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MULTI-CURRENCY CREDIT DEFAULT SWAPS

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Credit default swaps (CDS) on a reference entity may be traded in multiple currencies, in that, protection upon default may be offered either in the currency where the entity resides, or in a more liquid and global foreign currency. In this situation, currency fluctuations clearly introduce a source of risk on CDS spreads. For emerging markets, but in some cases even in well-developed markets, the risk of dramatic foreign exchange (FX)-rate devaluation in conjunction with default events is relevant. We address this issue by proposing and implementing a model that considers the risk of foreign currency devaluation that is synchronous with default of the reference entity. As a fundamental case, we consider the sovereign CDSs on Italy, quoted both in EUR and USD. Preliminary results indicate that perceived risks of devaluation can induce a significant basis across domestic and foreign CDS quotes. For the Republic of Italy, a USD CDS spread quote of 440 bps can translate into an EUR quote of 350 bps in the middle of the Euro-debt crisis in the first week of May 2012. More recently, from June 2013, the basis spreads between the EUR quotes and the USD quotes are in the range around 40 bps. We explain in detail the sources for such discrepancies. Our modeling approach is based on the reduced form framework for credit risk, where the default time is modeled in a Cox process setting with explicit diffusion dynamics for default intensity/hazard rate and exponential jump to default. For the FX part, we include an explicit default-driven jump in the FX dynamics. As our results show, such a mechanism provides a further and more effective way to model credit/FX dependency than the instantaneous correlation that can be imposed

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among the driving Brownian motions of default intensity and FX rates, as it is not possible to explain the observed basis spreads during the Euro-debt crisis by using the latter mechanism alone.

Keywords: Credit default swaps; intensity models; reduced form models; credit crisis; liquidity crisis; devaluation jump; FX devaluation; quanto credit effects; quanto CDS; multi currency CDS.

1. Introduction

The need for quanto default modeling arises naturally when pricing credit derivatives offering protection in multiple currencies.

Reasons for entering into *credit default swaps* (CDS) in different currencies can come from financial, economic, or even legislative considerations: they range from the composition of the portfolio that has to be hedged to the accounting rules in force in the country where the investor is based. In case the reference entity is sovereign, economic reasons play a major role since for an investor it might be more appealing to buy protection on, for example, Republic of Italy's default in USD rather than in EUR. Indeed, in the latter case the currency value itself is strongly related with the reference entity's default.

Figure 1 shows the time series of par-spreads for USD-denominated and EUR-denominated CDSs on Republic of Italy from the beginning of 2011 until the end of 2013. The time range has been chosen so as to include the 2011 Euro-debt crisis.

The difference between the par-spreads for USD-denominated and EUR-denominated CDSs is shown in the bottom chart. In order to build a model which accounts for the default information and generate the spreads in the two currencies, the joint evolution of the obligors hazard rate and of the foreign exchange (FX) rate between the two currencies must be modeled.

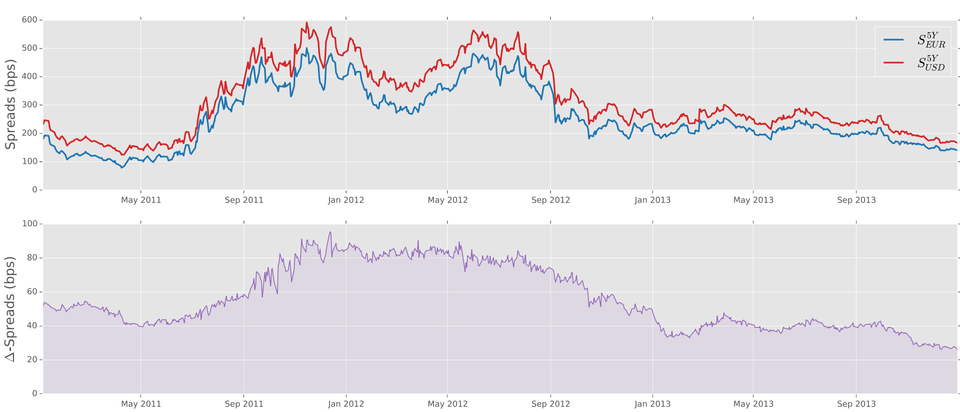


Fig. 1. The top chart contains the 5Y par-spread time series for USD-denominated CDSs, S^{5Y}_{USD}, and EUR-denominated CDSs, S^{5Y}_{EUR}, on Italy. The difference between the two par-spreads is shown in the bottom chart.

In the present paper, we show two ways to model the joint dynamics of credit and FX rates. In the first approach, the interaction between the credit and the FX component is entirely explained by an instantaneous correlation between the Brownian motions driving the stochastic hazard rate and the FX rate. In the second, more sophisticated modeling approach, an additional mechanism of dependence between the two components is introduced in the form of a conditional devaluation jump of the FX rate upon default of the reference obligor.

The diffusive approach emphasizes the limitations of confining the credit/market interaction to instantaneous correlation between hazard rate and market risk factors. As shown by comparing the model-implied quanto spreads in Fig. 5 with the observed quanto spreads in Fig. 1, instantaneous correlation alone is not able to explain the observed quanto spread. This phenomenon is akin to the pricing of credit correlation instruments where it has been observed that instantaneous correlation between hazard rates is unable to generate the sufficient level of dependence to hit the market spreads of index tranches (see, for example, Brigo *et al.* 2013, Brigo & Mercurio 2006, Cherubini *et al.* 2004).

Using the latter modeling approach, we will show how the introduction of jump-to-default effects achieves a much stronger FX/Credit dependence than correlated Brownian motions. In particular, the addition of FX jumps allows one to recover both the EUR and the USD spreads (see the results presented in Sec. 3.5). Furthermore, we show a powerful, yet simple, way of extracting the magnitude of currency devaluation upon default from the CDS market data (see Eq. (3.3)).

In addition to multi-currency CDSs, the quanto effect in credit modeling finds a natural application in the *credit valuation adjustment* (CVA) space. CVA is an adjustment to the fair value of a derivative contract that accounts for the expected loss due to the counterparty's default. We refer the interested reader to Brigo *et al.* (2013) for a comprehensive overview of CVA modeling and to Cherubini (2005) for specific discussions about collateral modeling. Modeling the dependence between credit and market risk factors is crucial to accurately calculate the CVA charge. One of the main challenges in calculating CVA is the lack of liquid CDS market data to calibrate model parameters. The calibration and approximation techniques showed in this paper to connect currency devaluation with multi-currency CDS par-spreads can as well be applied to CVA modeling — for example, to better reflect right-way or wrong-way risk. The resulting FX/Credit cross modeling improvement is crucial, especially in those cases where the interaction between the counterparty credit and the FX component is strong, i.e. with emerging market credits and systemically relevant counterparties.

In Sec. 2.5, we show how the introduction of default-driven FX jumps changes the dynamics of the stochastic hazard rate after a measure change. This happens because, from a mathematical perspective, the FX rate is a component of the Radon–Nikodym derivative that links the risk neutral probability measures associated to two different currencies. As stated by Girsanov theorem (see, for example, Jeanblanc *et al.* 2009), the dynamics of the compensated default process under different risk-neutral measures differ in their drift component. Such drift depends on the quadratic covariation between the FX rate and the default process (and it is zero when such covariation is null) and can be interpreted as the stochastic hazard rate of the reference entity.

The above result is strongly linked to another aspect of FX-rate modeling, which we will refer to as FX symmetry throughout this document (see the discussion in Sec. 2.2). Consistency between an FX-rate process and its reciprocal is not guaranteed under every possible distributional assumptions made on its dynamics. For example, in case of stochastic volatility FX modeling, the reciprocal FX rate would not necessarily have the same dynamics that one would expect given that the reciprocal FX rate is also a Radon–Nikodym derivative. For geometric Brownian motions, however, this consistency is guaranteed. Due to the change in the hazard rate in the second pricing measure induced by the jump-to-default feature of the FX-rate/Radon–Nikodym derivative process, we prove in Proposition 2 that the symmetry is preserved also for our specific FX model.

1.1. *Overview of the related work*

We refer to Bielecki *et al.* (2005) for an overview of the general problem of deducing a PDE to price defaultable claims and to Bielecki *et al.* (2008) for the specific problem of CDS hedging in a reduced-form framework.

For an introduction to the joint modeling of credit and FX in a reduced-form framework with application to quanto-CDS pricing, we refer to Ehlers & Schönbucher (2006), EL-Mohammadi (2009). Ehlers & Schönbucher (2006) propose the idea to link FX and hazard rate by considering a jump-diffusion model for the FX-rate process where the jump happens at the default time. Differently from the present work, no explicit derivation of the PDE is presented, as the focus is on affine processes modeling.

The same idea is presented and developed in EL-Mohammadi (2009). In that work it is shown how to calculate quanto-corrected survival probabilities using a PDE-based approach. In order to do that, the author deduces a Fokker–Planck equation for the joint distribution of FX and hazard rate.

The approach we present in Sec. 2 below is based on the same Jump-to-Default framework as the one used in the references above. In our case, however, the calculation of the quanto-corrected survival probabilities depends on solving a Feynman–Kac equation, the solution of which is a price, while in EL-Mohammadi (2009) a probability density distribution was calculated. At implementation level, the difference between the two approaches lies in the fact that in the latter case an additional integration step would be required to calculate a price. Additionally, the way we work out our main pricing equation makes clear what instruments and in what amounts one would need to effectively implement a delta-hedging strategy.

An algorithm using a fixed-point approach has been recently proposed to calculate CVA in Kim & Leung (2016).

The techniques showed in this paper seem particularly relevant for long-maturity trades, where the effects of idiosyncratic jump-to-default components on counterparty risk can be more pronounced and where, therefore, they can have a big impact on wrong-way risk estimation. For a relevant example of CVA calculations related to long-maturity trades, we refer to Biffis *et al.* (2016), where the cost of CVA and collateralization are calculated for longevity swaps.

The use of Lévy processes with local volatility to price options on defaultable assets has been recently explored in Lorig *et al.* (2015), where a family of asymptotic expansions for the transition density of the underlying is derived. Differently from the approach presented in this paper, in that case a single stochastic process drives both the default intensity and the option's underlying. On the other hand, being able to account for the implied volatility skew is feature currently missing from the framework presented in Sec. 2 and that will be explored in future works.

With respect to the Italy's test case that is presented in the results' Sec. 3, we note that the Euro-area situation presents interesting problems that go beyond the mere credit-FX interaction which is the focus of the present work. An additional layer of complexity is provided in this case by the interconnectedness between the credit risk of the different currencies.

Empirically, Germany quanto CDS basis tends to be more pronounced than the Greece one (see Pykthtin & Sokol 2013), reflecting higher correlation between EUR/USD and Germany hazard rate of default and higher EUR/USD devaluation upon Germany default.

Following the first publication of the present work as a pre-print, Itkin *et al.* (2017) extended the model presented in Sec. 2 by adding two stochastic processes to represent the domestic and the foreign-measure short rates. They added jump-at-default components to both FX and foreign (defaulted) interest rates. In their work, they show that devaluation of the foreign currency decreases the value of the foreign CDS, while the increase of the foreign interest rate tends to increase it. Turfus derived analytic formulae in a setting where one of the two interest rate components, that FX rate, and the stochastic credit intensity are stochastic. Survival probabilities are calculated using a perturbation expansion approach predicated on the basis that the foreign interest rate and the credit default intensity are small.

1.2. *Quanto CDS*

Quanto CDS are designed to provide protection upon default of a certain entity in a given currency. There are cases, like for sovereign entities or for systemically important companies, when an investor might prefer to buy protection on a currency other than the one in which the assets of the reference entity are denominated.

A typical reason for entering this type of trades would be to avoid the FX risk linked to the devaluation effect associated to the reference entity's default.

Alternatively, protection might be needed in a different currency from the one in which the assets of the reference entity are denominated because it serves as a hedge on a security denominated in that specific currency.

The discounted cashflows of the premium leg, Π^{Premium}, are given (as seen from the protection seller's perspective) by

$$\Pi^{\text{Premium}} = S^c \sum_{i=0}^{N} \mathbb{1}_{\tau > T_i} D_0^{\text{ccy}}(T_i), \tag{1.1}$$

where $(T_0, \ldots, T_N)$ is the set of quarterly spaced payment times, $D_t^{\text{ccy}}(T)$ is the stochastic discount factor for currency ccy at time t for maturity T, S^c is the contractual spread, and τ is the default time of the reference entity.

The protection leg is made of a single cash flow, $\Pi^{\text{Protection}}$, paid upon default of the reference entity on a reference obligation:

$$\Pi^{\text{Protection}} = \text{LGD}\, \mathbb{1}_{\tau \leq T_N} D_0^{\text{ccy}}(\tau), \tag{1.2}$$

where (LGD) is the *loss given default* related to the contract.

