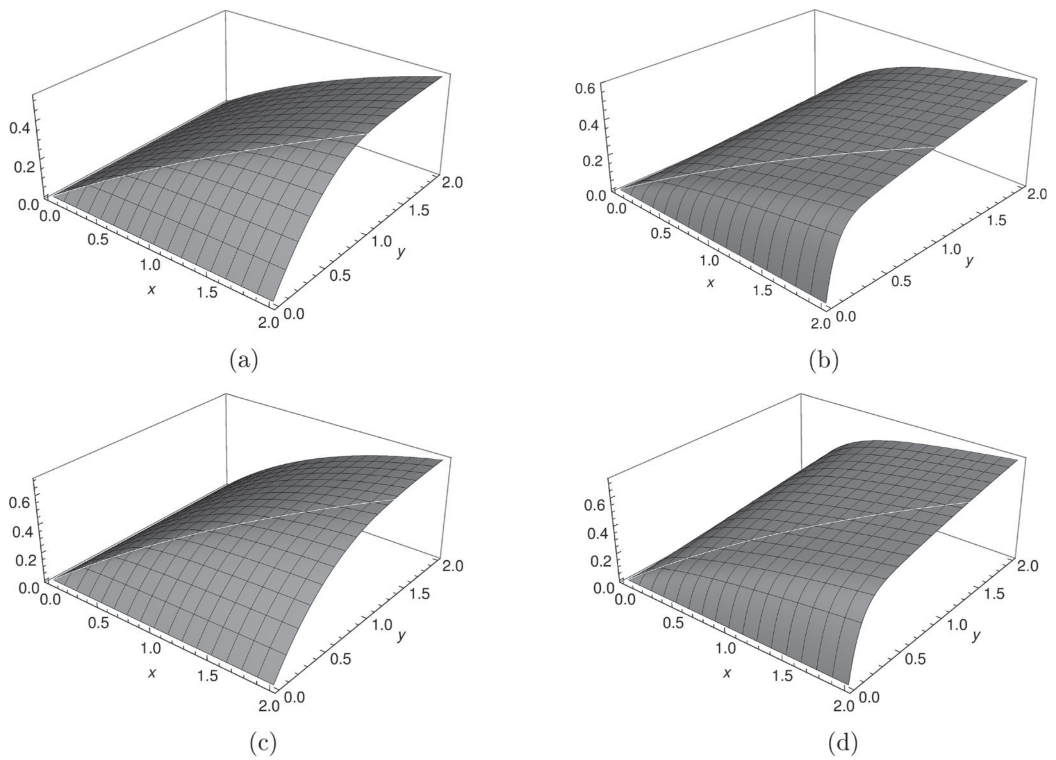


Figure 4. The value function for $\rho = 1$, $\bar{\mu} = 1$, different diffusion rates $\sigma_1, \sigma_2 > 0$ and different $\delta > -\frac{\bar{\mu}}{\sigma_1 + \sigma_2}$. (a) $\sigma_1 = 2$, $\sigma_2 = 1$, $\delta = -\frac{1}{4}$. (b) $\sigma_1 = 2$, $\sigma_2 = 1$, $\delta = 2$. (c) $\sigma_1 = \frac{3}{2}$, $\sigma_2 = 1$, $\delta = -\frac{1}{4}$ and (d) $\sigma_1 = \frac{3}{2}$, $\sigma_2 = 1$, $\delta = 2$.