The spread S^c that makes the expected value of the cash-flows in Eq. (1.1) equal to the expected value of the cash-flow in Eq. (1.2) is referred to as *par-spread* and we will usually use S to denote it. The existence of CDSs on the same reference entity whose premium and protection cashflows are paid in different currencies creates a basis spread between the par-spreads of these contracts. Figure 2 provides a schematic representation of two possible contracts settled in two different currencies.

We refer to Elizalde *et al.* (2010) and references therein for an overview on quanto CDS markets and for a thorough exposition of the rules governing these contracts. Here, we just note that:

(i) the standard contracts for sovereign CDS are denominated in USD. This means in particular that for countries of the EUR zone, like Italy, Greece or Germany, the modeling set up to use when including a devaluation approach is the one detailed in Sec. 2.5;

Fig. 2. Protection on a given reference entity can be bought by A from B in different currencies. The stream of payments in Eq. (1.1) is indicated by the solid arrow, while the dashed arrow is used for the contingent payment in Eq. (1.2). The LGD payment, albeit settled in different currencies, is the same percentage of the notional in the two contracts.

(ii) upon default of the reference entity, a common auction sets the LGD. The LGD so defined is valid for all the CDSs, irrespectively of the currency they are denominated in.

2. Model Description

We begin by considering a probability space $(\Omega, \mathcal{F}, \mathbb{Q}, (\mathcal{F}_t, t \geq 0))$ satisfying the usual hypotheses. In particular $(\mathcal{F}_t, t \geq 0)$ is a filtration under which the dynamics of the risk factors are adapted and under which the default time of the reference entity is a stopping-time. Depending on the specific examples, we will also consider spaces with a different equivalent measure, for example the risk neutral measure associated to the liquid money market or the risk neutral measure associated to the contractual currency money market.

We will be using a Cox process model for the default component and we will refer to the stochastic intensity of the default event simply as *hazard rate* or *intensity*, using the two terms interchangeably.

Unlike the usual approach followed in the so-called "reduced-form" framework for credit risk modeling (see Lando 2004, Brigo & Mercurio 2006), we do not introduce a second filtration with respect to which only the stochastic processes driving the market risk-factors are measurable. The total filtration $(\mathcal{F}_t, t \geq 0)$, inclusive of market and default risk, is the only filtration we will consider (that is called $(\mathcal{G}_t, t \geq 0)$ in Brigo & Mercurio 2006). The practical reason for considering a second, reduced, filtration is to apply theoretical results originally developed to price interest rates derivatives. Due to the specific model choices we make in the following, however, this would not present any real advantage, while, as shown in Secs. 2.4 and 2.5, working with a single filtration gives us the possibility to calculate the quanto adjustment using a PDE approach.

2.1. *The roles of the currencies*

For the pricing of any quanto CDS, we will be considering two relevant currencies. The first one is the *contractual currency*. This currency is a contract's attribute: it is the currency in which both premium leg and protection leg payments are settled. In applications to quanto CDS, for a given reference entity, we will be considering CDSs in at least two different contractual currencies. The second one is the *liquid currency*. For any given reference entity, this is the contractual currency of the most liquidly traded CDS. It is used to define a risk-neutral measure to price and calibrate the model.

We list here two examples to illustrate the use of the contractual and liquid currencies.

(1) the pricing in USD-measure of a CDS on Republic of Italy settled in EUR;
(2) the pricing in USD-measure of a CDS on Republic of Italy settled in USD.

We chose the test cases so that for all of them USD is the liquid currency, but this is not necessarily true for all CDS available in multiple currencies. It is worth noting that the test case 2 can be priced using a usual single currency approach. Test cases 1 and 2 will be used in Sec. 3.5 to illustrate the capability of the model specified in Sec. 2.5 to explain the currency basis observed in the market.

2.2. *Two markets measures*

Contractual and liquid currencies are *absolute* attributes of a CDS instrument and they are needed to correctly setup the calibration and pricing problems. An alternative currency classification is *relative* to the chosen pricing measure. From this perspective, we will be using *domestic* and *foreign* currency to denote the currency linked to the risk-neutral pricing measure and the currency of the contract we want to price, respectively.

Let us consider the corresponding money market accounts as the numeraires for both the economies. We will use a superscript $^{\mathrm{f}}$, to denote variables in the foreign economy, so that, for example, the two numeraires are $(B_t, t \geq 0)$ for the domestic economy and $(B_t^{\mathrm{f}}, t \geq 0)$ for the foreign economy. The money market account dynamics are given by

$$\mathrm{d}B_t = r_t B_t \,\mathrm{d}t, \quad B_0 = 1, \tag{2.1a}$$

$$\mathrm{d}B_t^{\mathrm{f}} = r_t^{\mathrm{f}} B_t^{\mathrm{f}} \,\mathrm{d}t, \quad B_0^{\mathrm{f}} = 1, \tag{2.1b}$$

where $(r_t, t \geq 0)$ and $(r_t^{\mathrm{f}}, t \geq 0)$ are the stochastic processes describing the short rates in the two economies.

Let us also consider an exchange rate $(Z_t^{\mathrm{f}}, t \geq 0)$ between the currencies of the two economies. Z_t^{f} is defined as the price of one unit of the liquid currency expressed as units of the contractual/foreign currency in a spot exchange at time t.

We are interested in finding an expression for the Radon–Nikodym derivative that changes the probability measure from $\mathbb{Q}^{\mathrm{f}}$ to $\mathbb{Q}$. This can be worked out by using the Change of Numeraire technique and a generic payoff denominated in the contractual currency, represented by the function ϕ_T^{f}. To do so, we consider, as said above, the contractual-currency money market account, $(B_t^{\mathrm{f}}, t \geq 0)$, as a numeraire for the measure $\mathbb{Q}^{\mathrm{f}}$, while for the measure $\mathbb{Q}$ we still use the liquid-currency money market account, but with value denominated in the contractual currency, $((Z^{\mathrm{f}}B)_t, t \geq 0)$. The price of the contractual currency payoff ϕ^{f} can be expressed in the two measures as

$$\mathbb{E}_0^{\mathrm{f}}\left[\frac{1}{B_T^{\mathrm{f}}}\phi_T^{\mathrm{f}}\right] = \mathbb{E}_0\left[\frac{Z_0^{\mathrm{f}}}{B_T Z_T^{\mathrm{f}}}\phi_T^{\mathrm{f}}\right]. \tag{2.2}$$

The $\mathbb{E}_t^{\mathrm{f}}[\cdot]$ expectation on the left-hand side, on the other hand, can be written as

$$\mathbb{E}_0^{\mathrm{f}}\left[\frac{1}{B_T^{\mathrm{f}}}\phi_T^{\mathrm{f}}\right] = \mathbb{E}_0^{\mathrm{f}}\left[\frac{B_T Z_T^{\mathrm{f}}}{B_T^{\mathrm{f}} Z_0^{\mathrm{f}}}\frac{Z_0^{\mathrm{f}}}{B_T Z_T^{\mathrm{f}}}\phi_T^{\mathrm{f}}\right] \tag{2.3}$$

and the two expressions above can be used to obtain the Radon–Nikodym derivative that defines the change of measure from $\mathbb{Q}^{\mathrm{f}}$ to $\mathbb{Q}$:

$$L^{\mathrm{f}}_T := \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{Q}^{\mathrm{f}}}\bigg|_{\mathcal{G}_T} = \frac{B_T Z^{\mathrm{f}}_T}{Z^{\mathrm{f}}_0 B^{\mathrm{f}}_T}, \quad L^{\mathrm{f}}_0 = 1. \tag{2.4}$$

In the following, we will be considering deterministic interest rates both for the liquid-currency and for the contractual-currency economy. This means that the money market accounts will be described by

$$\mathrm{d}B_t = r(t)B_t\,\mathrm{d}t, \quad B_0 = 1, \tag{2.5a}$$

$$\mathrm{d}B^{\mathrm{f}}_t = r^{\mathrm{f}}(t)B^{\mathrm{f}}_t\,\mathrm{d}t, \quad B^{\mathrm{f}}_0 = 1, \tag{2.5b}$$

in place of (2.1a) and (2.1b). To lighten the notation, in most cases we will drop the t-dependency for $r(t)$ and $r^{\mathrm{f}}(t)$ in the following equations.

The process defined in Eq. (2.4) has to be a martingale in the foreign measure. Given the assumption of deterministic interest rates, this condition can be used to determine the drift of $(Z^{\mathrm{f}}_t, t \geq 0)$. Due to Itô's formula, the dynamics of $(L^{\mathrm{f}}_t, t \geq 0)$ can be written as

$$\mathrm{d}{L^{\mathrm{f}}}_t = \mathrm{d}\left(\frac{B_t}{B^{\mathrm{f}}_t}\frac{Z^{\mathrm{f}}_t}{X_0}\right) = \frac{B_t}{B^{\mathrm{f}}_t Z^{\mathrm{f}}_0}(\mathrm{d}Z^{\mathrm{f}}_t + rZ^{\mathrm{f}}_t\,\mathrm{d}t - r^{\mathrm{f}}Z^{\mathrm{f}}_t\,\mathrm{d}t), \quad {L^{\mathrm{f}}}_0 = 1. \tag{2.6}$$

If, for example, we assume a lognormal dynamics for the FX rate

$$\mathrm{d}Z^{\mathrm{f}}_t = \mu^{\mathrm{f}} Z^{\mathrm{f}}_t\,\mathrm{d}t + \sigma Z^{\mathrm{f}}_t\,\mathrm{d}W^{\mathrm{f}}_t, \quad Z^{\mathrm{f}}_0 = x, \tag{2.7}$$

where $x \in \mathbb{R}^+$ is the value of the spot FX rate at time 0, then asking that $(L_t, t \geq 0)$ in Eq. (2.6) is a martingale brings to the familiar condition

$$\mu^{\mathrm{f}} = r^{\mathrm{f}} - r. \tag{2.8}$$

Remark 2.1. More generally, the same result holds true in case of a stochastic process $(X_t, t \geq 0)$ whose dynamics is given by

$$\mathrm{d}X_t = \mu^{\mathrm{f}} X_t\,\mathrm{d}t + \nu\,\mathrm{d}I_t, \quad X_0 = x, \tag{2.9}$$

where $(I_t, t \geq 0)$ is a generic $\mathbb{Q}^{\mathrm{f}}$-martingale and where $(\nu_t, t \geq 0)$ is an adapted and left-continuous process.

An equivalent argument would lead, starting from the foreign-currency measure and going to the pricing-currency one, to setting a drift condition for the process $(Z_t, t \geq 0)$ defined as $Z_t = \frac{1}{Z^{\mathrm{f}}_t}$. Following what was done with $(Z^{\mathrm{f}}_t, t \geq 0)$, one can define it as a geometric Brownian motion and use arbitrage considerations to determine the drift. The dynamics of $(Z_t, t \geq 0)$ would then be written as

$$\mathrm{d}Z_t = \mu Z_t\,\mathrm{d}t + \sigma Z_t\,\mathrm{d}W_t, \quad Z_0 = \frac{1}{x}. \tag{2.10}$$

The Radon–Nikodym measure in this case would be given by

$$L_t = \frac{Z_t B_t^{\mathrm{f}}}{Z_0 B_t}, \quad L_0 = 1. \tag{2.11}$$

Requiring that $(L_t, t \geq 0)$ has to be a martingale under the pricing-currency measure, would set the drift term as

$$\mu = r - r^{\mathrm{f}}. \tag{2.12}$$

Alternatively, one could deduce the dynamics for $(Z_t, t \geq 0)$ in $\mathbb{Q}$ starting from $(Z_t^{\mathrm{f}}, t \geq 0)$, whose dynamics is known in $\mathbb{Q}^{\mathrm{f}}$. Then, it is possible to deduce the dynamics of $(Z_t, t \geq 0)$ in $\mathbb{Q}^{\mathrm{f}}$ by applying Itô's formula to the process given by $Z_t = f(Z_t^{\mathrm{f}})$ where $f(x) = {}^1\!/\!_x$. Once its dynamics is known, the form of the driving martingales under $\mathbb{Q}$ can be worked out using Girsanov theorem. Under the log-normal dynamics chosen for the FX rates, this latter approach and the one starting from the Radon–Nikodym derivative in Eq. (2.11) lead to the same result. A detailed calculation in case the dynamics of the FX rate is subject also to jump-to-default effect, is presented in Sec. 2.5 below.

There are cases, for example Constant Elasticity of Variance or stochastic volatility FX-rate models, where the consistency between the arbitrage-free dynamics obtained under the two different specifications is not guaranteed. In these models, if one starts from $(Z_t^{\mathrm{f}}, t \geq 0)$ as a primitive modeling quantity, and then implies the distribution of $(Z_t, t \geq 0)$ at some time t from the law of Z_t^{f}, what will be obtained can be a different distribution from the one that one would have had by starting from $(Z_t, t \geq 0)$ as a primitive modeling quantity based on the same dynamical properties as $(Z_t^{\mathrm{f}}, t \geq 0)$.

In applications to quanto CDS pricing, where the FX rate is used in Eq. (2.2), and where depending on the circumstances, we might be interested in pricing or calibrating either under the liquid-currency measure or under the contractual-currency measure, there is a degree of arbitrariness in using one specification or the other. Having consistency between the two specifications is a desirable property to avoid results that depend on the aforementioned choice.

2.3. *Modeling framework for the quanto CDS correction*

In this section, we derive model-independent formulas to price contingent claims where contractual currency is different from the liquid currency used to define the pricing measure. In the next sections, we will show the application of these formulas under different dynamics assumptions for the main risk factors.

Let us start by calculating the value of a defaultable zero-coupon bond; it will be then used as a building block to calculate CDS values. To do so, we choose a payoff function $\phi_T^{\mathrm{f}} = \mathbb{1}_{\tau > T}$ in Eq. (2.3) and write

$$V_t^{\mathrm{f}}(T) = \mathbb{E}_t^{\mathrm{f}}\left[\frac{B_t^{\mathrm{f}}}{B_T^{\mathrm{f}}}\mathbb{1}_{\tau>T}\right] = \mathbb{E}_t\left[\frac{B_t^{\mathrm{f}}}{B_T^{\mathrm{f}}}\mathbb{1}_{\tau>T}\frac{\mathrm{d}\mathbb{Q}^{\mathrm{f}}}{\mathrm{d}\mathbb{Q}}\right]. \tag{2.13}$$

Given the Radon–Nikodym derivative from Eq. (2.11), the above expression can be written as

$$V_t^{\mathrm{f}}(T) = \frac{B_t}{Z_t}\mathbb{E}_t\left[\frac{Z_T}{B_T}\mathbb{1}_{\tau>T}\right], \tag{2.14}$$

which, under the assumption of deterministic interest rates, can be simplified into:

$$V_t^{\mathrm{f}}(T) = \frac{D(t,T)}{Z_t}\mathbb{E}_t[Z_T\mathbb{1}_{\tau>T}], \tag{2.15}$$

where $D(t,T) = B_t/B_T$ is the discount factor from time T to time $t \le T$.

It might be useful[a] to define the foreign currency survival probabilities as

$$q_t^{\mathrm{f}}(T) := \frac{V_t^{\mathrm{f}}(T)}{D^{\mathrm{f}}(t,T)}. \tag{2.16}$$

Let us now consider the price, expressed in the liquid currency, of the defaultable zero-coupon bond settled in the contractual currency, U. This is given by

$$U_t(T) = V_t^{\mathrm{f}}(T)Z_t = D(t,T)\mathbb{E}_t[Z_T\mathbb{1}_{\tau>T}]. \tag{2.17}$$

Being the $\mathbb{Q}$-price of a tradable asset, the drift of the process $(U_t, t \ge 0)$ has to be given by $r(t)U_t\,\mathrm{d}t$. Therefore, we can write a Feynman–Kac equation to calculate $U_t(T)$. Once $U_t(T)$ is known, $q_t^{\mathrm{f}}(T)$ can be calculated as

$$q_t^{\mathrm{f}}(T) = \frac{U_t(T)}{Z_t D^{\mathrm{f}}(t,T)}. \tag{2.18}$$

2.4. *A diffusive correlation model*: *Exponential OU/GBM*

In this section, we present a specific model to calculate U. We will be working with an intensity process and a FX-rate process which are defined and calibrated in the liquid measure.

Let us denote by $(\lambda_t, t \ge 0)$ a stochastic process given by $\lambda_t = e^{Y_t}$ where $(Y_t, t \ge 0)$ is an Ornstein–Uhlenbeck process defined as the solution of

$$\mathrm{d}Y_t = a(b - Y_t)\,\mathrm{d}t + \sigma^Y\,\mathrm{d}W_t^Y, \quad Y_0 = y, \tag{2.19}$$

where the parameters $(a, b, \sigma^Y, y) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$. Let us also consider a GBM process for the FX rate

$$\mathrm{d}Z_t = \mu^Z Z_t\,\mathrm{d}t + \sigma^Z Z_t\,\mathrm{d}W_t^Z \quad Z_0 = z, \tag{2.20}$$

where μ^Z is set by no-arbitrage considerations and it is given in this case by Eq. (2.12), and where $(\sigma^Z, z) \in \mathbb{R} \times \mathbb{R}^+$.

The dependence between FX rate and the default's intensity can be specified in this model through the instantaneous correlation between the two driving Brownian

[a] Mostly for computational reasons because such definition would easily allow CDS prices defined for single currency calculations to be re-used for quanto CDS pricing without further modifications.

motions, $\rho \in [-1, 1]$,

$$\mathrm{d}\,\langle W^Y, W^Z \rangle_t = \rho\,\mathrm{d}t. \tag{2.21}$$

Let finally $(D_t, t \geq 0)$ be the default process $D_t = \mathbb{1}_{\tau < t}$.

Remark 2.2. Due to the choice of modeling FX rates as geometric Brownian motions and to the symmetry relation that was discussed in Sec. 2.2, it does not matter if we choose to model $(Z_t, t \geq 0)$, or $(Z_t^{\mathrm{f}}, t \geq 0)$, as the two dynamics are consistent.

Remark 2.3. The choice of the (exponential OU and GBM) dynamics has been mainly driven by the need for the intensity process to stay positive. However, different intensities dynamics, possibly with local volatilities, can easily be accounted for using the same framework presented below as far as they are only driven by Wiener processes and no jump processes are involved. Extensive literature has been produced on the use of square root processes for the default intensity, mostly due to their tractability in obtaining closed form solutions for bonds, CDS and CDS options, see for example Brigo & El-Bachir (2010) and Brigo & Alfonsi (2005), where exact and closed form calibration to CDS curves is also discussed. For the FX-rate dynamics, instead, there is no such freedom of choice as the drift is given by no-arbitrage conditions, and introducing local or stochastic volatilities might break the symmetry relation between the FX rate and its reciprocal.

Pricing equation

In this section, we deduce a pricing equation to calculate the value of U. We follow the approach used in Bielecki *et al.* (2005). Given the strong Markov property of all the processes defined so far, $U_t(T)$ can be expressed as a function of t, Z_t, Y_t and D_t. Let us denote its value at t for $Z_t = z$, $Y_t = y$ and $D_t = d$ by $f(t, z, y, d)$. f is a function depending on both continuous and jump processes, and its Itô differential can be written as (see, for example, Jeanblanc *et al.* 2009)

$$\begin{aligned} \mathrm{d}f_t &= \partial_t f\,\mathrm{d}t + \partial_z f(\mu^Z z\,\mathrm{d}t + \sigma^Z z\,\mathrm{d}W_t^Z) + \partial_y f(a(b - Y_t)\,\mathrm{d}t + \sigma^Y\,\mathrm{d}W_t^Y) \\ &\quad + \frac{1}{2}(\sigma^Z z)^2 \partial_{zz} f\,\mathrm{d}t + \frac{1}{2}(\sigma^Y)^2 \partial_{yy} f\,\mathrm{d}t + \rho\sigma^Z\sigma^Y z\partial_{zy} f\,\mathrm{d}t + \Delta f\,\mathrm{d}D_t, \end{aligned} \tag{2.22}$$

where with some abuse of notation, we have defined the jump-to-default term as

$$\Delta f := f(t, Z_{t-}, Y_t, D_{t-} + \Delta D_t) - f(t, Z_{t-}, Y_t, D_{t-}). \tag{2.23}$$

Definition of $\boldsymbol{\Delta f}$. It must be noted that $(D_t, t \geq 0)$ starts at 0 and jumps to 1 at single time, τ, upon default. This means, in particular, that $D_{t-} + \Delta D_t$ takes a value different from zero only upon default, and that, for all the times previous to that, the following equation holds:

$$D_t = 0, \quad t < \tau. \tag{2.24}$$

The first term in Eq. (2.23) can then be rewritten as

$$f(t, Z_{t-}, Y_t, D_{t-} + \Delta D_t) = f(t, Z_{t-}, Y_t, \Delta D_t) \tag{2.25}$$

and the equation for Δf can be written as

$$\Delta f = f(t, Z_{t-}, Y_t, 1) - f(t, Z_{t-}, Y_t, 0). \tag{2.26}$$

Compensated martingale for $(D_t, t \geq 0)$. A compensator for $(D_t, t \geq 0)$ in the measure $\mathbb{Q}$ is defined as the process $(A_t, t \geq 0)$ such that $D_t - A_t$ is a $\mathbb{Q}$-martingale with respect to $(\mathcal{G}_t, t \geq 0)$. The compensator for $(D_t, t \geq 0)$ is given by

$$\mathrm{d}A_t = \mathbb{1}_{\tau > t} \lambda_t \,\mathrm{d}t, \quad A_0 = 0. \tag{2.27}$$

We define the resulting martingale as $(M_t, t \geq 0)$, where

$$M_t = D_t - A_t. \tag{2.28}$$

Consequently, the compensator of the term $\Delta f \Delta D_t$ in Eq. (2.22) can be written as

$$(1 - D_t) e^{Y_t} \Delta f \,\mathrm{d}t, \tag{2.29}$$

which conditional on $\mathcal{G}_t$, $D_t = d$, $Z_{t-} = z$, and $Y_t = y$, is equal to

$$(1 - d) e^{y} (f(t, z, y, 1) - f(t, z, y, 0)) \,\mathrm{d}t. \tag{2.30}$$

It is possible to write a Feynman–Kac type PDE to compute the value of $U_t(T)$. Indeed $(U_t, t \geq 0)$ is a $\mathbb{Q}$-price and, as such, it must locally grow at the rate r. Therefore, its drift must satisfy the following equation:

$$\begin{aligned} &\partial_t f + \mu^Z z \partial_z f + a(b - Y_t)\partial_y f + \frac{1}{2}(\sigma^Z z)^2 \partial_{zz} f \\ &\quad + \frac{1}{2}(\sigma^Y)^2 \partial_{yy} f + \rho \sigma^Z \sigma^Y z \partial_{zy} f + e^y (1 - d)\Delta f = r f \,\mathrm{d}t, \end{aligned} \tag{2.31}$$

where the explicit dependence of f on the state variables (x, y, t, d) has been omitted for clarity of reading. The last term in the left-hand side of the equation is the only one of the equation where the values $f(t, z, y, 0)$ and $f(t, z, y, 1)$ appear together (see Eq. (2.30)). It is therefore possible, by conditioning first on $d = 1$ and then on $d = 0$, to decouple the two functions

$$u(t, z, y) := f(t, z, y, 1), \tag{2.32a}$$

$$v(t, z, y) := f(t, z, y, 0), \tag{2.32b}$$

and to calculate them by solving iteratively two separate PDE problems. We first solve for u, as for $d = 1$ the last term does not appear in the equation, and, once u has been calculated, we use it to solve for v. Final conditions for the two functions are respectively given by

$$v(T, z, y) = f(T, z, y, 0) = z; \tag{2.33a}$$

$$u(T, z, y) = f(T, z, y, 1) = 0. \tag{2.33b}$$

The PDE problem that must be solved to obtain u is then given by

$$\partial_t u = ru - \mu^Z z\partial_z u - a(b-y)\partial_y u - \frac{1}{2}(\sigma^Z x)^2\partial_{zz}u$$
$$- \frac{1}{2}(\sigma^Y)^2\partial_{yy}u - \rho\sigma^Z\sigma^Y z\partial_{zy}u, \tag{2.34a}$$
$$u(T,z,y) = 0. \tag{2.34b}$$

The solution to this problem is $u \equiv 0$, therefore in this case one can solve directly the PDE for v, which is then given by

$$\partial_t v = rv - \mu^Z z\partial_z v - a(b-y)\partial_y v - \frac{1}{2}(\sigma^Z x)^2\partial_{zz}v$$
$$- \frac{1}{2}(\sigma^Y)^2\partial_{yy}v - \rho\sigma^Z\sigma^Y z\partial_{zy}v + e^y v, \tag{2.35a}$$
$$v(T,z,y) = z. \tag{2.35b}$$