$$
= \begin{cases} 1 - \exp\left(-2\min\left\{\frac{\mu_1 x}{\sigma_1^2}, \frac{\mu_2 y}{\sigma_2^2}\right\}\right), & \frac{\mu_1}{\sigma_1} = \frac{\mu_2}{\sigma_2}; \\ & \frac{\mu_1}{\sigma_1} > \frac{\mu_2}{\sigma_2}, \frac{x}{\sigma_1} \geq \frac{y}{\sigma_2}; \\ & \frac{\mu_1}{\sigma_1} < \frac{\mu_2}{\sigma_2}, \frac{x}{\sigma_1} \leq \frac{y}{\sigma_2}; \\ \Phi\left(\frac{|\mu_1 y - \mu_2 x|}{L}\right) - \exp\left(-\frac{2\mu_1 x}{\sigma_1^2}\right)\Phi\left(\frac{\mu_1 y + D(2,1)x}{L}\right) & \frac{\mu_1}{\sigma_1} > \frac{\mu_2}{\sigma_2}, \frac{x}{\sigma_1} < \frac{y}{\sigma_2}; \\ \quad - \exp\left(-\frac{2\mu_2 y}{\sigma_2^2}\right)\Phi\left(\frac{\mu_2 x + D(1,2)y}{L}\right) & \frac{\mu_1}{\sigma_1} < \frac{\mu_2}{\sigma_2}, \frac{x}{\sigma_1} > \frac{y}{\sigma_2}; \\ \quad + \exp\left(-\frac{2\mu_1 x}{\sigma_1^2} - \frac{2\mu_2 y}{\sigma_2^2} + \frac{4\min\{\mu_2 x, \mu_1 y\}}{\sigma_1 \sigma_2}\right) & \\ \quad \times \Phi\left(\frac{D(2,1)x + D(1,2)y + 2\min\{\mu_2 x, \mu_1 y\}}{L}\right), & \end{cases}
$$

where

$$
L = \sqrt{(\sigma_1 y - \sigma_2 x)(\mu_1\sigma_2 - \mu_2\sigma_1)}, \quad D(i,j) = \mu_i - \frac{2\mu_j\sigma_i}{\sigma_j}, \quad i,j \in \{1,2\}.
$$

The gain of collaboration is then given by the difference

$$
V(x,y;\sigma_1,\sigma_2) - \mathbb{P}[x + \mu_1 t + \sigma_1 W_t > 0,\ y + \mu_2 t + \sigma_2 W_t > 0 \text{ for all } t \in [0,\infty)].
$$

See Figure 5 for the gain of collaboration in case of different diffusion coefficients and drift rates.

Remark 3.6: Observe that in the extended (13) model we also relax the assumption on δ and do not assume that $\delta > -\min\{\frac{\mu_1}{\sigma_1}, \frac{\mu_2}{\sigma_2}\}$. For $\delta \in (-\frac{\bar{\mu}}{\sigma_1+\sigma_2}, -\min\{\frac{\mu_1}{\sigma_1}, \frac{\mu_2}{\sigma_2}\})$ the companies are forced to collaborate, in particular firm i with $\frac{\mu_i}{\sigma_i} > \frac{\mu_j}{\sigma_j}$, $j \neq i$, has to make payments to firm j at every point in time. Thus, the gain of collaboration can be negative, see Figure 6.

4. Perfectly negative correlation: $\rho = -1$

In this section, we focus on the case $\rho = -1$ and obtain a different characterization of the value function in terms of the probability that a reflected Brownian motion with drift never hits a specific line. Unfortunately, we cannot use similar arguments as in the case $\rho = 1$ to derive an explicit formula for the value function.

From Theorem 2.2, we already know that an optimal strategy for the transfer payments is given by

$$
u_s^* = (\bar{\mu} + \delta)\,\mathbb{1}_{\left\{X_s^{x,u^*} \leq Y_s^{y,u^*}\right\}} - \delta\,\mathbb{1}_{\left\{X_s^{x,u^*} > Y_s^{y,u^*}\right\}}.
$$

For perfectly negatively correlated Brownian motions and for the optimal strategy u^*, it holds that

$$
\begin{aligned}
Z_t^{(1)} &= X_t^{x,u^*} + Y_t^{y,u^*} = x + y + \bar{\mu}\, t, \\
Z_t^{(2)} &= Y_t^{y,u^*} - X_t^{x,u^*} = y - x - 2W_t + \int_0^t (\bar{\mu} - 2u_s^*)\, ds \\
&= y - x - 2W_t - \int_0^t (\bar{\mu} + 2\delta)\, \mathrm{sign}\big(Z_t^{(2)}\big)\, ds.
\end{aligned}
$$

Moreover,

$$
\tau(x,y;u^*) = \inf\left\{t \in [0,\infty) : \left|Z_t^{(2)}\right| \geq x + y + \bar{\mu}\, t\right\}
$$

and $V(x,y) = \mathbb{P}[\tau(x,y;u^*) = \infty]$.