Remark 2.4 (Interpretation of u and v). The functions u and v account for the post-default value and pre-default value, respectively, of a derivative with payoff $\phi(x,y,d)$. The price of this derivative can be written as

$$V_t = \mathbb{1}_{\tau>t-}\mathbb{E}_t[\phi(X_T,Y_T,D_T) \,|\, X_t = x, Y_t = y, D_t = d], \tag{2.36}$$

where due to the strong Markov property of the processes $(X_t, t \geq 0)$, $(Y_t, t \geq 0)$, and $(D_t, t \geq 0)$, the expected value on the right-hand side can be written as

$$f(t,x,y,d) = \mathbb{E}_t[\phi(X_T,Y_T,D_T) \,|\, X_t = x, Y_t = y, D_t = d]. \tag{2.37}$$

This can be decomposed as $f(t,x,y,d) = \mathbb{1}_{d=1}u(t,x,y) + \mathbb{1}_{d=0}v(t,x,y)$, where

$$v(t,x,y) := \mathbb{E}_t[\phi(X_T,Y_T,D_T) \,|\, X_t = x, Y_t = y, D_t = 0], \tag{2.38a}$$
$$u(t,x,y) := \mathbb{E}_t[\phi(X_T,Y_T,D_T) \,|\, X_t = x, Y_t = y, D_t = 1], \tag{2.38b}$$

in fact

$$f(t,x,y,d) = \mathbb{E}_t[\phi(X_T,Y_T,D_T) \,|\, X_t = x, Y_t = y, D_t = d] \tag{2.39a}$$
$$= \mathbb{1}_{\tau>t}\mathbb{E}_t[\phi(X_T,Y_T,D_T) \,|\, X_t = x, Y_t = y, D_t = 0]$$
$$+ \mathbb{1}_{\tau\leq t}\mathbb{E}_t[\phi(X_T,Y_T,D_T) \,|\, X_t = x, Y_t = y, D_t = 1] \tag{2.39b}$$
$$= \mathbb{1}_{\tau>t}v(t,x,y) + \mathbb{1}_{\tau\leq t}u(t,x,y) \tag{2.39c}$$

as both $\mathbb{1}_{\tau>t}$ and $\mathbb{1}_{\tau\leq t}$ are measurable in the $\mathcal{G}_t$ filtration. The derivative price can then be written as

$$V_t = \mathbb{1}_{\tau>t}v(t,X_t,Y_t) + \Delta D_t u(t,X_t,Y_t), \tag{2.40}$$

where we defined

$$\Delta D_t := \mathbb{1}_{\tau>t} - \mathbb{1}_{\tau>t-}. \tag{2.41}$$

2.5. *A jump-to-default framework*

The exponential OU-based model described in Sec. 2.4 can be extended by incorporating a devaluation mechanism in the FX-rate dynamics. By linking the devaluation to the default event, it is possible to introduce a further source of dependence between $(\lambda_t, t \geq 0)$ and $(Z_t, t \geq 0)$. The results shown in Sec. 3 further demonstrate that this is a more suitable mechanism to model the basis spread for quanto-CDS.

In this section, we will discuss in general how the dynamics of the risk factors are affected by the introduction of a jump-to-default effect on the FX component. Given that the Radon–Nikodym derivative depends on the FX rate, this change is expected to have an impact on all the risk factors whose dynamics has to be written in a measure different from the one in which they have been originally calibrated and, potentially, on the FX symmetry discussed is in Sec. 2.2. This is proven to hold true also in this new, more general, framework (see Proposition 2). Finally, we will apply the general results from the first subsections to the pricing of quanto CDS.

Risk factors dynamics

Let us then consider a jump-diffusion process for the FX rate in place of Eq. (2.20), while we will be keeping the same model choice for the intensity $\lambda_t = e^{Y_t}$:

$$\mathrm{d}Y_t = a(b - Y_t)\,\mathrm{d}t + \sigma^Y\,\mathrm{d}W_t^Y, \quad Y_0 = y, \tag{2.42a}$$

$$\mathrm{d}Z_t = \bar{\mu} Z_t\,\mathrm{d}t + \sigma^Z Z_t\,\mathrm{d}W_t^Z + \gamma^Z Z_{t-}\,\mathrm{d}D_t, \quad Z_0 = z, \tag{2.42b}$$

$$\mathrm{d}\langle W^Y, W^Z\rangle_t = \rho\,\mathrm{d}t, \tag{2.42c}$$

where as before, the parameters $(a, b, \sigma^Y, y) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$, $(\sigma^Z, z) \in \mathbb{R} \times \mathbb{R}^+$, $\rho \in [-1, 1]$, and where $\gamma^Z \in [-1, \infty)$ is the devaluation/revaluation rate of the FX process. From a mathematical point of view, γ^Z could lie in $[-1, \infty)$, but in this model we restrict it to be nonpositive for financial reasons. This devaluation factor is typically used in the case of reference entities whose default can negatively impact the value of their local currency. In the case of Italy, the expectation is that, in case of default, the FX rate used to exchange EUR in USD would fall.

We leave unspecified the drift term of $(Z_t, t \geq 0)$ and we simply use $\bar{\mu}$ for it in order to distinguish it from μ^Z. The introduction of the jump term leads to a result different from Eq. (2.12) if we want the process defined in Eq. (2.11) to still be a martingale — this is presented in Eq. (2.51).

Remark 2.5 (Jumps). The jump term in SDE for jump-diffusion processes can be described equivalently using $(D_t, t \geq 0)$ or the compensated process $(M_t, t \geq 0)$, the effect of using one term or the other being just a change in the drift term. We prefer using the noncompensated term when introducing the FX process in order to highlight the jump structure and hence the additional source of dependence between the FX and the credit component. On the other hand, the description in terms of the compensated martingale $(M_t, t \geq 0)$ will arise naturally every time the fundamental theorem of asset pricing will be used to derive no-arbitrage drift conditions, e.g. when Eq. (2.6) is used to deduce Eq. (2.51), or to derive the main pricing equation Eq. (2.62).

Hazard rate's and FX rate's dynamics in $\mathbb{Q}^{f}$

Given the dependence of $(L_t, t \geq 0)$ on $(D_t, t \geq 0)$ via $(Z_t, t \geq 0)$, in this case the change of measure modifies not only the expected value of $(W_t, t \geq 0)$, but also the expected value of $(M_t, t \geq 0)$ which was originally given by $\mathrm{d}M_t = \mathrm{d}D_t - (1 - D_t)\lambda_t \,\mathrm{d}t$ in $\mathbb{Q}$. However, Girsanov's theorem provides the adjustments for each of these processes needed to obtain a martingale in the new measure.

$$\mathrm{d}W_t^{\mathrm{f}} = \mathrm{d}W_t - \frac{\mathrm{d}\langle W, Z\rangle_t}{Z_t} = \mathrm{d}W_t - \sigma^Z \,\mathrm{d}t, \tag{2.43a}$$

$$\mathrm{d}M_t^{\mathrm{f}} = \mathrm{d}M_t - (1 - D_t)\gamma^Z \lambda_t \,\mathrm{d}t. \tag{2.43b}$$

The Wiener process decomposition in $\mathbb{Q}^{\mathrm{f}}$ is given by the same formula used in Sec. 2.4, while we derive the martingale decomposition for $(D_t, t \geq 0)$ as a result of the following proposition.

Proposition 1. *Let* $(M_t, t \geq 0)$ *be the martingale associated to the default process* $(D_t, t \geq 0)$ *in the domestic currency measure*

$$\mathrm{d}M_t = \mathrm{d}D_t - (1 - D_t)\lambda_t \,\mathrm{d}t, \tag{2.44}$$

then an application of the Girsanov theorem allows to write the correspondent martingale in the foreign measure $(M_t^f, t \geq 0)$ *as*

$$\begin{aligned}
\mathrm{d}M_t^f &= \mathrm{d}M_t - \frac{\mathrm{d}\,\langle M, L\rangle_t}{L_t} = \mathrm{d}M_t - \mathrm{d}\,\langle D, \gamma^Z D\rangle_t && (2.45\text{a}) \\
&= \mathrm{d}M_t - (1 - D_t)\gamma^Z \lambda_t \,\mathrm{d}t && (2.45\text{b}) \\
&= \mathrm{d}D_t - (1 - D_t)(1 + \gamma^Z)\lambda_t \,\mathrm{d}t, && (2.45\text{c})
\end{aligned}$$

where the dynamics of $(L_t, t \geq 0)$ *is defined by Eqs.* (2.11) *and* (2.42b). *Equation* (2.45c) *states that the intensity of the Poisson process driving the default event in the foreign currency is given by*

$$\lambda_t^f := (1 + \gamma^Z)\lambda_t. \tag{2.46}$$

Proof. Integration by parts gives

$$\mathrm{d}(M_t^{\mathrm{f}} L_t) = L_t\,\mathrm{d}M_t^{\mathrm{f}} + M_t^{\mathrm{f}}\,\mathrm{d}L_t + \mathrm{d}[M^{\mathrm{f}}, L]_t \tag{2.47a}$$

$$= L_t\,\mathrm{d}M_t^{\mathrm{f}} + M_t^{\mathrm{f}}\,\mathrm{d}L_t + \gamma^Z L_t\,\mathrm{d}D_t \tag{2.47b}$$

$$= L_t(\mathrm{d}M_t - (1 - D_t)\gamma^Z \lambda_t\,\mathrm{d}t) + M_t^{\mathrm{f}}\,\mathrm{d}\hat{L}_t + \gamma^Z L_t\,\mathrm{d}D_t \tag{2.47c}$$

$$= L_t\,\mathrm{d}M_t + M_t^{\mathrm{f}}\,\mathrm{d}L_t + \gamma^Z L_t\,\mathrm{d}M_t \tag{2.47d}$$

so the process $((LM^{\mathrm{f}})_t, t \geq 0)$ is a martingale in the domestic measure as it can be written as a sum of stochastic integrals on local martingales. As a consequence, the process $(M_t^{\mathrm{f}}, t \geq 0)$ is a local martingale in the foreign measure. □

Remark 2.6 (CDS par-spreads approximation). In all the cases where the well-known approximation

$$\lambda \approx \frac{S}{1 - R} \tag{2.48}$$

between hazard rates, CDS par-spreads, S, and recovery rates, R, holds, the relation in Eq. (2.46) can be written in terms of CDS par-spreads rather than hazard rates as

$$S^{\mathrm{f}} = (1 + \gamma^Z)S. \tag{2.49}$$

This happens, for example, when the hazard rate is constant in time and when the premium leg's cash-flows can be approximated by a stream of continuously compounded payments (see Brigo & Mercurio 2006).

FX-rates dynamics in the two measures and symmetry

The FX rate in this model is a jump-diffusion process, whose jumps are given by (see Eq. (2.42b))

$$\Delta Z_t = \gamma^Z Z_{t-} \Delta D_t. \tag{2.50}$$

Notice that also this specification of the FX rate is subject to arbitrage constraints such that the Radon–Nikodym derivative defined by Eq. (2.11) be a martingale. The condition equivalent to Eq. (2.12) in the case where the FX dynamics is given by Eq. (2.42b) is provided by

$$\bar{\mu} = \mu^Z - \lambda_t \gamma^Z \mathbb{1}_{\tau > t} = r - r^{\mathrm{f}} - \lambda_t \gamma^Z \mathbb{1}_{\tau > t}. \tag{2.51}$$

Despite the introduction of the jump in the FX-rate dynamics, the consistency highlighted in Sec. 2.2 between $(Z_t^{\mathrm{f}}, t \geq 0)$ and $(Z_t, t \geq 0)$ is maintained. From a practical point of view this means that we do not need to worry about which FX rate we use, as one can be obtained as a transformation of the first one and it is guaranteed to satisfy the no-arbitrage relations for the associated Radon–Nikodym derivative. This is proved in the next proposition.

Proposition 2 (FX-rates symmetry under devaluation jump to default). *Let us consider an FX-rate process whose dynamics in the domestic measure $\mathbb{Q}$ is specified by Eq.* (2.42b) *and whose drift is given by Eq.* (2.51). *Then the dynamics of the process* $(Z^f_t, t \geq 0)$ *where* $Z^f_t = {}^1\!/_{Z_t}$ *in the foreign measure* $\mathbb{Q}^f$ *is given by*

$$\mathrm{d}Z^f_t = (r^f - r)Z^f_t\,\mathrm{d}t - \sigma^Z Z^f_t\,\mathrm{d}W^{Z,f}_t + Z^f_{t-}\gamma^f\,\mathrm{d}M^f_t, \quad Z^f_0 = \frac{1}{z}, \tag{2.52}$$

where the devaluation rate for $(Z^f_t, t \geq 0)$ *is given by*

$$\gamma^f = -\frac{\gamma^Z}{1+\gamma^Z}. \tag{2.53}$$

In particular, (2.52) *is such that the Radon–Nikodym derivative defined by Eq.* (2.4) *is a* $\mathbb{Q}^f$*-martingale.*

Proof. The relation between Z and Z^{f} is given by $Z^{\mathrm{f}}_t = \phi(Z_t)$, where $\phi(x) = 1/x$. By applying Itô's rule for jump-diffusion processes (see, for example, Jeanblanc *et al.* 2009, Protter 2005), we obtain

$$\begin{aligned}
\mathrm{d}Z^{\mathrm{f}}_t &= \mathrm{d}\phi(Z_t) = \phi'(Z_{t-})\,\mathrm{d}Z_t + \frac{1}{2}\phi''(Z_{t-})\,\mathrm{d}[Z]^c_t \\
&\quad + \sum_{s\leq t}((\phi(Z_{s-} + \Delta Z_{s-}) - \phi(Z_{s-})) - \phi(Z_{s-})\Delta Z_{s-}) && \text{(2.54a)}\\
&= \mathrm{d}\left(\frac{1}{Z_t}\right) = -\frac{\mathrm{d}Z_t}{Z_t^2} + \frac{\mathrm{d}[Z]^c_t}{Z_t^3} + \left(\frac{1}{Z_{t-} + \gamma^Z Z_{t-}} - \frac{1}{Z_{t-}}\right)\mathrm{d}D_t + \frac{\Delta Z}{Z_t^2} && \text{(2.54b)}\\
&= -\bar{\mu}\frac{1}{Z_t}\,\mathrm{d}t - \sigma^Z\frac{1}{Z_t}\,\mathrm{d}W^Z_t - \gamma^Z\frac{1}{Z_{t-}}\,\mathrm{d}D_t + (\sigma^Z)^2\frac{1}{Z_t}\,\mathrm{d}t \\
&\quad + \frac{1}{Z_{t-}}\left(\frac{1}{1+\gamma^Z} - 1\right)\mathrm{d}D_t + \frac{\gamma^Z}{Z_{t-}}\,\mathrm{d}D_t && \text{(2.54c)}\\
&= -\bar{\mu}\frac{1}{Z_t}\,\mathrm{d}t - \sigma^Z\frac{1}{Z_t}\,\mathrm{d}W^Z_t + (\sigma^Z)^2\frac{1}{Z_t}\,\mathrm{d}t + \frac{1}{Z_{t-}}\gamma^X\,\mathrm{d}D_t && \text{(2.54d)}\\
&= -\bar{\mu}Z^{\mathrm{f}}_t\,\mathrm{d}t - \sigma^Z Z^{\mathrm{f}}_t\,\mathrm{d}W^Z_t + (\sigma^Z)^2 Z^{\mathrm{f}}_t\,\mathrm{d}t + Z^{\mathrm{f}}_{t-}\gamma^{\mathrm{f}}\,\mathrm{d}D_t, && \text{(2.54e)}
\end{aligned}$$

where we used γ^X to denote the jumps of $(Z^{\mathrm{f}}_t, t \geq 0)$, given by

$$\gamma^{\mathrm{f}} = -\frac{\gamma^Z}{1+\gamma^Z}. \tag{2.55}$$

We can now use Girsanov's theorem in the form of Eq. (2.43a) for $(W^Z_t, t \geq 0)$ and Eq. (2.45c) for $(D_t, t \geq 0)$ to decompose $(Z^{\mathrm{f}}_t, t \geq 0)$ in a sum of local martingales in the new measure $\mathbb{Q}^{\mathrm{f}}$. As a result

$$\begin{aligned}\mathrm{d}Z_t^{\mathrm{f}} &= -\bar{\mu} Z_t^{\mathrm{f}}\,\mathrm{d}t - \sigma^Z Z_t^{\mathrm{f}}(\mathrm{d}W_t^{Z,\mathrm{f}} - \sigma^Z\,\mathrm{d}t) + (\sigma^Z)^2 Z_t^{\mathrm{f}}\,\mathrm{d}t \\ &\quad + Z_{t-}^{\mathrm{f}}\gamma^{\mathrm{f}}(\mathrm{d}M_t^{\mathrm{f}} + (1+\gamma^Z)(1-D_t)\lambda_t\,\mathrm{d}t) \qquad (2.56\mathrm{a}) \\ &= -(\bar{\mu} - \gamma^{\mathrm{f}}(1+\gamma^Z)(1-D_t)\lambda_t\,\mathrm{d}t) Z_t^{\mathrm{f}}\,\mathrm{d}t - \sigma^Z Z_t^{\mathrm{f}}\,\mathrm{d}W_t^{Z,\mathrm{f}} + Z_{t-}^{\mathrm{f}}\gamma^{\mathrm{f}}\,\mathrm{d}M_t. \qquad (2.56\mathrm{b})\end{aligned}$$

Given that $\bar{\mu}$ is given by (see Eqs. (2.8) and (2.51)) $r - r^{\mathrm{f}} - (1-D_t)\gamma^Z\lambda_t$, the $\mathbb{Q}^{\mathrm{f}}$-dynamics of $(Z_t^{\mathrm{f}}, t \geq 0)$ can be written as

$$\begin{aligned}\mathrm{d}Z_t^{\mathrm{f}} &= (r^{\mathrm{f}} - r + \gamma^Z(1-D_t)\lambda_t + \gamma^{\mathrm{f}}(1+\gamma^Z)(1-D_t)\lambda_t) Z_t^{\mathrm{f}}\,\mathrm{d}t \\ &\quad - \sigma^Z Z_t^{\mathrm{f}}\,\mathrm{d}W_t^{Z,\mathrm{f}} + Z_{t-}^{\mathrm{f}}\gamma^{\mathrm{f}}\,\mathrm{d}M_t^{\mathrm{f}} \qquad (2.57\mathrm{a}) \\ &= (r^{\mathrm{f}} - r) Z_t^{\mathrm{f}}\,\mathrm{d}t - \sigma^Z Z_t^{\mathrm{f}}\,\mathrm{d}W_t^{Z,\mathrm{f}} + Z_{t-}^{\mathrm{f}}\gamma^{\mathrm{f}}\,\mathrm{d}M_t^{\mathrm{f}}. \qquad (2.57\mathrm{b})\end{aligned}$$

□

Alternatively, a representation where the jumps are highlighted can be used for the $\mathbb{Q}^{\mathrm{f}}$-dynamics of $(Z_t^{\mathrm{f}}, t \geq 0)$

$$\mathrm{d}Z_t^{\mathrm{f}} = (r^{\mathrm{f}} - r - (1-D_t)\gamma^{\mathrm{f}}\lambda_t) Z_t^{\mathrm{f}}\,\mathrm{d}t - \sigma^Z Z_t^{\mathrm{f}}\,\mathrm{d}W_t^{Z,\mathrm{f}} + Z_{t-}^{\mathrm{f}}\gamma^{\mathrm{f}}\,\mathrm{d}D_t, \quad Z_0^{\mathrm{f}} = \frac{1}{z}. \qquad (2.58)$$

Pricing equation

In this section, we consider the case where liquid currency and pricing currency coincide and are different from the contractual currency. As discussed in Sec. 1.2, this is the typical setup arising to price in the USD-market measure CDSs written on European Monetary Union countries, as the standard currency for them is USD. If one wants to price a EUR denominated contract for such reference entities in the USD measure, one has first to calibrate the default's intensity to USD-denominated contracts and then the pricing can be carried out using the equations derived in this section. This is also the procedure followed to produce the results showed in Sec. 3.5 below.

Without loss of generality, we will study the case of liquid currency and pricing currency associated to the domestic measure $\mathbb{Q}$.

$$\mathrm{d}Y_t = a(b - Y_t)\,\mathrm{d}t + \sigma^Y\,\mathrm{d}W_t^Y, \qquad (2.59\mathrm{a})$$

$$\mathrm{d}Z_t = \bar{\mu}_Z Z_t\,\mathrm{d}t + \sigma^Z Z_t\,\mathrm{d}W_t^Z + \gamma^Z Z_t\,\mathrm{d}D_t, \qquad (2.59\mathrm{b})$$

$$\mathrm{d}\langle W^Y, W^Z\rangle_t = \rho\,\mathrm{d}t, \qquad (2.59\mathrm{c})$$

with

$$\mathrm{d}M_t = \mathrm{d}D_t - (1-D_t)\lambda_t\,\mathrm{d}t, \qquad (2.60)$$

so that the no-arbitrage drift is given by (see Eq. (2.51))

$$\bar{\mu}_Z = r - r^{\mathrm{f}} - \gamma^Z (1 - D_t)\lambda_t. \tag{2.61}$$

An application of the generalized Itô formula (see, for example, Jeanblanc *et al.* 2009) allows us to write the $\mathbb{Q}$-dynamics of $(U_t, t \geq 0)$. Using $U_t = f(t, Z_t, Y_t, D_t)$:

$$\begin{aligned} \mathrm{d}f = {} & \partial_t f\,\mathrm{d}t + \partial_z f(\bar{\mu}_Z Z_{t-}\,\mathrm{d}t + \sigma^Z Z_t\,\mathrm{d}W_{t-}^Z + \gamma^Z Z_{t-}\,\mathrm{d}D_t) \\ & + \partial_y f(a(b - Y_t)\,\mathrm{d}t + \sigma^Y\,\mathrm{d}W_t^Y) + \frac{1}{2}(\sigma^Z z)^2 \partial_{zz} f\,\mathrm{d}t + \frac{1}{2}(\sigma^Y)^2 \partial_{yy} f\,\mathrm{d}t \\ & + \rho\sigma^Z\sigma^Y z\partial_{zy} f\,\mathrm{d}t + \Delta f\,\mathrm{d}D_t - \partial_z f \Delta Z_t. \end{aligned} \tag{2.62}$$

The pricing equation could be deduced by the f dynamics in the same way used to deduce Eq. (2.35) discussed in Sec. 2.4. In this case, the jump-to-default term is defined as

$$\Delta f := f(t, Z_{t-} + \Delta Z_t, Y_t, D_{t-} + \Delta D_t) - f(t, Z_{t-}, Y_t, D_{t-}). \tag{2.63}$$

Definition of $\boldsymbol{\Delta f}$. Differently from the pure diffusive case from Sec. 2.4, Δf depends in this case on both the jumps of $(Z_t, t \geq 0)$ and $(D_t, t \geq 0)$. The two jump components, however, are driven by a common jump driver ($(D_t, t \geq 0)$ itself, see Eq. (2.42b)), and the jumps in the FX-rate dynamics are given by Eq. (2.50).

Consequently, the equation for Δf can be written as

$$\Delta f = f(t, (1 + \gamma^Z) Z_{t-}, Y_t, 1) - f(t, Z_{t-}, Y_t, 0), \tag{2.64}$$

and the final conditions are given by

$$v(T, z, y) = f(T, z, y, 0) = z; \tag{2.65a}$$

$$u(T, z, y) = f(T, (1 + \gamma^Z) z, y, 1) = 0. \tag{2.65b}$$

As for the pure diffusive case, $u(t, x, y) \equiv 0$, so that there is only one PDE system for v to solve:

$$\begin{aligned} \partial_t v = {} & rv - (r - r^{\mathrm{f}}) z\partial_z v - a(b - y)\partial_y v - \frac{1}{2}(\sigma^Z z)^2 \partial_{zz} v \\ & - \frac{1}{2}(\sigma^Y)^2 \partial_{yy} v - \rho\sigma^Z\sigma^Y z\partial_{zy} v + e^y (v - \gamma^Z z\partial_z v) \end{aligned} \tag{2.66a}$$

$$v(T, z, y) = z. \tag{2.66b}$$

Inferring default probability devaluation factor from the FX-rate devaluation factor

It is possible to link the FX-rate devaluation factor introduced in Eq. (2.42b) with a probability rescaling factor. This is done in the following proposition.

Proposition 3 (Default probabilities devaluation). *Under the following hypotheses*:

(i) *small tenors*:

$$T \to 0, \tag{2.67}$$

(ii) *deterministic hazard rate*, $\lambda(\cdot)$,

the ratio of the quanto-corrected and single-currency default probabilities can be approximated through

$$\frac{1 - q_0^f(T)}{1 - q_0(T)} \approx 1 + \gamma^Z. \tag{2.68}$$

Proof. See Appendix A. □

Remark 2.7. Although the above approximation is only proven in the case of deterministic hazard-rate, results from the later sections suggest that, for small tenors and up to a certain level of credit risk, it can be applied to the case of stochastic processes as well.

We tested the behavior of this approximation with respect to different levels of credit risk in Sec. 3.3 on results generated using the model (2.42a)–(2.42c).

3. Results

3.1. *Numerical methods*

In order to produce the results presented in this section, the PDE-system (2.66) has been solved numerically, both for direct calculations of quanto-adjusted survival probabilities and for the calibration problems described later in Sec. 3.5.

For this purpose, we implemented a finite-difference method belonging to the family of *alternating-direction implicit* (ADI) schemes. The description of the scheme that has been used can be found in During *et al.* (2013).