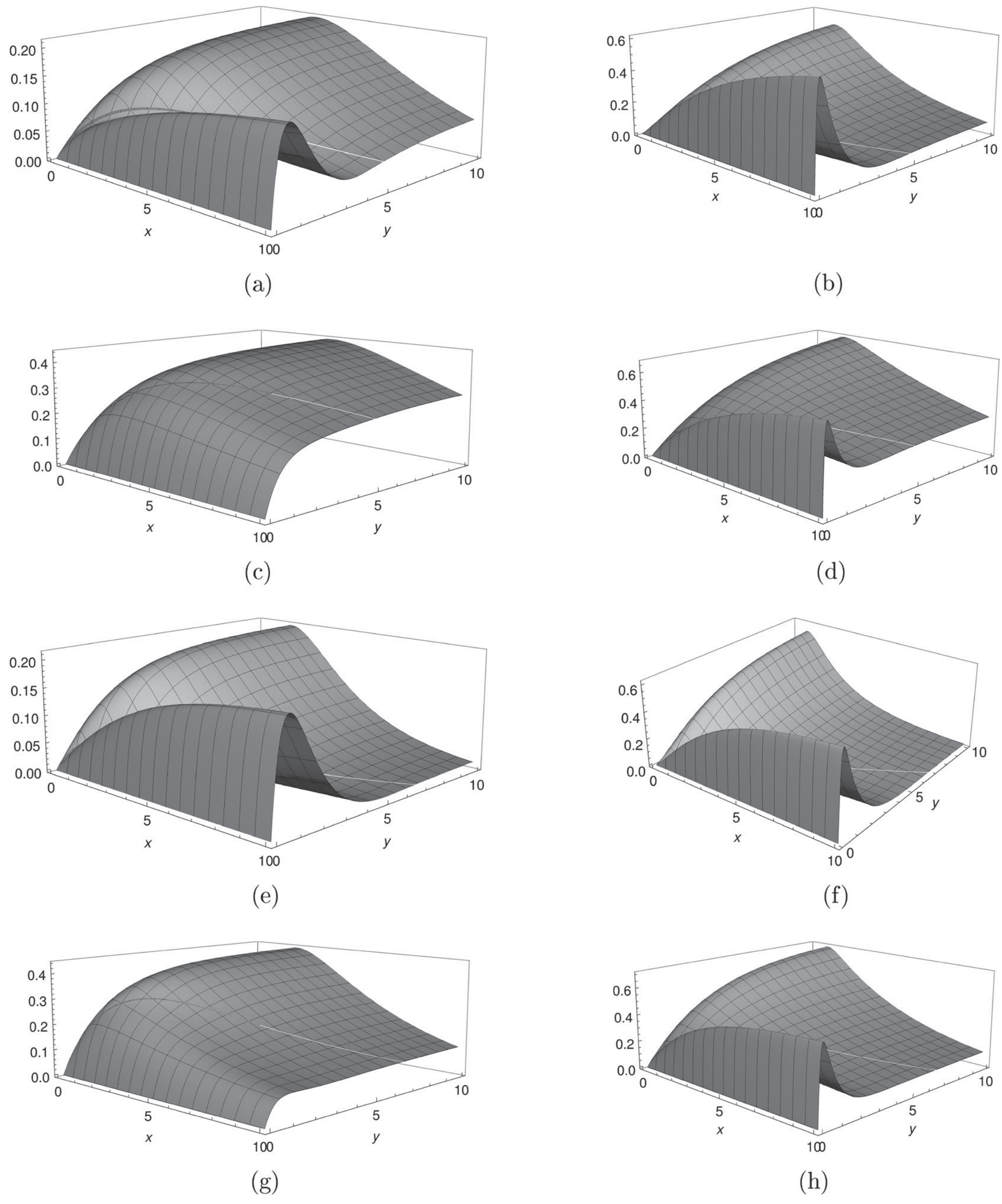


Figure 5. The gain of collaboration for $\rho = 1$, $\bar{\mu} = 1$ and different drift rates μ_1, $\mu_2 > 0$, diffusion rates $\sigma_1, \sigma_2 > 0$ and different $\delta > -\frac{\bar{\mu}}{\sigma_1+\sigma_2}$. (a) $\mu_1 = \mu_2 = \frac{1}{2}$, $\sigma_1 = 2$, $\sigma_2 = 1$, $\delta = -\frac{1}{10}$. (b) $\mu_1 = \mu_2 = \frac{1}{2}$, $\sigma_1 = 2$, $\sigma_2 = 1$, $\delta = 2$. (c) $\mu_1 = \frac{1}{4}$, $\mu_2 = \frac{3}{4}$, $\sigma_1 = 2$, $\sigma_2 = 1$, $\delta = -\frac{1}{10}$. (d) $\mu_1 = \frac{1}{4}$, $\mu_2 = \frac{3}{4}$, $\sigma_1 = 2$, $\sigma_2 = 1$, $\delta = 2$. (e) $\mu_1 = \mu_2 = \frac{1}{2}$, $\sigma_1 = \frac{3}{2}$, $\sigma_2 = 1$, $\delta = -\frac{1}{10}$. (f) $\mu_1 = \mu_2 = \frac{1}{2}$, $\sigma_1 = \frac{3}{2}$, $\sigma_2 = 1$, $\delta = 2$. (g) $\mu_1 = \frac{1}{4}$, $\mu_2 = \frac{3}{4}$, $\sigma_1 = \frac{3}{2}$, $\sigma_2 = 1$, $\delta = -\frac{1}{10}$ and (h) $\mu_1 = \frac{1}{4}$, $\mu_2 = \frac{3}{4}$, $\sigma_1 = \frac{3}{2}$, $\sigma_2 = 1$, $\delta = 2$.

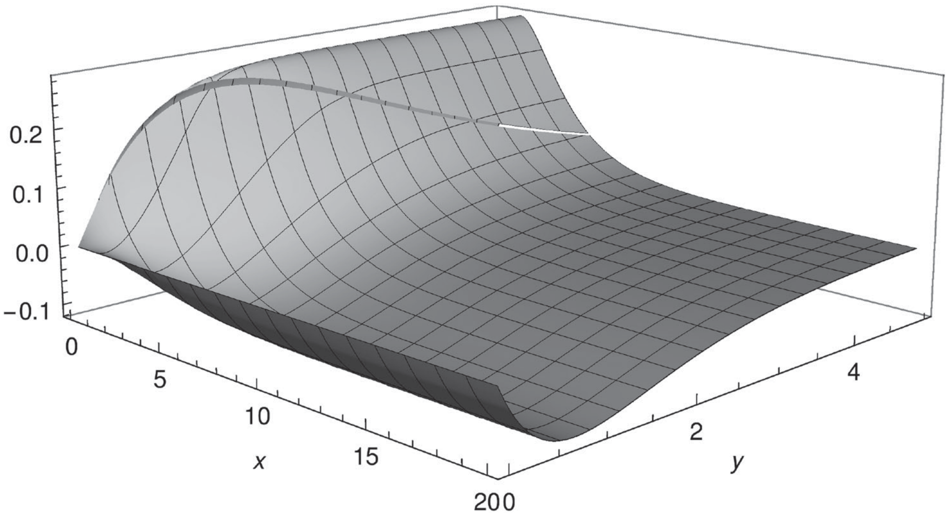


Figure 6. For $\mu_1 = \frac{1}{4}, \mu_2 = \frac{3}{4}, \sigma_1 = \sigma_2 = 1, \delta = -\frac{9}{20}$ the gain of collaboration can be negative.

The process $(\frac{1}{2}|Z_t^{(2)}|)_{t\in[0,\infty)}$ is a representation of a reflected Brownian motion with drift $-(\frac{\bar{\mu}}{2} + \delta)$, see Graversen & Shiryaev (2000). Therefore, the value function $V(x, y)$ can be interpreted as the probability that a reflected Brownian motion with drift $-(\frac{\bar{\mu}}{2} + \delta)$ never hits the linear barrier

$$b(t) = \frac{1}{2}(x + y + \bar{\mu}\, t).$$

To the best of our knowledge, no closed formula for the hitting probability is available in the literature.

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