Differently from what was suggested in that work, however, we did not use Neumann or Dirichlet types of boundary conditions — rather, the second derivative of the solution was set to zero on the boundaries. Consistently with this choice, we setup the numerical grid dimension sizes large enough such to avoid nonlinearities of the payoff on the boundaries.

3.2. *Quanto CDS par-spreads parameters dependence*

In this section, we show how the quanto-corrected CDS par-spreads are affected by changing the value of some of the parameters.

The parameter which affected the most the value of the spreads in this analysis is, as one expects, the devaluation rate, γ^Z (see Fig. 3). For the chosen value of the parameters, a change in the instantaneous correlation from its extreme values,

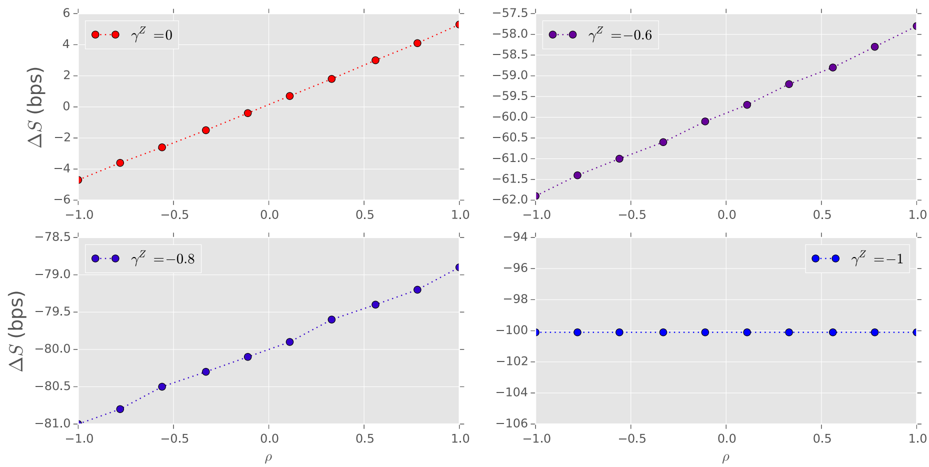


Fig. 3. 5Y CDS par-spread impact versus ρ and γ. The reference value for the par-spread is calculated using the parameter values in Table 1.

Table 1. Parameters used to produce the par-spreads impact in Fig. 3.

z	μ	σ^Z	a	b	y	σ^Y	T
0.8	0.0	0.1	0.0001	-210.0	-4.089	0.2	5.0

-1 and 1, can usually move the par spread of less than 10 bps, while moving the devaluation rate to its extreme value, 1, can bring to zero the level of the par spread.

Figure 4 shows that par-spread sensitivity to the volatility of the FX-rate process is slightly weaker than the one to the log-hazard rate's volatility for the chosen

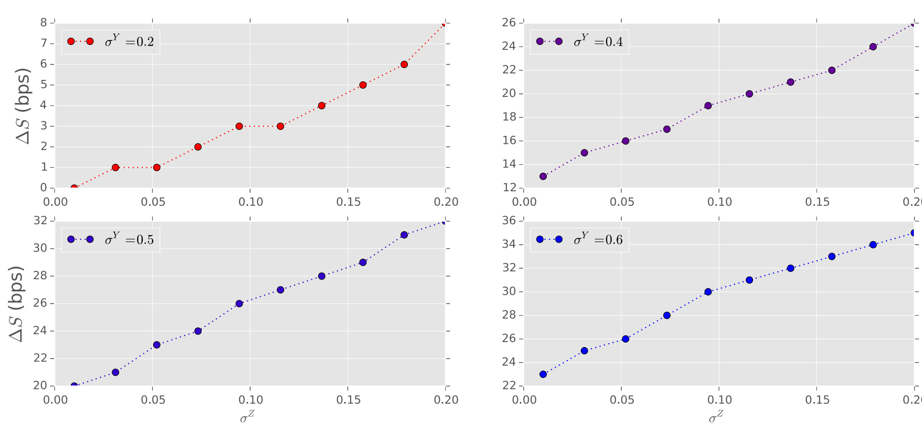


Fig. 4. 5Y CDS par-spread impact versus σ^Z and σ^Y. The reference value is produced using the parameter values in Table 2.

Table 2. Parameters used to produce the results shown in Fig. 4.

z	μ	ρ	a	b	y	T
0.8	0.0	0.5	0.0001	-210.0	-4.089	5.0

ranges of parameter values. In our example, a 5Y par-spread can change of around 10 bps with σ^Z ranging from 1% to 20%, while it can range up to 30 bps with σ^Y going from 20% to 70% and with σ^Z fixed at 20%.

In Fig. 5, we show the sensitivity of par-spreads to the value of diffusive correlation ρ. The dependence of par spreads on the correlation is extremely weak for values of σ^Y in the range of 20%. Around this level of log-hazard rate volatility, the maximum change that correlation can produce on the quanto-par spreads is 10 bps. From Fig. 5, a more realistic value of σ^Y of 60% is required to observe an impact of around 30 bps on the 5Y par-spread when changing the correlation from -1 to 1, showing the limits of a purely diffusive correlation model in explaining large differences between domestic and quanto-corrected CDS par-spreads.

There are circumstances where the basis between par-spreads of CDSs in different currencies can be sensibly higher than these values. In those cases, a purely

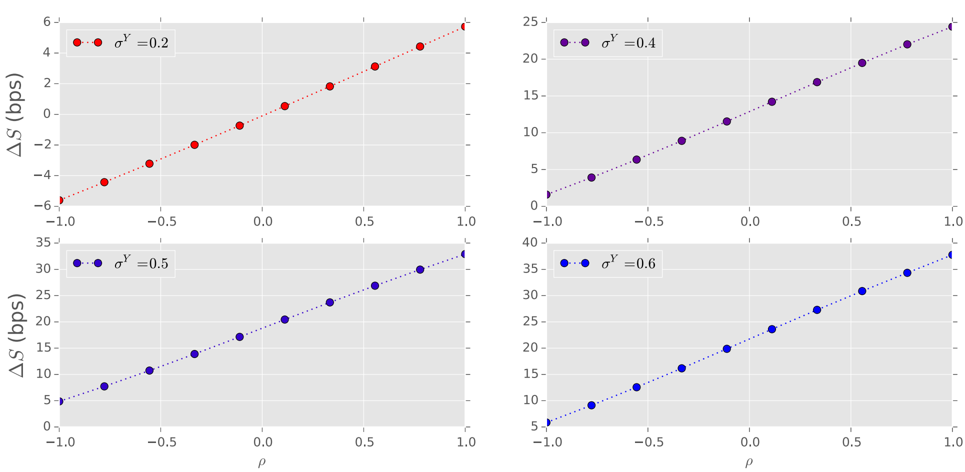


Fig. 5. 5Y CDS par-spread impact versus ρ and σ^Y. The reference value is produced using the parameter values in Table 3.

Table 3. Parameters used to produce the results shown in Fig. 5.

z	μ	σ^Z	a	b	y	T
0.8	0.0	0.1	0.0001	204.0	-4.089	5.0

diffusive model for the hazard rate is not sufficient to explain the observed basis and an approach where dependence is induced by devaluation jumps is required. As an example of an historical occurrence of such a wide basis, we show in Sec. 3.5 results of model calibrations to the time series of par-spreads for EUR-denominated and USD-denominated 5Y CDSs on Republic of Italy.

In the different context of impact of dependence on CDS credit valuation adjustments, even under collateralization, Brigo & Chourdakis (2009), Brigo *et al.* (2013, 2014) show that a copula function on the jump to default exponential thresholds may be necessary to obtain sizable effects when looking at credit-credit dependence, pure diffusive correlation not being enough.

3.3. *Test on the impact of tenor and credit worthiness on the quanto correction*

To test the relation given in Eq. (2.68), we set the diffusive correlation to zero and we chose the following set of log-hazard rate parameters:

$$a = 1 \times 10^{-4}, \quad b = -210.45, \quad \sigma = 0.2,$$

whereas we have produced low spread scenarios and high spread scenarios by choosing two different values for Y_0, the first one, $y^L = -4.089$, such that the resulting CDS par-spread term structure is flat at around 100 bps, and the second one, $y^H = -2.089$ such to produce a flat CDS par spread term structure with a value of around 740 bps.

Figures 6 and 7 show, in line with the nature of the approximation (2.68), that the approximation is less accurate for higher maturities, as evident in both charts

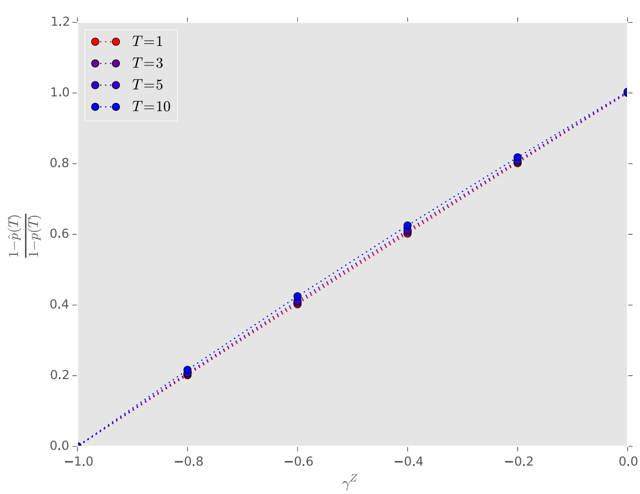


Fig. 6. Comparison of curves $1-\hat{P}_0(T)/1-P_0(T)$ for different maturities in a low spreads scenario, $Y_0 = y^L$.

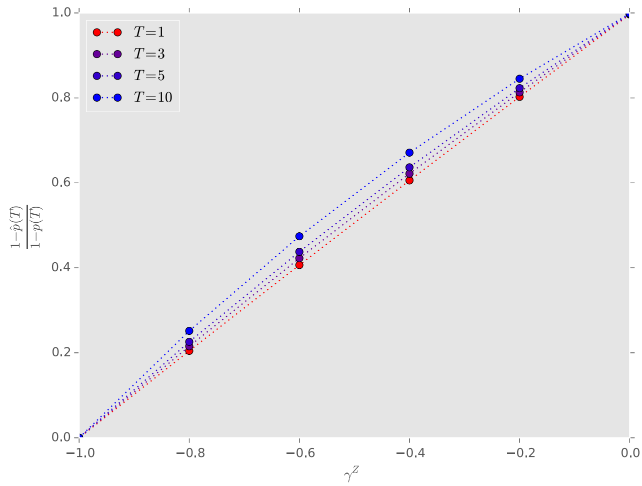


Fig. 7. Comparison of $1-\hat{P}_0(T)/1-P_0(T)$ for different maturities in a high spreads scenario, $Y_0 = y^H$.

by comparing blue lines (short maturities) with red lines (long maturities). It is also less accurate and for higher values of CDS spreads, as highlighted by the comparison between Fig. 7 (high spreads) and Fig. 6 (low spreads).

3.4. *Correlation impact on the short term versus long term*

We checked numerically the robustness of the theoretical relation between survival probabilities and γ^Z that was shown in Eq. (2.68). We calculated the ratio between the local and the quanto-corrected survival probability returned by the exponential OU model for different maturities and for different values of ρ. We can express this value as a function of the devaluation jump size, γ^Z, and of the maturity, T, and denote it by $k(\gamma, t)$. We then checked how this is affected by changes in γ and we compared k with the limit-case value provided by Eq. (2.68):

$$\tilde{k}(\gamma) := 1 + \gamma. \tag{3.1}$$

We used a set of parameters such to produce a flat CDS par-spread term-structure:

$$a = 1.00e - 004, \quad b = -210.45, \quad \sigma = 0.2, \quad Y_0 = -4.089.$$

The results are reported in Tables 4–6. They show that the approximation becomes less and less effective with increasing values of correlation or maturities. In the case of 10Y survival probabilities, $\tilde{k}$ doesn't provide a good approximation of k not even in case of null correlation. Conversely, for $\rho = -0.9$, the approximation fails to provide accurate values of k even in the case of 1Y maturity. This is in line with the results from Proposition 3, as in both cases the hypotheses under which the approximation was deduced are clearly not valid.

Table 4. Comparison between $\tilde{k}$ and k with $\rho = 0$.

		$k(\gamma, T)$			
γ	$\tilde{k}(\gamma)$	$T = 1$	$T = 3$	$T = 5$	$T = 10$
0.0	1.0	0.996	0.997	0.998	0.999
−0.2	0.8	0.802	0.813	0.823	0.845
−0.4	0.6	0.606	0.622	0.637	0.671
−0.6	0.4	0.407	0.423	0.438	0.474
−0.8	0.2	0.205	0.215	0.226	0.252
−1.0	0.0	0.000	0.000	0.000	0.000

Table 5. Comparison between $\tilde{k}$ and k with $\rho = -0.9$.

		$k(\gamma, T)$			
γ	$\tilde{k}(\gamma)$	$T = 1$	$T = 3$	$T = 5$	$T = 10$
0.0	1.0	0.989	0.980	0.971	0.956
−0.2	0.8	0.796	0.798	0.800	0.806
−0.4	0.6	0.601	0.610	0.619	0.638
−0.6	0.4	0.404	0.415	0.425	0.449
−0.8	0.2	0.203	0.211	0.219	0.238
−1.0	0.0	0.000	0.000	0.000	0.000

Table 6. Comparison between $\tilde{k}$ and k with $\rho = 0.9$.

		$k(\gamma, T)$			
γ	$\tilde{k}(\gamma)$	$T = 1$	$T = 3$	$T = 5$	$T = 10$
0.0	1.0	1.004	1.014	1.021	1.029
−0.2	0.8	0.808	0.827	0.841	0.867
−0.4	0.6	0.610	0.632	0.650	0.685
−0.6	0.4	0.410	0.429	0.446	0.481
−0.8	0.2	0.206	0.219	0.230	0.253
−1.0	0.0	0.000	0.000	0.000	0.000

Additionally, let us consider the value of quanto-adjusted survival probability obtained using different values of maturities, T, devaluation jump sizes, γ, and correlation ρ, and let us denote it by $q_0^{\mathrm{f}}(T; \gamma, \rho)$. In Table 7 and in Fig. 8, we reported the difference between the values of the quanto-corrected survival probabilities obtained using extreme values of correlation, $(T, \gamma) \mapsto q_0^{\mathrm{f}}(T; \gamma, -0.9) - q_0^{\mathrm{f}}(T; \gamma, 0.9)$.

They show, as hinted in EL-Mohammadi (2009), that correlation has a smaller impact on short term survival probabilities: moving the correlation between the values of -0.9 and 0.9 has an absolute impact of 0.13% in the case of zero devaluation jumps and for 1Y survival probabilities, whereas the impact for 10Y survival probabilities — also considered in the case of no devaluation jumps — can achieve 4.28%. They also show, as suggested by the results presented in Sec. 3.2, that with increasing values of the devaluation jump size, the correlation impact tends to become less and less material until it vanishes for the value $\gamma = -1$.

Table 7. Difference between $q_0^{\mathrm{f}}(T;\gamma,\rho=-0.9)$ and $q_0^{\mathrm{f}}(T;\gamma,\rho=0.9)$ for different values of devaluation jump size, γ, and maturity, T.

	$q_0^{\mathrm{f}}(T;\gamma,-0.9)-q_0^{\mathrm{f}}(T;\gamma,0.9)$			
γ	$T=1$ (%)	$T=3$ (%)	$T=5$ (%)	$T=10$ (%)
0.0	0.13	0.80	1.76	4.28
−0.2	0.10	0.66	1.45	3.60
−0.4	0.08	0.50	1.11	2.77
−0.6	0.05	0.34	0.75	1.85
−0.8	0.03	0.17	0.37	0.90
−1.0	0.00	0.00	0.00	0.00

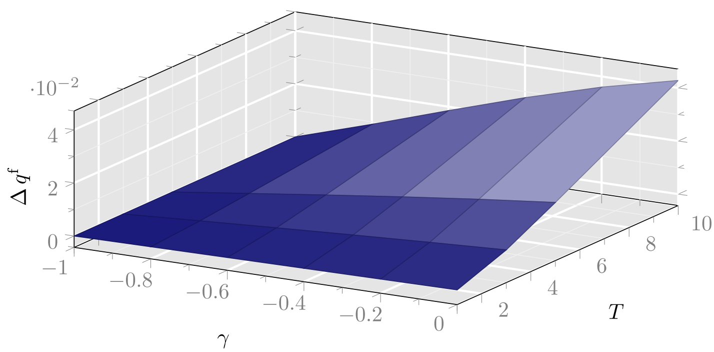


Fig. 8. Difference between $q_0^{\mathrm{f}}(T;\gamma,\rho=-0.9)$ and $q_0^{\mathrm{f}}(T;\gamma,\rho=0.9)$ for different values of devaluation jump size, γ, and maturity, T.

3.5. *Model calibration to market data for* 2011–2013

In this section, we present the results of the calibration of the model described in Sec. 2.5, where pricing currency and liquid currency coincide and are USD, and where we considered two contractual currencies, EUR and USD.

We used the observed CDSs spreads on Italy, both the USD-denominated ones and the EUR-denominated ones, to calibrate the model parameters. Given the current model parameterization (i.e. constant parameters), the model is not able to fit CDS par-spreads term-structures containing an arbitrary number of points. For this reason, we calibrated it using two USD-denominated and two EUR-denominated CDSs, choosing only the most liquid tenor points for this task (five and ten years). The model could be re-parameterized in a way to fit more granular CDS term-structures — for example, by introducing a time-dependency on the long-term mean log-intensity parameter, b.

In principle, also single-name CDS swaptions could be used in this calibration process (see Brigo & Alfonsi 2005), but, given the lack of liquidity on this instrument, we preferred proxying them with the at-the-money implied volatilities quoted for options on CDX.

Market data description

On each business day in the time range 2011–2013, we used a Levenberg Marquardt algorithm to calibrate the model to the market data. Let $\mathcal{T} = \{t_0, \ldots, t_N\}$ denote the dates in this sample period. We made the following assumptions on the market data:

(i) we consider the CDS par-spreads on Republic of Italy with 5 years and 10 years tenor, both in USD and in EUR;
(ii) we use the same short rate for domestic and foreign currency

$$r(t_i) = r^{\mathrm{f}}(t_i) = r, \quad t_i \in \mathcal{T}, \tag{3.2}$$

(iii) on every $t_i \in \mathcal{T}$ we assign the value of the at-the-money Black volatility from an option with 6 months expiry to σ^Z;
(iv) we keep the speed of mean reversion a of $(Y_t, t \geq 0)$ flat at the level 1×10^{-4};
(v) on every $t_i \in \mathcal{T}$ we calibrated σ^Y to the at-the-money option Black volatility for expiry one month.

Denoting by $p^Y := (b, y_0)$ the parameters to be calibrated for $(Y_t, t \geq 0)$ that are needed in single currency CDS pricing, and by $p := (b, y_0, \rho, \gamma)$ the set of parameters needed to price a quanto CDS, we adopted the following procedure to calibrate the model in Eqs. (2.59a)–(2.59c):

(i) first we calibrated p^Y to the USD-denominated par-spread for the given date. We kept the parameters a and σ^Y fixed at a level of 1×10^{-4} and 50% respectively;
(ii) we calibrated σ^Y to the CDS index option, keeping the p^Y at the level calibrated at the previous step. In order to do that, we looked for the value of σ^Y such that a single-name CDS options, would have a price such to deliver a Black volatility matching the one quoted for the CDS Index option. In order to calculate the price of the CDS option, no FX dynamics is needed, so we used a model specified through Eq. (2.59a) only and we solved the corresponding one-dimensional pricing equation using a Crank–Nicolson scheme.
(iii) we used the calibrated value of p^Y as a starting point in the iterative routine carried out to calibrate the set of model parameters p to both the EUR-denominated and the USD-denominated CDSs. The starting guess point to calibrate p can be written in terms of the calibrated point p^Y as $p_0 = (p_1^Y, p_2^Y, \gamma_0, \rho_0)$, where p_i^Y is the ith component of p^Y and where γ_0 and ρ_0 are the guess values for γ and ρ, respectively. We kept σ^Y fixed at the level calibrated at the previous step.

Results

The calibrated γ and ρ are showed in Fig. 9 together with the relevant market data used in calibration, EUR-denominated and USD-denominated CDS par-spreads

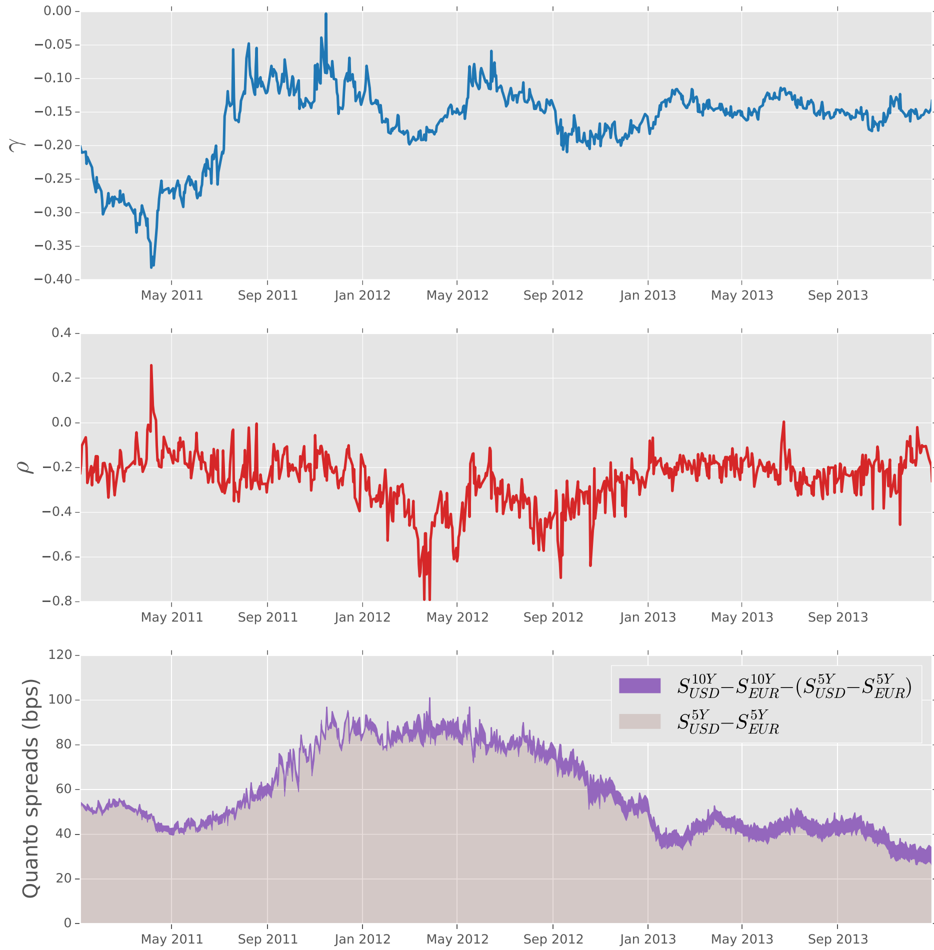


Fig. 9. The top chart shows the calibrated values of γ in $\mathcal{T}$. The middle chart shows the calibrated values of ρ in $\mathcal{T}$. The bottom chart shows the time series of corresponding quanto CDS par-spreads.

for 5 years maturities, $S^{5\mathrm{Y}}_{\mathrm{EUR}}$ and $S^{5\mathrm{Y}}_{\mathrm{USD}}$, and for 10 years maturities, $S^{10\mathrm{Y}}_{\mathrm{EUR}}$ and $S^{10\mathrm{Y}}_{\mathrm{USD}}$.

It is important to note that, even in periods of high quanto CDS basis spreads as the one taken in consideration, the value of γ^Z is always far from reaching its extreme limit, -1. This avoids the identification problem that would otherwise arise when trying to calibrate ρ. As noted when commenting the results presented in Fig. 3, under such an extreme γ^Z scenario, there would be no sensitivity to ρ, and, therefore, it would be not possible to calibrate it.

The aim of this section is to interpret the calibrated parameters in terms of market data. To do so, we will be relying on the theoretical results from the previous section.

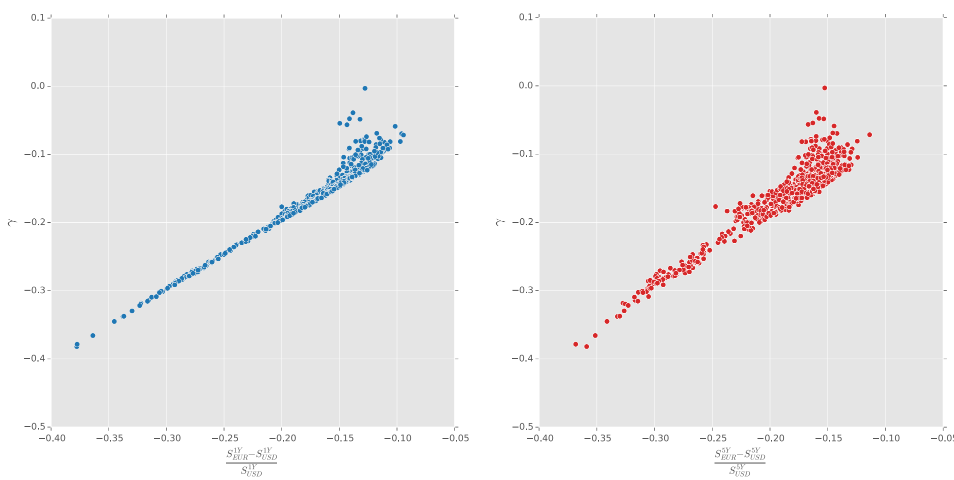
Interpretation of the devaluation factor γ^Z For the devaluation rate, γ^Z, we exploited the results from Proposition 3, and we used the relative basis spreads as an approximation:

$$\gamma^Z \approx \frac{S_{\mathrm{EUR}} - S_{\mathrm{USD}}}{S_{\mathrm{USD}}}. \tag{3.3}$$

This relation is similar to the one proved in Proposition 1 for hazard-rates. As noted in Remark 2.6, the result from that proposition could be transposed in terms of CDS par-spreads — and therefore be written as Eq. (3.3) — in cases where the stream of the premium leg's cash-flows could be approximated as continuously compounded stream of payments and in a setting where the hazard rate was modeled as a deterministic function of time and where the CDS par-spread term structure was flat.

We expect such conditions to be more easily satisfied by short-term CDS par-spread. Due to liquidity reasons, however, we had to use CDS par-spreads with 5 years and 10 years tenor for calibration, and these maturity values can be too large. Therefore, we used model-implied par-spreads for this test; in this way we have been able to use also short maturities, like one year, that are usually not very liquid in the market.

The comparison between γ^Z and its market-data approximation is showed in Fig. 10. In line with our expectations, the left-hand chart, where 1Y-spreads have been used to build the relative basis spread, shows a remarkably good agreement between the two variables. The same agreement does not hold for the right-hand chart, where 5Y-spreads were instead used.



(a) (b)

Fig. 10. Scatter plot comparing the calibrated values of γ^Z in ordinates with a relative basis spread in abscissas. (a) Relative basis spread for 1Y maturity CDSs and (b) relative basis spread for 5Y maturity CDSs.

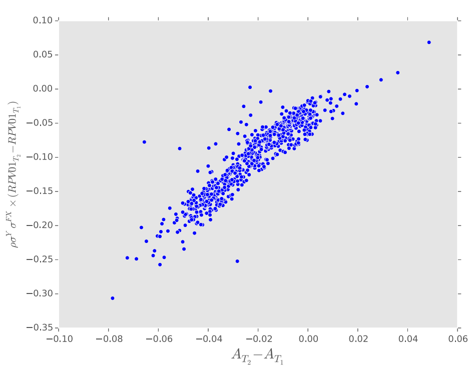
Interpretation of the instantaneous correlation parameter ρ In order to provide a similar assessment on the parameter ρ, we relied on some heuristic results derived in Elizalde *et al.* (2010). In that technical report, a simplified pricing formula based on cost of hedging arguments is presented for quanto CDS. Their result can be written in terms of the variable defined by our framework as

$$\frac{S_{\mathrm{EUR}}(T) - S_{\mathrm{USD}}(T)}{S_{\mathrm{USD}}(T)} \approx \gamma^Z + \sigma^Y \sigma^Z \rho \,\mathrm{RPV01}(T), \tag{3.4}$$

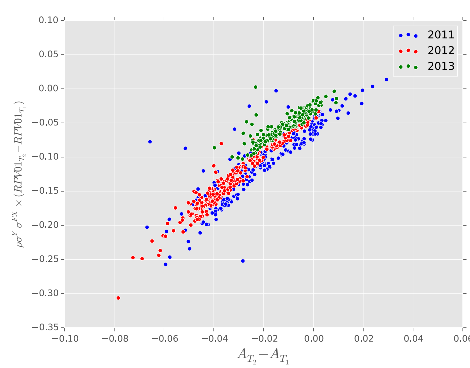
where $\mathrm{RPV01}(t)$ is the risky annuity of a CDS with tenor t years. We applied the formula above to two tenor points T_1 and T_2 obtaining two equations, one for each tenor. In order to test the values of ρ that we obtained in calibration, we worked out a single equation as a difference between the equations for the two tenor points:

$$\frac{S_{\mathrm{EUR}}(T_2) - S_{\mathrm{USD}}(T_2)}{S_{\mathrm{USD}}(T_2)} - \frac{S_{\mathrm{EUR}}(T_1) - S_{\mathrm{USD}}(T_1)}{S_{\mathrm{USD}}(T_1)}$$
$$\approx \sigma^Y \sigma^Z \rho(\mathrm{RPV01}(T_2) - \mathrm{RPV01}(T_1)). \tag{3.5}$$

Specifically, we chose $T_1 = 1$, $T_2 = 10$ and we used the model-implied values of $S_{\mathrm{EUR}}(T_1)$, $S_{\mathrm{EUR}}(T_2)$, $S_{\mathrm{USD}}(T_1)$, $S_{\mathrm{USD}}(T_2)$, $\mathrm{RPV01}(T_1)$ and $\mathrm{RPV01}(T_2)$. We further used the values σ^Z coming from the market while the values of σ^Y and ρ are the ones obtained in calibration. The results are presented in Fig. 11 and they show a scatterplot of the proposed relation between model parameters and market data. The data are reported for the whole time-range 2011–2013 in Fig. 11(a), while Fig. 11(b) contains the year-by-year plot. Due to the empirical nature of the Eq. (3.5), we didn't expect to find an exact relation between ρ and other model parameters and market data. Nonetheless, a clear pattern is exhibited and this



(a)



(b)

Fig. 11. Scatter plot comparing the product of the calibrated values of ρ, σ^Y,σ^Z, and the model-implied difference between risky annuities in ordinates with a difference of relative basis spread in abscissas (we used $A_T := \frac{S_{\mathrm{EUR}}(T) - S_{\mathrm{USD}}(T)}{S_{\mathrm{USD}}(T)}$). (a) Scatter plot for the time-range 2011–2013 and (b) split by year.

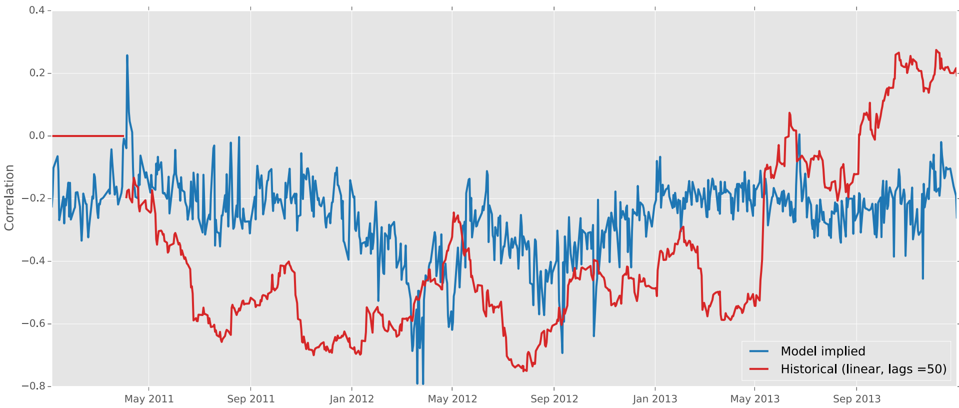


Fig. 12. Time series of the implied and historical correlation between EURUSD FX rate and Italy's CDS spread.

gives some confidence that such relation can be used to produce at least rough approximations for ρ by using observable market data.

Model-implied versus historical correlation In Fig. 12, we reported a comparison between the correlation parameter we obtained in calibration, ρ, and a historical estimator of correlation between daily log-returns of CDS par-spreads for one-year tenor contracts and daily log-returns of the FX spot rate. For assets where the market correctly prices gamma and cross-gamma risks, the basis between implied and realized covariance terms can be actually traded. This happens, for example, for implied and historical volatilities on equity indices.

In times where the values of implied and realized covariance terms diverge, the effect of such trading strategies is usually to bring them closer. We interpret the lack of evident convergence between implied and realized correlation in the chart in Fig. 12 as a signal of the lack of an efficient market for this correlation risk.

The fact that the implied correlation is generally smaller in absolute value than the realized one is consistent with our modeling choices and with the estimator used to calculate the realized correlation. The historical correlation has been estimated on a 50 days time-window using log-returns of the FX rate and this would neglect the impact of the jump term on its instantaneous volatility. Such an underestimation of the instantaneous volatility of the jump-diffusion process used in our modeling approach would result in an overestimation of the correlation with the credit component.

4. Conclusions

In this paper, we have investigated the impact on quanto CDS basis coming from different modeling assumptions about the FX rate/reference entity's credit worthiness dependence structure. In particular, we have discussed the effectiveness of the introduction of default-driven FX devaluation jumps to explain large basis spreads. We have considered CDSs on Republic of Italy denominated in EUR and USD — and the big change in quanto basis observed in 2011–2013 — as a testing case for our proposed modeling approach. We have found that the jump mechanism allows one to explain the size of the quoted basis, whereas pure shock correlation between FX rates and credit spread does not. Moreover, we have derived an approximated formula to calibrate the size of the FX devaluation jump from quoted CDS spreads and we have checked its performance using the real market data. Finally, we have investigated a symmetry property of FX-rate modeling, proving that it holds in the case of our proposed jump-to-default approach.

Appendix A. Proof of Proposition 3

Proof. Using the definition of quanto survival probability (see Eq. (2.16)), it is possible to write

$$1 - q_0^{\mathrm{f}}(T) = 1 - \frac{\mathbb{E}_0[Z_T \mathbb{1}_{\tau>T}]}{Z_0} \frac{B(T)}{B^{\mathrm{f}}(T)} \tag{A.1a}$$

$$= \frac{Z_0 - \dfrac{\mathbb{E}_0[Z_T \mathbb{1}_{\tau>T}]\, B(T)}{B^{\mathrm{f}}(T)}}{Z_0} \tag{A.1b}$$

$$= \frac{\mathbb{E}_0\left[\dfrac{Z_0 \mathbb{1}_{\tau\le T} + Z_0 \mathbb{1}_{\tau>T} - B(T)}{B^{\mathrm{f}}(T) Z_T \mathbb{1}_{\tau>T}}\right]}{Z_0} \tag{A.1c}$$

$$= \mathbb{E}_0[\mathbb{1}_{\tau\le T}] + \frac{\mathbb{E}_0\left[\mathbb{1}_{\tau>T}\left(\dfrac{Z_0 - B(T)}{B^{\mathrm{f}}(T) Z_T}\right)\right]}{Z_0}. \tag{A.1d}$$

An application of Bayes' formula allows one to re-write the second term on the right-hand side of the last equation as

$$\frac{\mathbb{E}_0\left[\mathbb{1}_{\tau>T}\left(\dfrac{Z_0 - B(T)}{B^{\mathrm{f}}(T) Z_T}\right)\right]}{Z_0} = \frac{\mathbb{E}_0\left[\left(\dfrac{Z_0 - B(T)}{B^{\mathrm{f}}(T) Z_T}\right) \middle| \mathbb{1}_{\tau>T}\right]}{Z_0} \mathbb{E}_0[\mathbb{1}_{\tau>T}]$$

$$= \left(1 - \frac{B(T)\mathbb{E}_0[Z_T \mid \mathbb{1}_{\tau>T}]}{B^{\mathrm{f}}(T) Z_0}\right) \mathbb{E}_0[\mathbb{1}_{\tau>T}]. \tag{A.2}$$

Let us now consider the term $\mathbb{E}_t[Z_T \mid \mathbb{1}_{\tau>T}]$. Under the assumption of deterministic hazard rate, this can be evaluated as

$$\mathbb{E}_t[Z_T \mid \mathbb{1}_{\tau>T}] = Z_0 e^{(r-r^{\mathrm{f}})(T-t) - \gamma^Z \int_t^T \lambda(s)\,\mathrm{d}s} \tag{A.3}$$

so that

$$\frac{B(T)\mathbb{E}_0[Z_T \mid \mathbb{1}_{\tau>T}]}{B^{\mathrm{f}}(T)Z_0} = e^{-\gamma^Z \int_0^T \lambda(s)\,\mathrm{d}s}. \tag{A.4}$$

Equation (A.1d) can then be re-written as

$$\frac{1-q_0^{\mathrm{f}}(T)}{1-q_0(T)} = e^{-\gamma^Z \int_0^T \lambda(s)\,\mathrm{d}s}. \tag{A.5}$$

When the integral $\int_t^T \lambda(s)\,\mathrm{d}s$ is small enough, the above can be approximated as γ^Z. This happens, in particular, in the limit of small maturities, from which the thesis follows. □

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